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RESEARCH ARTICLE

q-Spherical Fuzzy Rough Frank Aggregation Operators in AI Neural Networks: Applications in Military Transport Systems

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ABSTRACT This study introduces a novel neural network approach integrating q-spherical fuzzy rough Frank aggregation operators, aiming to enhance AI systems' resilience to uncertain and imprecise data in military transport systems. Three new operators are developed: q-spherical fuzzy rough Frank weighted averaging (q-SFRFWA), q-spherical fuzzy rough Frank ordered weighted averaging (q-SFRFOWA), and q-spherical fuzzy rough Frank hybrid weighted averaging (q-SFRFHWA), tailored to handle complex decision-making scenarios. We demonstrate their efficacy in multiple attribute decision-making using q-spherical fuzzy rough data, providing valuable insights and expanding the knowledge base in this domain. Through numerical examples, we illustrate the practical application of these operators, validating their effectiveness and relevance in real-world settings. Comparative and sensitivity analyses further corroborate the superiority of our proposed approach over existing methods. This research offers a robust decision-making framework equipped to manage intricate and unreliable data, promising significant advancements in military transport systems and beyond.

INDEX TERMS q-spherical fuzzy rough sets, Frank operators, neural network in artificial intelligence.

I. INTRODUCTION

In the realm of artificial intelligence (AI), the integration of neural networks with advanced mathematical concepts has indeed ushered in significant progress. However, despite these strides, there persist notable technical gaps, particularly in the realm of handling uncertainty and imprecision within decision-making scenarios. These gaps are particularly evident in domains such as military transport systems, where decisions frequently hinge on uncertain and ambiguous conditions, thereby posing substantial challenges to the efficacy of AI systems. Addressing these technical gaps necessitates a nuanced comprehension of the inherent challenges associated with managing ambiguous data, as well as an acknowledgment of the limitations of current approaches in adequately addressing uncertainty. Traditional neural network models, for instance, often encounter difficulties in effectively managing ambiguous or imprecise data, thereby resulting in suboptimal decision outcomes. Furthermore, existing AI methodologies lack the requisite sophistication to effectively tackle uncertainty within complex decision-making environments—a crucial aspect for ensuring reliable and robust decision-making in practical applications. By acknowledging and delineating these technical gaps, our study seeks to underscore the pressing need for novel approaches that can effectively address uncertainty and imprecision within decision-making scenarios. Through a comprehensive exploration of these challenges, we aim to

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lay the groundwork for the development of advanced AI systems capable of navigating the complexities of uncertain and ambiguous data, particularly within domains as critical as military transport systems. The current study endeavors to bridge these technical gaps by introducing q-spherical fuzzy rough Frank aggregation operators into neural network frameworks. This innovative approach seeks to augment AI systems' capabilities by leveraging neural networks' learning prowess alongside the advanced uncertainty handling techniques provided by q-spherical fuzzy rough Frank aggregation operators. By integrating these operators into neural network architectures, we aim to enhance the resilience and effectiveness of AI systems in navigating complex decision-making scenarios, particularly in domains characterized by uncertain and ambiguous data, such as military transport systems. The gap arises due to the limitations of traditional AI methodologies, including neural networks, in adequately addressing uncertainty and ambiguity. While neural networks demonstrate remarkable learning capabilities, they often struggle when confronted with uncertain or imprecise data, resulting in unreliable decision outcomes. Additionally, current AI techniques lack sophisticated mechanisms for comprehensively handling uncertainty, leaving decision-making processes vulnerable to errors and inaccuracies. Therefore, the research question emerges: How can AI systems be enhanced to better handle uncertainty and ambiguity in decision-making scenarios, particularly in domains like military transport systems? To address this question and bridge the identified gap, this study proposes the integration of q-spherical fuzzy rough Frank aggregation operators into neural network frameworks. These operators offer advanced techniques for handling uncertainty and ambiguity in data, providing a solid foundation for robust decision-making in complex scenarios. By leveraging the learning capabilities of neural networks alongside the specialized uncertainty handling techniques offered by q-spherical fuzzy rough Frank aggregation operators, this study aims to enhance AI systems' resilience and effectiveness in navigating the challenges posed by uncertain and ambiguous data, ultimately improving decision-making outcomes in critical domains like military transport systems. The motivation behind this study is rooted in the pressing need to address these technical gaps and empower AI systems with the ability to make informed, reliable, and robust decisions in real-world settings. The application of our proposed approach in military transport systems underscores the urgency of developing advanced decision-making frameworks capable of coping with uncertain and ambiguous data, thereby improving operational efficiency and effectiveness. Identifying and addressing these technical gaps and proposing a novel solution, our study aims to significantly advance the capabilities of AI systems and enhance their practical applicability in critical domains like military transport systems. We appreciate the reviewer's feedback and will ensure that the Introduction section adequately discusses the technical gaps associated with the problem at hand.

A. DECISION-MAKING PROCESS IN TRANFORT SYSTEM OF MILITARY ORGANIZATION

The American Army is part of the organizational structure that enables administration. Its organs, notably the traffic service organs, play an important role in decision-making. The Army's decision priority levels vary from everyday operations to strategic. Regardless of the decision level, the decision-making process remains relevant. Traffic support organs are sometimes compelled to select between accepting or rejecting a single action. However, traffic support organizations regularly face situations in which they must decide which of multiple proposed remedies is best and should be implemented. The ranking procedure comprises analyzing the various activities and selecting based on the best-demonstrated results of each activity. Such findings underline the need to consistently and methodically approach the decision-making process, regardless of the nature of the challenges at hand. This is because any wrong decision might jeopardize the American army's fighting readiness. Most of the decision-making inside a military organization occurs in the lack of relevant knowledge, in situations of more or lesser ambiguity about future environmental actions, the consequences of alternative decisions, and so on. Decision-making, as a leadership strategy, is distinct and more precise than other methods. It links the conclusion (decision) as the culmination of the conceptual process to action as the start of execution. Because of its importance, decision-making is classified into two categories: leadership preparation and action preparation. This implies that decision-making connects two dimensions of human activity: intellectual endeavor and material realization, theory and implementation. This suggests that decision-making plays a significant role in the leadership process [1], [2].

B. LITERATURE REVIEW

The decision-making process may not always follow this exact sequence. During the process, the influence of components is not restricted to certain phases; rather, it intertwines, repeats, and complements. Furthermore, the intensity varies according to the circumstance. The decision-making process in a military organization involves certain elements [3], including:

- Objectives
- Criteria
- Problem formulation
- Alternatives
- Modelling
- Implementation of decisions

Decisions are taken to accomplish specified goals. Determining the system's objectives is a difficult undertaking that often necessitates preliminary investigations of a variety of data. In the context of military organizations, objectives relate to the tasks that the observed military system, or military unit, must do. An aim might be, for example, the ability to do a given amount of damage to the opponent. A military organization, like any other, must work efficiently during decision-making. A criterion is a quantitative measure of success in achieving a certain objective. It might be claimed that every problem has optimal criteria. Because of the unpredictable nature of military systems, selecting the suitable criterion for varied decision-making conditions is a difficult challenge. Common criteria in military systems include task execution time, expected loss ratio, likelihood of target achievement, mathematical task expectation, and so on. Models are an important part of the decision-making process because they connect the objectives, alternatives, consequences, and criteria of a specific choice issue.

The implementation of the decision unmistakably shows the shortcomings of the choices made. However, whether the option is excellent or bad, inadequate execution will fail to achieve the expected results. This is especially true for military decisions concerning the structure and execution of combat operations. Artificial intelligence (AI) is a scientific field that studies computing systems for sensing, reasoning, and decision-making. Expert systems in artificial intelligence are a chain of knowledge connected by rules. During reasoning, search occurs in all directions, branching across the knowledge base's structure in a tree-like manner. As the depth of the search increases, so does the breadth of the "tree." Artificial intelligence is divided into various divisions and subcategories, the most noteworthy of which are fuzzy logic and artificial neural networks. Fuzzy logic simulates uncertainty and imprecision in systems, whereas artificial neural networks replicate the functioning of the human brain and are used for machine learning and pattern recognition. These methodologies are critical components of artificial intelligence, allowing computers to examine complex data and make decisions in ways similar to the human mind. The origins of fuzzy logic [4] may be traced back to the standard crisp set theory. In addition, Zadeh introduced the notion of a fuzzy set, which focuses solely on the grades that received a positive evaluation. Atanassov [5] introduced the intuitionistic fuzzy set (IFS) to expand upon the concept of fuzzy sets. The intuitionistic fuzzy set (IFS) incorporates both the positive and negative grades with the stipulation that their cumulative value should not exceed 1. Pythagorean fuzzy sets (PFS) were proposed by Yager et al. [6] in 2013. The authors proposed Pythagorean fuzzy sets as an extension of classical fuzzy sets to cope with ambiguity and vagueness in decision-making processes [1]. The Pythagorean fuzzy set architecture incorporates both membership and non-membership degrees, resulting in a more comprehensive representation of uncertainty than traditional fuzzy sets. In 2014, Coung and Kreinovich expanded upon the concepts of fuzzy sets and IFS introduced a ground-breaking notion known as the picture fuzzy set (PFS) [7]. This innovation provided a fresh perspective within this field of study. Within the framework of PFS, the author delved into the categorization of grades into positive, neutral, and negative classifications. Gündoğdu and Kahraman [8] propose a spherical fuzzy set (SFS) as a possible way to address this challenge. Scholarly interest in the subject of SFS has grown in recent years. Kahraman et al. and his research team [9] suggested the innovative conception of (q-SFS) in their determinations to concentrate the descendants of uncertainty. This innovative perception has provided evidence to be favorable in accompanying students in making well-informed varieties. The concept of rough sets (RS) was first introduced by Pawlak [10], [11] as a means of dealing with uncertainty. When examined from a mathematical perspective, this configuration demonstrates attributes that could be construed as vagueness and indeterminacy. Rough set theory (RST) is a modification of the traditional set theory, that uses the notion of connection to elucidate the operations of information systems. Researchers have acknowledged that the applicability of the equivalence relation in Pawlak's relational semantic theory is subject to notable constraints in a range of real-world situations, a point emphasized by multiple scholars. It is well acknowledged to initiate the concept of a "q-spherical fuzzy set." Every single element in the q-SFS framework is classified as either positive, neutral, or negative. The notions of q-SFRS were first presented by Azim et al. [12] in their research paper published in 2023. This fuzzy set combines the advantages inherent in both the RS and the q-SFS. This research introduces a practical approach to decision-making within the framework of q-spherical fuzzy rough sets, thereby expanding the existing knowledge in this field. Within q-SFRS, three distinct parameters involve lower and upper approximations. In the context of Industry 4.0, Azim and their team [13] proposed a project prioritization method using the q-SFR analytic hierarchy process in 2023. Similarly, Ali et al. [14] introduced the concept of averaging aggregation operators within the framework of q-ROPFStS in 2023, exploring their applications in multiple attribute decision-making (MADM). The work by Srinivasu et al. [27] proposes a novel approach for cluster head identification and data encryption in wireless sensor networks, leveraging probabilistic buckshot-driven methods. Ahmed et al. [28] present a software framework aimed at enhancing security measures for sensor data within the context of Ambient Assisted Technology. Krishna et al. [29] introduce a secure software framework tailored for mobile healthcare applications in the Internet of Medical Things (IoMT) domain. Rashid and Bhat [30] conduct a systematic review exploring the transition from topological methods to deep learning approaches for identifying influencers in online social networks. Srinivasu et al. [27] presented a method for probabilistic buckshot-driven cluster head identification and accumulative data encryption in wireless sensor networks (WSN). Kumar et al. [31] propose a method for secure data aggregation in wireless sensor networks through the use of homomorphic encryption. Westhoff, et al. [32] present a concealed data aggregation technique for reverse multicast traffic in sensor networks, addressing encryption, key distribution, and routing adaptation.

Vinodha and Anita [33] provide a comprehensive review of secure data aggregation techniques for wireless sensor networks, summarizing various approaches and their effectiveness. Ullah et al. [34] propose a scheme for secure critical data reclamation in isolated clusters within IoT-enabled wireless sensor networks, focusing on enhancing data security. The primary inquiry guiding this research is centered on enhancing the capabilities of AI systems to effectively navigate decision-making scenarios characterized by uncertainty and ambiguity, with a specific focus on domains such as military transport systems. This study seeks to address the following research question: What novel approaches can be devised to bolster AI systems' capacity to handle uncertain and ambiguous data within decision-making contexts, particularly in critical domains like military transport systems?

The motivation behind this article lies in the recognition of q-SFRS offering greater flexibility compared to PFS and SFS in studying decision-making (DM) problems. The article addresses the complexity of MADM problems influenced by imprecise factors within the q-SFRS environment. It highlights the limitations of existing operators and proposes a beyond-state-of-the-art method to overcome these limitations, providing excellent findings for various information categories represented by q-SFRS data. Frank aggregation operators are a versatile family of operators that are widely used in information fusion and decision-making, particularly in fuzzy collections. This essay explores their fundamental concepts, properties, and applications. In MADM and other disciplines requiring the aggregation of diverse information, Frank aggregation operators offer a great answer since they can used to provide different emphasis to different inputs based on their significance or expert opinion. These operators account for intrinsic ambiguity and vagueness and consider using fuzzy set representations or alternative uncertainty models. The integration of q-SFRSs through the Frank aggregation operators presents an intriguing potential for analysis in the dynamic field of decision-making. This combination aims to enhance the flexibility and precision of decision-making across various domains. The utilization of q-spherical fuzzy rough CODAS to address uncertainty and hesitancy in real-world decision situations has contributed to its popularity. The flexible background offered by q-SFRS theory, capable of managing and interpreting vague information, facilitates a more representative categorization of complex decision environments. Frank aggregation operators (FAOs) have emerged as a valuable tool for multi-attribute group decision-making (MAGDM) and information fusion, particularly in cases of uncertainty and imprecision. This is demonstrated by their growing application in a variety of industries, as indicated by: Wang et al. [15] proposed an improved MAGDM approach that uses FAOs to handle reluctant fuzzy data, resulting in more robust and adaptable decision-making. Du et al. [16] improved FAOs to accommodate complex q-rung ortho pair fuzzy information, making them more effective in information fusion tasks with incomplete or ambiguous data. Ullah et al. [17] developed a multi-attribute decision-making process based on T-spherical fuzzy frank prioritized aggregation operators, proving its utility in complex decision-making scenarios. Yahya et al. [18] employed FAOs in a probabilistic hesitant fuzzy multiple attribute decision-making approach. Liu et al. [19] employed FAOs in conjunction with neutrosophic sets to improve information in the multi-attribute decision-making process. These experiments demonstrate how adaptable and effective FAOs are in MAGDM and information fusion, particularly when faced with uncertainty and imprecision. Their ability to handle many data types, apply weights, and manage compensatory effects makes them a viable option for a wide range of decision-making and information-processing tasks.

Tang et al. [20] employed a multiple-attribute decision-making approach based on dual hesitant fuzzy Frank aggregation operators. By leveraging the synergies between q-SFRSs and FAOs, this research endeavors to equip decision-makers with a tool capable of navigating the intricacies and nuances in complex decision-making scenarios. The ultimate objective is to foster a comprehensive and adaptable decision validation framework for the advancement of decision knowledge.

1. Using q-spherical fuzzy rough Frank aggregation operators to demonstrate how these sets may deal with ambiguity and uncertainty in actual decision-making settings.
2. Describing the adaptable framework provided by q-spherical fuzzy rough set theory for dealing with and modeling misinformation. Emphasizing the need to provide a more genuine picture of the decision's complicated environment.
3. Recognizing the weaknesses of standard decision models, particularly in the context of imprecise and random data, and providing a rationale for exploring novel techniques, such as merging Frank aggregation operators with q-spherical fuzzy rough sets. The subsequent sections of the study are structured as follows:

Section II offers a comprehensive overview of various concepts, including FS, IFS PFS, SFS, q-SFS, RS, and q-SFRS, providing a foundational understanding for the subsequent sections. In section III, we delve into the operational laws governing the q-SFR framework, focusing on Frank aggregation operators and their results based on section II. Section IV presents the applications and the proposed algorithm for solving the MCDM problem. Section V shows the use of the algorithm with the help of numerical examples. Section V offers the comparative analysis, sensitivity analysis, advantages, and limitations for the proposed operators. Section VI offers concluding remarks and future direction. To provide a structured and cohesive exploration of these concepts, our manuscript follows a systematic organization, as depicted in Figure 1.

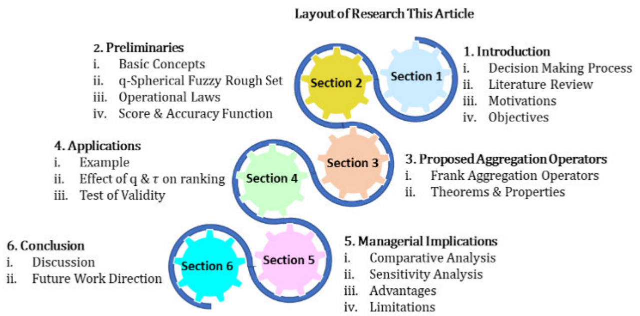

FIGURE 1. Structure of the research article.

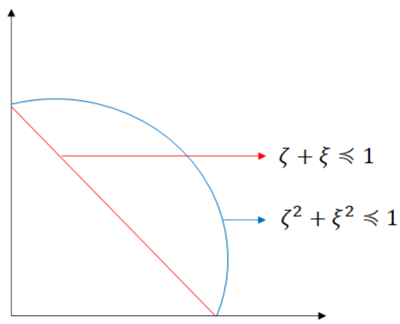

FIGURE 3. A comparison of the differences between Pythagorean and intuitionistic fuzzy spaces.

II. PRELIMINARIES

This section will look at various mathematical ideas, beginning with an in-depth review of FS, IFS, PFS, SPS, q-SFS, and RS.

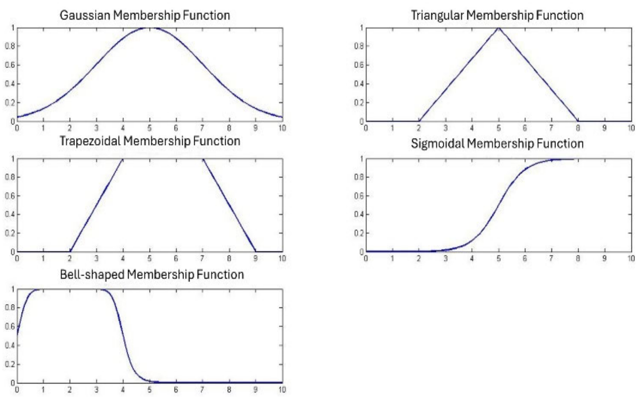

FIGURE 2. Some graphical representations of fuzzy spaces.

Definition 1: In 1965, Zadeh [4] proposed the idea of a fuzzy set as an extension of the conventional crisp set. The formal definition of a fuzzy set can be represented mathematically as follows:

$$\mathcal{A} = \{\langle \mathfrak{x}, \zeta_{\mathcal{A}}(\mathfrak{x})\rangle : \mathfrak{x} \in \mathcal{X}\} \tag{1}$$

where $0 \leq \zeta_{\mathcal{A}}(\mathfrak{x}) \leq 1$.

Definition 2: In 1986, Atanassov [5] proposed the intuitionistic fuzzy set (IFS) as an extension of the fuzzy set. The formal mathematical representation of an IFS is as follows:

$$\mathcal{A} = \{\langle \mathfrak{x}, \zeta_{\mathcal{A}}(\mathfrak{x}), \xi_{\mathcal{A}}(\mathfrak{x})\rangle : \mathfrak{x} \in \mathcal{X}\} \tag{2}$$

where $0 \leq \zeta_{\mathcal{A}}(\mathfrak{x}) + \xi_{\mathcal{A}}(\mathfrak{x}) \leq 1$.

Definition 3 ([6]): Let $\mathcal{X}$ be a non-empty finite set. A PyFS $\mathcal{A}$ over $\mathfrak{x} \in \mathcal{X}$ is defined as follows:

$$\mathcal{A} = \{\langle \mathfrak{x}, \zeta_{\mathcal{A}}(\mathfrak{x}), \xi_{\mathcal{A}}(\mathfrak{x})\rangle : \mathfrak{x} \in \mathcal{X}\} \tag{3}$$

where $\zeta_{\mathcal{A}}(\mathfrak{x})$ and $\xi_{\mathcal{A}}(\mathfrak{x})$ represent the MD and NMD of $\mathcal{A}$ respectively such that $\zeta_{\mathcal{A}}(\mathfrak{x}), \xi_{\mathcal{A}}(\mathfrak{x}) \in [0, 1]$ and where $0 \leq (\zeta_{\mathcal{A}}(\mathfrak{x}))^2 + (\xi_{\mathcal{A}})^2 \leq 1$.

Definition 4: Building on the fundamental principles of FSs and IFSs, Cuong and his team [7] introduced the idea of a picture fuzzy set in 2014. Its definition can be expressed mathematically as follows:

$$\mathcal{A} = \{\langle \mathfrak{x}, \zeta_{\mathcal{A}}(\mathfrak{x}), \eta_{\mathcal{A}}(\mathfrak{x}), \xi_{\mathcal{A}}(\mathfrak{x})\rangle : \mathfrak{x} \in \mathcal{X}\} \tag{4}$$

where $0 \leq \zeta_{\mathcal{A}}(\mathfrak{x}) + \eta_{\mathcal{A}}(\mathfrak{x}) + \xi_{\mathcal{A}}(\mathfrak{x}) \leq 1$.

The following symbols represent the representation of the membership functions for a fuzzy set in this situation, which includes positive, neutral, and negative aspects: $\zeta_{\mathcal{A}}(\mathfrak{x})(\mathfrak{x}) : X \to [0, 1]$, $\eta_{\mathcal{A}}(\mathfrak{x})$: $\mathcal{X} \to [0, 1]$ and $\xi_{\mathcal{A}}(\mathfrak{x})$: $\mathcal{X} \to [0, 1]$ respectively.

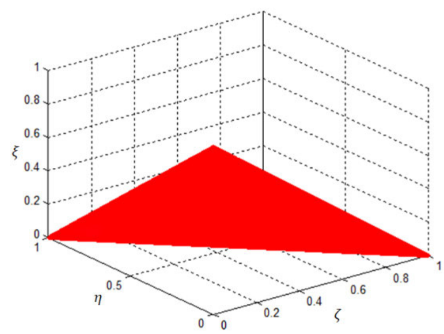

FIGURE 4. Picture membership grade space.

Definition 5: Gündoğdu et al. [8] introduced the idea of a spherical fuzzy set in 2019, further advancing the picture fuzzy set framework. The concept can be expressed in the following way from a mathematical standpoint:

$$\mathcal{A} = \{\langle \mathfrak{x}, \zeta_{\mathcal{A}}(\mathfrak{x})(\mathfrak{x}), \eta_{\mathcal{A}}(x), \xi_{\mathcal{A}}(\mathfrak{x})\rangle : \mathfrak{x} \in \mathcal{X}\} \tag{5}$$

where $0 \leq (\zeta_{\mathcal{A}}(\mathfrak{x}))^2 + (\eta_{\mathcal{A}}(\mathfrak{x}))^2 + (\xi_{\mathcal{A}}(\mathfrak{x}))^2 \leq 1$.

Where the positive, neutral, and negative membership function for a fuzzy set is represented by $\zeta_{\mathcal{A}}(\mathfrak{x})$: $\mathcal{X} \to [0, 1]$, $\eta_{\mathcal{A}}(\mathfrak{x})$: $\mathcal{X} \to [0, 1]$ and $\xi_{\mathcal{A}}(\mathfrak{x})$: $\mathcal{X} \to [0, 1]$ respectively.

Definition 6: The idea of a q-SFS was introduced by Kahraman et al. [9] in the year 2020, as an extension of the existing notion of a spherical fuzzy set. Mathematically, the concept may be formally defined in the following manner.

$$\mathcal{A} = \{\langle \mathfrak{x}, \zeta_{\mathcal{A}}(\mathfrak{x})(\mathfrak{x}), \eta_{\mathcal{A}}(\mathfrak{x}), \xi_{\mathcal{A}}(\mathfrak{x})\rangle : x \in \mathcal{X}\} \tag{6}$$

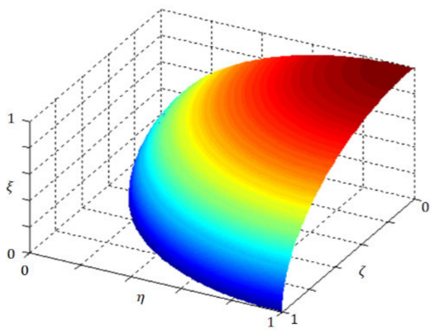

FIGURE 5. The condition $0 \leq (\zeta_{\mathcal{A}}(\mathfrak{x}))^2 + (\eta_{\mathcal{A}}(\mathfrak{x}))^2 + (\xi_{\mathcal{A}}(\mathfrak{x}))^2 \leq 1$ describes a spherical fuzzy set in three-dimensional space.

Such that $0 \leq (\zeta_{\mathcal{A}}(x))^q + (\eta_{\mathcal{A}}(x))^q + (\xi_{\mathcal{A}}(x))^q \leq 1$ for all $q \geq 1$.

Where $\zeta_{\mathcal{A}}: \mathcal{X} \to [0,1]$, $\eta_{\mathcal{A}}: \mathcal{X} \to [0,1]$ and $\xi_{\mathcal{A}}: \mathcal{X} \to [0,1]$ correspond to the positive, neutral, and negative membership functions, respectively.

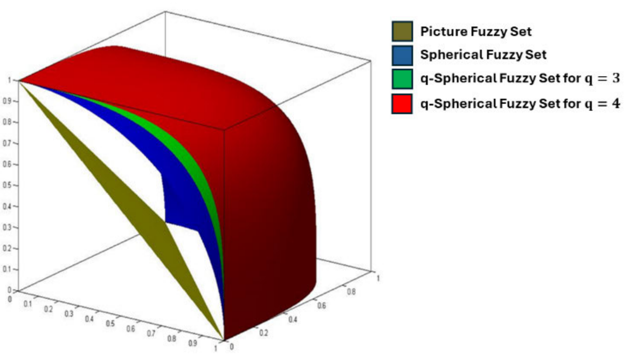

FIGURE 6. Graphical representation of the difference between picture fuzzy set, spherical fuzzy set and q-spherical fuzzy set.

Definition 7: Pawlak [10] introduced the notion of RS in back 1982. The definition of rough set is as follows: The triplet $(\mathcal{G}_1, \mathcal{G}_2, \mathfrak{R})$ is referred to as an approximation space when considering an arbitrary binary relation $\mathfrak{R}$ on $\mathcal{G}_1 \times \mathcal{G}_2$. The $\underline{\mathfrak{R}}(\mathcal{A})$ and $\overline{\mathfrak{R}}(\mathcal{A})$ are defined for sets $\mathcal{X} \subseteq \mathcal{G}_1$ and $\mathcal{A} \subseteq \mathcal{G}_2$.

$$\begin{pmatrix} \underline{\mathfrak{R}}(\mathcal{A}) = \{\mathfrak{x} \in \mathcal{G}_1 : [\mathfrak{x}]_{\mathcal{A}} \subseteq \mathcal{X}\} \\ \overline{\mathfrak{R}}(\mathcal{A}) = \left\{\mathfrak{x} \in \mathcal{G}_1 : [\mathfrak{x}]_{\mathcal{A}} \bigcap \mathcal{X} \neq \phi\right\} \end{pmatrix} \tag{7}$$

where $[\mathfrak{x}]_{\mathcal{A}}$ represents the idea of indiscernibility.

The set $\left(\mathfrak{R}(\mathcal{A}), \overline{\mathfrak{R}}(\mathcal{A})\right)$ is sometimes referred to as a rough set.

Definition 8 ([12]): A q-spherical fuzzy relation $\mathfrak{R}$ in is a q-spherical fuzzy subset of $\mathcal{G}_1 \times \mathcal{G}_2$. and is given by

$$\mathfrak{R} = \left\{ \langle (\mathfrak{w}, \mathfrak{x}) : \zeta_R(\mathfrak{w}, \mathfrak{x}), \eta_{\mathfrak{R}}(\mathfrak{w}, \mathfrak{x}), \xi_{\mathfrak{R}}(r, s) \rangle : \left((\zeta_{\mathfrak{R}}(\mathfrak{w}, \mathfrak{x}))^q + (\eta_{\mathfrak{R}}(\mathfrak{w}, \mathfrak{x}))^q + (\xi_R(\mathfrak{w}, \mathfrak{x}))^q \right) \leq 1 : \forall \mathfrak{w} \in \mathcal{G}_1, \mathfrak{x} \in \mathcal{G}_2 \right\},$$

where $\zeta_R: \mathcal{G}_1 \times \mathcal{G}_2 \to [0,1]$, $\eta_R: \mathcal{G}_1 \times \mathcal{G}_2 \to [0,1]$ and $\xi_R: \mathcal{G}_1 \times \mathcal{G}_2 \to [0,1]$.

Definition 9: Azim et al. [12] introduced the concept of a q-spherical fuzzy rough set, which is defined as:

For a universal set $\mathcal{G}_1$ and $\mathcal{G}_2$ is a set of attributes. Let $\mathfrak{R}$ be a q-SF relation from $\mathcal{G}_1$to$\mathcal{G}_2$. Then the triplet $(\mathcal{G}_1, \mathcal{G}_2, \mathfrak{R})$ is called q-SF approximation space. Now for any element $\mathfrak{w} \in \mathcal{G}_1$, the lower and upper approximation space of $\mathfrak{w}$ w.r.t approximation space $(\mathcal{G}_1, \mathcal{G}_2, \mathfrak{R})$ are presented and given as:

$$\mathcal{A} = \left(\underline{\mathcal{A}}, \overline{\mathcal{A}}\right) = \left\{ \mathfrak{w}, \begin{pmatrix} \underline{\zeta}_{\mathcal{A}}(\mathfrak{w}), \underline{\zeta}_{\mathcal{A}}(\mathfrak{w}), \underline{\zeta}_{\mathcal{A}}(\mathfrak{w}), \\ \overline{\zeta}_{\mathcal{A}}(\mathfrak{w}), \overline{\zeta}_{\mathcal{A}}(\mathfrak{w}), \overline{\zeta}_{\mathcal{A}}(w) \end{pmatrix} : \mathfrak{w} \in \mathcal{G}_1 \right\} \tag{8}$$

where,

$$\begin{aligned} \underline{\zeta}_{\mathcal{A}}(\mathfrak{w}) &= \wedge_{x \in \mathcal{G}_2} \{\zeta_R(\mathfrak{w}, x) \wedge \zeta_A(x)\}, \\ \underline{\eta}_{\mathcal{A}}(\mathfrak{w}) &= \vee_{x \in \mathcal{G}_2} \{\eta_R(\mathfrak{w}, x) \vee \eta_A(x)\}, \\ \underline{\xi}_{\mathcal{A}}(\mathfrak{w}) &= \vee_{x \in \mathcal{G}_2} \{\xi_R(\mathfrak{w}, x) \vee \xi_A(x)\}, \\ \overline{\zeta}_{\mathcal{A}}(\mathfrak{w}) &= \vee_{x \in \mathcal{G}_2} \{\zeta_R(\mathfrak{w}, x) \vee \zeta_A(x)\}, \\ \overline{\eta}_{\mathcal{A}}(\mathfrak{w}) &= \wedge_{x \in U_2} \{\eta_R(\mathfrak{w}, x) \wedge \eta_A(x)\}, \\ \overline{\xi}_{\mathcal{A}}(\mathfrak{w}) &= \wedge_{x \in \mathcal{G}_2} \{\xi_R(\mathfrak{w}, x) \wedge \xi_A(x)\}, \end{aligned}$$

with the condition that $(0 \leq \underline{\zeta}^q_{\mathcal{A}}(\mathfrak{w}) + \underline{\eta}^q_{\mathcal{A}}(\mathfrak{w}) + \underline{\xi}^q_{\mathcal{A}}(\mathfrak{w}) \leq 1))$ and $\left(0 \leq \overline{\zeta}^q_{\mathcal{A}}(\mathfrak{w}) + \overline{\eta}^q_{\mathcal{A}}(\mathfrak{w}) + \overline{\xi}^q_{\mathcal{A}}(\mathfrak{w}) \leq 1\right)$.

The q-SFRS is defined as a pair of q-SFSs, where $\underline{\mathcal{A}}$ is distinct from $\overline{\mathcal{A}}$. To facilitate comprehension, we will denote the given concept as $\mathcal{A} = (\underline{\mathcal{A}}, \overline{\mathcal{A}})$, which is referred to as a q-spherical fuzzy rough number. The notation $\mathcal{A}_i$ represents the set that encompasses all q-SFR numbers.

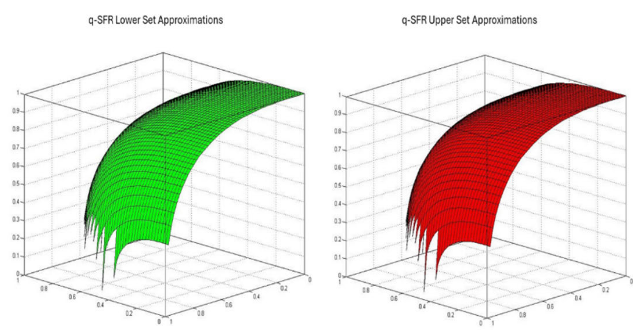

FIGURE 7. Graphical representation of q-spherical fuzzy rough set-in three-dimensional space.

Definition 10 ([12]): Let $\mathcal{A}_1 = (\underline{\zeta}_1, \underline{\eta}_1, \underline{\xi}_1, \overline{\zeta}_1, \overline{\eta}_1, \overline{\xi}_1)$, $\mathcal{A}_2 = (\underline{\zeta}_2, \underline{\eta}_2, \underline{\xi}_2, \overline{\zeta}_2, \overline{\eta}_2, \overline{\xi}_2)$ and $\mathcal{A} = (\underline{\zeta}, \underline{\eta}, \underline{\xi}, \overline{\zeta}, \overline{\eta}, \overline{\xi})$ be any three q-SFRNs, and $\omega > 0$, then,

1. As shown in the equation at the bottom of the next page.
2. As shown in the equation at the bottom of the next page.
3.

$$\mathcal{A}^{\omega} = \left\langle \frac{\underline{\zeta}^{\omega}, \sqrt[q]{1-(1-\underline{\eta}^q)^{\omega}}, \sqrt[q]{(1-\underline{\eta}^q)^{\omega} - \left(1-\underline{\eta}^q - \underline{\xi}^q\right)^{\omega}},}{\overline{\zeta}^{\omega}, \sqrt[q]{1-(1-\overline{\eta}^q)^{\omega}}, \sqrt[q]{(1-\overline{\eta}^q)^{\omega} - \left(1-\overline{\eta}^q - \overline{\xi}^q\right)^{\omega}}} \right\rangle,$$

4.

$$\omega\mathcal{A}=\left\langle\frac{\sqrt[q]{1-(1-\underline{\zeta}^{q})^{\omega}},\underline{\eta}^{\omega},\sqrt[q]{\left(1-\underline{\zeta}^{q}-\underline{\xi}^{q}\right)^{\omega}},}{\sqrt[q]{1-(1-\overline{\zeta}^{q})^{\omega}},\overline{\eta}^{\omega},\sqrt[q]{\left(1-\overline{\zeta}^{q}-\overline{\xi}^{q}\right)^{\omega}}}\right\rangle,$$

5. $\mathcal{A}_1=\mathcal{A}_2$ if and only if $\underline{\zeta}_1=\underline{\zeta}_2, \underline{\eta}_1=\underline{\eta}_2 \underline{\xi}_1=\underline{\xi}_2$ and $\overline{\zeta}_1=\overline{\zeta}_2, \overline{\eta}_1=\overline{\eta}_2 \overline{\xi}_1=\overline{\xi}_2$.

Definition 11 ([12]): Let $\mathcal{A}=(\underline{\zeta},\underline{\eta},\underline{\xi},\overline{\zeta},\overline{\eta},\overline{\xi})$ be a q-SFRN. Then the score value which is denoted as A_Q can be determined by the following function.

$$Sco(\mathcal{A})=\frac{2+\left(\underline{\zeta}\right)^{q}+(\overline{\zeta})^{q}-\left(\underline{\eta}\right)^{q}-(\overline{\eta})^{q}-\left(\underline{\xi}\right)^{q}-(\overline{\xi})^{q}}{3} \tag{9}$$

where,

$$0\leq Sco\,(\mathcal{A})\leq 1.$$

The notation $Sco(\mathcal{A})$ introduced in equation (9) represents the score value associated with a q-SFRN $\mathcal{A}=(\underline{\zeta},\underline{\eta},\underline{\xi},\overline{\zeta},\overline{\eta},\overline{\xi})$. In this context, "Sco" stands for "score," and it serves as a metric to quantify the characteristics or properties of the q-SFRN. The function $Sco(\mathcal{A})$ is derived from a specific formulation, as described in equation (9), where the components of the q-SFRN are raised to the power of q, and certain arithmetic operations are performed. The resulting score value $Sco\,(\mathcal{A})$ is bounded between 0 and 1, reflecting the normalized assessment of the q-SFRN's attributes. In essence, $Sco(\mathcal{A})$ provides a quantitative measure of the q-SFRN's features or attributes, enabling comparisons and evaluations within the framework of q-spherical fuzzy rough sets. We appreciate your inquiry regarding the notation, and we hope this clarification adequately addresses your question.

Definition 12 ([12]): Let $\mathcal{A}=(\underline{\zeta},\underline{\eta},\underline{\xi},\overline{\zeta},\overline{\eta},\overline{\xi})$ be a q-SFRN. The accuracy of $\mathcal{A}$ is calculated by using the formula mentioned in Equation No. 10.

$$Acc\,(\mathcal{A})=\frac{\left(\underline{\zeta}\right)^{q}+\left(\underline{\zeta}\right)^{q}-\left(\underline{\xi}\right)^{q}-(\overline{\xi})^{q}}{2} \tag{10}$$

where $-1\leq Acc\,(\mathcal{A})\leq 1$.

Definition 13 ([12]): Let $\mathcal{A}_1=(\underline{\zeta}_1,\underline{\eta}_1,\underline{\xi}_1,\overline{\zeta}_1,\overline{\eta}_1,\overline{\xi}_1)$ and $\mathcal{A}_2=(\underline{\zeta}_2,\underline{\eta}_2,\underline{\xi}_2,\overline{\zeta}_2,\overline{\eta}_2,\overline{\xi}_2)$ are two q-SFRNs, then

1) If $Sco(\mathcal{A}_1)<Sco(\mathcal{A}_2)$ then $\mathcal{A}_1<\mathcal{A}_2$,
2) If $Sco(\mathcal{A}_1)>Sco(\mathcal{A}_2)$ then $\mathcal{A}_1>\mathcal{A}_2$,
3) If $Sco\,(\mathcal{A}_1)=Sco(\mathcal{A}_2)$ then
   - If $Acc(\mathcal{A}_1)<Acc(\mathcal{A}_2)$ then $\mathcal{A}_1<\mathcal{A}_2$,
   - If $Acc(\mathcal{A}_1)>Acc(\mathcal{A}_2)$ then $\mathcal{A}_1>\mathcal{A}_2$,
   - If $Acc(\mathcal{A}_1)=Acc(\mathcal{A}_2)$ then $\mathcal{A}_1=\mathcal{A}_2$.

Definition 14 ([12]): Let $\mathcal{A}_1=(\underline{\zeta}_1,\underline{\eta}_1,\underline{\xi}_1,\overline{\zeta}_1,\overline{\eta}_1,\overline{\xi}_1)$ and $\mathcal{A}_2=(\underline{\zeta}_2,\underline{\eta}_2,\underline{\xi}_2,\overline{\zeta}_2,\overline{\eta}_2,\overline{\xi}_2)$ and $\mathcal{A}=(\underline{\zeta},\underline{\eta},\underline{\xi},\overline{\zeta},\overline{\eta},\overline{\xi})$ be any three q-SFRNs, and ω, ω_1 and ω_2 are any positive integers then the following properties are held.

1. $\mathcal{A}_1\oplus\mathcal{A}_2=\mathcal{A}_2\oplus\mathcal{A}_1$,
2. $\mathcal{A}_1\otimes\mathcal{A}_2=\mathcal{A}_2\otimes\mathcal{A}_1$
3. $\omega\left(\mathcal{A}_1\oplus\mathcal{A}_2\right)=\omega\mathcal{A}_1\oplus\omega\mathcal{A}_2$,
4. $\omega_1\mathcal{A}\oplus\omega_2\mathcal{A}=\left(\omega_1+\omega_2\right)\mathcal{A}$,
5. $\left(\mathcal{A}_1\otimes\mathcal{A}_2\right)^{\omega}=\mathcal{A}_1^{\omega}\otimes\mathcal{A}_2^{\omega}$,
6. $\mathcal{A}^{\omega_1}\otimes\mathcal{A}^{\omega_2}=\mathcal{A}^{\omega_1+\omega_2}$.

Definition 15 ([25]): The Frank t-norm and t-conorm are mathematical operations defined for real numbers r and s, where r and s fall in the interval [0, 1]. The formulas for these operations are as follows:

Frank t-norm (Fr):

$$Fr\,(r,s)=log_{\tau}\left(1+\frac{(\tau^{r}-1)\,(\tau^{s}-1)}{\tau-1}\right) \tag{11}$$

Frank t-conorm (Fr'):

$$Fr'\,(r,s)=1-log_{T}\left(1+\frac{\left(\tau^{1-r}-1\right)\left(\tau^{1-s}-1\right)}{T-1}\right) \tag{12}$$

Here $(r,s)\,\epsilon\,[0,1]\times[0,1]$ and τ is any value except 1.

By applying limit theory, the following is derived by Wang et al. [26] in 2009.

As τ approaches 1, the Frank t-conorm tends to $r+s-rs$ and the Frank t-norm tends to rs.

As T approaches infinity, the Frank t-conorm tends to $\min\{r+s,1\}$ and Frank t-norm tends to $\max\{0,r+s-1\}$.

III. PROPOSED OPERATIONAL LAWS FOR $q-SFRNs$

In this section, we develop a set of operational laws using Equations (11) and (12) in the context of q-spherical fuzzy rough numbers. Using these established operational laws, we provide a diversified collection of Aggregation Operators (AOs) designed specifically for the integration of q-spherical

$$\mathcal{A}_1\oplus\mathcal{A}_2=\left\langle\frac{\sqrt[q]{\underline{\zeta}_1^{q}+\underline{\zeta}_2^{q}-\underline{\zeta}_1^{q}\underline{\zeta}_2^{q}},\underline{\eta}_1^{q}\underline{\eta}_2^{q},\sqrt[q]{\left(1-\underline{\zeta}_2^{q}\underline{\xi}_1^{q}+1-\underline{\zeta}_1^{q}\underline{\xi}_2^{q}\right)-\underline{\xi}_1^{q}\underline{\xi}_2^{q}},}{\sqrt[q]{\overline{\zeta}_1^{q}+\overline{\zeta}_2^{q}-\overline{\zeta}_1^{q}\overline{\zeta}_2^{q}},\overline{\eta}_1^{q}\overline{\eta}_2^{q},\sqrt[q]{\left(1-\overline{\zeta}_2^{q}\overline{\xi}_1^{q}+1-\overline{\zeta}_1^{q}\overline{\xi}_2^{q}\right)-\overline{\xi}_1^{q}\overline{\xi}_2^{q}}}\right\rangle,$$

$$\mathcal{A}_1\otimes\mathcal{A}_2=\left\langle\frac{\underline{\zeta}_1^{q}\underline{\zeta}_2^{q},\sqrt[q]{\underline{\eta}_1^{q}+\underline{\eta}_2^{q}-\underline{\eta}_1^{q}\underline{\eta}_2^{q}},\sqrt[q]{\left(1-\underline{\eta}_2^{q}\underline{\xi}_1^{q}+1-\underline{\eta}_1^{q}\underline{\xi}_2^{q}\right)-\underline{\xi}_1^{q}\underline{\xi}_2^{q}},}{\overline{\zeta}_1^{q}\overline{\zeta}_2^{q},\sqrt[q]{\overline{\eta}_1^{q}+\overline{\eta}_2^{q}-\overline{\eta}_1^{q}\overline{\eta}_2^{q}},\sqrt[q]{\left(1-\overline{\eta}_2^{q}\overline{\xi}_1^{q}+1-\overline{\eta}_1^{q}\overline{\xi}_2^{q}\right)-\overline{\xi}_1^{q}\overline{\xi}_2^{q}}}\right\rangle$$

fuzzy rough information. This technique greatly increases the flexibility and accuracy of aggregation procedures within the stated framework, resulting in higher decision-making efficacy in complex settings.

A. OPERATIONAL LAWS

Definition 16: Let $\mathcal{A}_1 = (\underline{\zeta}_1, \underline{\eta}_1, \underline{\xi}_1, \overline{\zeta}_1, \overline{\eta}_1, \overline{\xi}_1)$, $\mathcal{A}_2 = (\underline{\zeta}_2, \underline{\eta}_2, \underline{\xi}_2, \overline{\zeta}_2, \overline{\eta}_2, \overline{\xi}_2)$ and $\mathcal{A} = (\underline{\zeta}, \underline{\eta}, \underline{\xi}, \overline{\zeta}, \overline{\eta}, \overline{\xi})$ be any three q-SFRNs, where $\omega > 0$. let τ be any real number except 1. The essential Frank's operations for q-SFRNs are presented as follows:

$$\text{(i)}\ .\mathcal{A}_1 \oplus \mathcal{A}_2 = \left[\left\langle \begin{array}{l} \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\underline{\zeta}_1^q}-1\right)\left(\tau^{1-\underline{\zeta}_2^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\underline{\eta}_1^q}-1\right)\left(\tau^{\underline{\eta}_2^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\underline{\xi}_1^q}-1\right)\left(\tau^{\underline{\xi}_2^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\overline{\zeta}_1^q}-1\right)\left(\tau^{1-\overline{\zeta}_2^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\overline{\eta}_1^q}-1\right)\left(\tau^{\overline{\eta}_2^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\overline{\xi}_1^q}-1\right)\left(\tau^{\overline{\xi}_2^q}-1\right)}{\tau-1}\right)} \end{array}\right\rangle\right]$$

$$\text{(ii)}\ .\mathcal{A}_1 \otimes \mathcal{A}_2 = \left[\left\langle \begin{array}{l} \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\underline{\zeta}_1^q}-1\right)\left(\tau^{\underline{\zeta}_2^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\underline{\eta}_1^q}-1\right)\left(\tau^{1-\underline{\eta}_2^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{\underline{\xi}_1^q}-1\right)\left(\tau^{\underline{\xi}_2^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{1-\overline{\zeta}_1^q}-1\right)\left(\tau^{1-\overline{\zeta}_2^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\overline{\eta}_1^q}-1\right)\left(\tau^{1-\tau^{\overline{\eta}_2^q}}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\overline{\xi}_1^q}-1\right)\left(\tau^{1-\tau^{\overline{\xi}_2^q}}-1\right)}{\tau-1}\right)} \end{array}\right\rangle\right]$$

$$\text{(iii)}\ .\omega\mathcal{A} = \left[\left\langle \begin{array}{l} \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\underline{\zeta}^q}-1\right)^\omega}{(\tau-1)^{\omega-1}}\right)} \\ \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\underline{\eta}^q}-1\right)^\omega}{(\tau-1)^{\omega-1}}\right)}, , \\ \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\underline{\xi}^q}-1\right)^\omega}{(\tau-1)^{\omega-1}}\right)}, , \\ \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\overline{\zeta}^q}-1\right)^\omega}{(\tau-1)^{\omega-1}}\right)}, \\ \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\overline{\eta}^q}-1\right)^\omega}{(\tau-1)^{\omega-1}}\right)}, \\ \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\overline{\xi}^q}-1\right)^\omega}{(\tau-1)^{\omega-1}}\right)} \end{array}\right\rangle\right]$$

$$(iv)\ .\mathcal{A}^\omega = \left[\left\langle \begin{array}{l} \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\underline{\zeta}^q}-1\right)^\omega}{(\tau-1)^{\omega-1}}\right)}, \\ \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\underline{\eta}^q}-1\right)^\omega}{(\tau-1)^{\omega-1}}\right)}, \\ \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\underline{\xi}^q}-1\right)^\omega}{(\tau-1)^{\omega-1}}\right)}, \\ \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\overline{\zeta}^q}-1\right)^\omega}{(\tau-1)^{\omega-1}}\right)}, \\ \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\overline{\eta}^q}-1\right)^\omega}{(\tau-1)^{\omega-1}}\right)}, \\ \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\overline{\xi}^q}-1\right)^\omega}{(\tau-1)^{\omega-1}}\right)} \end{array}\right\rangle\right]$$

Theorem 1: Let ω, ω_1, ω_2 be any three positive real numbers . Then for three q-SFRNs $\mathcal{A}_1 = (\underline{\zeta}_1, \underline{\eta}_1, \underline{\xi}_1, \overline{\zeta}_1, \overline{\eta}_1, \overline{\xi}_1)$, $\mathcal{A}_2 = (\underline{\zeta}_2, \underline{\eta}_2, \underline{\xi}_2, \overline{\zeta}_2, \overline{\eta}_2, \overline{\xi}_2)$ and $\mathcal{A} = (\underline{\zeta}, \underline{\eta}, \underline{\xi}, \overline{\zeta}, \overline{\eta}, \overline{\xi})$, the FTN and FTCN are defined as follows:

I. $\mathcal{A}_1 \oplus \mathcal{A}_2 = \mathcal{A}_2 \oplus \mathcal{A}_1$
II. $\mathcal{A}_1 \otimes \mathcal{A}_2 = \mathcal{A}_2 \otimes \mathcal{A}_1$
III. $g\left(\mathcal{A}_1 \oplus \mathcal{A}_2\right) = \omega\mathcal{A}_1 \oplus \omega\mathcal{A}_2$
IV. $\mathcal{A}\left(\omega_1 + \omega_2\right) = \omega_1\mathcal{A} \oplus \omega_2\mathcal{A}$
V. $\left(\mathcal{A}_1 \otimes \mathcal{A}_2\right)^\omega = \mathcal{A}_1^\omega \otimes \mathcal{A}_2^\omega$
VI. $\mathcal{A}^{\omega_1} \otimes \mathcal{A}^{\mathfrak{g}_2} = \mathcal{A}^{\omega_1+\omega_2}$.

Proof: Using Definition 16, we get

$$\mathcal{A}_1 \oplus \mathcal{A}_2 = \left[\left\langle \begin{array}{l} \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\underline{\zeta}_1^q}-1\right)\left(\tau^{1-\underline{\zeta}_2^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\underline{\eta}_1^q}-1\right)\left(\tau^{\underline{\eta}_2^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\underline{\xi}_1^q}-1\right)\left(\tau^{\underline{\xi}_2^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\overline{\zeta}_1^q}-1\right)\left(\tau^{1-\overline{\zeta}_2^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\overline{\eta}_1^q}-1\right)\left(\tau^{\overline{\eta}_2^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\overline{\xi}_1^q}-1\right)\left(\tau^{\overline{\xi}_2^q}-1\right)}{\tau-1}\right)} \end{array} \right\rangle\right]$$

$$= \left[\left\langle \begin{array}{l} \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\underline{\zeta}_2^q}-1\right)\left(\tau^{1-\underline{\zeta}_1^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\underline{\eta}_2^q}-1\right)\left(\tau^{\underline{\eta}_1^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\underline{\xi}_2^q}-1\right)\left(\tau^{\underline{\xi}_1^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\overline{\zeta}_2^q}-1\right)\left(\tau^{1-\overline{\zeta}_1^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\overline{\eta}_2^q}-1\right)\left(\tau^{\overline{\eta}_1^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\overline{\xi}_2^q}-1\right)\left(\tau^{\overline{\xi}_1^q}-1\right)}{\tau-1}\right)} \end{array} \right\rangle\right]$$

$$= \mathcal{A}_2 \oplus \mathcal{A}_1.$$

Similarly,

$$\mathcal{A}_1 \otimes \mathcal{A}_2$$

$$= \left[\left\langle \begin{array}{l} \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\underline{\zeta}_1^q}-1\right)\left(\tau^{\underline{\zeta}_2^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\underline{\eta}_1^q}-1\right)\left(\tau^{1-\underline{\eta}_2^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\underline{\xi}_1^q}-1\right)\left(\tau^{1-\underline{\xi}_2^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\overline{\zeta}_1^q}-1\right)\left(\tau^{\overline{\zeta}_2^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\overline{\eta}_1^q}-1\right)\left(\tau^{1-\tau^{\overline{\eta}_2^q}}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\overline{\xi}_1^q}-1\right)\left(\tau^{1-\tau^{\overline{\xi}_2^q}}-1\right)}{\tau-1}\right)} \end{array} \right\rangle\right]$$

$$= \left[\left\langle \begin{array}{l} \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\underline{\zeta}_2^q}-1\right)\left(\tau^{\underline{\zeta}_1^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\underline{\eta}_2^q}-1\right)\left(\tau^{1-\underline{\eta}_1^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\underline{\xi}_2^q}-1\right)\left(\tau^{1-\underline{\xi}_1^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{log_\tau\left(1 + \frac{\left(\tau^{\overline{\zeta}_2^q}-1\right)\left(\tau^{\overline{\zeta}_1^q}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\overline{\eta}_2^q}-1\right)\left(\tau^{1-\tau^{\overline{\eta}_1^q}}-1\right)}{\tau-1}\right)}, \\ \sqrt[q]{1 - log_\tau\left(1 + \frac{\left(\tau^{1-\overline{\xi}_2^q}-1\right)\left(\tau^{1-\tau^{\overline{\xi}_1^q}}-1\right)}{\tau-1}\right)} \end{array} \right\rangle\right]$$

$$= \mathcal{A}_2 \otimes \mathcal{A}_1.$$

$$
\begin{aligned}
&\omega\left(\mathcal{A}_1 \oplus \mathcal{A}_2\right) \\
&= \omega \left\langle \begin{bmatrix}
\sqrt[q]{1 - log_{\tau}\left(1 + \frac{\left(\tau^{1-\underline{\zeta}_1^q}-1\right)\left(\tau^{1-\underline{\zeta}_2^q}-1\right)}{\tau-1}\right)}, \\
\sqrt[q]{log_{\tau}\left(1 + \frac{\left(\tau^{\underline{\eta}_1^q}-1\right)\left(\tau^{\underline{\eta}_2^q}-1\right)}{\tau-1}\right)}, \\
\sqrt[q]{log_{\tau}\left(1 + \frac{\left(\tau^{\underline{\xi}_1^q}-1\right)\left(\tau^{\underline{\xi}_2^q}-1\right)}{\tau-1}\right)}, \\
\sqrt[q]{1 - log_{\tau}\left(1 + \frac{\left(\tau^{1-\overline{\zeta}_1^q}-1\right)\left(\tau^{1-\overline{\zeta}_2^q}-1\right)}{\tau-1}\right)}, \\
\sqrt[q]{log_{\tau}\left(1 + \frac{\left(\tau^{\overline{\eta}_1^q}-1\right)\left(\tau^{\overline{\eta}_2^q}-1\right)}{\tau-1}\right)}, \\
\sqrt[q]{log_{\tau}\left(1 + \frac{\left(\tau^{\overline{\xi}_1^q}-1\right)\left(\tau^{\overline{\xi}_2^q}-1\right)}{\tau-1}\right)}
\end{bmatrix} \right\rangle
\end{aligned}
$$

$$
= \left\langle \begin{bmatrix}
\sqrt[q]{1 - log_{\tau}\left(1 + \frac{\left(\tau^{1-\underline{\zeta}_1^q}-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}, \\
\sqrt[q]{log_{\tau}\left(1 + \frac{\left(\tau^{\underline{\eta}_1^q}-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}, \\
\sqrt[q]{log_{\tau}\left(1 + \frac{\left(\underline{\xi}_1^q-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}, \\
\sqrt[q]{1 - log_{\tau}\left(1 + \frac{\left(\tau^{1-\overline{\zeta}_1^q}-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}, \\
\sqrt[q]{log_{\tau}\left(1 + \frac{\left(\tau^{\overline{\eta}_1^q}-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}, \\
\sqrt[q]{log_{\tau}\left(1 + \frac{\left(\tau^{\overline{\xi}_1^q}-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}
\end{bmatrix} \right\rangle
$$

$$
= \left\langle \begin{bmatrix}
\sqrt[q]{1 - log_{\tau}\left(1 + \frac{\left(\tau^{1-\underline{\zeta}_1^q}-1\right)^{\omega}\left(\tau^{1-\underline{\zeta}_2^q}-1\right)^{\omega}}{(\tau-1)^{2\omega-1}}\right)}, \\
\sqrt[q]{log_{\tau}\left(1 + \frac{\left(\tau^{\underline{\eta}_1^q}-1\right)^{\omega}\left(\tau^{\underline{\eta}_2^q}-1\right)^{\omega}}{(\tau-1)^{2\omega-1}}\right)}, \\
\sqrt[q]{log_{\tau}\left(1 + \frac{\left(\tau^{\underline{\xi}_1^q}-1\right)^{\omega}\left(\tau^{\underline{\xi}_2^q}-1\right)^{\omega}}{(\tau-1)^{2\omega-1}}\right)}, \\
\sqrt[q]{1 - log_{\tau}\left(1 + \frac{\left(\tau^{1-\overline{\zeta}_1^q}-1\right)^{\omega}\left(\tau^{1-\overline{\zeta}_2^q}-1\right)^{\omega}}{(\tau-1)^{2\omega-1}}\right)}, \\
\sqrt[q]{log_{\tau}\left(1 + \frac{\left(\tau^{\overline{\eta}_1^q}-1\right)^{\omega}\left(\tau^{\overline{\eta}_2^q}-1\right)^{\omega}}{(\tau-1)^{2\omega-1}}\right)}, \\
\sqrt[q]{log_{\tau}\left(1 + \frac{\left(\tau^{\overline{\xi}_1^q}-1\right)^{\omega}\left(\tau^{\overline{\xi}_2^q}-1\right)^{\omega}}{(\tau-1)^{2\omega-1}}\right)}
\end{bmatrix} \right\rangle.
$$

$$
\begin{aligned}
&\oplus \left\langle \begin{bmatrix}
\sqrt[q]{1 - log_{\tau}\left(1 + \frac{\left(\tau^{1-\underline{\zeta}_2^q}-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}, \\
\sqrt[q]{log_{\tau}\left(1 + \frac{\left(\tau^{\underline{\eta}_2^q}-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}, \\
\sqrt[q]{log_{\tau}\left(1 + \frac{\left(\underline{\xi}_2^q-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}, \\
\sqrt[q]{1 - log_{\tau}\left(1 + \frac{\left(\tau^{1-\overline{\zeta}_2^q}-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}, \\
\sqrt[q]{log_{\tau}\left(1 + \frac{\left(\tau^{\overline{\eta}_2^q}-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}, \\
\sqrt[q]{log_{\tau}\left(1 + \frac{\left(\tau^{\overline{\xi}_2^q}-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}
\end{bmatrix} \right\rangle \\
&= \omega\mathcal{A}_1 \oplus \omega\mathcal{A}_2
\end{aligned}
$$

Therefore, $\omega\left(\mathcal{A}_1 \oplus \mathcal{A}_2\right)=\omega \mathcal{A}_1 \oplus \mathfrak{g} \mathcal{A}_2$.

$$\omega_1 \mathcal{A} \oplus \omega_2 \mathcal{A}$$

$$=\left[\left\langle\begin{array}{c}
\sqrt[q]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\underline{\zeta}^q}-1\right)^{\omega_1}}{(\tau-1)^{\omega_1-1}}\right)}, \\
\sqrt[q]{\log_\tau\left(1+\frac{\left(\tau^{\underline{\eta}^q}-1\right)^{\omega_1}}{(\tau-1)^{\omega_1-1}}\right)}, \\
\sqrt[q]{\log_\tau\left(1+\frac{\left(\tau^{\underline{\xi}^q}-1\right)^{\omega_1}}{(\tau-1)^{\omega_1-1}}\right)}, \\
\sqrt[q]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\overline{\zeta}^q}-1\right)^{\omega_1}}{(\tau-1)^{\omega_1-1}}\right)}, \\
\sqrt[q]{\log_\tau\left(1+\frac{\left(\tau^{\overline{\eta}^q}-1\right)^{\omega_1}}{(\tau-1)^{\omega_1-1}}\right)}, \\
\sqrt[q]{\log_\tau\left(1+\frac{\left(\tau^{\overline{\xi}^q}-1\right)^{\omega_1}}{(\tau-1)^{\omega_1-1}}\right)}
\end{array}\right\rangle\right]$$

$$\oplus\left[\left\langle\begin{array}{c}
\sqrt[q]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\underline{\zeta}^q}-1\right)^{\omega_2}}{(\tau-1)^{\omega_2-1}}\right)}, \\
\sqrt[q]{\log_\tau\left(1+\frac{\left(\tau^{\underline{\eta}^q}-1\right)^{\omega_2}}{(\tau-1)^{\omega_2-1}}\right)}, \\
\sqrt[q]{\log_\tau\left(1+\frac{\left(\tau^{\underline{\xi}^q}-1\right)^{\omega_2}}{(\tau-1)^{\omega_2-1}}\right)}, \\
\sqrt[q]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\overline{\zeta}^q}-1\right)^{\omega_2}}{(\tau-1)^{\omega_2-1}}\right)}, \\
\sqrt[q]{\log_\tau\left(1+\frac{\left(\tau^{\overline{\eta}^q}-1\right)^{\omega_2}}{(\tau-1)^{\omega_2-1}}\right)}, \\
\sqrt[q]{\log_\tau\left(1+\frac{\left(\tau^{\overline{\xi}^q}-1\right)^{\omega_2}}{(\tau-1)^{\omega_2-1}}\right)}
\end{array}\right\rangle\right]$$

$$=\left[\left\langle\begin{array}{c}
\sqrt[q]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\underline{\zeta}^q}-1\right)^{\omega_1+\omega_2}}{(\tau-1)^{\omega_1+\omega_2-1}}\right)}, \\
\sqrt[q]{\log_\tau\left(1+\frac{\left(\tau^{\underline{\eta}^q}-1\right)^{\omega_1+\omega_2}}{(\tau-1)^{\omega_1+\omega_2-1}}\right)}, \\
\sqrt[q]{\log_\tau\left(1+\frac{\left(\tau^{\underline{\xi}^q}-1\right)^{\omega_1+\omega_2}}{(\tau-1)^{\omega_1+\omega_2-1}}\right)}, \\
\sqrt[q]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\overline{\zeta}^q}-1\right)^{\omega_1+\omega_2}}{(\tau-1)^{\omega_1+\omega_2-1}}\right)}, \\
\sqrt[q]{\log_\tau\left(1+\frac{\left(\tau^{\overline{\eta}^q}-1\right)^{\omega_1+\omega_2}}{(\tau-1)^{\omega_1+\omega_2-1}}\right)}, \\
\sqrt[q]{\log_\tau\left(1+\frac{\left(\tau^{\overline{\xi}^q}-1\right)^{\omega_1+\omega_2}}{(\tau-1)^{\omega_1+\omega_2-1}}\right)}
\end{array}\right\rangle\right]$$

$$=\mathcal{A}\left(\omega_1 \oplus \omega_2\right).$$

$$\left(\mathcal{A}_1 \otimes \mathcal{A}_2\right)^\omega$$

$$=\left[\left\langle\begin{array}{c}
\sqrt[q]{\log_\tau\left(1+\frac{\left(\tau^{\underline{\zeta}_1^q}-1\right)\left(\tau^{\underline{\zeta}_2^q}-1\right)}{\tau-1}\right)}, \\
\sqrt[q]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\underline{\eta}_1^q}-1\right)\left(\tau^{1-\underline{\eta}_2^q}-1\right)}{\tau-1}\right)}, \\
\sqrt[q]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\underline{\xi}_1^q}-1\right)\left(\tau^{1-\underline{\xi}_2^q}-1\right)}{\tau-1}\right)}, \\
\sqrt[q]{\log_\tau\left(1+\frac{\left(\tau^{\overline{\zeta}_1^q}-1\right)\left(\tau^{\overline{\zeta}_2^q}-1\right)}{\tau-1}\right)}, \\
\sqrt[q]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\overline{\eta}_1^q}-1\right)\left(\tau^{1-\overline{\eta}_2^q}-1\right)}{\tau-1}\right)}, \\
\sqrt[q]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\overline{\xi}_1^q}-1\right)\left(\tau^{1-\tau^{\overline{\xi}_2^q}}-1\right)}{\tau-1}\right)}
\end{array}\right\rangle\right]^\omega$$

$$
= \left\langle \begin{bmatrix}
\sqrt[q]{log_{\tau}\left(1+\frac{\left(\tau^{\underline{\zeta}_1^q}-1\right)^{\omega}\left(\tau^{\underline{\zeta}_2^q}-1\right)^{\omega}}{(\tau-1)^{2\omega-1}}\right)}, \\
\sqrt[q]{1-log_{\tau}\left(1+\frac{\left(\tau^{1-\underline{\eta}_1^q}-1\right)^{\omega}\left(\tau^{1-\underline{\eta}_2^q}-1\right)^{\omega}}{(\tau-1)^{2\omega-1}}\right)}, \\
\sqrt[q]{1-log_{\tau}\left(1+\frac{\left(\tau^{1-\underline{\xi}_1^q}-1\right)^{\omega}\left(\tau^{1-\underline{\xi}_2^q}-1\right)^{\omega}}{(\tau-1)^{2\omega-1}}\right)}, \\
\sqrt[q]{log_{\tau}\left(1+\frac{\left(\tau^{\overline{\zeta}_1^q}-1\right)^{\omega}\left(\tau^{\overline{\zeta}_2^q}-1\right)^{\omega}}{(\tau-1)^{2\omega-1}}\right)}, \\
\sqrt[q]{1-log_{\tau}\left(1+\frac{\left(\tau^{1-\overline{\eta}_1^q}-1\right)^{\omega}\left(\tau^{1-\tau^{\overline{\eta}_2^q}}-1\right)^{\omega}}{(\tau-1)^{2\omega-1}}\right)}, \\
\sqrt[q]{1-log_{\tau}\left(1+\frac{\left(\tau^{1-\overline{\xi}_1^q}-1\right)^{\omega}\left(\tau^{1-\tau^{\overline{\xi}_2^q}}-1\right)^{\omega}}{(\tau-1)^{2\omega-1}}\right)}
\end{bmatrix} \right\rangle
$$

$$
= \left\langle \begin{bmatrix}
\sqrt[q]{log_{\tau}\left(1+\frac{\left(\tau^{\underline{\zeta}_1^q}-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}, \\
\sqrt[q]{1-log_{\tau}\left(1+\frac{\left(\tau^{1-\underline{\eta}_1^q}-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}, \\
\sqrt[q]{1-log_{\tau}\left(1+\frac{\left(\tau^{1-\underline{\xi}_1^q}-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}, \\
\sqrt[q]{log_{\tau}\left(1+\frac{\left(\tau^{\overline{\zeta}_1^q}-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}, \\
\sqrt[q]{1-log_{\tau}\left(1+\frac{\left(\tau^{1-\overline{\eta}_1^q}-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}, \\
\sqrt[q]{1-log_{\tau}\left(1+\frac{\left(\tau^{1-\overline{\xi}_1^q}-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}
\end{bmatrix} \right\rangle
$$

$$
\otimes \left\langle \begin{bmatrix}
\sqrt[q]{log_{\tau}\left(1+\frac{\left(\tau^{\underline{\zeta}_2^q}-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}, \\
\sqrt[q]{1-log_{\tau}\left(1+\frac{\left(\tau^{1-\underline{\eta}_2^q}-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}, \\
\sqrt[q]{1-log_{\tau}\left(1+\frac{\left(\tau^{\underline{\xi}_2^q}-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}, \\
\sqrt[q]{log_{\tau}\left(1+\frac{\left(\tau^{1-\overline{\zeta}_2^q}-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}, \\
\sqrt[q]{1-log_{\tau}\left(1+\frac{\left(\tau^{1-\overline{\eta}_2^q}-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}, \\
\sqrt[q]{1-log_{\tau}\left(1+\frac{\left(\tau^{1-\overline{\xi}_2^q}-1\right)^{\omega}}{(\tau-1)^{\omega-1}}\right)}
\end{bmatrix} \right\rangle
$$

$$
= \mathcal{A}_1^{\omega} \otimes \mathcal{A}_2^{\omega}.
$$

$$
\mathcal{A}^{\omega_1} \otimes \mathcal{A}^{\omega_2}
$$

$$
= \left\langle \begin{bmatrix}
\sqrt[q]{log_{\tau}\left(1+\frac{\left(\tau^{\underline{\zeta}^q}-1\right)^{\omega_1}}{(\tau-1)^{\omega_1-1}}\right)}, \\
\sqrt[q]{1-log_{\tau}\left(1+\frac{\left(\tau^{1-\underline{\eta}^q}-1\right)^{\omega_1}}{(\tau-1)^{\omega_1-1}}\right)}, \\
\sqrt[q]{1-log_{\tau}\left(1+\frac{\left(\tau^{1-\underline{\xi}^q}-1\right)^{\omega_1}}{(\tau-1)^{\omega_1-1}}\right)}, \\
\sqrt[q]{log_{\tau}\left(1+\frac{\left(\tau^{\overline{\zeta}^q}-1\right)^{\omega_1}}{(\tau-1)^{\omega_1-1}}\right)}, \\
\sqrt[q]{1-log_{\tau}\left(1+\frac{\left(\tau^{1-\overline{\eta}^q}-1\right)^{\omega_1}}{(\tau-1)^{\omega_1-1}}\right)}, \\
\sqrt[q]{1-log_{\tau}\left(1+\frac{\left(\tau^{1-\overline{\xi}^q}-1\right)^{\omega_1}}{(\tau-1)^{\omega_1-1}}\right)}
\end{bmatrix} \right\rangle
$$

$$\otimes\left\langle\begin{bmatrix}\sqrt[q]{\log_{\tau}\left(1+\frac{\left(\tau^{\underline{\zeta}^{q}}-1\right)^{\omega_2}}{(\tau-1)^{\omega_2-1}}\right)},\\ \sqrt[q]{1-\log_{\tau}\left(1+\frac{\left(\tau^{1-\underline{\eta}^{q}}-1\right)^{\omega_2}}{(\tau-1)^{\omega_2-1}}\right)},\\ \sqrt[q]{1-\log_{\tau}\left(1+\frac{\left(\tau^{1-\underline{\xi}^{q}}-1\right)^{\omega_2}}{(\tau-1)^{\omega_2-1}}\right)},\\ \sqrt[q]{\log_{\tau}\left(1+\frac{\left(\tau^{\overline{\zeta}^{q}}-1\right)^{\omega_2}}{(\tau-1)^{\omega_2-1}}\right)},\\ \sqrt[q]{1-\log_{\tau}\left(1+\frac{\left(\tau^{1-\overline{\eta}^{q}}-1\right)^{\omega_2}}{(\tau-1)^{\omega_2-1}}\right)},\\ \sqrt[q]{1-\log_{\tau}\left(1+\frac{\left(\tau^{1-\overline{\xi}^{q}}-1\right)^{\omega_2}}{(\tau-1)^{\omega_2-1}}\right)}\end{bmatrix}\right\rangle$$

$$=\left\langle\begin{bmatrix}\sqrt[q]{\log_{\tau}\left(1+\frac{\left(\tau^{\underline{\zeta}^{q}}-1\right)^{\omega_1+\omega_2}}{(\tau-1)^{\omega_1+\omega_2-1}}\right)},\\ \sqrt[q]{1-\log_{\tau}\left(1+\frac{\left(\tau^{1-\underline{\eta}^{q}}-1\right)^{\omega_1+\omega_2}}{(\tau-1)^{\omega_1+\omega_2-1}}\right)},\\ \sqrt[q]{1-\log_{\tau}\left(1+\frac{\left(\tau^{1-\underline{\xi}^{q}}-1\right)^{\omega_1+\omega_2}}{(\tau-1)^{\omega_1+\omega_2-1}}\right)},\\ \sqrt[q]{\log_{\tau}\left(1+\frac{\left(\tau^{\overline{\zeta}^{q}}-1\right)^{\omega_1+\omega_2}}{(\tau-1)^{\omega_1+\omega_2-1}}\right)},\\ \sqrt[q]{1-\log_{\tau}\left(1+\frac{\left(\tau^{1-\overline{\eta}^{q}}-1\right)^{\omega_1+\omega_2}}{(\tau-1)^{\omega_1+\omega_2-1}}\right)},\\ \sqrt[q]{1-\log_{\tau}\left(1+\frac{\left(\tau^{1-\overline{\xi}^{q}}-1\right)^{\omega_1+\omega_2}}{(\tau-1)^{\omega_1+\omega_2-1}}\right)}\end{bmatrix}\right\rangle$$

$$=\mathcal{A}^{\omega_1+\omega_2}.$$

B. q-SPHERICAL FUZZY ROUGH FRANK AVERAGING AGGREGATION OPERATORS

We present a variety of averaging operators in this section that use the rules from Section III. These operators can effectively combine and simplify information, which facilitates data analysis and decision-making.

Definition 17: Let $\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right)$ (i = 1, 2, ..., n) be a set of q-SFRNs with their corresponding weight vector ω_{i} (i = 1, 2, ..., n) such that $\sum_{\mathrm{i}=1}^{\mathrm{n}}\omega_{\mathrm{i}} = 1$ and $\omega_{\mathrm{i}} \in [0, 1]$. Then the operator $\mathrm{q-SFRFWA}: \mathcal{A}^{\mathrm{n}} \rightarrow \mathcal{A}$ is defined as

$$\mathrm{q-SFRFWA}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{\mathrm{n}}\right) = \oplus_{\mathrm{i}=1}^{\mathrm{n}}\omega_{\mathrm{i}}\mathcal{A}_{\mathrm{i}}$$
$$=\left\langle\begin{bmatrix}\sqrt[q]{1-\log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{1-\underline{\zeta}_i^{q}}-1\right)^{\omega_{\mathrm{i}}}\right)},\\ \sqrt[q]{\log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\underline{\eta}_i^{q}}-1\right)^{\omega_{\mathrm{i}}}\right)},\\ \sqrt[q]{\log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\underline{\xi}_i^{q}}-1\right)^{\omega_{\mathrm{i}}}\right)},\\ \sqrt[q]{1-\log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{1-\overline{\zeta}_i^{q}}-1\right)^{\omega_{\mathrm{i}}}\right)},\\ \sqrt[q]{\log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\overline{\eta}_i^{q}}-1\right)^{\omega_{\mathrm{i}}}\right)},\\ \sqrt[q]{\log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\overline{\xi}_i^{q}}-1\right)^{\omega_{\mathrm{i}}}\right)}\end{bmatrix}\right\rangle \quad (13)$$

Theorem 2: The aggregated value obtained by $q-SFRFWA$ operator of $q-SFRNs$ is still a $q-SFRN$, and

$$\mathrm{q-SFRFWA}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{\mathrm{n}}\right) = \oplus_{\mathrm{i}=1}^{\mathrm{n}}\omega_{\mathrm{i}}\mathcal{A}_{\mathrm{i}}$$
$$=\left\langle\begin{bmatrix}\sqrt[q]{1-\log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{1-\underline{\zeta}_i^{q}}-1\right)^{\omega_{\mathrm{i}}}\right)},\\ \sqrt[q]{\log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\underline{\eta}_i^{q}}-1\right)^{\omega_{\mathrm{i}}}\right)},\\ \sqrt[q]{\log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\underline{\xi}_i^{q}}-1\right)^{\omega_{\mathrm{i}}}\right)},\\ \sqrt[q]{1-\log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{1-\overline{\zeta}_i^{q}}-1\right)^{\omega_{\mathrm{i}}}\right)},\\ \sqrt[q]{\log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\overline{\eta}_i^{q}}-1\right)^{\omega_{\mathrm{i}}}\right)},\\ \sqrt[q]{\log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\overline{\xi}_i^{q}}-1\right)^{\omega_{\mathrm{i}}}\right)}\end{bmatrix}\right\rangle \quad (14)$$

Proof: This theorem is established by using mathematical induction.

Step 1. For n = 2, we have

$$\mathrm{q-SFRFWA}\left(\mathcal{A}_1, \mathcal{A}_2\right) = \omega_1\mathcal{A}_1 \oplus \omega_2\mathcal{A}_2$$
$$\left\langle\begin{bmatrix}\sqrt[q]{1-\log_{\tau}\left(1+\left(\tau^{1-\underline{\zeta}_1^{q}}-1\right)^{\omega_1}\right)},\\ \sqrt[q]{\log_{\tau}\left(1+\left(\tau^{\underline{\eta}_1^{q}}-1\right)^{\omega_1}\right)},\\ \sqrt[q]{\log_{\tau}\left(1+\left(\underline{\xi}_1^{q}-1\right)^{\omega_1}\right)},\\ \sqrt[q]{1-\log_{\tau}\left(1+\left(\tau^{1-\overline{\zeta}_1^{q}}-1\right)^{\omega_1}\right)},\\ \sqrt[q]{\log_{\tau}\left(1+\left(\tau^{\overline{\eta}_1^{q}}-1\right)^{\omega_1}\right)},\\ \sqrt[q]{\log_{\tau}\left(1+\left(\tau^{\overline{\xi}_1^{q}}-1\right)^{\omega_1}\right)}\end{bmatrix}\right\rangle$$

$$\oplus\left[\left\langle\begin{array}{l}\sqrt[q]{1-log_{\tau}\left(1+\left(\tau^{1-\underline{\zeta}_2^q}-1\right)^{\omega_2}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\left(\tau^{\underline{\eta}_2^q}-1\right)^{\omega_2}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\left(\underline{\xi}_2^q-1\right)^{\omega_2}\right)},\\ \sqrt[q]{1-log_{\tau}\left(1+\left(\tau^{1-\overline{\zeta}_2^q}-1\right)^{\omega_2}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\left(\tau^{\overline{\eta}_2^q}-1\right)^{\omega_2}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\left(\tau^{\overline{\xi}_2^q}-1\right)^{\omega_2}\right)}\end{array}\right\rangle\right]$$

$$=\left[\left\langle\begin{array}{l}\sqrt[q]{1-log_{\tau}\left(1+\left(\tau^{1-\underline{\zeta}_1^q}-1\right)^{\omega_1}\left(\tau^{1-\underline{\zeta}_2^q}-1\right)^{\omega_2}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\left(\tau^{\underline{\eta}_1^q}-1\right)^{\omega_1}\right)\left(1+\left(\tau^{\underline{\eta}_2^q}-1\right)^{\omega_2}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\left(\tau^{\underline{\xi}_1^q}-1\right)^{\omega_1}\right)\left(1+\left(\tau^{\underline{\xi}_2^q}-1\right)^{\omega_2}\right)}\\ \sqrt[q]{1-log_{\tau}\left(1+\left(\tau^{1-\overline{\zeta}_1^q}-1\right)^{\omega_1}\left(\tau^{1-\overline{\zeta}_2^q}-1\right)^{\omega_2}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\left(\tau^{\overline{\eta}_1^q}-1\right)^{\omega_1}\left(\tau^{\overline{\eta}_2^q}-1\right)^{\omega_2}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\left(\tau^{\overline{\xi}_1^q}-1\right)^{\omega_1}\left(\tau^{\overline{\xi}_2^q}-1\right)^{\omega_2}\right)}\end{array}\right\rangle\right]$$

$$=\left[\left\langle\begin{array}{l}\sqrt[q]{1-log_{\tau}\left(1+\prod_{i=1}^{2}\left(\tau^{1-\underline{\zeta}_i^q}-1\right)^{\omega_i}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{2}\left(\tau^{\underline{\eta}_i^q}-1\right)^{\omega_i}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{2}\left(\tau^{\underline{\xi}_i^q}-1\right)^{\omega_i}\right)},\\ \sqrt[q]{1-log_{\tau}\left(1+\prod_{i=1}^{2}\left(\tau^{1-\overline{\zeta}_i^q}-1\right)^{\omega_i}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{2}\left(\tau^{\overline{\eta}_i^q}-1\right)^{\omega_i}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{2}\left(\tau^{\overline{\xi}_i^q}-1\right)^{\omega_i}\right)}\end{array}\right\rangle\right].$$

where $\sum_{i=1}^{2}\omega_i=1$.

Consequently, when n equals 2, the assertion holds.

Step 2. Assume the validity of the result for $n = k$, i.e.,

$$q-\text{SFRFWA}\left(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_k\right)=\oplus_{i=1}^{k}\omega_i\mathcal{A}_i$$
$$=\left[\left\langle\begin{array}{l}\sqrt[q]{1-log_{\tau}\left(1+\prod_{i=1}^{k}\left(\tau^{1-\underline{\zeta}_i^q}-1\right)^{\omega_k}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{k}\left(\tau^{\underline{\eta}_i^q}-1\right)^{\omega_k}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{k}\left(\tau^{\underline{\xi}_i^q}-1\right)^{\omega_k}\right)},\\ \sqrt[q]{1-log_{\tau}\left(1+\prod_{i=1}^{k}\left(\tau^{1-\overline{\zeta}_i^q}-1\right)^{\omega_k}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{k}\left(\tau^{\overline{\eta}_i^q}-1\right)^{\omega_k}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{k}\left(\tau^{\overline{\xi}_i^q}-1\right)^{\omega_k}\right)}\end{array}\right\rangle\right]$$

Step 3. To demonstrate Equation (14) is true for the case where $n = k + 1$ i.e.,

$$q-\text{SFRFWA}\left(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_k\oplus\mathcal{A}_{k+1}\right)$$
$$=\omega_1\mathcal{A}_1\oplus\omega_2\mathcal{A}_2\oplus\ldots\oplus\omega_k\mathcal{A}_k\oplus\omega_{k+1}\mathcal{A}_{k+1}$$

$$=\left[\left\langle\begin{array}{l}\sqrt[q]{1-log_{\tau}\left(1+\prod_{i=1}^{k}\left(\tau^{1-\underline{\zeta}_i^q}-1\right)^{\omega_k}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{k}\left(\tau^{\underline{\eta}_i^q}-1\right)^{\omega_k}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{k}\left(\tau^{\underline{\xi}_i^q}-1\right)^{\omega_k}\right)},\\ \sqrt[q]{1-log_{\tau}\left(1+\prod_{i=1}^{k}\left(\tau^{1-\overline{\zeta}_i^q}-1\right)^{\omega_k}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{k}\left(\tau^{\overline{\eta}_i^q}-1\right)^{\omega_k}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{k}\left(\tau^{\overline{\xi}_i^q}-1\right)^{\omega_k}\right)}\end{array}\right\rangle\right]$$

$$\oplus\left[\left\langle\begin{array}{l}\sqrt[q]{1-log_{\tau}\left(1+\left(\tau^{1-\underline{\zeta}_{k+1}^q}-1\right)^{\omega_{k+1}}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\left(\tau^{\underline{\eta}_{k+1}^q}-1\right)^{\omega_{k+1}}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\left(\tau^{\underline{\xi}_{k+1}^q}-1\right)^{\omega_{k+1}}\right)},\\ \sqrt[q]{1-log_{\tau}\left(1+\left(\tau^{1-\overline{\zeta}_{k+1}^q}-1\right)^{\omega_{k+1}}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\left(\tau^{\overline{\eta}_{k+1}^q}-1\right)^{\omega_{k+1}}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\left(\tau^{\overline{\xi}_{k+1}^q}-1\right)^{\omega_{k+1}}\right)}\end{array}\right\rangle\right]$$

$$=\left[\left\langle\begin{array}{l}\sqrt[q]{1-log_{\tau}\left(1+\prod_{i=1}^{k}\left(\tau^{1-\underline{\zeta}_k^q}-1\right)^{\omega_k}\left(\tau^{1-\underline{\zeta}_{k+1}^q}-1\right)^{\omega_{k+1}}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{k}\left(\tau^{\underline{\eta}_k^q}-1\right)^{\omega_k}\left(\tau^{\underline{\eta}_{k+1}^q}-1\right)^{\omega_{k+1}}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{k}\left(\tau^{\underline{\xi}_k^q}-1\right)^{\omega_k}\left(\tau^{\underline{\xi}_{k+1}^q}-1\right)^{\omega_{k+1}}\right)},\\ \sqrt[q]{1-log_{\tau}\left(1+\prod_{i=1}^{k}\left(\tau^{1-\overline{\zeta}_k^q}-1\right)^{\omega_k}\left(\tau^{1-\overline{\zeta}_{k+1}^q}-1\right)^{\omega_{k+1}}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{k}\left(\tau^{\overline{\eta}_k^q}-1\right)^{\omega_k}\left(\tau^{\overline{\eta}_{k+1}^q}-1\right)^{\omega_{k+1}}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{k}\left(\tau^{\overline{\xi}_k^q}-1\right)^{\omega_k}\left(\tau^{\overline{\xi}_{k+1}^q}-1\right)^{\omega_{k+1}}\right)}\end{array}\right\rangle\right]$$

$$=\left[\left\langle\begin{array}{l}\sqrt[q]{1-log_{\tau}\left(1+\prod_{i=1}^{k+1}\left(\tau^{1-\underline{\zeta}_i^q}-1\right)^{\omega_i}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{k+1}\left(\tau^{\underline{\eta}_i^q}-1\right)^{\omega_i}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{k+1}\left(\tau^{\underline{\xi}_i^q}-1\right)^{\omega_i}\right)},\\ \sqrt[q]{1-log_{\tau}\left(1+\prod_{i=1}^{k+1}\left(\tau^{1-\overline{\zeta}_i^q}-1\right)^{\omega_i}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{k+1}\left(\tau^{\overline{\eta}_i^q}-1\right)^{\omega_i}\right)},\\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{k+1}\left(\tau^{\overline{\xi}_i^q}-1\right)^{\omega_i}\right)}\end{array}\right\rangle\right]$$

where $\sum_{i=1}^{k+1}\omega_i=1$.

Thus, Equation (14) holds for $k+1$. By mathematical induction, we conclude that the result is true for all values of n.

Theorem 3 (Property of Idempotency): If the $q-SFRNs$ $\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right)(i=1,2,\ldots,n)$ are identical, i.e., be a $\mathcal{A}_i = \mathcal{A}$ for all i, where $\mathcal{A} = (\underline{\zeta}, \underline{\eta}, \underline{\xi}, \overline{\zeta}, \overline{\eta}, \overline{\xi})$, then $q-SFRFWA(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathcal{A}$.

Proof. As $\mathcal{A}_\mathrm{i} = \mathcal{A}$, for all i, then we obtain

$$q-\mathrm{SFRFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_\mathrm{n})$$

$$= \left[\left\langle \begin{array}{l} \sqrt[q]{1-\log_\tau\left(1+\prod_{i=1}^n\left(\tau^{1-\underline{\zeta}_i^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{\log_\tau\left(1+\prod_{i=1}^n\left(\tau^{\underline{\eta}_i^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{\log_\tau\left(1+\prod_{i=1}^n\left(\tau^{\underline{\xi}_i^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{1-\log_\tau\left(1+\prod_{i=1}^n\left(\tau^{1-\overline{\zeta}_i^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{\log_\tau\left(1+\prod_{i=1}^n\left(\tau^{\overline{\eta}_i^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{\log_\tau\left(1+\prod_{i=1}^n\left(\tau^{\overline{\xi}_i^q}-1\right)^{\omega_i}\right)} \end{array} \right\rangle\right]$$

$$= \left[\left\langle \begin{array}{l} \sqrt[q]{1-\log_\tau\left(1+\prod_{i=1}^n\left(\tau^{1-\underline{\zeta}_i^q}-1\right)^{\sum_{i=1}^{k+1}\omega_i}\right)}, \\ \sqrt[q]{\log_\tau\left(1+\prod_{i=1}^n\left(\tau^{\underline{\eta}_i^q}-1\right)^{\sum_{i=1}^{k+1}\omega_i}\right)}, \\ \sqrt[q]{\log_\tau\left(1+\prod_{i=1}^n\left(\tau^{\underline{\xi}_i^q}-1\right)^{\sum_{i=1}^{k+1}\omega_i}\right)}, \\ \sqrt[q]{1-\log_\tau\left(1+\prod_{i=1}^n\left(\tau^{1-\overline{\zeta}_i^q}-1\right)^{\sum_{i=1}^{k+1}\omega_i}\right)}, \\ \sqrt[q]{\log_\tau\left(1+\prod_{i=1}^n\left(\tau^{\overline{\eta}_i^q}-1\right)^{\sum_{i=1}^{k+1}\omega_i}\right)}, \\ \sqrt[q]{\log_\tau\left(1+\prod_{i=1}^n\left(\tau^{\overline{\xi}_i^q}-1\right)^{\sum_{i=1}^{k+1}\omega_i}\right)} \end{array} \right\rangle\right]$$

$$= \left[\left\langle \begin{array}{l} \sqrt[q]{1-\log_\tau\left(1+\left(\tau^{1-\underline{\zeta}^q}-1\right)\right)}, \\ \sqrt[q]{\log_\tau\left(1+\left(\tau^{\underline{\eta}^q}-1\right)\right)}, \\ \sqrt[q]{\log_\tau\left(1+\left(\tau^{\underline{\xi}^q}-1\right)\right)}, \\ \sqrt[q]{1-\log_\tau\left(1+\left(\tau^{1-\overline{\zeta}^q}-1\right)\right)}, \\ \sqrt[q]{\log_\tau\left(1+\left(\tau^{\overline{\eta}^q}-1\right)\right)}, \\ \sqrt[q]{\log_\tau\left(1+\left(\tau^{\overline{\xi}^q}-1\right)\right)} \end{array} \right\rangle\right]$$

$$= \left(\underline{\zeta}, \underline{\eta}, \underline{\xi}, \overline{\zeta}, \overline{\eta}, \overline{\xi}\right) = \mathcal{A}$$

Therefore, the result can be derived from the information provided.

Theorem 4 (Property of Boundedness): Assuming $\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right)(i=1,2,\ldots,n)$ be a family of $q-SFRNs$. If

$\mathcal{A}^- = min\{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_\mathrm{n}\}$ and

$\mathcal{A}^+ = max\{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_\mathrm{n}\}$, then

$\mathcal{A}^- \leq q-SFREWG(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \mathcal{A}^+$.

Proof: Let $\mathcal{A}^- = \left(\underline{\zeta}^-, \underline{\eta}^-, \underline{\xi}^-, \overline{\zeta}^-, \overline{\eta}^-, \overline{\xi}^-\right)$ and $\mathcal{A}^+ = \left(\underline{\zeta}^+, \underline{\eta}^+, \underline{\xi}^+, \overline{\zeta}^+, \overline{\eta}^+, \overline{\xi}^+\right)$. Therefore, we have

$$\underline{\zeta}^- = \min_i\left\{\underline{\zeta}_i\right\}, \underline{\eta}^- = \max_i\left\{\underline{\eta}_i\right\}, \underline{\xi}_\mathrm{i} = \max_i\left\{\underline{\xi}_i\right\},$$
$$\overline{\zeta}^- = \min_i\{\overline{\zeta}_\mathrm{i}\}, \overline{\eta}^- = \max_i\{\overline{\eta}_i\}, \overline{\xi}^- = \max_i\{\overline{\xi}_\mathrm{i}\},$$
$$\underline{\zeta}^+ = \max_i\left\{\underline{\zeta}_i\right\}, \underline{\eta}^+ = \min_i\left\{\underline{\eta}_i\right\}, \underline{\xi}^+ = \min_i\left\{\underline{\xi}_i\right\},$$
$$\overline{\zeta}^+ = \max_i\{\overline{\zeta}_\mathrm{i}\}, \overline{\eta}^+ = \min_i\{\overline{\eta}_i\}, \overline{\xi}^+ = \min_i\{\overline{\xi}_\mathrm{i}\}$$

$$\sqrt[q]{1-log_\tau\left(1+\prod_{i=1}^n\left(\tau^{1-\underline{\zeta}^{-q}}-1\right)^{\omega_i}\right)}$$
$$\leq \sqrt[q]{1-log_\tau\left(1+\prod_{i=1}^n\left(\tau^{1-\underline{\zeta}_i^q}-1\right)^{\omega_i}\right)}$$
$$\leq \sqrt[q]{1-log_\tau\left(1+\prod_{i=1}^n\left(\tau^{1-\underline{\zeta}^{+q}}-1\right)^{\omega_i}\right)},$$
$$\sqrt[q]{log_\tau\left(1+\prod_{i=1}^n\left(\tau^{\underline{\eta}^{-q}}-1\right)^{\omega_i}\right)}$$
$$\geq \sqrt[q]{log_\tau\left(1+\prod_{i=1}^n\left(\tau^{\underline{\eta}_i^q}-1\right)^{\omega_i}\right)}$$
$$\geq \sqrt[q]{log_\tau\left(1+\prod_{i=1}^n\left(\tau^{\underline{\eta}^{+q}}-1\right)^{\omega_i}\right)},$$
$$\sqrt[q]{log_\tau\left(1+\prod_{i=1}^n\left(\tau^{\underline{\xi}^{-q}}-1\right)^{\omega_i}\right)}$$
$$\geq \sqrt[q]{log_\tau\left(1+\prod_{i=1}^n\left(\tau^{\underline{\xi}_i^q}-1\right)^{\omega_i}\right)}$$
$$\geq \sqrt[q]{log_\tau\left(1+\prod_{i=1}^n\left(\tau^{\underline{\xi}^{+q}}-1\right)^{\omega_i}\right)},$$
$$\sqrt[q]{1-log_\tau\left(1+\prod_{i=1}^n\left(\tau^{1-\overline{\zeta}^{-q}}-1\right)^{\omega_i}\right)}$$
$$\leq \sqrt[q]{1-log_\tau\left(1+\prod_{i=1}^n\left(\tau^{1-\overline{\zeta}_i^q}-1\right)^{\omega_i}\right)}$$
$$\leq \sqrt[q]{1-log_\tau\left(1+\prod_{i=1}^n\left(\tau^{1-\overline{\zeta}^{+q}}-1\right)^{\omega_i}\right)},$$
$$\sqrt[q]{log_\tau\left(1+\prod_{i=1}^n\left(\tau^{\overline{\eta}^{-q}}-1\right)^{\omega_i}\right)}$$
$$\geq \sqrt[q]{log_\tau\left(1+\prod_{i=1}^n\left(\tau^{\overline{\eta}_i^q}-1\right)^{\omega_i}\right)}$$
$$\geq \sqrt[q]{log_\tau\left(1+\prod_{i=1}^n\left(\tau^{\overline{\eta}^{+q}}-1\right)^{\omega_i}\right)},$$
$$\sqrt[q]{log_\tau\left(1+\prod_{i=1}^n\left(\tau^{\overline{\xi}^{-q}}-1\right)^{\omega_i}\right)}$$

$$\geq \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\overline{\xi}_i^q}-1\right)^{\omega_i}\right)}$$
$$\geq \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\overline{\xi}^{+q}}-1\right)^{\omega_i}\right)}.$$

Therefore,

$$\mathcal{A}^{-} \leq q-SFRFWA\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) \leq \mathcal{A}^{+}.$$

Theorem 5 (Property of Monotonicity): Assuming $\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right)$ $(i = 1, 2, \ldots, n)$ and $\mathcal{A}_i^{*} = \left(\underline{\zeta}_i^{*}, \underline{\eta}_i^{*}, \underline{\xi}_i^{*}, \overline{\zeta}_i^{*}, \overline{\eta}_i^{*}, \overline{\xi}_i^{*}\right)$ $(i = 1, 2, \ldots, n)$ be a collection of two $q-SFRNs$ such that $\mathcal{A}_i \leq \mathcal{A}_i^{*}$ for all i, then $q-SFRFWA\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) \leq q-SFRFWA\left(\mathcal{A}_1^{*}, \mathcal{A}_2^{*}, \ldots, \mathcal{A}_n^{*}\right)$.

Proof: As, $\underline{\zeta}_i \leq \underline{\zeta}_i^{*}, \underline{\eta}_i \geq \underline{\eta}_i^{*}, \underline{\xi}_i \geq \underline{\xi}_i^{*}, \overline{\zeta}_i \leq \overline{\zeta}_i^{*}, \overline{\eta}_i \geq \overline{\eta}_i^{*}$ *and*, $\overline{\xi}_i \geq \overline{\xi}_i^{*}$

$$\text{So}\sqrt[q]{1-\log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{1-\underline{\zeta}_i^q}-1\right)^{\omega_i}\right)}$$
$$\leq \sqrt[q]{1-\log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{1-\underline{\zeta}_i^{*q}}-1\right)^{\omega_i}\right)}$$
$$\sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\underline{\eta}_i^q}-1\right)^{\omega_i}\right)}$$
$$\geq \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\underline{\eta}_i^{*q}}-1\right)^{\omega_i}\right)}$$
$$\sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\underline{\xi}_i^q}-1\right)^{\omega_i}\right)}$$
$$\geq \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\underline{\xi}_i^{*q}}-1\right)^{\omega_i}\right)}$$
$$\sqrt[q]{1-\log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{1-\overline{\zeta}_i^q}-1\right)^{\omega_i}\right)}$$
$$\leq \sqrt[q]{1-\log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{1-\overline{\zeta}_i^{*q}}-1\right)^{\omega_i}\right)}$$
$$\sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\overline{\eta}_i^q}-1\right)^{\omega_i}\right)}$$
$$\geq \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\overline{\eta}_i^{*q}}-1\right)^{\omega_i}\right)}$$

and

$$\sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\overline{\xi}_i^q}-1\right)^{\omega_i}\right)}$$
$$\geq \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\overline{\xi}_i^{*q}}-1\right)^{\omega_i}\right)}$$

Hence

$$q-SFRFWA\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right)$$
$$\leq q-SFRFWA\left(\mathcal{A}_1^{*}, \mathcal{A}_2^{*}, \ldots, \mathcal{A}_n^{*}\right).$$

C. $q-SFRFOWAOPERATOR$

Assuming $\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right)$ $(i = 1, 2, \ldots, n)$ ne a collection of q-SFRNs, the q-spherical fuzzy rough Frank ordered averaging operator (q-SFRFOWA) operator is defined as a mapping $q-SFRFOWA : \mathcal{A}^n \longrightarrow \mathcal{A}$ associated with the weight vector $(\omega_1, \omega_2, \ldots, \omega_n)^T$ adhering the condition $\omega_i > 0$ and the constrain $\sum_{i=1}^{n}\omega_i = 1$.

$$q-SFRFOWA\left(\mathcal{A}_{\delta(1)}, \mathcal{A}_{\delta(2)}, \ldots, \mathcal{A}_{\delta(n)}\right)$$
$$= \mathcal{A}_{\delta(1)} \oplus \mathcal{A}_{\delta(2)}, \ldots, \oplus\mathcal{A}_{\delta(n)} = \oplus_{i=1}^{n}\left(\omega_i\mathcal{A}_{\delta(i)}\right)$$

where $\delta(1), \delta(2), \ldots, \delta(n)$ is a permutation of $(1, 2, 3, .., n)$ such that $\mathcal{A}_{\delta(1)} \leq \mathcal{A}_{\delta(i-1)}$ for all $i = 1, 2, 3, .., n$.

Now,

Assuming $\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right)$ $(i = 1, 2, \ldots, n)$ be a collection of q-SFRNs and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector adhering to the condition $\omega_i > 0$ and the constrain $\sum_{i=1}^{n}\omega_i = 1$. Then

$$q-SFRFOWA\left(\mathcal{A}_{\delta(1)}, \mathcal{A}_{\delta(2)}, \ldots, \mathcal{A}_{\delta(n)}\right)$$
$$= \begin{bmatrix} \left\langle \begin{array}{l} \sqrt[q]{1-log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{1-\underline{\zeta}_{\delta(i)}^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\underline{\eta}_{\delta(i)}^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\underline{\xi}_{\delta(i)}^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{1-log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{1-\overline{\zeta}_{\delta(i)}^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\overline{\eta}_{\delta(i)}^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\overline{\xi}_{\delta(i)}^q}-1\right)^{\omega_i}\right)} \end{array} \right\rangle \end{bmatrix}$$

Theorem 6: Assuming $\mathcal{A}_i = (\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i)(i = 1, 2, \ldots, n)$ be a collection of q-SFRNs and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector adhering to the condition $\omega_i > 0$ and the constrain $\sum_{i=1}^{n}\omega_i = 1.1$ If it meets the requirements, it is known as $q-SFRFOWA$operator.

$$q-SFRFOWA\left(\mathcal{A}_{\delta(1)}, \mathcal{A}_{\delta(2)}, \ldots, \mathcal{A}_{\delta(n)}\right)$$
$$= \begin{bmatrix} \left\langle \begin{array}{l} \sqrt[q]{1-log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{1-\underline{\zeta}_{\delta(i)}^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\underline{\eta}_{\delta(i)}^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\underline{\xi}_{\delta(i)}^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{1-log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{1-\overline{\zeta}_{\delta(i)}^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\overline{\eta}_{\delta(i)}^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{log_{\tau}\left(1+\prod_{i=1}^{n}\left(\tau^{\overline{\xi}_{\delta(i)}^q}-1\right)^{\omega_i}\right)} \end{array} \right\rangle \end{bmatrix}$$

Proof: The proof is the same as Theorem 2.

Theorem 7 (Idempotency): Assuming $\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right)$ $(i = 1, 2, \ldots, n)$ be a collection of $q-SFRNs$ and $(\omega_1, \omega_2, \ldots, \omega_n)^T$ signifies the weight vector adhering to the condition $\omega_i > 0$ and the constrain $\sum_{i=1}^{n}\omega_i = 1$. $\mathcal{A}_i$ $(i = 1, 2, \ldots, n)$ are the same $\forall i$, then

$$q-SFRFOWA\left(\mathcal{A}_{\delta(1)}, \mathcal{A}_{\delta(2)}, \ldots, \mathcal{A}_{\delta(n)}\right) = \mathcal{A}$$

Proof: The proof is the same as Theorem 3.

Theorem 8 (Boundness): Assuming $\mathcal{A}_i = (\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i)$ $(i = 1, 2, \ldots, n)$ be a collection of $q-SFRNs$ and $(\omega_1, \omega_2, \ldots, \omega_n)^T$ signifies the weight vector adhering to the condition $\omega_i > 0$ and the constrain $\sum_{i=1}^{n} \omega_i = 1$.Let$\mathcal{A}^- = min\{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ and $\mathcal{A}^+ = max\{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$, then

Then

$$\mathcal{A}^- \leq q-SFRFOWA\left(\mathcal{A}_{\delta(1)}, \mathcal{A}_{\delta(2)}, \ldots, \mathcal{A}_{\delta(n)}\right) \leq \mathcal{A}^+$$

Proof: The proof is the same as Theorem 4.

Theorem 9 (Monotonicity): Assuming

$$\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right) (i = 1, 2, \ldots, n) \text{ and}$$

$\mathcal{A}_i^* = \left(\underline{\zeta}_i^*, \underline{\eta}_i^*, \underline{\xi}_i^*, \overline{\zeta}_i^*, \overline{\eta}_i^*, \overline{\xi}_i^*\right)$ $(i = 1, 2, \ldots, n)$ be a collection of two $q-SFRNs$ such that $\mathcal{A}_i \leq \mathcal{A}_i^*$ for all I, then $q-SFRFOWA\left(\mathcal{A}_{\delta(1)}, \mathcal{A}_{\delta(2)}, \ldots, \mathcal{A}_{\delta(n)}\right)$
$\leq q-SFRFOWA\left(\mathcal{A}_{\delta(1)}^*, \mathcal{A}_{\delta(2)}^*, \ldots, \mathcal{A}_{\delta(n)}^*\right)$.

Proof: The proof is the same as Theorem 5.

D. $q-SFRFHWA$OPERATOR

Assuming $\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right)$ $(i = 1, 2, \ldots, n)$ ne a collection of q-SFRNs, the q-spherical fuzzy rough Frank hybrid averaging operator (q-SFRFHWA) operator is defined as a mapping $q-SFRFHWA : \mathcal{A}^n \longrightarrow \mathcal{A}$ associated with the weight vector $(\omega_1, \omega_2, \ldots, \omega_n)^T$ adhering the condition $\omega_i > 0$ and the constrain $\sum_{i=1}^{n} \omega_i = 1$.

$$\begin{aligned} &q-SFRFHWA\left(\mathcal{A}_{\alpha(1)}, \mathcal{A}_{\alpha(2)}, \ldots, \mathcal{A}_{\alpha(n)}\right)\\ &= \mathcal{A}_{\alpha(1)} \oplus \mathcal{A}_{\alpha(2)}, \ldots, \oplus \mathcal{A}_{\alpha(n)} = \oplus_{i=1}^{n}\left(\omega_i \mathcal{A}_{\alpha(i)}\right) \end{aligned}$$

where $\alpha(1), \alpha(2), \ldots, \alpha(n)$ is a permutation of (1, 2, 3, .., n) such that $\mathcal{A}_{\alpha(1)} \leq \mathcal{A}_{\alpha(i-1)}$ for all $i = 1, 2, 3, .., n$.

Now,

Assuming $\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right)$ $(i = 1, 2, \ldots, n)$ be a collection of q-SFRNs and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector adhering to the condition $\omega_i > 0$ and the constrain $\sum_{i=1}^{n} \omega_i = 1$. Then

$$q-SFRFHWA\left(\mathcal{A}_{\alpha(1)}, \mathcal{A}_{\alpha(2)}, \ldots, \mathcal{A}_{\alpha(n)}\right) = \left[\left\langle \begin{array}{l} \sqrt[q]{1-log_\tau\left(1+\prod_{i=1}^{n}\left(\tau^{1-\underline{\zeta}_{\alpha(i)}^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{log_\tau\left(1+\prod_{i=1}^{n}\left(\tau^{\underline{\eta}_{\alpha(i)}^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{log_\tau\left(1+\prod_{i=1}^{n}\left(\tau^{\underline{\xi}_{\alpha(i)}^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{1-log_\tau\left(1+\prod_{i=1}^{n}\left(\tau^{1-\overline{\zeta}_{\alpha(i)}^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{log_\tau\left(1+\prod_{i=1}^{n}\left(\tau^{\overline{\eta}_{\delta(i)}^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{log_\tau\left(1+\prod_{i=1}^{n}\left(\tau^{\overline{\xi}_{\alpha(i)}^q}-1\right)^{\omega_i}\right)} \end{array} \right\rangle\right]$$

where $\alpha(i)$, the *ith* greatest value as determined by the overall order of $\mathcal{A}_{\alpha(1)} \geq \mathcal{A}_{\alpha(2)} \geq \cdots \geq \mathcal{A}_{\alpha(n)}$ where $\mathcal{A}_{\alpha(i)}$ has the *ith* highest weighted value.

Theorem 10: Assuming $\mathcal{A}_i = (\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i)(i = 1, 2, \ldots, n)$ be a collection of q-SFRNs and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector adhering to the condition $\omega_i > 0$ and the constrain $\sum_{i=1}^{n} \omega_i = 1.1$ If it meets the requirements, it is known as $q-SFRFHWA$operator.

$$q-SFRFHWA\left(\mathcal{A}_{\alpha(1)}, \mathcal{A}_{\alpha(2)}, \ldots, \mathcal{A}_{\alpha(n)}\right) = \left[\left\langle \begin{array}{l} \sqrt[q]{1-log_\tau\left(1+\prod_{i=1}^{n}\left(\tau^{1-\underline{\zeta}_{\alpha(i)}^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{log_\tau\left(1+\prod_{i=1}^{n}\left(\tau^{\underline{\eta}_{\alpha(i)}^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{log_\tau\left(1+\prod_{i=1}^{n}\left(\tau^{\underline{\xi}_{\alpha(i)}^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{1-log_\tau\left(1+\prod_{i=1}^{n}\left(\tau^{1-\overline{\zeta}_{\alpha(i)}^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{log_\tau\left(1+\prod_{i=1}^{n}\left(\tau^{\overline{\eta}_{\delta(i)}^q}-1\right)^{\omega_i}\right)}, \\ \sqrt[q]{log_\tau\left(1+\prod_{i=1}^{n}\left(\tau^{\overline{\xi}_{\alpha(i)}^q}-1\right)^{\omega_i}\right)} \end{array} \right\rangle\right]$$

Proof: The proof is the same as Theorem 2.

Theorem 11 (Idempotency): Assuming $\mathcal{A}_i = (\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i)(i = 1, 2, \ldots, n)$ be a collection of $q-SFRNs$and $(\omega_1, \omega_2, \ldots, \omega_n)^T$ signifies the weight vector adhering to the condition $\omega_i > 0$ and the constrain $\sum_{i=1}^{n} \omega_i = 1.\mathcal{A}_i\,(i = 1, 2, \ldots, n)$ are the same $\forall i$, then

$$q-SFRFHWA\left(\mathcal{A}_{\alpha(1)}, \mathcal{A}_{\alpha(2)}, \ldots, \mathcal{A}_{\alpha(n)}\right) = \mathcal{A}$$

Proof: The proof is the same as Theorem 3.

Theorem 12 (Boundness): Assuming $\mathcal{A}_i = (\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i)(i = 1, 2, \ldots, n)$ be a collection of $q-SFRNs$ and $(\omega_1, \omega_2, \ldots, \omega_n)^T$ signifies the weight vector adhering to the condition $\omega_i > 0$ and the constrain $\sum_{i=1}^{n} \omega_i = 1$.Let$\mathcal{A}^- = min\{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ and $\mathcal{A}^+ = max\{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$, then

Then $\mathcal{A}^- \leq q-SFRFHWA(\mathcal{A}_{\alpha(1)}, \mathcal{A}_{\alpha(2)}, \ldots, \mathcal{A}_{\alpha(n)}) \leq \mathcal{A}^+$

Proof: The proof is the same as Theorem 4.

Theorem 13 (Monotonicity): Assuming $\mathcal{A}_i = (\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i)$ $(i = 1, 2, \ldots, n)$ and $\mathcal{A}_i^* = (\underline{\zeta}_i^*, \underline{\eta}_i^*, \underline{\xi}_i^*, \overline{\zeta}_i^*, \overline{\eta}_i^*, \overline{\xi}_i^*)$ $(i = 1, 2, \ldots, n)$ be a collection of two $q-SFRNs$ such that $\mathcal{A}_i \leq \mathcal{A}_i^*$ for all i, then

$$\begin{aligned} &q-SFRFHWA\left(\mathcal{A}_{\alpha(1)}, \mathcal{A}_{\alpha(2)}, \ldots, \mathcal{A}_{\alpha(n)}\right)\\ &\leq q-SFRFHWA\left(\mathcal{A}_{\alpha(1)}^*, \mathcal{A}_{\alpha(2)}^*, \ldots, \mathcal{A}_{\alpha(n)}^*\right). \end{aligned}$$

Proof: The proof is the same as Theorem 5.

IV. APPLICATIONS

This section focuses on solving MADM problems using the previously described operators and q-SFR numbers. An example is presented to demonstrate the effectiveness and use of these operators in real-world scenarios.

Consider the following sets: $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \ldots, \mathcal{V}_m\}$ for the m alternatives, $\mathcal{J} = \{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \ldots, \mathcal{J}_n\}$ for n criteria, and $\mathcal{D} = \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, .., \mathcal{D}_k\}$ for k experts.

Consider the corresponding weight vector for criteria as $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$. The weight vectors meet the identical requirements and are in the closed interval [0, 1], with their sum equal to one. Let $\mathcal{A}_{ij} = (\underline{\zeta}_{ij}, \underline{\eta}_{ij}, \underline{\xi}_{ij}, \overline{\zeta}_{ij}, \overline{\eta}_{ij}, \overline{\xi}_{ij})(i = 1, 2, \ldots, n)$ and $(j = 1, 2, \ldots, m)$, where $(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i)$ and $(\overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i)$ represents lower set approximation and upper aet approximation, subject to the constraint $(0 \leq \underline{\zeta}_{ij}^q + \underline{\eta}_{ij}^q + \underline{\xi}_{ij}^q \leq 1)$ and $(0 \leq \overline{\zeta}_{ij}^q + \overline{\eta}_{ij}^q + \overline{\xi}_{ij}^q \leq 1)$. Following is the procedure to solve an MCDM problem.

Step 1: Construct $\mathcal{D}^{(k)} = \left[\left(\mathcal{A}_{ij}^{(k)}\right)\right]_{m\times n} (k = 1, 2, 3, \ldots, d)$ for decision.

These matrices can be seen as the input features for the neural network.

Step 2: This step is focused on handling different types of criteria, especially distinguishing between benefit criteria and cost criteria, in the context of a neural network-based decision-making process. If the criteria have two types, such as benefit criteria and cost criteria, the $\mathcal{D}^{(k)} = \left[\left(\mathcal{A}_{ij}^{(k)}\right)\right]_{m\times n} (k = 1, 2, 3, \ldots, d)$ can be converted into the normalized decision matrices $\mathcal{R}^{(s)} = (s = 1, 2, 3, \ldots, t)$ where

$$r^{(s)} = \begin{cases} \mathcal{A}_{ij}^{(k)} \text{ for benefit type of criteria} \\ \left[\left(\mathcal{A}_{ij}^{(k)}\right)\right]^C \text{ for cost type of criteria} \end{cases}$$

where $\left[\left(\mathcal{A}_{ij}^{(k)}\right)\right]^C$ is a complement of $\mathcal{A}_{ij}^{(k)}$.

Step 3: This step aligns with the neural network training process, with the proposed operators. The aggregation of matrices into R can be seen as the neural network output. Utilize the proposed operators to aggregate $\mathcal{R}^{(k)} = \left[\left(\mathcal{A}_{ij}^{(k)}\right)\right]_{m\times n}$ into $\mathcal{R} = \left[\mathcal{A}_{ij}\right]_{m\times n}$

Step 4: After obtaining the neural network output ($\mathcal{R}$), this step corresponds to evaluating the score of different alternatives based on the calculated values from the neural network.

Step 5: This final step aligns with the decision-making outcome, where the alternative with the highest score, derived from the neural network, is chosen as the best solution.

The flow chart of the proposed model is shown in Figure 8.

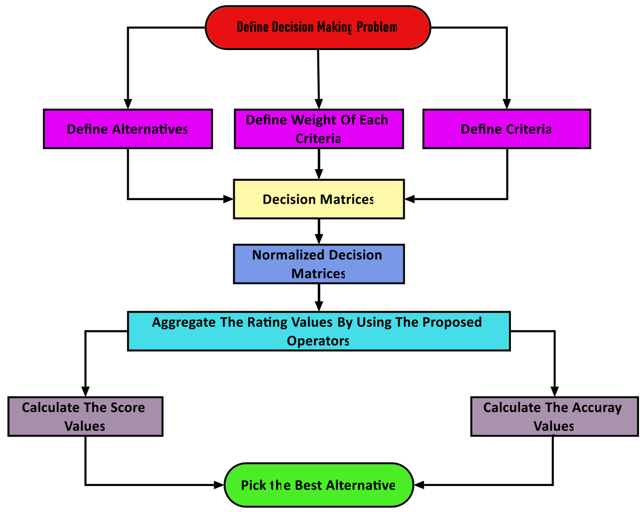

FIGURE 8. Flow chart of the proposed model.

A. NUMERICAL EXAMPLE

To elucidate and illustrate the suggested technique, we offer an example in this section. In this scenario, the American army is tasked with selecting a new transport vehicle for its ground forces. The decision-making process involves evaluating four alternative vehicles based on four criteria. The criteria are: $\mathcal{J}_1$ = Mobility, $\mathcal{J}_2$ = Payload capacity, $\mathcal{J}_3$ = Fuel efficiency and $\mathcal{J}_4$ = Tactical versatility. The alternatives are $\mathcal{V}_1$ = M1 Abrams tank transporter, $\mathcal{V}_2$ = Oshkosh defense JLTV (Joint Light Tactical Vehicle), $\mathcal{V}_3$ = Humvee (High Mobility Multipurpose Wheeled Vehicle) and $\mathcal{V}_4$ = Boing CH-47 Chinook Helicopter. A group of four experts assigns weights to these criteria $\omega = (0.3, 0.1, 0.4, 0.2)^T$. The weight vector presents the importance of each criterion as determined by a group of experts in the context of the neural network approach in artificial intelligence.

The sum of the components of a weight vector being 1 is a fundamental requirement in many decision-making and optimization techniques, including those utilizing fuzzy sets and rough sets. This normalization ensures that the weights assigned to different attributes or criteria collectively represent the entire decision space without bias or distortion. The normalization process helps in standardizing the weights, making them comparable across different attributes or criteria. It ensures that the influence of each attribute on the decision-making process is appropriately balanced and proportionate. Additionally, a weight vector with components summing to 1 simplifies the interpretation of the weights, as they can be directly interpreted as percentages or proportions of importance. The reason behind this normalization is to avoid any disproportionate influence of individual attributes on the decision outcome, thereby promoting fairness and consistency in decision-making. By ensuring that the sum of weights is 1, decision-makers can more accurately assess the relative importance of each attribute and make informed decisions based on a comprehensive understanding of the entire decision space.

Now let's create a decision matrix $\mathcal{D}^{(k)} = (\alpha_{ij}^{(k)})(k = 1, 2, 3, 4)$ for each transport vehicle. Tables 1-4 provide the decision matrix for the neural network approach in AI. The goal is to rank these cars and select the most suitable new transport vehicle for the ground forces of the American army. Figure 9 illustrates a decision tree used for the neural networks in AI for selecting new transport vehicles for the ground forces of the American army.

The values depicted in these (Tables 1-4) were indeed generated randomly and served as inputs for the neural network utilized in our study.

Step 1: Construct the decision matrices (Tables 1-4).

Step 2: Construct the normalized decision matrices. Since none of the above criteria (mobility, payload capacity, fuel

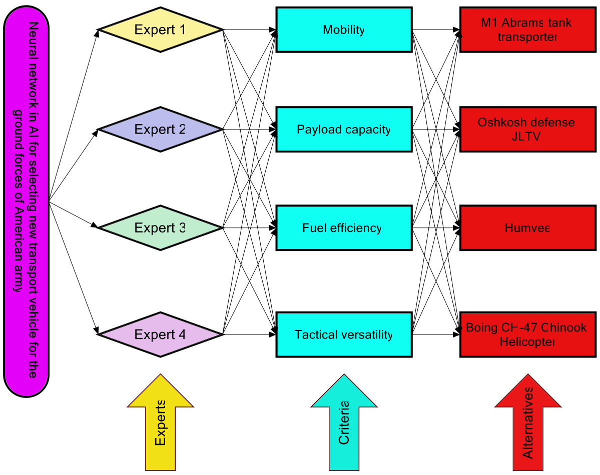

FIGURE 9. Neural network in AI for selecting new transport vehicles for the ground forces of the American army.

TABLE 1. Decision matrix $\mathcal{D}^1$.

Alternatives	$\mathcal{J}_1$	$\mathcal{J}_2$
$\mathcal{V}_1$	(0.58,0.27,0.74, 0.67,0.76,0.74)	(0.29,0.65,0.65, 0.28,0.08,0.36)
$\mathcal{V}_2$	(0.25,0.25,0.85, 0.79,0.97,0.97)	(0.05,0.06,0.26, 0.56,0.06,0.98)
$\mathcal{V}_3$	(0.24,0.32,0.73, 0.24,0.25,0.98)	(0.98,0.68,0.19, 0.68,0.28,0.79)
$\mathcal{V}_4$	(0.19,0.96,0.69, 0.27,0.79,0.78)	(0.98,0.95,0.15, 0.95,0.75,0.15)
Alternatives	$\mathcal{J}_3$	$\mathcal{J}_4$
$\mathcal{V}_1$	(0.95,0.16,0.95, 0.05,0.16,0.65)	(0.96,0.75,0.79, 0.32,0.73,0.63)
$\mathcal{V}_2$	(0.95,0.45,0.35, 0.19,0.45,0.55)	(0.45,0.66,0.96, 0.64,0.54,0.23)
$\mathcal{V}_3$	(0.15,0.28,0.23, 0.31,0.25,0.18)	(0.75,0.46,0.35, 0.86,0.35,0.83)
$\mathcal{V}_4$	(0.26,0.83,0.16, 0.29,0.93,0.86)	(0.96,0.36,0.13, 0.85,0.25,0.75)

TABLE 2. Decision matrix $\mathcal{D}^2$.

Alternatives	$\mathcal{J}_1$	$\mathcal{J}_2$
$\mathcal{V}_1$	(0.24,0.83,0.27, 0.71,0.25,0.25)	(0.24,0.17,0.23, 0.20,0.94,0.22)
$\mathcal{V}_2$	(0.39,0.32,0.59, 0.47,0.62,0.92)	(0.95,0.36,0.82, 0.76,0.69,0.47)
$\mathcal{V}_3$	(0.36,0.62,0.59, 0.56,0.35,0.94)	(0.57,0.73,0.27, 0.73,0.69,0.22)
$\mathcal{V}_4$	(0.17,0.45,0.99, 0.69,0.57,0.97)	(0.27,0.86,0.27, 0.92,0.26,0.53)
Alternatives	$\mathcal{J}_3$	$\mathcal{J}_4$
$\mathcal{V}_1$	(0.82,0.83.92, 0.85,0.39,0.80)	(0.62,0.36,0.35, 0.75,0.52,0.65)
$\mathcal{V}_2$	(0.29,0.61,0.56, 0.20,0.86,0.25)	(0.38,0.65,0.72, 0.82,0.25,0.32)
$\mathcal{V}_3$	(0.66,0.37,0.35, 0.26,0.86,0.96)	(0.79,0.23,0.55, 0.96,0.86,0.29)
$\mathcal{V}_4$	(0.88,0.65,0.23, 0.12,0.55,0.49)	(0.69,0.67,0.59, 0.82,0.63,0.31)

TABLE 3. Decision matrix $\mathcal{D}^3$.

Alternatives	$\mathcal{J}_1$	$\mathcal{J}_2$
$\mathcal{V}_1$	(0.36,0.36,0.32 0.69,0.36,0.24)	(0.85,0.10,0.56, 0.48,0.08,0.39)
$\mathcal{V}_2$	(0.47,0.98,0.74, 0.36,0.78,0.27)	(0.48,0.29,0.52, 0.68,0.78,0.35)
$\mathcal{V}_3$	(0.39,0.79,0.74, 0.27,0.78,0.63)	(0.28,0.69,0.57, 0.69,0.85,0.32)
$\mathcal{V}_4$	(0.27,0.58,0.25, 0.79,0.25,0.78)	(0.62,0.69,0.58, 0.96,0.85,0.35)
Alternatives	$\mathcal{J}_3$	$\mathcal{J}_4$
$\mathcal{V}_1$	(0.54,0.89,0.32, 0.89,0.56,0.23)	(0.89,0.30,0.59, 0.25,0.50,0.20)
$\mathcal{V}_2$	(0.36,0.74,0.56, 0.23,0.89,0.25)	(0.65,0.30,0.58, 0.39,0.51,0.70)
$\mathcal{V}_3$	(0.78,0.63,0.74, 0.23,0.65,0.25)	(0.23,0.30,0.90, 0.39,0.50,0.39)
$\mathcal{V}_4$	(0.32,0.56,0.23, 0.25,0.58,0.52)	(0.32,0.30,0.54, 0.56,0.85,0.85)

efficiency, or tactical versatility) are identified as cost. Cost criteria are frequently features for which lower values are favored, albeit this varies depending on the context of the decision-making. So, there is no need for normalization.

Step 3. Utilize the q-SFRFWA operator to derive the overall aggregated values $\mathcal{R}$.

Utilize the q-SFRFWA operator once again to derive the overall preference values.

Calculate the overall preference values $\mathcal{B}_i$ $(i = 1, 2, 3, 4)$ for the alternative $\mathcal{V}_i (i = 1, 2, 3, 4)$ using the given data and the $q - SFRFHWA$ operator as shown below:

$$\mathcal{B}_1 = \begin{pmatrix} 0.5287, 0.5648, 0.3256, \\ 0.2546, 0.5467, 0.5468 \end{pmatrix},$$

$$\mathcal{B}_2 = \begin{pmatrix} 0.8547, 0.6534, 0.7458, \\ 0.2546, 0.3654, 0.8547 \end{pmatrix},$$

$$\mathcal{B}_3 = \begin{pmatrix} 0.2568, 0.4527, 0.7485, \\ 0.7429, 0.8546, 0.6589 \end{pmatrix} \text{ and}$$

$$\mathcal{B}_4 = \begin{pmatrix} 0.8547, 0.6523, 0.5874, \\ 0.4578, 0.4256, 0.3258 \end{pmatrix}.$$

Step 4: By using Equation (9) we get $Sco(\mathcal{B}_1) = 0.5409$, $Sco\,(\mathcal{B}_2) = 0.4246$, $Sco(\mathcal{B}_3) = 0.3349$ and $Sco(\mathcal{B}_4) = 0.7095$.

Step 5: Using the score values, we can establish the ranking order of the available alternatives as follows: $\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$. Hence $\mathcal{V}_4 =$ Boing CH-47 Chinook Helicopter is the best alternative.

By utilizing the q-SFRFOWA operator overall preference values $\mathcal{B}_i$ $(i = 1, 2, 3, 4)$ for the alternative $\mathcal{V}_i (i = 1, 2, 3, 4)$ using the given data and the $q-SFRFOWA$ operator

TABLE 4. Decision matrix $\mathcal{D}^4$.

Alternatives	$\mathcal{J}_1$	$\mathcal{J}_2$	$\mathcal{J}_3$	$\mathcal{J}_4$
$\mathcal{V}_1$	(0.27,0.63,0.36, 0.69,0.96,0.25)	(0.54,0.37,0.53, 0.39,0.52,0.37)	(0.57,0.38,0.55, 0.39,0.53,0.37)	(0.56,0.31,0.56, 0.34,0.53,0.34)
$\mathcal{V}_2$	(0.27,0.85,0.52, 0.27,0.89,0.78)	(0.56,0.36,0.54, 0.36,0.57,0.36)	(0.60,0.36,0.56, 0.80,0.54,0.36)	(0.56,0.36,0.52, 0.80,0.54,0.34)
$\mathcal{V}_3$	(0.47,0.39,0.25, 0.74,0.78,0.96)	(0.52,0.34,0.54, 0.39,0.57,0.36)	(0.30,0.36,0.53, 0.39,0.54,0.34)	(0.50,0.39,0.55, 0.39,0.51,0.35)
$\mathcal{V}_4$	(0.12,0.59,0.32, 0.25,0.74,0.74)	(0.97,0.36,0.57, 0.66,0.57,0.35)	(0.54,0.36,0.52, 0.33,0.57,0.34)	(0.60,0.34,0.55, 0.34,0.55,0.34)

TABLE 5. Aggregated decision matrix $\mathcal{R}$.

$\mathcal{V}_i$	$\mathcal{J}_1$	$\mathcal{J}_2$	$\mathcal{J}_3$	$\mathcal{J}_4$
$\mathcal{V}_1$	(0.5486,0.5478,0.6598, 0.2485,0.2546,0.5468)	(0.2587,0.2456,0.8569, 0.6987,0.2546,0.8576)	(0.4854,0.8547,0.8596, 0.2546,0.5246,0.6523)	(0.8547,0.2569,0.7489, 0.2365,0.8546,0.7455)
$\mathcal{V}_2$	(0.6524,0.2546,0.4785, 0.7485,0.4189,0.2546)	(0.2546,0.7485,0.2536, 0.2569,0.5236,0.7489)	(0.8745,0.8547,0.7458, 0.2549,0.5246,0.2536)	(0.2546,0.2418,0.8546, 0.2478,0.2365,0.8745)
$\mathcal{V}_3$	(0.6523,0.3456,0.2356, 0.9674,0.2546,0.4579)	(0.2453,0.7423,0.2543, 0.2548,0.7485,0.8596)	(0.5694,0.8574,0.7485, 0.2456,0.3256,0.2536)	(0.2548,0.7413,0.8563, 0.2586,0.9643,0.2564)
$\mathcal{V}_4$	(0.3256,0.9874,0.2452, 0.6547,0.2549,0.4785)	(0.2457,0.2369,0.7425, 0.7485,0.7428,0.2543)	(0.2436,0.7459,0.7458, 0.7425,0.2453,0.8546)	(0.2549,0.2546,0.3256, 0.2587,0.7489,0.3274)

as shown below:

$$\mathcal{B}_1 = \begin{pmatrix} 0.5246, 0.3697, 0.2546, \\ 0.3879, 0.2489, 0.2437 \end{pmatrix},$$

$$\mathcal{B}_2 = \begin{pmatrix} 0.4569, 0.2463, 0.4897, \\ 0.3658, 0.2543, 0.7458 \end{pmatrix},$$

$$\mathcal{B}_3 = \begin{pmatrix} 0.2543, 0.2436, 0.8746, \\ 0.2785, 0.2356, 0.5489 \end{pmatrix} \text{ and}$$

$$\mathcal{B}_4 = \begin{pmatrix} 0.9856, 0.2551, 0.2463, \\ 0.2451, 0.7486, 0.5674 \end{pmatrix}.$$

By using Equation (9) we get $Sco(\mathcal{B}_1) = 0.7019$, $Sco(\mathcal{B}_2) = 0.5269$, $Sco(\mathcal{B}_3) = 0.3920$ and $Sco(\mathcal{B}_4) = 0.7795$. Using the score values, we can establish the ranking order of the available alternatives as follows: $\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$. Hence $\mathcal{V}_4$ = Boing CH-47 Chinook Helicopter is the best alternative.

By utilizing the q-SFRFHWA operator overall preference values $\mathcal{B}_i$ $(i = 1, 2, 3, 4)$ for the alternative $\mathcal{V}_i (i = 1, 2, 3, 4)$ using the given data and the $q-SFRFHWA$ operator as shown below:

$$\mathcal{B}_1 = \begin{pmatrix} 0.1596, 0.8462, 0.3529, \\ 0.8524, 0.5749, 0.25463 \end{pmatrix},$$

$$\mathcal{B}_2 = \begin{pmatrix} 0.2549, 0.4785, 0.2439, \\ 0.2874, 0.5283, 0.7534 \end{pmatrix},$$

$$\mathcal{B}_3 = \begin{pmatrix} 0.7481, 0.4726, 0.1569, \\ 0.5246, 0.9854, 0.6289 \end{pmatrix} \text{ and}$$

$$\mathcal{B}_4 = \begin{pmatrix} 0.2547, 0.2789, 0.5467, \\ 0.7412, 0.3678, 0.5824 \end{pmatrix}.$$

By using Equation (9) we get $Sco(\mathcal{B}_1) = 0.5890$, , $Sco(\mathcal{B}_2) = 0.4470$, $Sco(\mathcal{B}_3) = 0.4160$ and $Sco(\mathcal{B}_4) = 0.6638$. Using the score values, we can establish the ranking order of the available alternatives as follows: $\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$. Hence $\mathcal{V}_4$ = Boing CH-47 Chinook Helicopter is the best alternative.

Table 6 provides a succinct illustration of the score's values utilizing the $q-SFRFWA$, $q-SFRFOWA$, and $q-SFRFHWA$ operators.

TABLE 6. Alternatives scores and sequence of ranking.

Operators	Score values			
	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$
$q-SFRFWA$	0.5409	0.4246	0.3349	0.7095
$q-SFRFOWA$	0.7019	0.5269	0.3920	0.7795
$q-SFRFHWA$	0.5890	0.4470	0.4160	0.6638

Table 7 provides a succinct illustration of the ranking order utilizing the $q-SFRFWA$, $q-SFRFOWA$ and $q-SFRFHWA$ operators.

TABLE 7. The alternative sequence of ranking.

Operators	Ranking
$q-SFRFWA$	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$
$q-SFRFOWA$	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$
$q-SFRFHWA$	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$

The graphical representation of score values is shown in Figure 10.

This study also investigates the applicability of these operators in cases where decision-makers seek to adjust their choice aggregation approaches to their preferences. Table 8 shows the results when various operators are employed, demonstrating how decision-makers may improve their decisions by considering both assigned values and expert opinions at the same time. The previous discussion shows that the proposed aggregation operators offer decision-makers a more adaptive framework for selecting viable choices. Furthermore, as compared to conventional aggregation methods, these operators offer more flexibility.

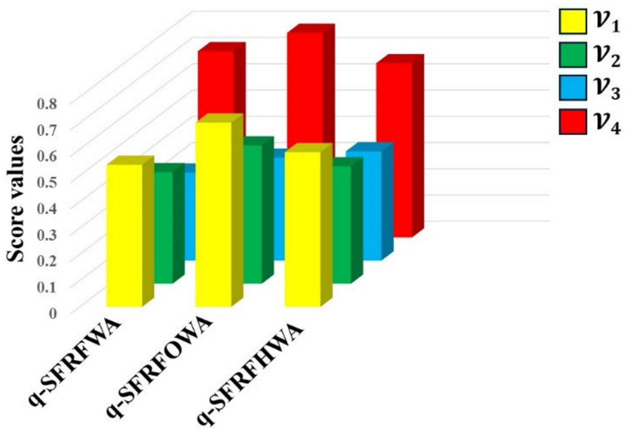

FIGURE 10. Graphical representation of score values of q-SFRFWA, q-SFRFOWA and q-SFRFHWA.

This shows that the proposed operators can handle a broader range of decision-making scenarios while also providing better flexibility and relevance in several settings. By providing a more adaptive and inclusive framework, these aggregation operators enable decision-makers to make informed decisions that are in line with their needs and preferences. Furthermore, the generalizability of these operators assures their usefulness across a wide range of decision domains, hence improving the overall robustness and reliability of the decision-making process.

B. EFFECT OF q ON RANKING ORDER AND SCORE VALUES

To fulfill the constraint requirement $(0 \leq \underline{\zeta}^q_{\mathcal{A}}(\mathfrak{w}) + \underline{\eta}^q_{\mathcal{A}}(\mathfrak{w}) + \underline{\xi}^q_{\mathcal{A}}(\mathfrak{w}) \leq 1)$ and $(0 \leq \overline{\zeta}^q_{\mathcal{A}}(\mathfrak{w}) + \overline{\eta}^q_{\mathcal{A}}(\mathfrak{w}) + \overline{\xi}^q_{\mathcal{A}}(\mathfrak{w}) \leq 1)$, and then by examining the attribute values, the decision maker is capable of identifying a minimum numerical parameter q. For example, while evaluating an alternative, if the attribute values are (0.54,0.25,0.54,0.96,0.78,0.69), one should choose q as 3 or q as 4, as both configurations meet the criterion. However, we employed several values of q in Step 3 of the novel approach to solve the case to fully evaluate the effect of parameter q on the experimental results. Table 9 presents the results of these modifications and indicates that $\mathcal{V}_4$ is at the top, followed by $\mathcal{V}_1$, $\mathcal{V}_2$, and finally, $\mathcal{V}_3$. Notable is the relevance of the best alternative and the unchanging ranking. Table 10 illustrates this point. Specifically, when q equals 1. The alternatives and ratings offered do not adhere to the requirements of either 1 (i.e., under PFRS environment $\left(0 \leq \underline{\zeta}_{\mathcal{A}}(\mathfrak{w}) + \underline{\eta}_{\mathcal{A}}(\mathfrak{w}) + \underline{\xi}_{\mathcal{A}}(\mathfrak{w}) \leq 1\right)$ and $\left(0 \leq \overline{\zeta}_{\mathcal{A}}(\mathfrak{w}) + \overline{\eta}_{\mathcal{A}}(\mathfrak{w}) + \overline{\xi}_{\mathcal{A}}(\mathfrak{w}) \leq 1\right)$) or 2 (i.e., under SFRS environment $(0 \leq \underline{\zeta}^2_{\mathcal{A}}(\mathfrak{w}) + \underline{\eta}^2_{\mathcal{A}}(\mathfrak{w}) + \underline{\xi}^2_{\mathcal{A}}(\mathfrak{w}) \leq 1)$ and $(0 \leq \overline{\zeta}^2_{\mathcal{A}}(\mathfrak{w}) + \overline{\eta}^2_{\mathcal{A}}(\mathfrak{w}) + \overline{\xi}^2_{\mathcal{A}}(\mathfrak{w}) \leq 1)$.

Table 8 illustrates the consistent consistency in the ranking order of alternatives at different q-parameter values. This enduring stability of the hierarchy offers decision-makers a reliable framework for evaluating test alternatives within a limited set. It establishes a safe and flexible environment for careful examination and informed decision-making based on defined parameters.

TABLE 8. Sorting alternatives according to their respective parameter q values.

Parameter q	Ranking order	Best alternative
$q = 1$	Unable to determine	---
$q = 2$	Unable to determine	---
$q = 3$	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_4$
$q = 4$	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_4$
$q = 5$	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_4$
$q = 6$	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_4$
$q = 7$	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_4$
$q = 8$	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_4$
$q = 9$	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_4$
$q = 10$	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_4$
$q = 11$	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_4$

TABLE 9. Ranking result for different values of τ using the q-SFRFWA.

Parameter	Score values				Ranking
	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	
$\tau = 3$	0.5409	0.4246	0.3349	0.7095	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$
$\tau = 5$	0.4965	0.3752	0.2856	0.6585	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$
$\tau = 7$	0.4327	0.3125	0.2274	0.5985	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$
$\tau = 9$	0.3674	0.2459	0.1524	0.5248	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$
$\tau = 12$	0.2854	0.1697	0.0798	0.4678	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$

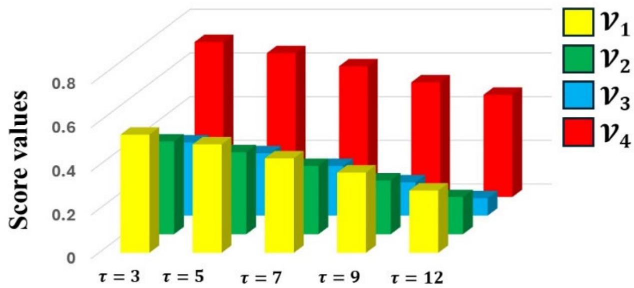

FIGURE 11. Graphical representation of ranking result for different values of τ using the q-SFRFWA.

TABLE 10. Ranking result for different values of τ using the q-SFRFOWA.

Parameter	Score values				Ranking
	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	
$\tau = 3$	0.7019	0.5269	0.3920	0.7795	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$
$\tau = 5$	0.6127	0.4389	0.3052	0.6874	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$
$\tau = 7$	0.5367	0.3596	0.2274	0.6025	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$
$\tau = 9$	0.4658	0.2874	0.1528	0.5386	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$
$\tau = 12$	0.4027	0.2254	0.0963	0.4785	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$

C. TEST OF VALIDITY

To demonstrate the versatility of the proposed technique in diverse contexts, we use the evaluation protocol introduced by Wang and Trianafilo [21] as follows:

Step 1: Changing the ranking values of sub-optimal alternatives that indicate inferior quality is not expected to affect

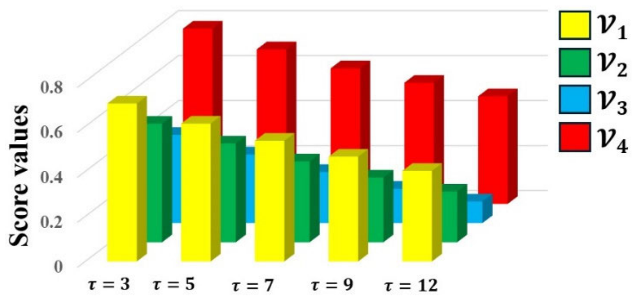

FIGURE 12. Graphical representation of ranking result for different values of τ using the q-SFRFOWA.

TABLE 11. Ranking result for different values of τ using the q-SFRFHWA.

Parameter	Score values				Ranking
	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	
$\tau = 3$	0.5890	0.4470	0.4160	0.6638	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$
$\tau = 5$	0.5147	0.3952	0.3025	0.6585	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$
$\tau = 7$	0.4725	0.3574	0.2674	0.6329	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$
$\tau = 9$	0.4287	0.3028	0.2145	0.5987	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$
$\tau = 12$	0.3654	0.2474	0.1574	0.5396	$\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$

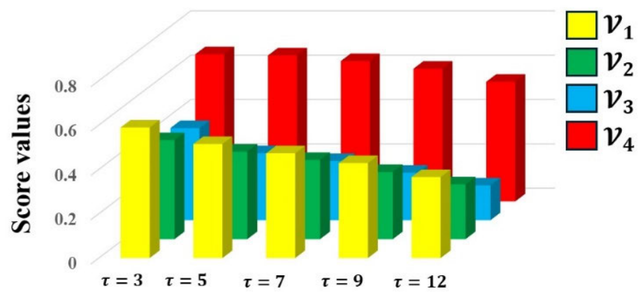

FIGURE 13. Graphical representation of ranking result for different values of τ using the q-SFRFHWA.

the identification of optimal alternatives. It preserves the highest-ranked choice, assuming a constant relative weight for the criteria.

Step 2. Transitivity should be followed in the procedure.

Step 3. When using the same decision-making process for a given problem that has been broken into smaller ones, the initial ranking of the alternatives should be preserved.

Test of validity utilizing criteria 1.

The alternatives ranked by using our suggested method are $\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$. Based on test criteria 1, we replaced the non-optimal alternative A_1 with the lowest alternative A_1^* to evaluate the stability of the suggested method. (0.25,0.85,0.25,0.63,0.96,0.25), (0.25,0.85,0.74,0.74,0.58,0.87), and (0.24,0.97,0.12,0.85, 0.78,0.36) were used as the rating values of $\mathcal{V}_3^*$. The aggregated score values for the alternatives were as follows after we used our suggested methodology: $Sco(\mathcal{V}_1) = 0.7523$, $Sco(\mathcal{V}_2) = 0.6951$, $Sco(\mathcal{V}_3^*) = 0.6589$, and $Sco(\mathcal{V}_4) = 0.8954$. As a result, $\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3^*$ is the new ranking order, and the best alternative still adheres to the first suggested strategy. Consequently, our method meets test requirement 1 by producing a consistent result.

Test of validity employing criteria 2 and 3.

The fragmented decision-making subcases are regarded as $\{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_4\}$, $\{\mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4\}$ and $\{\mathcal{V}_1, \mathcal{V}_3, \mathcal{V}_4\}$ to assess the validity based on criteria 2 and 3. They rank in the following sequence via the procedures mentioned: $\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2$, $\mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_3$ and $\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_3$. After combining all the findings, the overall ranking appears as $\mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3$, This is perfectly consistent with the results of the initial decision-making process. Consequently, our proposed strategy meets the criteria stated in requirements 2 and 3.

As a researcher, it is crucial to make our findings accessible and reproducible for fellow academics. Therefore, authors must provide a structured tabular representation elucidating the neural network hyperparameters utilized in their study. This should include essential parameters such as Network architecture (e.g., number of layers, type of layers) Activation functions Learning rate Batch size Optimizer algorithm Regularization techniques (e.g., dropout rate) Loss function Training epochs Providing this information in a clear and organized format would not only aid in replicating the experiments but also allow for a better understanding of how different hyperparameters may impact the results. Additionally, the researchers need to address the technical gaps identified in the introduction by proposing novel approaches to handle uncertainty and ambiguity within decision-making scenarios. By integrating q-spherical fuzzy rough Frank aggregation operators into neural network frameworks, the study aims to bridge these gaps and enhance the resilience and effectiveness of AI systems in navigating complex decision-making scenarios, particularly in domains like military transport systems.

TABLE 12. Neural network hyperparameters.

Hyperparameter	**Value/Description**
Network architecture	Multi-layer perceptron (MLP)
Number of layers	3 (input layer, hidden layer, output layer)
Hidden layer size	128 neurons
Activation function	ReLU (Rectified Linear Unit)
Learning rate	0.001
Batch size	32
Optimizer algorithm	Adam
Regularization	Dropout (0.2)
Loss function	Mean Squared Error (MSE)
Training epochs	100

This table provides a clear overview of the key hyperparameters used in the neural network architecture, including details on the network's structure, activation functions, learning rate, batch size, optimizer algorithm, regularization technique, loss function, and training epochs.

V. MANAGERIAL IMPLICATIONS

The framework shows remarkable flexibility and efficacy in a variety of decision-making contexts. Executives in a variety of fields can effectively use the q-SFR Frank aggregation operators for a range of objectives. For example, it demonstrates its value in the neural network in the AI selection procedure by subsidiary the estimation of various considerations of the procedure to establish the best beneficial transport supply. Moreover, it might assist in the choice of conservation performances, granting supervisors to select the beneficial approach to preserve their tools or approach. The additional region where the demonstration may be worked is in the assessment of machines in developed situations, which can assist supervisors in deciding the effectiveness and pertinence of numerous automated organizations. It may also be consumed in the collection of substantial operating tackle, and supplementary executives in making sophisticated findings observing the best and highest beneficial tools for their requirements. It is significant, nonetheless, to identify that the executive construction procedure within this structure is manipulated by the inclinations of specialists and participants. While the template suggests a systematic and methodical methodology for decision-making, the assumptions and standings are determined by the decision-makers' conclusions and inclinations. As an outcome, involving professionals and participants is significant in confirming the legitimacy and reputation of the conclusions. Two most important assessments are conceded out to enhance the trustworthiness and sturdiness of the attained outcomes:

Comparative analysis is an effective tool for decision-makers to rank, consider, and judge conclusions from several alternatives, each explored using separate conditions. It promotes a better knowledge of trade agreements and allows for more informed decision-making by highlighting the advantages and disadvantages of each possibility. By performing a sensitivity analysis, important insights are gained about the stability and sensitivity of the results. Decision-makers can examine how various factors influence their choices, enhancing their ability to make adaptive decisions in a dynamic environment. Incorporating this analysis into the decision-making process enables managers to increase reliability and confidence in their strategic decisions. The q-SFR Frank aggregation operators, combined with comparative and sensitivity analysis, offers a comprehensive framework that equips managers in diverse industries and applications with the tools needed to make informed and flexible decisions.

A. COMPARATIVE ANALYSIS

A comparative study is conducted to validate the robustness and effectiveness of this research against other contemporary multiple criteria decision-making (MCDM) methods. To achieve this goal, the problem is solved using eleven different MCDM models that operate within the framework of T-spherical fuzzy methods. The models selected for comparison include T-SFEHIA [22], T-SFEHIG [22], T-SFHHA [23], T-SFHHG [23], T-SFFHA [24], T-SFFHG [24], CT-SFWA [35], CT-SFWG [35], T-SFG [36], TSFPWA [37] and TSFPWG [37].

Table 13 presents the evaluation rankings derived from both the proposed model and the existing eleven MCDM models, highlighting comparative evaluations in different decision-making frameworks.

TABLE 13. Score values of different approaches.

Approaches	Score values			
	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$
T-SFEHIA [22]	−0.0207	−0.2146	−0.9255	0.0230
T-SFEHIG [22]	0.0296	−0.0676	−0.1309	0.1639
T-SFHHA [23]	0.1525	0.1048	0.1091	0.3257
T-SFHHG [23]	−0.1520	−0.1529	−0.2851	−0.0041
T-SFFHA [24]	0.1171	0.0615	0.0391	0.3218
T-SFFHG [24]	−0.1160	−0.1195	−0.2451	0.0619
CT-SFWA [35]	0.8120	0.4800	0.8120	0.7400
CT-SFWG [35]	0.8340	0.4830	0.8340	0.7270
T-SFG [36]	0.7645	0.252	−0.2115	−0.1833
TSFPWA [37]	−0.31708	0.09386	0.055878	0.016373
TSFPWG [37]	−0.41794	0.200728	0.438338	0.458634
q-SFFWA [This Paper]	0.5409	0.4246	0.3349	0.7095
q-SFFOWA [This Paper]	0.7019	0.5269	0.3920	0.7795
q-SFFHWA [This Paper]	0.5890	0.4470	0.4160	0.6638

The graphical representation of the score values of different approaches is shown in Figure. 14.

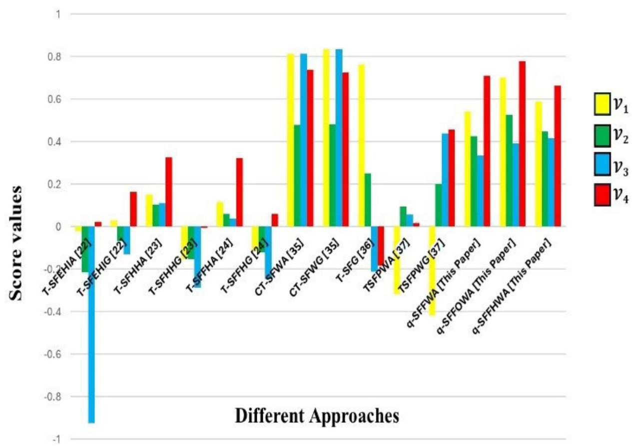

FIGURE 14. Graphical representation of different approaches and their score values.

Based on the propositions, calculations, and applications discussed above, the following comparative remarks and advantages of employing the notion of q-spherical fuzzy rough sets emerge:

1. Traditional fuzzy sets and intuitionistic fuzzy sets exhibit limitations as they may fail to capture complete information specifications in certain scenarios. The conditions of membership degree and non-membership degree may not always be satisfied, restricting decision makers from expressing opinions freely.

2. To address these limitations, Yager proposed Pythagorean fuzzy sets, extending the representation to $\zeta^2 + \xi^2 \leq 1$., enabling a wider range of applications.
3. In contexts involving uncertain information, such as voting systems, the introduction of "degree of refusal" necessitates the utilization of picture fuzzy sets. However, this approach presents its limitations in accommodating decision-maker flexibility.
4. Spherical fuzzy numbers offer a solution, capable of representing diverse information sets without exceeding the bounds of unity. This flexibility empowers decision-makers to allocate membership values according to their preferences.
5. The utilization of q-spherical fuzzy rough sets and associated algorithms, as demonstrated in selection processes, provides a generalized framework for impactful applications.
6. The proposed aggregation operators effectively handle imprecise information with a degree of refusal, offering superior reliability compared to existing approaches, as delineated in Table 12 and Table 13.
7. The applicability of q-spherical fuzzy rough sets extends to various domains, including stock investment analysis, airline service quality evaluation, investment banking authority selection, and electronic learning factor assessment, indicating their broad utility and relevance.
8. By leveraging the advantages of q-spherical fuzzy rough sets, decision-makers can navigate complex decision landscapes with greater confidence and precision.

Table 14 represents the pros and cons of the proposed operators and existing operators along with their year of publications.

TABLE 14. The pros and cons of both the proposed and existing operators.

Operators	Approximations Set	Parameter q	Year
T-SFEHIA [22]	×	×	2020
T-SFEHIG [22]	×	×	2020
T-SFHHA [23]	×	×	2020
T-SFHHG [23]	×	×	2020
CT-SFWA [35]	×	×	2020
CT-SFWG [35]	×	×	2020
T-SFG [36]	×	×	2020
T-SFFHA [24]	×	×	2021
T-SFFHG [24]	×	×	2021
TSFPWA [37]	×	×	2021
TSFPWG [37]	×	×	2021
q-SFFWA [This Paper]	✓	✓	2024
q-SFFOWA [This Paper]	✓	✓	2024
q-SFFHWA [This Paper]	✓	✓	2024

It is important to recognize that each approach comes with its own set of limitations. For example, the FFRS technique allows decision-makers to rank alternatives within the constraints of $(0 \leq \underline{\zeta}^3_{\mathcal{A}}(\mathfrak{w}) + \underline{\eta}^3_{\mathcal{A}}(\mathfrak{w}) + \underline{\xi}^3_{\mathcal{A}}(\mathfrak{w}) \leq 1)$ and $(0 \leq \overline{\zeta}^3_{\mathcal{A}}(\mathfrak{w}) + \overline{\eta}^3_{\mathcal{A}}(\mathfrak{w}) + \overline{\xi}^3_{\mathcal{A}}(\mathfrak{w}) \leq 1)$. To overcome these limitations, the proposed approach provides a more flexible environment for decision-makers. By reducing these constraints, decision-makers can provide more accurate classifications and make well-informed decisions. The unique features of different techniques including the comparison of the proposed approach are presented in Table 15.

TABLE 15. Comparison of characteristics between different methods.

Approaches	MD	Neu − MD	Non − MD	q
PFS	Yes	Yes	Yes	No
SFS	Yes	Yes	Yes	No
$q-SFS$	Yes	Yes	Yes	Yes
$q-SFRS$	Yes	Yes	Yes	Yes

B. SENSITIVITY ANALYSIS

In this study, to validate the developed model, two separate sensitivity analyses concerning changes in criteria and decision-making weights on the final ranking are presented. In this first study, a temporal sensitivity analysis is performed based on each criterion. For this purpose, the weight values of the reference criteria, i.e., high importance, equal importance, and low importance, are determined to see the effect of changing the criteria weight on the final ranking. Then, assigning these reference values to each criterion one by one, the model is run, and the alternatives are ranked. The results obtained according to the total 16 scenarios thus obtained are presented in Figure 15.

In all scenarios, alternative $\mathcal{V}_4$ ranks first and alternative $\mathcal{V}_3$ ranks last. Even with extreme values, altering the criterion weights has little effect on model output. In the second analysis, the weights of the decision makers are significantly changed, and 15 different scenarios are obtained based on different values of the weights. Figure 16 presents the final ranking of the decision makers' weight distribution. Alternative $\mathcal{V}_4$ is the best choice in all scenarios, while alternative $\mathcal{V}_3$ is the worst choice. Although the ranking order of the two alternatives may vary depending on the combination of weights used, the proposed approach generally produces reliable results and has reasonable consistency across different decision-weighting scenarios.

C. ADVANTAGES

The proposed technique has various benefits:

1. The addition of parameter q to the aggregation operators gives decision-makers a great deal of freedom. This versatility allows them to tailor the settings to the individual needs and preferences of the decision-making scenarios. The decision process's versatility allows for varying degrees of membership and non-membership,

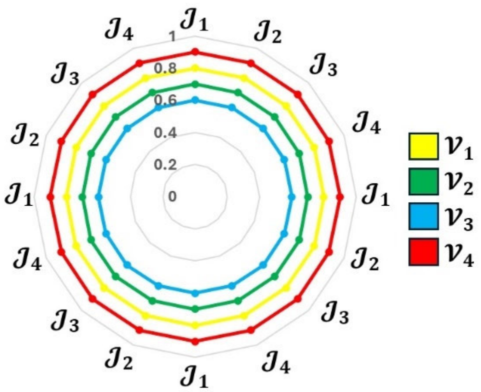

FIGURE 15. Alternative classification considering variations in criteria weights.

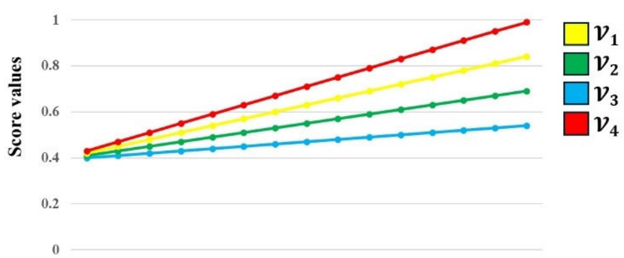

FIGURE 16. Alternative rankings in response to adjustments in decision-making weights.

making it appropriate for a broad range of real-world scenarios.

2. The parametric character of the suggested operators enables decision-makers to fine-tune the impact of membership and non-membership degrees. This degree of control enables decision-makers to accurately tailor the aggregation process to their preferences and the unique aspects of the situation at hand.
3. The symmetry of the suggested aggregation operators concerning the parameter ensures that the ranking orders of alternatives stay generally consistent across parameter values. This stability is critical in decision-making because it prevents the outcomes from being impacted by the decision-makers' pessimism or optimism.

D. LIMITATIONS

Every research endeavor inherently has limitations, and the methodology proposed in this study is no exception. Below is a discussion of these constraints:

1. The applicability of the proposed technique may be limited to specific domains or decision contexts. Understanding these limitations is critical to determining the optimal use of the recommended strategy.
2. As with any research approach, the proposed method relies on certain assumptions and simplifications to facilitate analysis. It is important to recognize that these assumptions may not align perfectly with real-world scenarios, potentially limiting the broad or practical applicability of the results.
3. The accomplishment of the suggested framework is established through a case study including four alternatives and four characteristics. It is critical to identify that the pattern may be expanded to integrate more possibilities and abilities in future efforts.
4. For several values of the parameter q, alternative ranking orders are calculated. It is important to note that more investigations might be conducted to investigate the hierarchical order for other values of these considerations.

TABLE 16. Explanation of abbreviations.

Abbreviations	Explanation
FS	Fuzzy set
IFS	Intuitionistic fuzzy set
PFS	Picture fuzzy set
SFS	Spherical fuzzy sets
q-SFS	q-spherical fuzzy sets
RS	Rough set
q-SFRS	q-spherical fuzzy rough set
IFRS	Intuitionistic fuzzy rough set
PyFRS	Pythagorean fuzzy rough set
MADM	Multi-attribute decision-making
q-ROFRS	q-rung orthopair fuzzy rough set
PFRS	Picture fuzzy rough set
q-ROFS	q-rung orthopair fuzzy set
AOs	Aggregation operators
AI	Artificial Intelligence
PyFS	Pythagorean fuzzy set
MD	Membership Degree
Neu-MD	Membership Degree
Non-MD	Neutral Membership Degree
MCDM	Multi-criteria decision-making
q-ROPFStS	q-rung orthopair picture fuzzy soft set
q-SFRNs	q-spherical fuzzy rough number
MAGDM	Multi-attribute group decision-making

VI. CONCLUSION

In conclusion, this paper presents a novel class of aggregation operators, namely the q-SFRFWA, q-SFRFOWA, and q-SFRFHWA operators, which seamlessly integrate the advantages of Frank operations with the flexibility of q-SFRSs. Through extensive experimentation and comparative analysis with existing MCDM models, the effectiveness and robustness of the proposed operators have been demonstrated. These operators exhibit superior data aggregation precision, making them valuable assets in various domains, including data analysis and decision-making processes. Furthermore, the study recognizes the evolving landscape of artificial intelligence (AI) and proposes the incorporation of a neural network approach as a promising avenue for future research. By leveraging neural network models, the capabilities of the proposed aggregation operators can be further enhanced, offering dynamic and flexible frameworks for decision-making tasks. Future research directions include exploring the synergy between neural networks and aggregation operators to address complex uncertainties and indeterminacies in real-world scenarios. The introduction of the q-SFRFWA, q-SFRFOWA, and q-SFRFHWA operators represents a significant contribution to the field of multiple criteria decision-making. These operators not only enhance data aggregation precision but also pave the way for innovative applications in various domains. Embracing the potential of neural network integration offers exciting opportunities for advancing decision-making processes and coping with the challenges of modern decision landscapes.

CONFLICT OF INTEREST

The authors confirm that they do not possess any discernible conflicting financial interests or personal relationships that could appear to impact the research detailed in this paper.

APPENDIX

See Table 16.

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