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RESEARCH ARTICLE

Novel Results on Fixed-Time Complex Projective Lag Synchronization for Fuzzy Complex-Valued Neural Networks With Inertial Item

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ABSTRACT This paper focuses on investigating fuzzy complex-valued neural networks with inertial item. By utilizing fixed-time stability theory and inequality techniques, we designed two types of feedback controllers and obtained some new criteria to ensure that the system achieves fixed-time complex projective lag synchronization(CPLS). Compared with previous works, we study the complex-value system as a whole, and compared to ordinary synchronization, CPLS has a broader range of applications. Finally, we provide numerical simulations to verify the effectiveness of the theoretical results.

INDEX TERMS Fixed-time complex projective lag synchronization (FXCPLS), fuzzy neural networks, complex-valued neural networks, inertial item.

I. INTRODUCTION

In practical applications, uncertainty, approximation, and fuzziness are inevitably encountered. In order to deal with fuzzy or uncertain situations, Yang et al. [1], [2] first introduced fuzzy AND operators and OR operators for research based on traditional cellular neural networks(NNs), and proposed the concept of fuzzy neural networks(FNNs). Compared with general NNs, FNNs have better robustness and adaptability, and can be applied to complex practical application scenarios, such as pattern classification, associative memory and parallel processing [3], [4], [5]. In addition, some researchers have demonstrated that fuzzy logic can be used to approximate any nonlinear function. Thus, it has broad application prospects and research value, and many scholars studied FNNs and achieved many excellent results [6], [7], [8].

NNs are generally described by first-order differential equations, such as the FNNs mentioned above. In real systems, there are a large number of second-order dynamical phenomena, such as variable-speed operation of information vectors constructed from network topologies. Babcock and Westervelt [9] introduced the concept of inertial neural networks(INNs) in 1986 by introducing inductors in neural circuits to display inertial features. INNs are described by for second order differential equations. This feature can be well applied in generating pseudo random sequences and image information processing. In addition, the equivalent circuit of inductance can simulate the synapses of squid and the membrane semicircular canals of animal hair cells [10], [11]. When we combine the INNs with the FNNs, we get the fuzzy inertial neural networks(FINNs). In recent years, many scholars studied FINNs and achieved some meaningful results. Yang and Zhang [12] constructed a novel controller using maximum analysis method and studied the global asymptotic synchronization problem of FINNs.

We note that the NNs discussed above is a real valued NNs. With the continuous development of computer hardware and software technology, the problem of complex signals has been involved in many industrial production engineering processes. There is an increasing interest among

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researchers in the application and study of complex-valued neural networks (CVNNs). The state variables, connection weight matrices, and activation functions of CVNNs are all complex valued, therefore, CVNNs have more diverse dynamic behaviors. Currently, CVNNs is utilized in various fields, like language recognition, signal processing and pattern recognition. Reference [13] generalized the universal approximation theorem of NNs to CVNNs. Cheng et al. [14] investigated the fixed-time problem for fractional-order CVNNs and obtained corresponding sufficient conditions. There are many literatures [15], [16], [17], and [18] studies on CVNNs by separating systems into real and imaginary systems. However, the approach increases system's dimensionality and complicates the calculation process, resulting in conservative outcomes. Moreover, the separation method may pose practical challenges in implementing it in real-world applications. Therefore, it is more practical to analyse the dynamic behavior for fuzzy inertial complex-valued neural networks (FICVNNs) using a non-separation method.

Synchronization refers to adjusting the response system through a controller, ultimately achieving consistent dynamic behavior between the response system and the driving system. In recent years, synchronization has become an important research direction in NNs due to its widespread application in secure communication, associative memory, image processing and information science [19], [20]. The synchronization of NNs attract the attention of scholars in various fields, and there have many excellent achievements in this field to date [21], [22], [23], [24]. Complex projective synchronization can reflect the proportional relationship between synchronization states by introducing a scaling factor. When our scaling factors are 0, -1, and 1, they correspond to stabilization, anti-synchronization, and complete synchronization. Therefore, complex projective synchronization more general.

We generalize complex projective synchronization to obtain complex projective lag synchronization(CPLS) [25]. CPLS is a special synchronization method that introduces a lag term on the basis of complex projective synchronization, which can better handle the delay problems in actual systems. The significance of CPLS lies in the fact that in some practical applications, such as control systems and communication systems, there is a common occurrence of time delay between systems. If synchronization is required between these systems, traditional complex projective synchronization methods may cause instability and errors in the synchronization state due to delay. CPLS can solve this problem by introducing a lag term, thereby making the synchronization state between systems more accurate and stable.

Settling-time(ST) is an important evaluation indicator for the speed of system synchronization, and existing research mainly focuses on finite-time stability [26], [27], [28]. The ST of finite-time stability is contingent upon the initial condition, and if the initial condition is unknown

TABLE 1. The meaning of symbols in the article.

Notation	Meaning
S	Set $S = 1, 2, 3 \ldots \eta$
R	real number domains
C	complex number domains
C^η	$\eta - \mathit{dimensional}$ complex number space
i	imaginary unit and $\sqrt{-1} = i$
$\overline{\lambda}$	conjugation of λ, $\lambda = a + bi \in C, \overline{\lambda} = a - bi$
α_ι, ℓ_ι	positive constant
$\partial_{\iota\rho}, \hbar_{\iota\rho}$	complex-valued neuron connection weights
$d_{\iota\rho}$	the elements of feed-forward templates
$\varsigma_{\iota\rho}/\varpi_{\iota\rho}$	the elements of fuzzy feed-forward MIN/MAX templates
$r_{\iota\rho}/\delta_{\iota\rho}$	fuzzy feed-forward MIN/MAX templates
$f_\rho(\cdot), g_\rho(\cdot)$	complex-valued activation function
τ_ρ	discrete delay
$\sharp$	$max_{\rho\in S}\{\tau_\rho\}$
$\lvert\lambda\rvert_1$	$\lvert\lambda\rvert_1 = \lvert a\rvert + \lvert b\rvert$
$\lvert\lambda\rvert_2$	$\lvert\lambda\rvert_2 = \sqrt{\lambda\overline{\lambda}}$

or cannot be provided, it becomes impossible to accurately estimate the ST. Therefore, finite-time stability has limitations.

Polyakov [29] introduced the concept of fixed-time stability, which ensures that the ST of a system is unrelated to its initial value. He also estimated the upper limit of the system stability time. Currently, there are numerous studies exploring the theory of fixed-time stability. In order to save control costs, Zhang et al. [30] designed an event triggered control scheme for fixed-time synchronization(FXS) and stabilization of discontinuous NNs. Liu and Zhang [31] designed two control strategies and obtained sufficient conditions to ensure that FICVNNs achieve fixed-time lag synchronization(FXLS).

Inspired by the above content, this paper explores the fixed-time complex projective lag synchronization(FXCPLS) for FICVNNs using a non-separation approach. The key contributions of this paper can be summarized as follows:

1. Our model is more versatile than previous models in [12] and [23] as it integrates fuzzy logic, inertial item, and complex numbers into the NNs.

2. On the base of the Lyapunov stability theory, we obtain sufficient conditions to ensure that FICVNNs achieve FXCPLS. Meanwhile, by selecting different lag constants and projective parameters, our conclusion can be extended to fixed-time complex projective synchronization(FXCPS), FXCL along with fixed-time anti-synchronization(FXAS).

3. By designing a complex-valued feedback controller, we implemented FXCPLS for FICVNNs and calculated the upper limit of ST.

The remaining sections of this paper are arranged as shown below. Section II provides some preliminaries, including model description, etc. Section III introduces novel findings on FXCPLS of FICVNNs. In section IV, we validate the effectiveness of our results through numerical simulations. Finally, section V provides a comprehensive summary of the entire article and provides future research directions.

II. PRELIMINARIES

The FICVNNs is:

$$\begin{aligned}\ddot{\lambda}_\iota(t) &= -\alpha_\iota\lambda_\iota(t) - \ell_\iota\dot{\lambda}_\iota(t) + \sum_{\rho=1}^{\eta}\partial_{\iota\rho}f_\rho(\lambda_\rho(t)) + \sum_{\rho=1}^{\eta}\hbar_{\iota\rho} \\ &\quad \times g_\rho(\lambda_\rho(t-\tau_\rho)) + \sum_{\rho=1}^{\eta}d_{\iota\rho}H_\rho(t) + \bigwedge_{\rho=1}^{\eta} r_{\iota\rho}H_\rho(t) \\ &\quad + \bigvee_{\rho=1}^{\eta}\delta_{\iota\rho}H_\rho(t) + \bigwedge_{\rho=1}^{\eta}\varsigma_{\iota\rho}g_\rho(\lambda_\rho(t-\tau_\rho)) \\ &\quad + \bigvee_{\rho=1}^{\eta}\varpi_{\iota\rho}g_\rho(\lambda_\rho(t-\tau_\rho)), t \geq 0, \iota \in S, \qquad (1)\end{aligned}$$

where $\iota \in S$, $\lambda_\iota(t) \in C$ means the ιth state at time t, $H_\rho(t)$ is the input of the ιth neuron. The initial conditions of FICVNNs (1) are given as $\lambda_\iota(\phi) = \prime_\iota(\phi)$, $\dot{\lambda}_\iota(\phi) = ¶_\iota(\phi)$, $\phi \in [-\sharp, 0]$. The meanings of other symbols are shown in Table 1.

To simplify the analysis of FICVNNs (1), an intermediate variable $v_\iota(t) = \dot{\lambda}_\iota(t) + \lambda_\iota(t)$ is introduced. FICVNNs (1) is expressed as follows:

$$\begin{aligned}\dot{\lambda}_\iota(t) &= -\lambda_\iota(t) + v_\iota(t), \\ \dot{v}_\iota(t) &= -\xi_\iota v_\iota(t) - h_\iota\lambda_\iota(t) + \sum_{\rho=1}^{\eta}\partial_{\iota\rho}f_\rho(\lambda_\rho(t)) + \sum_{\rho=1}^{\eta}\hbar_{\iota\rho} \\ &\quad \times g_\rho(\lambda_\rho(t-\tau_\rho)) + \sum_{\rho=1}^{\eta}d_{\iota\rho}H_\rho(t) + \bigwedge_{\rho=1}^{\eta} r_{\iota\rho}H_\rho(t) \\ &\quad + \bigvee_{\rho=1}^{\eta}\delta_{\iota\rho}H_\rho(t) + \bigwedge_{\rho=1}^{\eta}\varsigma_{\iota\rho}g_\rho(\lambda_\rho(t-\tau_\rho)) \\ &\quad + \bigvee_{\rho=1}^{\eta}\varpi_{\iota\rho}g_\rho(\lambda_\rho(t-\tau_\rho)), t \geq 0, \iota \in S, \qquad (2)\end{aligned}$$

where $\xi_\iota = \ell_\iota - 1$, and $h_\iota = \alpha_\iota - \xi_\iota$. The drive FICVNNs are represented by Eq. (1), and the response FICVNNs can be translated as shown below:

$$\begin{aligned}\dot{\kappa}_\iota(t) &= -\kappa_\iota(t) + w_\iota(t), \\ \dot{w}_\iota(t) &= -\xi_\iota w_\iota(t) - h_\iota\kappa_\iota(t) + \sum_{\rho=1}^{\eta}\partial_{\iota\rho}f_\rho(\kappa_\rho(t)) + \sum_{\rho=1}^{\eta}\hbar_{\iota\rho} \\ &\quad \times g_\rho(\kappa_\rho(t-\tau_\rho)) + \sum_{\rho=1}^{\eta}d_{\iota\rho}H_\rho(t) + \bigwedge_{\rho=1}^{\eta} r_{\iota\rho}H_\rho(t) \\ &\quad + \bigvee_{\rho=1}^{\eta}\delta_{\iota\rho}H_\rho(t) + \bigwedge_{\rho=1}^{\eta}\varsigma_{\iota\rho}g_\rho(\kappa_\rho(t-\tau_\rho)) \\ &\quad + \bigvee_{\rho=1}^{\eta}\varpi_{\iota\rho}g_\rho(\kappa_\rho(t-\tau_\rho)) + \tilde{\upsilon}_\iota(t), t \geq 0, \iota \in S, \quad (3)\end{aligned}$$

with initial state $\kappa_\iota(\phi) = \tilde{\prime}_\iota(\phi)$, $w_\iota(\phi) = \tilde{¶}_\iota(\phi)$, where $\tilde{\upsilon}_\iota(t)$ denote controllers. To obtain the error system from Eqs. (2) and (3), $\Xi_\iota(t) = \kappa_\iota(t) - m\lambda_\iota(t-\gamma)$, $\Theta_\iota(t) = w_\iota(t) - mv_\iota(t-\gamma)$, where $\gamma \geq 0$ is lag-constant, $m \in C$ is projective factor. Next, the error system as follows:

$$\begin{aligned}\dot{\Xi}_\iota(t) &= -\Xi_\iota(t) + \Theta_\iota(t), \\ \dot{\Theta}_\iota(t) &= -\xi_\iota\Theta_\iota(t) - h_\iota\Xi_\iota(t) + \sum_{\rho=1}^{\eta}\Big(\partial_{\iota\rho}f_\rho\big(\kappa_\rho(t)\big) - m\partial_{\iota\rho} \\ &\quad \times f_\rho\big(\lambda_\rho(t-\gamma)\big)\Big) + \sum_{\rho=1}^{\eta}\Big(\hbar_{\iota\rho}g_\rho(\kappa_\rho(t-\tau_\rho)) - m \\ &\quad \hbar_{\iota\rho}g_\rho(\lambda_\rho(t-\tau_\rho-\gamma))\Big) + \bigwedge_{\rho=1}^{\eta}\varsigma_{\iota\rho}g_\rho(\kappa_\rho(t-\tau_\rho)) \\ &\quad - \bigwedge_{\rho=1}^{\eta} m\varsigma_{\iota\rho}g_\rho(\lambda_\rho(t-\tau_\rho-\gamma)) + \bigvee_{\rho=1}^{\eta}\varpi_{\iota\rho}g_\rho(\kappa_\rho(t \\ &\quad -\tau_\rho)) - \bigvee_{\rho=1}^{\eta} m\varpi_{\iota\rho}g_\rho(\lambda_\rho(t-\tau_\rho-\gamma)) + \tilde{\upsilon}_\iota(t), \\ &\quad t \geq 0, \iota \in S. \qquad (4)\end{aligned}$$

Remark 1: Unlike [17], [18], the model (1) we constructed not only considers inertial item but also introduces fuzzy logic. Compared to the CVNNs discussed in previous literature, the model we discussed is more general.

Hypothesis 1: For $\lambda_1, \lambda_2 \in C$, $\lambda_1 \neq \lambda_2$, there are constants $L_\iota, \tilde{L}_\iota, M_\iota, \tilde{M}_\iota > 0 (\iota \in S)$, then

$$\begin{aligned}|f_\iota(\lambda_1) - f_\iota(\lambda_2)|_1 &\leq L_\iota|\lambda_1 - \lambda_2|_1, \\ |g_\iota(\lambda_1) - g_\iota(\lambda_2)|_1 &\leq M_\iota|\lambda_1 - \lambda_2|_1, \\ |f_\iota(\lambda_1) - f_\iota(\lambda_2)|_2 &\leq \tilde{L}_\iota|\lambda_1 - \lambda_2|_2, \\ |g_\iota(\lambda_1) - g_\iota(\lambda_2)|_2 &\leq \tilde{M}_\iota|\lambda_1 - \lambda_2|_2.\end{aligned}$$

Definition 1 [32]: Systems (2) and (3) can achieve FXCPLS if there exists a fixed-time T_f and the ST function $T(\wp(0))$, then:

$$\begin{cases}\lim\limits_{t \to T(\wp(0))}|\wp(t)|_p = 0, \\ \forall t \geq T(\wp(0)), \wp(t) = 0, \\ T(\wp(0)) \leq T_f,\end{cases}$$

where $p = 1 or\ 2$, $\wp(t) \in C^{2\eta}$, $\wp(t) = \big(\Xi_1(t), \Xi_2(t), \ldots, \Xi_\eta(t), \Theta_1(t), \Theta_2(t), \ldots, \Theta_\eta(t)\big)$.

Lemma 1 [30]: Assuming $V(\cdot)$ is a radial unbounded function, and $V(\wp(t)) = 0 \Leftrightarrow \wp(t) = 0$, and the given relationship holds

$$D^+V(t) \leq \begin{cases}-kV(t) - \Omega_1V^\beta(t) - \zeta, & V(t) \in (0,1), \\ -kV(t) - \Omega_2V^\beta(t) - \zeta, & V(t) \geq 1,\end{cases}$$

then, the system (2) and (3) can achieve FXCPLS, in which $k > 0, \Omega_1 > 0, \Omega_1 > 0, k < min\{\Omega_1, \Omega_2\}$, $\beta = \Phi + sign\big(V(t) - 1\big)$, $1 < \Phi < 2$. The upper bound of ST is estimated to be $T_f = \frac{1}{(k+\zeta)(\Phi-2)}ln\frac{\Omega_1}{\Omega_1+\zeta+k} - \frac{1}{\Phi k}ln\frac{\Omega_2}{\Omega_2+k}$.

Lemma 2 [32]: Let $\lambda_\iota \geq 0, \iota = 1, 2, 3 \cdots \eta; 0 < \mho_1 \leq 1, \mho_2 > 1$ one has

$$\sum_{\iota=1}^{\eta} \lambda_\iota^{\mho_1} \geq (\sum_{\iota=1}^{\eta} \lambda_\iota)^{\mho_1}, \sum_{\iota=1}^{\eta} \lambda_\iota^{\mho_2} \geq \eta^{1-\mho_2}(\sum_{\iota=1}^{\eta} \lambda_\iota)^{\mho_2}.$$

Lemma 3 [33]: For any $\vartheta(t) : R \rightarrow C$, the following formula holds

$$(1)\overline{[\omega(t)]}\vartheta(t) + [\vartheta(t)]\overline{\vartheta(t)} = 2|\vartheta(t)|_1.$$
$$(2)\overline{[\vartheta(t)]}[\vartheta(t)] = |[\vartheta(t)]|_1.$$
$$(3)D^+|\vartheta(t)|_1 = \frac{1}{2}(\overline{[\vartheta(t)]}D^+\vartheta(t) + \vartheta(t)D^+\overline{\vartheta(t)}).$$

Hence, D^+ represents the Dini derivative.

Lemma 4 [33]: For any $\vartheta(t) : R \rightarrow C$ and there exists measurable selection $\omega(t) \in \bar{co}(\vartheta(t))$, where

$$\bar{co}(\vartheta(t)) = \bar{co}(sign(Re(\vartheta(t)))) + i\bar{co}(sign(Im(\vartheta(t)))),$$

then the following formulas hold:

$$(1)\overline{[\vartheta(t)]}\omega(t) + \overline{\omega}[\vartheta(t)] = 2|[\vartheta(t)]|_1.$$
$$(2)\omega(t)\overline{\vartheta(t)} + \omega(t)\vartheta(t) = 2|\vartheta(t)|_1.$$

Hence, D^+ represents the Dini derivative.

III. MAIN RESULTS

A. DESIGNING A CONTROLLER BASED ON 1-NORM

Construct the controller as follows

$$\begin{cases} \tilde{\upsilon}_\iota(t) = \upsilon_\iota^*(t) + \grave{\upsilon}_{1\iota}(t), & \Theta_\iota(t)\overline{\Theta_\iota(t)} \neq 0, \\ \tilde{\upsilon}_\iota(t) = 0, & \Theta_\iota(t)\overline{\Theta_\iota(t)} = 0, \end{cases} \tag{5}$$

where

$$\begin{aligned} \upsilon_\iota^*(t) = &\sum_{\rho=1}^{\eta} \Big(m\partial_{\iota\rho} f_\rho(\lambda_\rho(t-\gamma)) - \partial_{\iota\rho} f_\rho(m\lambda_\rho(t-\gamma))\Big) \\ &+ \sum_{\rho=1}^{\eta} \Big(m\hbar_{\iota\rho} f_\rho(\lambda_\rho(t-\tau_\rho-\gamma)) - \hbar_{\iota\rho} f_\rho(m\lambda_\rho(t-\tau_\rho \\ &-\gamma))\Big) + \bigwedge_{\rho=1}^{\eta} m\varsigma_{\iota\rho} g_\rho(\lambda_\rho(t-\tau_\rho-\gamma)) - \bigwedge_{\rho=1}^{\eta} \varsigma_{\iota\rho} g_\rho \\ &\times (m\lambda_\rho(t-\tau_\rho-\gamma)) + \bigvee_{\rho=1}^{\eta} m\varpi_{\iota\rho} g_\rho(\lambda_\rho(t-\tau_\rho-\gamma)) \\ &- \bigvee_{\rho=1}^{\eta} \varpi_{\iota\rho} g_\rho(m\lambda_\rho(t-\tau_\rho-\gamma)), \end{aligned}$$

$$\begin{aligned} \grave{\upsilon}_{1\iota}(t) = &-[\Theta_\iota(t)]\Big(\zeta_{1\iota}|\Xi_\iota(t)|_1 + \zeta_{2\iota}|\Theta_\iota(t)|_1 + |\Xi_\iota(t)|_1^\beta \\ &+ |\Theta_\iota(t)|_1^\beta + \zeta_{3\iota} + \sum_{\rho=1}^{\eta} \varphi_{\iota\rho}|\Xi(t-\tau_\rho)|_1\Big). \end{aligned}$$

Noting that $\grave{\upsilon}_{1\iota}(t)$ is discontinuous, one has

$$\begin{aligned} \grave{\upsilon}_{1\iota}(t) = &-co([\Theta_\iota(t)])\Big(\zeta_{1\iota}|\Xi_\iota(t)|_1 + \zeta_{2\iota}|\Theta_\iota(t)|_1 + |\Xi_\iota(t)|_1^\beta \\ &+ |\Theta_\iota(t)|_1^\beta + \zeta_{3\iota} + \sum_{\rho=1}^{\eta} \varphi_{\iota\rho}|\Xi(t-\tau_\rho)|_1\Big). \end{aligned}$$

By the measurable selcetion theorem, there exists a function $\omega_\iota(t) \in co([\Theta_\iota(t)])$ such that

$$\begin{aligned} \grave{\upsilon}_{1\iota}(t) = &-\omega_\iota(t)\Big(\zeta_{1\iota}|\Xi_\iota(t)|_1 + \zeta_{2\iota}|\Theta_\iota(t)|_1 + |\Xi_\iota(t)|_1^\beta \\ &+ |\Theta_\iota(t)|_1^\beta + \zeta_{3\iota} + \sum_{\rho=1}^{\eta} \varphi_{\iota\rho}|\Xi(t-\tau_\rho)|_1\Big). \end{aligned}$$

Theorem 1: Under Hypothesis 1, the FICVNNs (2)-(3) can achieve FXCPLS under Eq. (5) if the following condition holds

$$-\varphi_{\iota\rho} + \big(|\hbar_{\iota\rho}|_1 + |\varsigma_{\iota\rho}|_1 + |\varpi_{\iota\rho}|_1\big)M_\rho \leq 0, \tag{6}$$

and $T_{1f} = \frac{1}{(k+\zeta)(\phi-2)} ln\frac{\Omega_1}{\Omega_1+\zeta+k} - \frac{1}{\phi k} ln\frac{\Omega_2}{\Omega_2+k}$.

Proof: Define the Lyapunov function as:

$$V_1(t) = \sum_{\iota=1}^{\eta} \Big(|\Xi_\iota(t)|_1 + |\Theta_\iota(t)|_1\Big). \tag{7}$$

Take the derivative of $V_1(t)$ along the trajectory of Eq.(4),

$$\begin{aligned} &D^+V_1(t) \\ &= \frac{1}{2}\sum_{\iota=1}^{\eta}\Big(\overline{[\Xi_\iota(t)]}\dot{\Xi}_\iota(t) + [\Xi_\iota(t)]\overline{\dot{\Xi}_\iota(t)} + \overline{[\Theta_\iota(t)]}\dot{\Theta}_\iota(t) \\ &\quad + [\Theta_\iota(t)]\overline{\dot{\Theta}_\iota(t)}\Big) \\ &\leq \sum_{\iota=1}^{\eta}\Big(-|\Xi_\iota(t)|_1 + \frac{1}{2}\big(\overline{[\Xi_\iota(t)]}\Theta_\iota(t) + [\Xi_\iota(t)]\overline{\Theta_\iota(t)}\big) \\ &\quad - \xi_\iota|\Theta_\iota(t)|_1 - \frac{1}{2}h_\iota\big(\overline{[\Theta_\iota(t)]}\Xi_\iota(t) + [\Theta_\iota(t)]\overline{\Xi_\iota(t)}\big)\Big) \\ &\quad + \frac{1}{2}\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}\Big(\overline{[\Theta_\iota(t)]}\big(\partial_{\iota\rho} f_\rho(\kappa_\rho(t)) - \partial_{\iota\rho} \\ &\quad \times f_\rho(m\lambda_\rho(t-\gamma))\big) + [\Theta_\iota(t)]\overline{\partial_{\iota\rho} f_\rho(\kappa_\rho(t)) - \partial_{\iota\rho}} \\ &\quad \overline{\times f_\rho(m\lambda_\rho(t-\gamma))} + \overline{[\Theta_\iota(t)]}\big(\hbar_{\iota\rho} g_\rho(\kappa_\rho(t-\tau_\rho)) \\ &\quad - \hbar_{\iota\rho} g_\rho(m\lambda_\rho(t-\tau_\rho-\gamma))\big) + [\Theta_\iota(t)]\overline{\hbar_{\iota\rho}(\kappa_\rho(t-\tau_\rho))} \\ &\quad \times \overline{-\hbar_{\iota\rho} g_\rho(m\lambda_\rho(t-\tau_\rho-\gamma))}\Big) \\ &\quad + \frac{1}{2}\sum_{\iota=1}^{\eta}\Big(\overline{[\Theta_\iota(t)]}\big(\bigwedge_{\rho=1}^{\eta}\varsigma_{\iota\rho} g_\rho(\kappa_\rho(t-\tau_\rho)) - \bigwedge_{\rho=1}^{\eta} \\ &\quad \times \varsigma_{\iota\rho} g_\rho(m\lambda_\rho(t-\tau_\rho-\gamma))\big) + [\Theta_\iota(t)]\overline{\bigwedge_{\rho=1}^{\eta}\varsigma_{\iota\rho}} \\ &\quad \overline{\times g_\rho(\kappa_\rho(t-\tau_\rho)) - \bigwedge_{\rho=1}^{\eta}\varsigma_{\iota\rho} g_\rho(m\lambda_\rho(t-\tau_\rho-\gamma))} \\ &\quad + \overline{[\Theta_\iota(t)]}\big(\bigvee_{\rho=1}^{\eta}\varpi_{\iota\rho} g_\rho(\kappa_\rho(t-\tau_\rho)) - \bigvee_{\rho=1}^{\eta}\varpi_{\iota\rho} \end{aligned}$$

$$\times g_\rho(m\lambda_\rho(t-\tau_\rho-\gamma))) + [\Theta_\iota(t)]\overline{\bigvee_{\rho=1}^{\eta}\varpi_{\iota\rho}} \\ \overline{\times g_\rho(\kappa_\rho(t-\tau_\rho)) - \bigvee_{\rho=1}^{\eta}\varpi_{\iota\rho}g_\rho(m\lambda_\rho(t-\tau_\rho-\gamma))}\Big) \\ + \frac{1}{2}\sum_{\iota=1}^{\eta}\Big(\overline{[\Theta_\iota(t)]}\grave{\upsilon}_{1\iota}(t) + [\Theta_\iota(t)]\overline{\grave{\upsilon}_{1\iota}(t)}\Big). \tag{8}$$

According to the definition of $[\Xi_\iota(t)]$, we get

$$\frac{1}{2}\Big(\overline{[\Xi_\iota(t)]}\Theta_\iota(t) + [\Xi_\iota(t)]\overline{\Theta_\iota(t)}\Big) \\ = sign\big(Re(\Xi_\iota(t))\big)Re(\Theta_\iota(t)) + sign\big(Im(\Xi_\iota(t))\big)Im(\Theta_\iota(t)) \\ \le |\Theta_\iota(t)|_1. \tag{9}$$

Based on the above, similarly, there is

$$-\frac{1}{2}h_\iota\Big(\overline{[\Theta_\iota(t)]}\Xi_\iota(t) + [\Theta_\iota(t)]\overline{\Xi_\iota(t)}\Big) \\ \le h_\iota|\Xi_\iota(t)|_1. \tag{10}$$

Based on Hypothesis 1, We get

$$\frac{1}{2}\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}\Big(\overline{[\Theta_\iota(t)]}\big(\partial_{\iota\rho}f_\rho(\kappa_\rho(t)) - \partial_{\iota\rho}f_\rho(m\lambda_\rho(t-\gamma))\big) \\ + [\Theta_\iota(t)]\overline{\partial_{\iota\rho}f_\rho(\kappa_\rho(t)) - \partial_{\iota\rho}f_\rho(m\lambda_\rho(t-\gamma))}\Big) \\ \le \sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}|\partial_{\iota\rho}f_\rho(\kappa_\rho(t)) - \partial_{\iota\rho}f_\rho(m\lambda_\rho(t-\gamma))|_1 \\ \le \sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}|\partial_{\iota\rho}|_1L_\rho|\kappa_\rho(t) - m\lambda_\rho(t-\gamma)|_1 \\ \le \sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}|\partial_{\rho\iota}|_1L_\iota|\Xi_\iota(t)|_1. \tag{11}$$

Similarly,

$$\frac{1}{2}\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}\Big(\overline{[\Theta_\iota(t)]}\hbar_{\iota\rho}\big(g_\rho(\kappa_\rho(t-\tau_\rho)) - g_\rho(m\lambda_\rho(t-\tau_\rho-\gamma))\big) \\ + [\Theta_\iota(t)]\overline{\hbar_{\iota\rho}g_\rho(\kappa_\rho(t-\tau_\rho)) - \hbar_{\iota\rho}g_\rho(m\lambda_\rho(t-\tau_\rho-\gamma))}\Big) \\ \le \sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}|\hbar_{\iota\rho}|_1M_\rho|\Xi_\rho(t-\tau_\rho)|_1. \tag{12}$$

Based on Hypothesis 1 and Lemma 2 in [8], there is

$$\frac{1}{2}\sum_{\iota=1}^{\eta}\Big(\overline{[\Theta_\iota(t)]}\Big(\bigwedge_{\rho=1}^{\eta}\varsigma_{\iota\rho}g_\rho(\kappa_\rho(t-\tau_\rho)) - \bigwedge_{\rho=1}^{\eta}\varsigma_{\iota\rho} \\ \times g_\rho(m\lambda_\rho(t-\tau_\rho-\gamma))) + [\Theta_\iota(t)]\overline{\bigwedge_{\rho=1}^{\eta}\varsigma_{\iota\rho}} \\ \overline{\times g_\rho(\kappa_\rho(t-\tau_\rho)) - \bigwedge_{\rho=1}^{\eta}\varsigma_{\iota\rho}g_\rho(m\lambda_\rho(t-\tau_\rho-\gamma))}\Big) \\ \le \sum_{\iota=1}^{\eta}|\bigwedge_{\rho=1}^{\eta}\varsigma_{\iota\rho}g_\rho(\kappa_\rho(t-\tau_\rho)) - \bigwedge_{\rho=1}^{\eta}\varsigma_{\iota\rho} \\ \times g_\rho(m\lambda_\rho(t-\tau_\rho-\gamma))|_1 \\ \le \sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}|\varsigma_{\iota\rho}|_1|g_\rho(\kappa_\rho(t-\tau_\rho)) - m\lambda_\rho(t-\tau_\rho-\gamma)|_1 \\ \le \sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}|\varsigma_{\iota\rho}|_1M_\rho|\Xi_\rho(t-\tau_\rho)|_1. \tag{13}$$

Likewise,

$$\frac{1}{2}\sum_{\iota=1}^{\eta}\Big(\overline{[\Theta_\iota(t)]}\Big(\bigvee_{\rho=1}^{\eta}\varpi_{\iota\rho}g_\rho(\kappa_\rho(t-\tau_\rho)) - \bigvee_{\rho=1}^{\eta}\varpi_{\iota\rho} \\ \times g_\rho(m\lambda_\rho(t-\tau_\rho-\gamma))) + [\Theta_\iota(t)]\overline{\bigvee_{\rho=1}^{\eta}\varpi_{\iota\rho}} \\ \overline{\times g_\rho(\kappa_\rho(t-\tau_\rho)) - \bigvee_{\rho=1}^{\eta}\varpi_{\iota\rho}g_\rho(m\lambda_\rho(t-\tau_\rho-\gamma))}\Big) \\ \le \sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}|\varpi_{\iota\rho}|_1M_\rho|\Xi_\rho(t-\tau_\rho)|_1. \tag{14}$$

Substituting Eqs. (9)-(14) into Eq. (8) yields

$$D^+V_1(t) \le \sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}(-1 + h_\iota + |\partial_{\rho\iota}|_1L_\iota)|\Xi_\iota(t)|_1 \\ + \sum_{\iota=1}^{\eta}(-\xi_\iota+1)|\Theta_\iota(t)|_1 + \sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}\big(|\hbar_{\iota\rho}|_1 + |\varsigma_{\iota\rho}|_1 \\ + |\varpi_{\iota\rho}|_1\big)M_\rho|\Xi_\rho(t-\tau_\rho)|_1 + \frac{1}{2}\sum_{\iota=1}^{\eta}\Big(\overline{[\Theta_\iota(t)]}\grave{\upsilon}_{1\iota}(t) \\ + [\Theta_\iota(t)]\overline{\grave{\upsilon}_{1\iota}(t)}\Big). \tag{15}$$

From Eq. (5) we get

$$\frac{1}{2}\sum_{\iota=1}^{\eta}\Big(\overline{[\Theta_\iota(t)]}\grave{\upsilon}_{1\iota}(t) + [\Theta_\iota(t)]\overline{\grave{\upsilon}_{1\iota}(t)}\Big) \\ = -\frac{1}{2}\sum_{\iota=1}^{\eta}\zeta_{1\iota}\Big(\overline{[\Theta_\iota(t)]}\omega_\iota(t) + [\Theta_\iota(t)]\overline{\omega_\iota(t)}\Big)|\Xi_\iota(t)|_1 \\ - \frac{1}{2}\sum_{\iota=1}^{\eta}\zeta_{2\iota}\Big(\overline{[\Theta_\iota(t)]}\omega_\iota(t) + [\Theta_\iota(t)]\overline{\omega_\iota(t)}\Big)|\Theta_\iota(t)|_1 \\ - \frac{1}{2}\sum_{\iota=1}^{\eta}\Big(\overline{[\Theta_\iota(t)]}\omega_\iota(t) + [\Theta_\iota(t)]\overline{\omega_\iota(t)}\Big)|\Xi_\iota(t)|_1^\beta \\ - \frac{1}{2}\sum_{\iota=1}^{\eta}\Big(\overline{[\Theta_\iota(t)]}\omega_\iota(t) + [\Theta_\iota(t)]\overline{\omega_\iota(t)}\Big)|\Theta_\iota(t)|_1^\beta \\ - \frac{1}{2}\sum_{\iota=1}^{\eta}\zeta_{3\iota}\Big(\overline{[\Theta_\iota(t)]}\omega_\iota(t) + [\Theta_\iota(t)]\overline{\omega_\iota(t)}\Big) \\ - \frac{1}{2}\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}\varphi_{\iota\rho}\Big(\overline{[\Theta_\iota(t)]}\omega_\iota(t) + [\Theta_\iota(t)]\overline{\omega_\iota(t)}\Big)|\Xi(t-\tau_\rho)|_1$$

$$\le -\sum_{\iota=1}^{\eta}\zeta_{1\iota}|\Xi_\iota(t)|_1-\sum_{\iota=1}^{\eta}\zeta_{2\iota}|\Theta_\iota(t)|_1-\sum_{\iota=1}^{\eta}|\Xi_\iota(t)|_1^\beta$$
$$-\sum_{\iota=1}^{\eta}|\Theta_\iota(t)|_1^\beta-\sum_{\iota=1}^{\eta}\zeta_{3\iota}-\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}\varphi_{\iota\rho}|\Xi(t-\tau_\rho)|_1. \tag{16}$$

By substituting Eq. (16) into Eq. (15), we get

$$\begin{aligned}D^+V_1(t)\le&\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}(-\zeta_{1\iota}-1+h_\iota+|\partial_{\rho\iota}|_1L_\iota)|\Xi_\iota(t)|_1\\&+\sum_{\iota=1}^{\eta}(-\zeta_{2\iota}-\xi_\iota+1)|\Theta_\iota(t)|_1+\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}\big(-\varphi_{\iota\rho}\\&+|\hbar_{\iota\rho}|_1+|\varsigma_{\iota\rho}|_1+|\varpi_{\iota\rho}|_1\big)M_\rho|\Xi_\rho(t-\tau_\rho)|_1\\&-\sum_{\iota=1}^{\eta}|\Xi_\iota(t)|_1^\beta-\sum_{\iota=1}^{\eta}|\Theta_\iota(t)|_1^\beta-\sum_{\iota=1}^{\eta}\zeta_{3\iota}\\\le&-\sum_{\iota=1}^{\eta}k_{1\iota}|\Xi_\iota(t)|_1-\sum_{\iota=1}^{\eta}k_{2\iota}|\Theta_\iota(t)|_1-\sum_{\iota=1}^{\eta}|\Xi_\iota(t)|_1^\beta\\&-\sum_{\iota=1}^{\eta}|\Theta_\iota(t)|_1^\beta-\zeta.\end{aligned}\tag{17}$$

where $k_{1\iota}=-(-\zeta_{1\iota}-1+h_\iota+|\partial_{\rho\iota}|_1L_\iota), k_{2\iota}=-(-\zeta_{2\iota}-\xi_\iota+1), \zeta=\sum_{\iota=1}^{\eta}\zeta_{3\iota}$.

By application of Lemma 2 we have:

(1)If $V_1(t)\in(0,1)$

$$\begin{aligned}&-\sum_{\iota=1}^{\eta}|\Xi_\iota(t)|_1^\beta-\sum_{\iota=1}^{\eta}|\Theta_\iota(t)|_1^\beta\\&\le-\Big(\sum_{\iota=1}^{\eta}|\Xi_\iota(t)|_1\Big)^\beta-\Big(\sum_{\iota=1}^{\eta}|\Theta_\iota(t)|_1\Big)^\beta\\&\le-\Omega_1V_1^\beta(t).\end{aligned}\tag{18}$$

(1)If $V_1(t)\ge1$,

$$\begin{aligned}&-\sum_{\iota=1}^{\eta}|\Xi_\iota(t)|_1^\beta-\sum_{\iota=1}^{\eta}|\Theta_\iota(t)|_1^\beta\\&\le-\eta^{1-\beta}\Big(\sum_{\iota=1}^{\eta}|\Xi_\iota(t)|_1\Big)^\beta-\eta^{1-\beta}\Big(\sum_{\iota=1}^{\eta}|\Theta_\iota(t)|_1\Big)^\beta\\&\le-\Omega_2V_1^\beta(t).\end{aligned}\tag{19}$$

From eqs. (18), (19), we obtain

$$D^+V_1(t)\le\begin{cases}-kV_1(t)-\Omega_1V_1^\beta(t)-\zeta, & V_1(t)\in(0,1),\\ -kV_1(t)-\Omega_2V_1^\beta(t)-\zeta, & V_1(t)\ge1,\end{cases}\tag{20}$$

where $k=min\{k_{1\iota},k_{2\iota}\}$, $\Omega_1=1$, $\Omega_2=-\eta^{1-\beta}2^{1-\beta}$.

Remark 2: Notice that the controller (5) contains the sign function $sign(\cdot)$ and therefore the controller (5) is discontinuous. However, in some cases, continuity is necessary and we can use $tanh(\cdot)$ approximation instead of $sign(\cdot)$.

B. DESIGNING A CONTROLLER BASED ON 2-NORM

Next, the controller will be designed based on the 2-norm of complex numbers to enable FICVNNs to implement FXCPLS. In this section, the continuous controller is designed as follows:

$$\begin{cases}\tilde{\upsilon}_\iota(t)=\upsilon_\iota^*(t)+\grave{\upsilon}_{2\iota}(t), & \Theta_\iota(t)\overline{\Theta_\iota(t)}\ne0,\\ \tilde{\upsilon}_\iota(t)=0, & \Theta_\iota(t)\overline{\Theta_\iota(t)}=0,\end{cases}\tag{21}$$

where

$$\begin{aligned}\grave{\upsilon}_{2\iota}(t)=&-\frac{1}{\overline{\Theta_\iota(t)}}\big(\tilde{\zeta}_{1\iota}|\Xi_\iota(t)|_2^2+(\frac{1}{2}|\Xi_\iota(t)|_2^2)^\beta+\tilde{\varphi}_\iota|\Xi_\iota(t-\tau_\rho)|_2^2\\&+\tilde{\zeta}_{3\iota}\big)-\tilde{\zeta}_{2\iota}|\Theta_\iota(t)|_2^2-(\frac{1}{2}|\Theta_\iota(t)|_2^2)^\beta.\end{aligned}$$

Theorem 2: Under Hypothesis 1 and controller (21), the FICVNNs (2)-(3) can achieve FXCPLS if the following condition holds

$$-\varphi_\iota+\frac{1}{2}\sum_{\rho=1}^{\eta}\big(|\hbar_{\iota\rho}|_2+|\varsigma_{\iota\rho}|_2+|\varpi_{\iota\rho}|_2\big)\tilde{M}_\rho\le0.$$

In addition, the upper bound on the ST is estimated to be $T_{2f}=\frac{1}{(\tilde{k}+\tilde{\zeta})(\tilde{\phi}-2)}ln\frac{\tilde{\Omega}_1}{\tilde{\Omega}_1+\tilde{\zeta}+\tilde{k}}-\frac{1}{\tilde{\phi}\tilde{k}}ln\frac{\tilde{\Omega}_2}{\tilde{\Omega}_2+\tilde{k}}$.

Proof: Define the Lyapunov function as:

$$\begin{aligned}V_2(t)&=\frac{1}{2}\sum_{\iota=1}^{\eta}\big(|\Xi_\iota(t)|_2^2+|\Theta_\iota(t)|_2^2\big)\\&=\frac{1}{2}\sum_{\iota=1}^{\eta}\Big(\Xi_\iota(t)\overline{\Xi_\iota(t)}+\Theta_\iota(t)\overline{\Theta_\iota(t)}\Big).\end{aligned}\tag{22}$$

Take the derivative of $V_2(t)$ along the trajectory of Eq.(4),

$$\begin{aligned}&D^+V_2(t)\\&=\frac{1}{2}\sum_{\iota=1}^{\eta}\Big(\dot{\Xi}_\iota(t)\overline{\Xi_\iota(t)}+\Xi_\iota(t)\overline{\dot{\Xi}_\iota(t)}+\dot{\Theta}_\iota(t)\overline{\Theta_\iota(t)}\\&\quad+\Theta_\iota(t)\overline{\dot{\Theta}_\iota(t)}\Big)\\&=\sum_{\iota=1}^{\eta}Re\Big(\dot{\Xi}_\iota(t)\overline{\Xi_\iota(t)}+\dot{\Theta}_\iota(t)\overline{\Theta_\iota(t)}\Big)\\&=\sum_{\iota=1}^{\eta}\Big(-Re\big(\Xi_\iota(t)\overline{\Xi_\iota(t)}\big)+Re\big(\Theta_\iota(t)\overline{\Xi_\iota(t)}\big)\\&\quad-Re\big(\xi_\iota\Theta_\iota(t)\overline{\Theta_\iota(t)}\big)-Re\big(h_\iota\Xi_\iota(t)\overline{\Theta_\iota(t)}\big)\\&\quad+\sum_{\rho=1}^{\eta}Re\Big(\partial_{\iota\rho}f_\rho\big(\kappa_\rho(t)\big)\overline{\Theta_\iota(t)}\Big)-\sum_{\rho=1}^{\eta}Re\Big(\partial_{\iota\rho}f_\rho\big(m\lambda_\rho(t\\&\quad-\gamma)\big)\overline{\Theta_\iota(t)}\Big)+\sum_{\rho=1}^{\eta}Re\Big(\hbar_{\iota\rho}g_\rho\big(\kappa_\rho(t-\tau_\rho)\big)\overline{\Theta_\iota(t)}\Big)\end{aligned}$$

$$-\sum_{\rho=1}^{\eta} Re\Big(\hbar_{\iota\rho}g_{\rho}\big(m\lambda_{\rho}(t-\tau_{\rho}-\gamma)\big)\overline{\Theta_{\iota}(t)}\Big)$$
$$+Re\Big(\bigwedge_{\rho=1}^{\eta}\varsigma_{\iota\rho}g_{\rho}\big(\kappa_{\rho}(t-\tau_{\rho})\big)\overline{\Theta_{\iota}(t)}\Big)$$
$$-Re\Big(\bigwedge_{\rho=1}^{\eta}\varsigma_{\iota\rho}g_{\rho}\big(m\lambda_{\rho}(t-\tau_{\rho}-\gamma)\big)\overline{\Theta_{\iota}(t)}\Big)$$
$$+Re\Big(\bigvee_{\rho=1}^{\eta}\varpi_{\iota\rho}g_{\rho}\big(\kappa_{\rho}(t-\tau_{\rho})\big)\overline{\Theta_{\iota}(t)}\Big)$$
$$-Re\Big(\bigvee_{\rho=1}^{\eta}\varpi_{\iota\rho}g_{\rho}\big(m\lambda_{\rho}(t-\tau_{\rho}-\gamma)\big)\overline{\Theta_{\iota}(t)}\Big)$$
$$+Re\big(\grave{\upsilon}_{2\iota}(t)\overline{\Theta_{\iota}(t)}\big)\Big). \tag{23}$$

On the basis of Hypothesis 1 and the properties of the inequality, we get:

$$\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}Re\Big(\partial_{\iota\rho}f_{\rho}\big(\kappa_{\rho}(t)\big)\overline{\Theta_{\iota}(t)}\Big)-\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}Re\Big(\partial_{\iota\rho}f_{\rho}\big(m \times \lambda_{\rho}(t-\gamma)\big)\overline{\Theta_{\iota}(t)}\Big)$$
$$\leq\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}\left|\partial_{\iota\rho}f_{\rho}\big(\kappa_{\rho}(t)\big)\overline{\Theta_{\iota}(t)}-\partial_{\iota\rho}f_{\rho}\big(m\lambda_{\rho}(t-\gamma)\big)\overline{\Theta_{\iota}(t)}\right|_2$$
$$\leq\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}|\partial_{\iota\rho}|_2|\overline{\Theta_{\iota}(t)}|_2|f_{\rho}\big(\kappa_{\rho}(t)\big)-f_{\rho}\big(m\lambda_{\rho}(t-\gamma)\big)|_2$$
$$\leq\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}|\partial_{\iota\rho}|_2\tilde{L}_{\rho}|\overline{\Theta_{\iota}(t)}|_2|\Xi_{\rho}(t)|_2$$
$$\leq\frac{1}{2}\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}\Big(|\partial_{\iota\rho}|_2\tilde{L}_{\rho}|\Theta_{\iota}(t)|_2^2+|\partial_{\rho\iota}|\tilde{L}_{\iota}|\Xi_{\iota}(t)|_2^2\Big). \tag{24}$$

$$\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}Re\Big(\hbar_{\iota\rho}g_{\rho}\big(\kappa_{\rho}(t-\tau_{\rho})\big)\overline{\Theta_{\iota}(t)}\Big)-\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}Re\Big(\hbar_{\iota\rho} \times g_{\rho}\big(m\lambda_{\rho}(t-\tau_{\rho}-\gamma)\big)\overline{\Theta_{\iota}(t)}\Big)$$
$$\leq\frac{1}{2}\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}\Big(|\hbar_{\iota\rho}|_2\tilde{M}_{\rho}|\Theta_{\iota}(t)|_2^2+|\hbar_{\rho\iota}|_2\tilde{M}_{\iota}|\Xi_{\iota}(t-\tau_{\rho})|_2^2\Big). \tag{25}$$

$$\sum_{\iota=1}^{\eta}Re\Big(\bigwedge_{\rho=1}^{\eta}\varsigma_{\iota\rho}g_{\rho}\big(\kappa_{\rho}(t-\tau_{\rho})\big)\overline{\Theta_{\iota}(t)}\Big)-\sum_{\iota=1}^{\eta}Re\Big(\bigwedge_{\rho=1}^{\eta}\varsigma_{\iota\rho} \times g_{\rho}\big(m\lambda_{\rho}(t-\tau_{\rho}-\gamma)\big)\overline{\Theta_{\iota}(t)}\Big)$$
$$\leq\sum_{\iota=1}^{\eta}\Big|\bigwedge_{\rho=1}^{\eta}\varsigma_{\iota\rho}g_{\rho}\big(\kappa_{\rho}(t-\tau_{\rho})\big)\overline{\Theta_{\iota}(t)}-\bigwedge_{\rho=1}^{\eta}\varsigma_{\iota\rho}g_{\rho}\big(m\lambda_{\rho}(t-\tau_{\rho}-\gamma)\big)\overline{\Theta_{\iota}(t)}\Big|_2$$
$$\leq\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}|\varsigma_{\iota\rho}|_2\Big|g_{\rho}\big(\kappa_{\rho}(t-\tau_{\rho})\big)-g_{\rho}\big(m\lambda_{\rho}(t-\tau_{\rho}-\gamma)\big)\Big|_2 \times|\overline{\Theta_{\iota}(t)}|_2$$
$$\leq\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}|\varsigma_{\iota\rho}|_2\tilde{M}_{\rho}|_2\Xi_{\rho}(t-\tau_{\rho})||\overline{\Theta_{\iota}(t)}|_2$$
$$\leq\frac{1}{2}\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}\Big(|\varsigma_{\iota\rho}|_2\tilde{M}_{\rho}|\Theta_{\iota}(t)|_2^2+|\varsigma_{\iota\rho}|_2\tilde{M}_{\iota}|\Xi_{\iota}(t-\tau_{\rho})|_2^2\Big). \tag{26}$$

$$\sum_{\iota=1}^{\eta}Re\Big(\bigvee_{\rho=1}^{\eta}\varpi_{\iota\rho}g_{\rho}\big(\kappa_{\rho}(t-\tau_{\rho})\big)\overline{\Theta_{\iota}(t)}\Big)-\sum_{\iota=1}^{\eta}Re\Big(\bigvee_{\rho=1}^{\eta}\varpi_{\iota\rho} \times g_{\rho}\big(m\lambda_{\rho}(t-\tau_{\rho}-\gamma)\big)\overline{\Theta_{\iota}(t)}\Big)$$
$$\leq\frac{1}{2}\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}\Big(|\varpi_{\iota\rho}|_2\tilde{M}_{\rho}|\Theta_{\iota}(t)|_2^2+|\varpi_{\iota\rho}|_2\tilde{M}_{\iota}|_2\Xi_{\iota}(t-\tau_{\rho})|_2^2\Big). \tag{27}$$

By adding (24)-(27) to (23), we obtain

$$D^{+}V_2(t)$$
$$\leq\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}\Big(\big(-1+\frac{1}{2}|1-h_{\iota}|_2+\frac{1}{2}|\partial_{\rho\iota}|_2\tilde{L}_{\iota}\big)|\Xi_{\iota}(t)|_2^2$$
$$+\big(-\xi_{\iota}+\frac{1}{2}(|1-h_{\iota}|_2+|\partial_{\iota\rho}|_2\tilde{L}_{\rho}+|\hbar_{\iota\rho}|_2\tilde{M}_{\rho}$$
$$+|\varsigma_{\iota\rho}|_2\tilde{M}_{\rho}+|\varpi_{\iota\rho}|_2\tilde{M}_{\rho})\big)|\Theta_{\iota}(t)|_2^2+\frac{1}{2}(|\hbar_{\iota\rho}|_2$$
$$+|\varsigma_{\iota\rho}|_2+|\varpi_{\iota\rho}|_2)\tilde{M}_{\rho}|\Xi_{\iota}(t-\tau_{\rho})|_2^2\Big)$$
$$+\sum_{\iota=1}^{\eta}Re\big(\grave{\upsilon}_{2\iota}(t)\overline{\Theta_{\iota}(t)}\big). \tag{28}$$

Form Eq. (21), then

$$D^{+}V_2(t)$$
$$\leq\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}\Big(\big(-\tilde{\zeta}_{1\iota}-1+\frac{1}{2}|1-h_{\iota}|_2+\frac{1}{2}|\partial_{\rho\iota}|_2\tilde{L}_{\iota}\big)$$
$$\times|\Xi_{\iota}(t)|_2^2+\big(-\tilde{\zeta}_{2\iota}-\xi_{\iota}+\frac{1}{2}(|1-h_{\iota}|_2+|\partial_{\iota\rho}|_2\tilde{L}_{\rho}$$
$$+|\hbar_{\iota\rho}|_2\tilde{M}_{\rho}+|\varsigma_{\iota\rho}|_2\tilde{M}_{\rho}+|\varpi_{\iota\rho}|_2\tilde{M}_{\rho})\big)|\Theta_{\iota}(t)|_2^2$$
$$+\big(-\tilde{\varphi}_{\iota}+\frac{1}{2}(|\hbar_{\iota\rho}|_2+|\varsigma_{\iota\rho}|_2+|\varpi_{\iota\rho}|_2)\tilde{M}_{\rho}\big)$$
$$\times|\Xi_{\iota}(t-\tau_{\rho})|_2^2\Big)-\frac{1}{2}\sum_{\iota=1}^{\eta}(|\Xi_{\iota}(t)|_2^2)^{\beta}$$
$$-\frac{1}{2}\sum_{\iota=1}^{\eta}(|\Theta_{\iota}(t)|_2^2)^{\beta}-\sum_{\iota=1}^{\eta}\tilde{\zeta}_{3\iota}$$

$$
\begin{aligned}
&\leq -\sum_{\iota=1}^{\eta}\big(k_{1\iota}|\Xi_\iota(t)|_2^2+k_{2\iota}|\Theta_\iota(t)|_2^2\big)+\sum_{\iota=1}^{\eta}\sum_{\rho=1}^{\eta}\big(-\varphi_\iota\\
&\quad+\frac{1}{2}(|\hbar_{\iota\rho}|_2+|\varsigma_{\iota\rho}|_2+|\varpi_{\iota\rho}|_2)\tilde{M}_\rho\big)|\Xi_\iota(t-\tau_\rho)|_2^2\\
&\quad-\frac{1}{2}\sum_{\iota=1}^{\eta}(|\Xi_\iota(t)|_2^2)^\beta-\frac{1}{2}\sum_{\iota=1}^{\eta}(|\Theta_\iota(t)|_2^2)^\beta-\sum_{\iota=1}^{\eta}\tilde{\zeta}_{3\iota}\\
&\leq -\sum_{\iota=1}^{\eta}min\{k_{1\iota},k_{2\iota}\}\big(|\Xi_\iota(t)|_2^2+|\Theta_\iota(t)|_2^2\big)\\
&\quad-\frac{1}{2}\sum_{\iota=1}^{\eta}(|\Xi_\iota(t)|_2^2)^\beta-\frac{1}{2}\sum_{\iota=1}^{\eta}(|\Theta_\iota(t)|_2^2)^\beta-\sum_{\iota=1}^{\eta}\tilde{\zeta}_{3\iota}\\
&\leq -\tilde{k}V_2(t)-\frac{1}{2}\sum_{\iota=1}^{\eta}(|\Xi_\iota(t)|_2^2)^\beta-\frac{1}{2}\sum_{\iota=1}^{\eta}(|\Theta_\iota(t)|_2^2)^\beta\\
&\quad-\sum_{\iota=1}^{\eta}\tilde{\zeta}_{3\iota},
\end{aligned}
\tag{29}
$$

where $\tilde{k}_{1\iota}=-\sum_{\rho=1}^{\eta}\big(-\tilde{\xi}_{1\iota}-1+\frac{1}{2}|1-h_\iota|_2+\frac{1}{2}|\partial_{\rho\iota}|_2\tilde{L}_\iota\big)$, $\tilde{k}_{2\iota}=-\sum_{\rho=1}^{\eta}\big(-\tilde{\zeta}_{2\iota}-\xi_\iota+\frac{1}{2}(|1-h_\iota|_2+|\partial_{\iota\rho}|_2\tilde{L}_\rho+|\hbar_{\iota\rho}|_2\tilde{M}_\rho+|\varsigma_{\iota\rho}|_2\tilde{M}_\rho+|\varpi_{\iota\rho}|_2\tilde{M}_\rho)\big)$, $\tilde{k}=2min\{\tilde{k}_{1\iota},\tilde{k}_{2\iota}\}$.

By application of Lemma 2 we have:

(1) If $V_2(t)\in(0,1)$,

$$
\begin{aligned}
&-\frac{1}{2}\sum_{\iota=1}^{\eta}(|\Xi_\iota(t)|_2^2)^\beta-\frac{1}{2}\sum_{\iota=1}^{\eta}(|\Theta_\iota(t)|_2^2)^\beta\\
&\leq -(\frac{1}{2})^{1-\beta}(\frac{1}{2}\sum_{\iota=1}^{\eta}|\Xi_\iota(t)|_2^2)^\beta-(\frac{1}{2})^{1-\beta}(\frac{1}{2}\sum_{\iota=1}^{\eta}|\Theta_\iota(t)|_2^2)^\beta\\
&\leq -(\frac{1}{2})^{1-\beta}V_2^\beta(t).
\end{aligned}
\tag{30}
$$

(2) If $V_2(t)>1$,

$$
\begin{aligned}
&-\frac{1}{2}\sum_{\iota=1}^{\eta}(|\Xi_\iota(t)|_2^2)^\beta-\frac{1}{2}\sum_{\iota=1}^{\eta}(|\Theta_\iota(t)|_2^2)^\beta\\
&\leq -(\frac{1}{2}\eta)^{1-\beta}(\frac{1}{2}\sum_{\iota=1}^{\eta}|\Xi_\iota(t)|_2^2)^\beta-(\frac{1}{2}\eta)^{1-\beta}(\frac{1}{2}\sum_{\iota=1}^{\eta}|\Theta_\iota(t)|_2^2)^\beta\\
&\leq -\eta^{1-\beta}V_2^\beta(t).
\end{aligned}
\tag{31}
$$

From eqs. (14), (15), we obtain

$$
D^+V_2(t)\leq\begin{cases}-\tilde{k}V_2(t)-\tilde{\Omega}_1V_2^\beta(t)-\tilde{\zeta}, & V_2(t)\in(0,1),\\ -\tilde{k}V_2(t)-\tilde{\Omega}_2V_2^\beta(t)-\tilde{\zeta}, & V_2(t)\geq 1,\end{cases}
\tag{32}
$$

where $\tilde{\Omega}_1=(\frac{1}{2})^{1-\beta}$, $\tilde{\Omega}_2=\eta^{1-\beta}$, $\tilde{\zeta}=\sum_{\iota=1}^{\eta}\tilde{\zeta}_{3\iota}$.

TABLE 2. Comparisons between recent works and this article.

References	FNNs	Inertial items	CVNNs	FXCPLS
[4], [7], [8]	√	×	×	×
[6], [12], [39]	√	√	×	×
[17], [18], [40]	×	×	√	×
[35], [41]–[45]	×	√	√	×
[22], [31], [46]	√	√	×	×
[47], [48]	√	√	√	×
This paper	√	√	√	√

Remark 3: In [34], [35], [36], the controllers designed contain the sign function $sign(\cdot)$, so the controllers are discontinuous. However, in some cases, continuity of the controller is necessary. Therefore, the controllers in this article does not include symbol functions, which can effectively avoid unnecessary oscillations.

Remark 4: This article studies the CPLS problem of FICVNNs, which combines complex projective synchronization and lag synchronization. Complex projective synchronization can also improve the level of secure communication. Due to the time required for signal transmission, lag synchronization is a reasonable solution for driving response systems. Therefore, CPLS can better transmit information and has practical value.

When the values of m and γ are given, the result of Theorem 1 can be generalized to the following conclusion.

Corollary 1: Under the conditions of Theorem 1 and controller (5), the following results hold

(1) if $m\neq 0,\gamma=0$, drive-response system (2) and (3) obtain fixed-time CPS;

(2) if $m=1,\gamma>0$, drive-response system (2) and (3) satisfy FXLS;

(3) if $m=1,\gamma=0$, drive-response system (2) and (3) obtain FXS.

Corollary 2: Under Theorem 1 and control Eq.(5), if $f_\rho(\cdot)$, $g_\rho(\cdot)$, $\rho\in S$ are odd functions, there are true that

(1) if $m=-1,\gamma>0$, drive-response system (2) and (3) get fixed-time lag anti-synchronization(FXLAS) at T_{1f}.

(2) if $m=-1,\gamma=0$, drive-response system (2) and (3) fulfil FXAS at T_{1f}.

Corollary 3: If $f_\rho(\cdot)$, $g_\rho(\cdot)$ satisfy $f_\rho(0)=g_\rho(0)=0$, $m=\gamma=0$, Eq. (2) achieves fixed-time stabilization under $\grave{\upsilon}_{1\iota}(t)$ and settling time is T_{1f}.

Remark 5: Similarly, if the conditions of Theorem 2 and controller (21) are established, results similar to Corollary 1-3 can also be obtained. Therefore, the results of this article are generalizable.

Remark 6: Previous research results only considered fixed-time projective synchronization(FXPS) or FXLS [31], [37], [38]. But FXPS, FXLS and anti-synchronization are special cases of our results. It is not difficult to see that the research results of this article are relatively comprehensive.

Remark 7: We present the differences between this article and other literature in Table 2.

IV. NUMERICAL EXAMPLES

Example: Let us consider the following FICVNNs:

$$\ddot{\lambda}_\iota(t) = -\alpha_\iota\lambda_\iota(t) - \ell_\iota\dot{\lambda}_\iota(t) + \sum_{\rho=1}^{\eta}\partial_{\iota\rho}f_\rho(\lambda_\rho(t)) + \sum_{\rho=1}^{\eta}\hbar_{\iota\rho} \times g_\rho(\lambda_\rho(t-\tau_\rho)) + \sum_{\rho=1}^{\eta}d_{\iota\rho}H_\rho(t) + \bigwedge_{\rho=1}^{\eta}r_{\iota\rho}H_\rho(t) + \bigvee_{\rho=1}^{\eta}\delta_{\iota\rho}H_\rho(t) + \bigwedge_{\rho=1}^{\eta}\varsigma_{\iota\rho}g_\rho(\lambda_\rho(t-\tau_\rho)) + \bigvee_{\rho=1}^{\eta}\varpi_{\iota\rho}g_\rho(\lambda_\rho(t-\tau_\rho)), t \geq 0, \iota \in S, \tag{33}$$

where $\iota, \rho = 1, 2$, $\alpha_1 = 1.4$, $\alpha_2 = 2$, $\ell_1 = 2$, $\ell_2 = 1.4$, $r_{\iota\rho} = \delta_{\iota\rho} = H_\rho(t) = 1$. $\tau_\rho = 1, f_\rho(\cdot) = g_\rho(\cdot) = tanh(Re(\cdot)) + tanh\big(Im(\cdot)\big)i$, $d_{\iota\rho} = 0, \iota, \rho = 1, 2$, $\partial_{11} = 1.5 + 2.6i$, $\partial_{12} = -2.0 + 1.7i$, $\partial_{21} = 1.2 + 0.6i$, $\partial_{22} = 1.0 - 1.5i$, $\hbar_{11} = 2.5 + 1.5i$, $\hbar_{12} = 1.0 - 1.2i$, $\hbar_{21} = -1.7 - 2.7i$, $\hbar_{22} = -2.4 - 1.6i$, $\varsigma_{11} = 0.4 - 1.0i$, $\varsigma_{12} = -2 - 0.3i$, $\varsigma_{21} = 1.5 - 1.6i$, $\varsigma_{22} = 1.1 - 1.6i$, $\varsigma_{11} = 0.4 - 1.0i$, $\varsigma_{12} = -2 - 0.3i$, $\varsigma_{21} = 1.5 - 1.6i$, $\varsigma_{22} = 1.1 - 1.6i$, $\varpi_{11} = 0.2 - 2.5i$, $\varpi_{12} = 2 - 1.4i$, $\varpi_{21} = 0.5 - 1.6i$, $\varpi_{22} = 2.8 + 0.6i$. The initial conditions of FICVNNs are chosen as $\lambda_1(\phi) = 1.8 + 1.7i$, $\dot{\lambda}_1(\phi) = 2.1 - 1.4i$, $\lambda_2(\phi) = -2.5 - 1.4i$, $\dot{\lambda}_2(\phi) = -1.8 + 1.0i$, $\phi \in [-1, 0]$. Then, the phase plot and the state trajectory of variables $\lambda_1(t)$, $\lambda_2(t)$ of FICVNNs are obtained and shown respectively in Figs. 1 and 2.

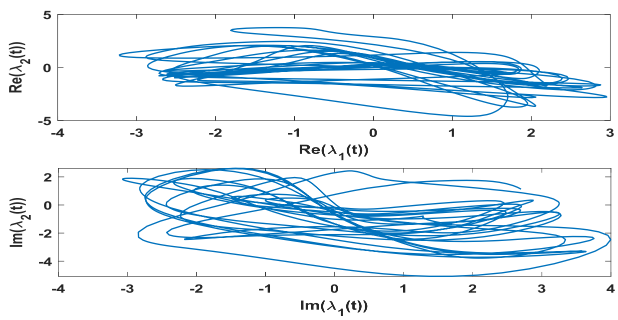

FIGURE 1. The phase plot of FICVNNs Eq. (21).

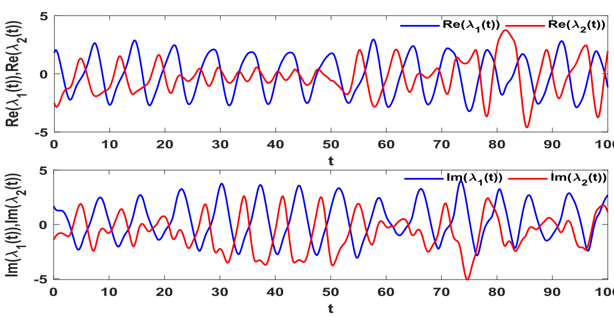

FIGURE 2. The state trajectories $\lambda_1(t), \lambda_2(t)$ for drive system (33).

Furthermore, let $v_\iota(t) = \dot{\lambda}_\iota(t) + \lambda_\iota(t)$, then FICVNNs (32) could be given by the following form

$$\dot{\lambda}_\iota(t) = -\lambda_\iota(t) + v_\iota(t),$$
$$\dot{v}_\iota(t) = -\xi_\iota v_\iota(t) - h_\iota\lambda_\iota(t) + \sum_{\rho=1}^{\eta}\partial_{\iota\rho}f_\rho(\lambda_\rho(t)) + \sum_{\rho=1}^{\eta}\hbar_{\iota\rho} \times g_\rho(\lambda_\rho(t-\tau_\rho)) + \sum_{\rho=1}^{\eta}d_{\iota\rho}H_\rho(t) + \bigwedge_{\rho=1}^{\eta}r_{\iota\rho}H_\rho(t) + \bigvee_{\rho=1}^{\eta}\delta_{\iota\rho}H_\rho(t) + \bigwedge_{\rho=1}^{\eta}\varsigma_{\iota\rho}g_\rho(\lambda_\rho(t-\tau_\rho)) + \bigvee_{\rho=1}^{\eta}\varpi_{\iota\rho}g_\rho(\lambda_\rho(t-\tau_\rho)), t \geq 0, \iota \in S. \tag{34}$$

Set the system (33) to be drive system and respond system is:

$$\dot{\kappa}_\iota(t) = -\kappa_\iota(t) + w_\iota(t),$$
$$\dot{w}_\iota(t) = -\xi_\iota w_\iota(t) - h_\iota\kappa_\iota(t) + \sum_{\rho=1}^{\eta}\partial_{\iota\rho}f_\rho(\kappa_\rho(t)) + \sum_{\rho=1}^{\eta}\hbar_{\iota\rho} \times g_\rho(\kappa_\rho(t-\tau_\rho)) + \sum_{\rho=1}^{\eta}d_{\iota\rho}H_\rho(t) + \bigwedge_{\rho=1}^{\eta}r_{\iota\rho}H_\rho(t) + \bigvee_{\rho=1}^{\eta}\delta_{\iota\rho}H_\rho(t) + \bigwedge_{\rho=1}^{\eta}\varsigma_{\iota\rho}g_\rho(\kappa_\rho(t-\tau_\rho)) + \bigvee_{\rho=1}^{\eta}\varpi_{\iota\rho} \times g_\rho(\kappa_\rho(t-\tau_\rho)) + \tilde{v}_\iota(t), t \geq 0, \iota \in S. \tag{35}$$

Here, the initial values of response system (35) are respectively taken as $\kappa_1(\phi) = 1.3 + 1.1i$, $w_1(\phi) = 3.0 - 1.8i$, $\kappa_2(\phi) = -2.0 - 1.2i$, $w_2(\phi) = -1.0 + 1.3i$. Fig.3 depicts the graph of the error state without a controller.

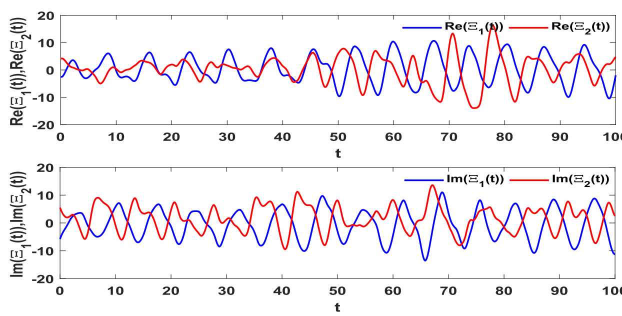

FIGURE 3. Error states without control.

A. FXCPLS UNDER THE CONTROLLER (5)

The continuous functions $f_\rho(\cdot)$ and $g_\rho(\cdot)$ in system (21) satisfies Hypothesis 1, and $L_\rho = M_\rho = 1$. In order to achieve FXCPLS between systems (33) and (34), we choose the parameters of the controller (5) as $\Phi = 1.6$, $\zeta_{11} = 25.6$, $\zeta_{12} = 29.1$, $\zeta_{21} = 15$, $\zeta_{22} = 0.7$, $\zeta_{31} = \zeta_{32} = 0.05$, $\varphi_{11} = 65 > \big(|\hbar_{11}|_1 + |\varsigma_{11}|_1 + |\varpi_{11}|_1\big)M_1 = 5.8$, $\varphi_{12} = 80 > \big(|\hbar_{12}|_1 + |\varsigma_{12}|_1 + |\varpi_{12}|_1\big)M_2 = 7.9$,

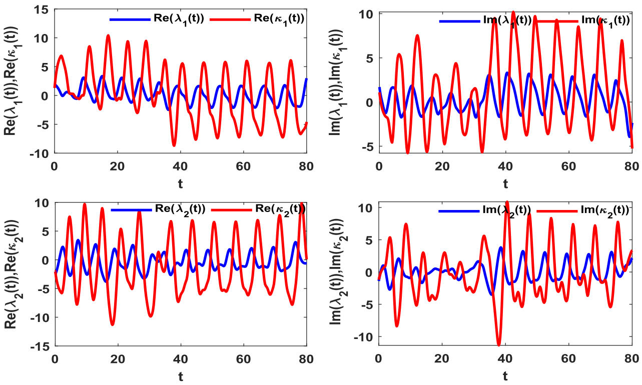

FIGURE 4. Trajectories of states $\lambda_\iota(t), \kappa_\iota(t)$ under control Eq.(5).

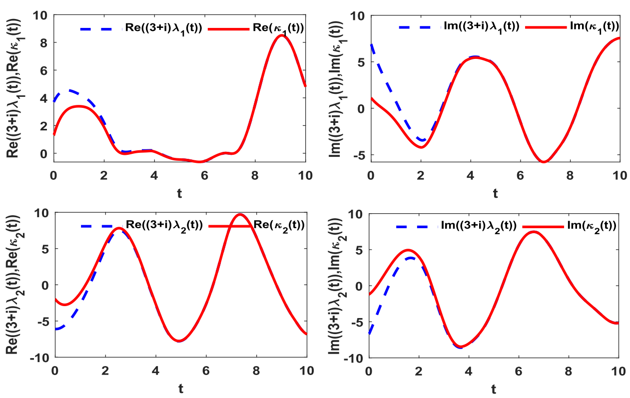

FIGURE 5. $(3+i)\lambda_\iota(t-2), \kappa_\iota(t)$ under Eq.(5).

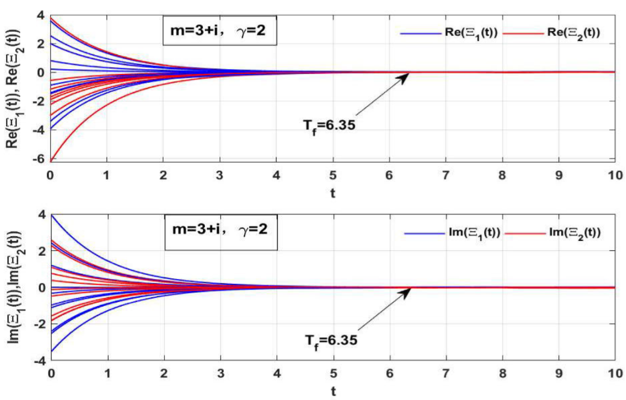

FIGURE 6. The error states and $m = 3+i, \gamma = 2$ with Eq. (5).

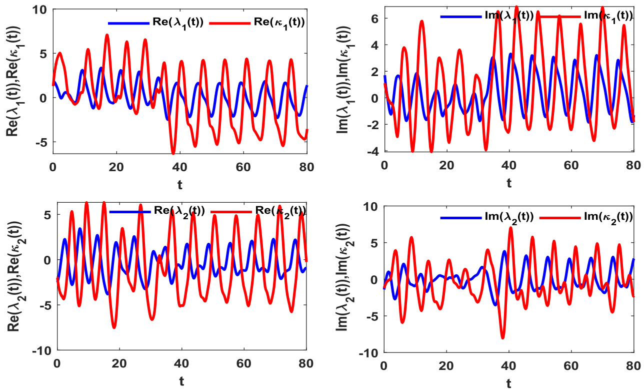

FIGURE 7. Trajectories of states $\lambda_\iota(t), \kappa_\iota(t)$ under control Eq.(21).

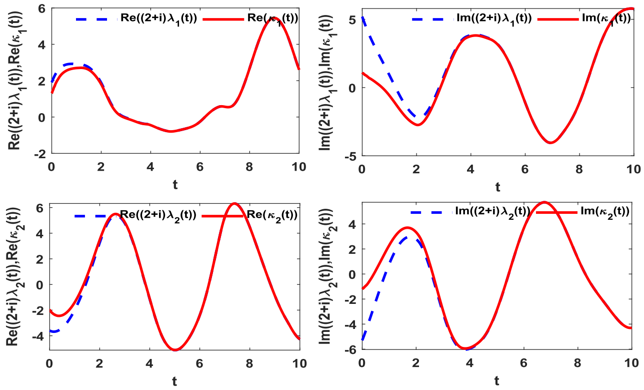

FIGURE 8. $(2+i)\lambda_\iota(t-2), \kappa_\iota(t)$ under Eq.(21).

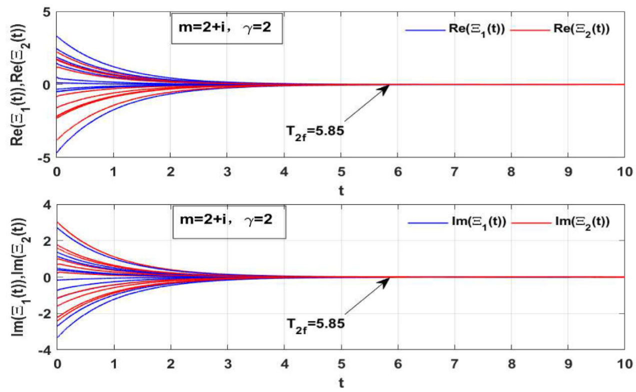

FIGURE 9. The error states and $m = 2+i, \gamma = 2$ with Eq. (21).

$\varphi_{21} = 64.4 > \big(|\hbar_{21}|_1 + |\varsigma_{21}|_1 + |\varpi_{21}|_1\big)M_1 = 9.6$, $\varphi_{22} = 80.5 > \big(|\hbar_{22}|_1 + |\varsigma_{22}|_1 + |\varpi_{22}|_1\big)M_2 = 10.1$. By simple calculation we get $k_{11} = 20.3, k_{12} = 22.3, k_{21} = 15, k_{22} = 0.1$, $k = min\{k_{1\iota}, k_{2\iota}\} = 0.1$, And $\Omega_1 = 1$, $\Omega_2 = 0.1088$, $\zeta = 0.1$. Therefore, the conditions of Theorem 1 are satisfied. According to Theorem 1, FICVNNs implements FXCPLS at $T_{1f} = 6.35$.

To verify the correctness of our theoretical results in the future, we set $m = 3+i, \gamma = 2$, under Theorem 1 and Eq.(5), drive-response systems (34) - (35) implement FXCPLS. Figs.4 and 5 show the state trajectories. And the error trajectory is drawn in Fig.6.

B. FXCPLS UNDER THE CONTROLLER (21)

Next, we verify the validity of the controller (21). First, we assign values to the parameters in the controller (21). Let $\Phi = 1.7$, $\tilde{\zeta}_{11} = 9.4716$, $\tilde{\zeta}_{12} = 1.52385$, $\tilde{\zeta}_{21} = 14.5762$, $\tilde{\zeta}_{22} = 18.54715$, $\tilde{\zeta}_{31} = 0.015$, $\tilde{\zeta}_{32} = 0.005$, $\varphi_1 = 6.3 > \frac{1}{2}\big(|\hbar_{11}|_2 + |\hbar_{12}|_2 + |\varsigma_{11}|_2 + |\varsigma_{12}|_2 + |\varpi_{11}|_2 + |\varpi_{11}|_2\big)\tilde{M}_\rho = 6.267$, $\varphi_2 = 7.4 > \frac{1}{2}\big(|\hbar_{21}|_2 + |\hbar_{22}|_2 + |\varsigma_{21}|_2 + |\varsigma_{22}|_2 + |\varpi_{21}|_2 + |\varpi_{21}|_2\big)\tilde{M}_\rho = 7.37495$. After calculation, one has $\tilde{k}_{11} = 8, \tilde{k}_{12} = 0.01, \tilde{k}_{21} = 6.2, \tilde{k}_{22} = 10, \tilde{k} = 2min\{\tilde{k}_{1\iota}, \tilde{k}_{2\iota}\} = 0.02, \tilde{\Omega}_1 = 0.813, \tilde{\Omega}_2 = 0.3078, \tilde{\zeta} = 0.02$. Therefore, the conditions of Theorem 2 are satisfied, systems (34) - (35) implement FXCPLS at $T_{2f} = 5.85$.

Similarly, let $m = 2 + i, \gamma = 2$, the drive-response system (34)-(35) realise FXCPLS at $\tilde{T}_{2f} = 5.85$, as shown in Figs. 7 and 8. Fig.9 depicts the trend of the error state under the controller (21).

V. CONCLUSION

Based on recent articles [25], [26], [49], we studied the FXCPLS problems of a class of CVNNs with inertial terms and fuzzy logic through non separation methods. Utilising fixed-time control theory, we have developed new criteria to guarantee that the FICVNNs (4) fulfils FXCPLS. Our conclusion can also be extended to other forms of synchronization, such as fixed-time LS, fixed-time AS, FXCPS, etc. At the same time, We conducted numerical simulations using Matlab to verify the theoretical results of this paper.

In order to obtain the stability and synchronization of NNs, most current control methods are time-triggered control, but the control cost is relatively high. Event-triggered control will only update when the measurement error exceeds the predesigned trigger condition threshold, which not only saves costs but also The efficiency is improved. Therefore, using event-triggered control to study complex-valued inertial neural networks is worthy of further research.

AUTHOR CONTRIBUTIONS

Yu Yao: Writing-original draft and software; Jing Han: Validation, and supervision; Guodong Zhang: Conceptualization, methodology, and Junhao Hu: Writing-review.

DECLARATION OF COMPETING INTEREST

None.

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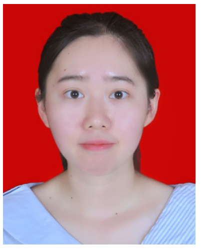

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