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RESEARCH ARTICLE

On Similarity Measures of Complex Picture Fuzzy Sets With Applications in the Field of Pattern Recognition

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ABSTRACT Despite the significant advancements in fuzzy set theory, existing similarity measures for complex picture fuzzy sets (*CPFSs*) often result in impractical results in real-world scenarios. This presents a critical gap in accurately modeling and analyzing CPFSs, particularly in applications like pattern recognition and medical diagnosis. The present work addresses this problem by introducing various novel similarity measures for CPFSs, accompanied by rigorous axiomatic validation and a thorough discussion of their properties. Different sets of CPFSs have been empirically evaluated using both existing and proposed similarity measures, demonstrating the practical applicability and superiority of the latter. Based on the principle of maximum similarity, a comprehensive methodology involving these proposed measures has been illustrated, along with their implementation in solving different problems in pattern recognition and medical diagnosis. Additionally, a comparative analysis has been conducted to provide better clarity and understanding of the effectiveness of these measures. The results indicate that the proposed similarity measures offer significant advantages and improved accuracy for pattern recognition and medical diagnosis problems.

INDEX TERMS Picture fuzzy sets, complex picture fuzzy sets, similarity measures, pattern recognition, medical diagnosis.

I. INTRODUCTION

In day-to-day real-life situations various decision-making strategies need the techniques of pattern recognition for a consistent realization in order to handle the uncertainty and imprecision [1], [2], [3]. The automated recognition of patterns is widely applicable in the field of engineering science, network analysis and the problem of medical diagnosis [4], [5]. The systematic study of pattern recognition problems has many implementations in the field of signals and systems, image processing, assembly robot design evaluation [6], [7], [8], bio-informatics and machine learning. In spite of having various extensions of fuzzy sets such as intuitionistic fuzzy sets (*IFSs*), Pyhtagorean fuzzy sets (*PyFSs*), and picture fuzzy sets (*PFSs*), these extensions are not sufficient to address the uncertainty and vagueness occurring due to periodicity in the data. Such complex data sets normally get originated in the field of medical research, biometric/facial recognition, audio/images which encounter deviations under the given phase of time. In order to handle such circumstances so that the best precise information can be projected in one go for avoiding the loss of information content, we should suitably signify the amplitude/phase term using the complex extensions of fuzzy sets [9], [10]. The application in the field of electromagnetic signals and solar activities can been by the cohesiveness of the fuzzy sets [11].

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To measure uncertainty/fuzziness associated with any real-life modeling, the role of different types of similarity/distance measures and entropy/inclusion measures has been explored by different researchers in different capacities. It may be noticed that similarity/distance measures mathematically provide the degree of similarity/discrimination among two sets of information [12]. For the sake of a brief review of the literature available, Szmidt [13] provided some distance/similarity measures for the intuitionistic fuzzy information set up and utilized in the field of decision-making problems. Further, Shen et al. [14] incorporated a new distance measure for intuitionistic fuzzy sets with its different properties and proposed a ''modified technique for order preference by similarity to ideal solution (*TOPSIS*)''. Next, Ye [15] presented two novel similarity measures on the basis of cosine functions very effectively and compared them with the existing other trigonometric measures. In further deliberations, on the basis of direct operations of membership/non-membership/hesitation, Song et al. [16] proposed a similarity measure for *IFSs* with application to the problem of medical diagnosis. Also, Singh and Kumar [17] put forward a new intuitionistic fuzzy similarity measure with applications to the field of pattern/face recognition problems.

Upon the introduction of picture fuzzy sets (*PFSs*) in literature, Cuong and Kreinovich [18] proposed the Hamming/Euclidean distance measures for the picture fuzzy environment. Next, Son [19] presented the generalization of some distance measures between *PFSs* with application in the field of clustering analysis while Dutta [20] marked out that Son's measure has some flaw and provided a new measure. Further, the normalized and parametric form of the Hamming/Euclidean/Haussdorff distance measures for PFSs were proposed by Singh et al. [21] with application to risk analysis of flood disaster. Also, Dinh and Thao [22] proposed some new distance/dissimilarity measures between *PFSs* and utilized them in the problems of ''pattern recognition and decision-sciences''. Also, Akram et al., [23] presented the modified version of MARCOS decision-making techniques under q-rung picture fuzzy environment to overcome the limitations of other fuzzy extensions. Further, a new parametric information measure was given on the q-rung picture fuzzy set which was applied in the green supply chain management sector [24]. Wei [25] provided the concept of picture fuzzy cross-entropy, and further proposed picture fuzzy cosine similarity measure [26] and utilized them accordingly/respectively in the decision-making problems. Akram et al., [27] studied the Dijkstra algorithm for a network for the best possible optimization study in trapezoidal picture fuzzy environment. Further, for solving the building materials recognition problems, Wei and Gao [28] presented the normalized/parametric form of the dice similarity measure for the *PFSs*. Also, based on the cosine function & the four standard uncertainty components of picture fuzzy information, Wei [29] presented a new similarity measure for dealing with the problems of ''strategic decision making''. The parameterizations of the attributes have also been successfully implemented in the renewable energy source selection problem with the incorporation of the matrix theory of picture fuzzy information [30]. In literature, the applications of distance/similarity measures are widely popularized by various researchers in the field of ''data-mining, medical diagnosis, decision-making and pattern recognition''. On the basis of cosine/cotangent functions, [29] have given some similarity measures for picture fuzzy sets dealing with the ''strategic decision-making problem''. Further, Wei [31] discussed some new cosine/weighted cosine, set-theoretic/grey similarity measures with their weighted form and incorporated into the problem of pattern recognition. An integrated ($EDAS/ELECTRE$) decision-making approach with the picture fuzzy information in the field of green production of gold mines have been presented by Liang et al. [32]. Different researchers utilized distance/entropy/dissimilarity measures for picture fuzzy information differently in linear-programming based *TOPSIS* method [33], in multi-attribute, decision-making problems [34], [35], [36] projection model [26], [37] etc. In the field of ''clustering analysis'', ''pattern recognition'' and ''medical diagnosis'', Son [38] discussed the generalized picture distance/association measures and Ganei et al. [39] introduced some new correlation coefficients in the picture fuzzy information setup. The parametric form of information measures has also been studied in the picture fuzzy setup and applied in the technology of hydrogen fuel cells while deploying *VIKOR* and *TOPSIS* decision-making techniques [40], [41].

In view of the understanding at this stage, it is being observed that the extended versions of fuzzy sets and picture fuzzy sets can address the inconsistency, vagueness/ambiguity of the information in the application fields but the non-inclusiveness in the time period and robustness of the data sometimes would not be possible to deal with these kinds of models. However, the periodicity/repeatedness and the uncertainty/inexactness of the data can be dealt with the help of the complex number framework at the same time. Initially, the concept of a complex fuzzy set was given by Ramot et al. [42] for handling such situations. It was proposed that the value of the membership degree range be extended to the unit disk of the complex plane. A variety of applications of *CFSs* have been brought into the literature by different researchers in the field of audio databases, biometric/medical studies, etc. In a systematic extension of complex fuzzy sets, the literature comes across the complex intuitionistic fuzzy set (*CIFS*) [43] and complex picture fuzzy set (*CPFS*) [44]. The complex neutral membership degree widens the applicability of uncertain information and keeps it more manageable for an expert. Akram et al. [44] proposed the concept of CPFS and introduced new Hamacher weighted/ordered weighted averaging aggregation operators along with geometric operators and application to some *MCDM* problems. Mahmood et al. [45] joined the notion of *CPFS* and N-soft sets to introduce the concept of complex picture fuzzy N-soft sets along with various algebraic operations and solved the problem of "e-waste recycling program and prediction about the FIFA world cup championship through audience poll". Also, Mahmood et al. [46] presented the new concept of "complex picture fuzzy soft aggregation operators" and applied in the problem of "multi-attribute decision-making (*MADM*)" problems. Further, Liu et al., [47] presented the complex picture fuzzy based power aggregation operators for a multi-criteria decision-making problem to cover the time period parameter of data. In addition to this, various decision-making models have been given for complex intuitionistic and complex Pythagorean fuzzy environments with the inclusion of Hamacher and Yager aggregation operators [48], [49].

Certainly, in some real-life day-to-day situations, it becomes very important to append the other dimension to the natural degree of membership/non-membership so that the complete information may be incorporated to avoid the natural loss of information. For this, the notion of phase term in the information becomes significant. We can illustrate this with an example. Let us suppose that there is an operational unit where the fingerprint attendance management machines (FAMMs) are functional in their offices. Suppose that the unit manager gets into consultation with an expert who provides necessary inputs for the models of FAMMs & their manufacturing period. The manager is supposed to identify the most optimal/dependable model of FAMMs in view of its manufacturing period at the same time. Such problems are two-dimensional which cannot be properly and accurately modeled with picture fuzzy sets with the twin-dimensionality at the same time. In such a situation, the optimal/best approach to encounter/address the complete information to the expert would be by utilizing the notion of complex picture fuzzy sets (*CPFSs*). The amplitude term in the *CPFS* may be utilized to represent the final plan of the company regarding the FAMMs and the phase term could be utilized to address the manager's judgment in view of the manufacturing period of the FAMMs.

A. NOVELTY

The novelties of the presented study lie in proposing a new kind of similarity measure for complex picture fuzzy sets for the modeled decision-making problems. Presently, there is no study available for the similarity measures of complex picture fuzzy sets. The proposed similarity measures for complex picture fuzzy sets are successfully applied in the decision making problems of pattern-recognition. The addition of uncertainty components and time period factor involvement in the complex picture fuzzy sets makes this set robust enough to cover uncertainty problems.

B. MOTIVATION & RESEARCH GAP

In view of the existing literature and characteristics of the *CPFSs*, the prime objective of the manuscript would be to introduce some similarity measures for the first time under a complex picture fuzzy environment for addressing the widely applicable pattern recognition problems having multi-dimensional complex nature of the information. The motivations behind the present research work are enumerated below:

- Develop and introduce the theory of similarity measures for *CPFSs* with their proof of validation and algebraic properties.
- To have a suitable comparison between the existing similarity measures vs the proposed similarity measures under a picture fuzzy environment and present their advantages.
- Implementation of the proposed similarity measures in different pattern recognition problems with due comparative analysis in view of existing literature.

The proposed contribution in this communication has been developed as follows: In Section II, some fundamental definitions/preliminaries in accordance with the present work have been provided. The formal definitions and notion of similarity measures (seven different) for the complex picture fuzzy sets have been introduced with proof of their validity in III along with some important properties. In Section IV, a characteristic and empirical comparative analysis of some existing similarity measures with the proposed one has been suitably carried out in detail for establishing the prominency of the proposed one with the help of deterministic values. On the basis of these deliberations and the principle of maximum similarity, Section V presents the utilization of the proposed similarity measures in different problems of pattern recognition and medical diagnosis. In addition to this, a similar comparative analysis with the help of these problems has been accordingly presented and discussed for better understanding and clarity to the readers. Also, we precisely present the advantages and effectiveness of the proposed work under the discussion section (Section VI). Finally, the paper has been duly concluded with the scope of future work in Section VII.

II. PRELIMINARIES

In this section, we revisit some fundamental notions in connection with *PFSs* and *CPFSs* with some of their binary operations over the universe of discourse U.

Definition 1: **Picture Fuzzy Set (PFS)** [18] "A PFS A on U is defined as $A = \{< u, \rho_A(u), \tau_A(u), \omega_A(u) >| u \in U\}$, where $\rho_A : U \rightarrow [0, 1]$, $\tau_A : U \rightarrow [0, 1]$ and $\omega_A : U \rightarrow [0, 1]$ is the degree of membership, degree of neutral membership (abstain) and degree of non-membership respectively and satisfies the condition $\rho_A(u) + \tau_A(u) + \omega_A(u) \leq 1$ with the degree of refusal given by $\pi_A(u) = 1 - (\rho_A(u) + \tau_A(u) + \omega_A(u))$."

Definition 2: **Complex Fuzzy Set (CFS)** [42] "A CFS A on U is a set of ordered pairs, which is given $A = \{(u, \rho_A(u)) \mid u \in U\}$, where $\rho_A : U \rightarrow \{c : c \in C, |c| \leq 1\}$ is a complex-valued membership function and for $u \in U$, the value of $\rho_A(u)$ is given as $\rho_A(u) = s_A(u)e^{i\theta_{sA}(u)}$, where $i = \sqrt{-1}, 0 \leq s_A(u) \leq 1$ *and* $0 \leq \theta_{sA}(u) \leq 2\pi$."

Definition 3: **Complex Picture Fuzzy Set (CPFS)** [44] "A CPFS A defined on U is defined as $A = \{(u, \rho_A(u), \tau_A(u), \omega_A(u)) \mid u \in U\}$, where $\rho_A, \tau_A(u), \omega_A(u) : U \rightarrow \{c : c \in C, |c| \leq 1\}$ is a complex-valued membership, neutral and non-membership functions, respectively and are given as $\rho_A(u) = s_A(u)e^{i\theta_{sA}(u)}$, $\tau_A(u) = t_A(u)e^{i\theta_{tA}(u)}$, $\omega_A(u) = m_A(u)e^{i\theta_{mA}(u)}$, where $i = \sqrt{-1}, 0 \leq s_A(u), t_A(u), m_A(u) \leq 1; 0 \leq s_A(u) + t_A(u) + m_A(u) \leq 1$ *and* $0 \leq \theta_{sA}(u), \theta_{tA}(u), \theta_{mA}(u) \leq 2\pi; 0 \leq \theta_{sA}(u)+\theta_{tA}(u)+\theta_{mA}(u) \leq 2\pi$ for all $u \in U$."

Note: For the ease of computations, we shall denote *CPFS* A as $\{(u, (s_A(u), \theta_{sA}(u)), (t_A(u), \theta_{tA}(u)), (m_A(u), \theta_{mA}(u))) \mid u \in U\}$.

Definition 4: [44] "Let $A = (u, (s_A(u), \theta_{sA}(u)), (t_A(u), \theta_{tA}(u)), (m_A(u), \theta_{mA}(u))) \mid u \in U$ and $B = \{(u, (s_B(u), \theta_{sB}(u)), (t_B(u), \theta_{tB}(u)), (m_B(u), \theta_{mB}(u))) \mid u \in U\}$ be two *CPFSs* defined on U. Then,

- $A \subseteq B$ if and only if $s_A(u) \leq s_B(u)$, $t_A(u) \leq t_B(u)$, $m_A(u) \leq m_B(u)$ for amplitude terms and $\theta_{sA}(u) \leq \theta_{sB}(u)$, $\theta_{tA}(u) \leq \theta_{tB}(u)$, $\theta_{mA}(u) \leq \theta_{mB}(u)$,
- $A^c = \{(u, ((m_A(u), \theta_{mA}(u)), (t_A(u), \theta_{tA}(u)), (s_A(u), \theta_{sA}(u)))), \mid u \in U\}$,
- $A \cup B = \{(u, (\max(s_A(u), s_B(u)), \max(\theta_{sA}(u), \theta_{sB}(u))), (\min(t_A(u), t_B(u)), \min(\theta_{tA}(u), \theta_{tB}(u))), (\min(m_A(u), m_B(u)), \min(\theta_{mA}(u), \theta_{mB}(u)))) \mid u \in U\}$,
- $A \cap B = \{(u, (\min(s_A(u), s_B(u)), \min(\theta_{sA}(u), \theta_{sB}(u))), (\max(t_A(u), t_B(u)), \max(\theta_{tA}(u), \theta_{tB}(u))), (\max(m_A(u), m_B(u)), \max(\theta_{mA}(u), \theta_{mB}(u)))) \mid u \in U\}$."

Definition 5: [44] "Let $A = \{(u, (s_A(u), \theta_{sA}(u)), (t_A(u), \theta_{tA}(u)), (m_A(u), \theta_{mA}(u))) \mid u \in U\}$ and $B = \{(u, (s_B(u), \theta_{sB}(u)), (t_B(u), \theta_{tB}(u)), (m_B(u), \theta_{mB}(u))) \mid u \in U\}$ be two *CPFSs* defined on U. Then,

- $A+B = \{(u, (s_A(u)+s_B(u)-s_A(u)s_B(u), \theta_{sA}(u)+\theta_{sB}(u)-\frac{\theta_{sA}(u)\theta_{sB}(u)}{2\pi}), (t_A(u)t_B(u), \frac{\theta_{tA}(u)\theta_{tB}(u)}{2\pi}), (m_A(u)m_B(u), \frac{\theta_{mA}(u)\theta_{mB}(u)}{2\pi})) \mid u \in U\}$,
- $A.B = \{(u, (s_A(u)s_B(u), \frac{\theta_{sA}(u)\theta_{sB}(u)}{2\pi}), (t_A(u) + t_B(u) - t_A(u)t_B(u), \theta_{tA}(u) + \theta_{tB}(u) - \frac{\theta_{tA}(u)\theta_{tB}(u)}{2\pi}), (m_A(u) + m_B(u) - m_A(u)m_B(u), \theta_{mA}(u) + \theta_{mB}(u) - \frac{\theta_{mA}(u)\theta_{mB}(u)}{2\pi})) \mid u \in U\}$."

III. SIMILARITY MEASURES FOR COMPLEX PICTURE FUZZY SETS

In this section, we introduce the notion of similarity measures for *CPFSs* for the first time and discuss their important properties. Further, we shall denote $\Theta(U)$ be the collection of all the nonzero *CPFSs* defined on the universe of discourse $U = \{u_1, u_2, u_3, \ldots, u_n, \}$.

Definition 6: For any two sets A and B $\in \Theta(U)$, similarity measure $\mathbb{S} : \Theta(U) \times \Theta(U) \longrightarrow [0, 1]$ is a real-valued which satisfies the following conditions:

(a) $0 \leq \mathbb{S}(A, B) \leq 1$,
(b) $\mathbb{S}(A, B) = 1$ if $A = B$,
(c) $\mathbb{S}(A, B) = \mathbb{S}(B, A)$,
(d) If $A \subseteq B \subseteq C$, then $\mathbb{S}(A, C) \leq \mathbb{S}(A, B)$ and $\mathbb{S}(A, C) \leq \mathbb{S}(B, C)$, where $C \in \Theta(U)$.

Definition 7: Let $A = \{(u, (s_A(u), \theta_{sA}(u)), (t_A(u), \theta_{tA}(u)), (m_A(u), \theta_{mA}(u))) \mid u \in U\}$ and $B = \{(u, (s_B(u), \theta_{sB}(u)), (t_B(u), \theta_{tB}(u)), (m_B(u), \theta_{mB}(u))) \mid u \in U\}$ be two *CPFSs* on U. Based on various parameters, we propose some of the similarity measures as follows:

(i) $\mathbb{S}_1(A, B)$

$$= \frac{1}{n}\sum_{j=1}^{n}\left\{\frac{\left\{\begin{array}{c}\min(s_A(u_j), s_B(u_j)) + \frac{1}{2\pi}\min(\theta_{sA}(u_j), \theta_{sB}(u_j))\\ +\min(t_A(u_j), t_B(u_j)) + \frac{1}{2\pi}\min(\theta_{tA}(u_j), \theta_{tB}(u_j))\\ + \min(m_A(u_j), m_B(u_j)) + \frac{1}{2\pi}\min(\theta_{mA}(u_j), \theta_{mB}(u_j))\end{array}\right\}}{\left\{\begin{array}{c}\max(s_A(u_j), s_B(u_j)) + \frac{1}{2\pi}\max(\theta_{sA}(u_j), \theta_{sB}(u_j))\\ +\max(t_A(u_j), t_B(u_j)) + \frac{1}{2\pi}\max(\theta_{tA}(u_j), \theta_{tB}(u_j))\\ + \max(m_A(u_j), m_B(u_j)) + \frac{1}{2\pi}\max(\theta_{mA}(u_j), \theta_{mB}(u_j))\end{array}\right\}}\right\}, \tag{1}$$

(ii) $\mathbb{S}_2(A, B)$

$$= \frac{\sum_{j=1}^{n}\left\{\begin{array}{c}\left\{\min(s_A(u_j), s_B(u_j))+ \frac{1}{2\pi}\min(\theta_{sA}(u_j), \theta_{sB}(u_j))\right.\\ +\min(t_A(u_j), t_B(u_j)) + \frac{1}{2\pi}\min(\theta_{tA}(u_j), \theta_{tB}(u_j))\\ \left.+ \min(m_A(u_j), m_B(u_j))+ \frac{1}{2\pi}\min(\theta_{mA}(u_j), \theta_{mB}(u_j))\right\}\end{array}\right\}}{\sum_{j=1}^{n}\left\{\begin{array}{c}\left\{\max(s_A(u_j), s_B(u_j)) + \frac{1}{2\pi}\max(\theta_{sA}(u_j), \theta_{sB}(u_j))\right.\\ +\max(t_A(u_j), t_B(u_j)) + \frac{1}{2\pi}\max(\theta_{tA}(u_j), \theta_{tB}(u_j))\\ \left.+ \max(m_A(u_j), m_B(u_j))+ \frac{1}{2\pi}\max(\theta_{mA}(u_j), \theta_{mB}(u_j))\right\}\end{array}\right\}}, \tag{2}$$

(iii) $\mathbb{S}_3(A, B)$

$$= \frac{1}{n}\sum_{j=1}^{n}\left\{\frac{\left\{\begin{array}{c}\min(s_A(u_j), s_B(u_j))\\ +\frac{1}{2\pi}\min(\theta_{sA}(u_j), \theta_{sB}(u_j))\\ + \min(1 - t_A(u_j), 1 - t_B(u_j))\\ +\frac{1}{2\pi}\min(2\pi - \theta_{tA}(u_j), 2\pi - \theta_{tB}(u_j))\\ + \min(1 - m_A(u_j), 1 - m_B(u_j))\\ +\frac{1}{2\pi}\min(2\pi - \theta_{mA}(u_j), 2\pi - \theta_{mB}(u_j))\end{array}\right\}}{\left\{\begin{array}{c}\max(s_A(u_j), s_B(u_j))\\ +\frac{1}{2\pi}\max(\theta_{sA}(u_j), \theta_{sB}(u_j))\\ + \max(1 - t_A(u_j), 1 - t_B(u_j))\\ +\frac{1}{2\pi}\max(2\pi - \theta_{tA}(u_j), 2\pi - \theta_{tB}(u_j))\\ + \max(1 - m_A(u_j), 1 - m_B(u_j))\\ +\frac{1}{2\pi}\max(2\pi - \theta_{mA}(u_j), 2\pi - \theta_{mB}(u_j))\end{array}\right\}}\right\}, \tag{3}$$

(iv) $\mathbb{S}_4(A,B)$

$$=\frac{\sum_{j=1}^{n}\left\{\begin{array}{c}\{\min(s_A(u_j),s_B(u_j))\\+\frac{1}{2\pi}\min(\theta_{sA}(u_j),\theta_{sB}(u_j))\\+\min(1-t_A(u_j),1-t_B(u_j))\\+\frac{1}{2\pi}\min(2\pi-\theta_{tA}(u_j),2\pi-\theta_{tB}(u_j))\\+\min(1-m_A(u_j),1-m_B(u_j))\\+\frac{1}{2\pi}\min(2\pi-\theta_{mA}(u_j),2\pi-\theta_{mB}(u_j))\}\end{array}\right\}}{\sum_{j=1}^{n}\left\{\begin{array}{c}\{\max(s_A(u_j),s_B(u_j))\\+\frac{1}{2\pi}\max(2\pi-\theta_{sA}(u_j),2\pi-\theta_{sB}(u_j))\\+\max(1-t_A(u_j),1-t_B(u_j))\\+\frac{1}{2\pi}\max(2\pi-\theta_{tA}(u_j),2\pi-\theta_{tB}(u_j))\\+\max(1-m_A(u_j),1-m_B(u_j))\\+\frac{1}{2\pi}\max(2\pi-\theta_{mA}(u_j),2\pi-\theta_{mB}(u_j))\}\end{array}\right\}},\tag{4}$$

(v) $\mathbb{S}_5(A,B)$

$$=1-\frac{1}{6n}\sum_{j=1}^{n}\left\{\begin{array}{c}\{|s_A(u_j)-s_B(u_j)|+\frac{1}{2\pi}|\theta_{sA}(u_j)-\theta_{sB}(u_j)|\\+|t_A(u_j)-t_B(u_j)|+\frac{1}{2\pi}|\theta_{tA}(u_j)-\theta_{tB}(u_j)|\\+|m_A(u_j)-m_B(u_j)|+\frac{1}{2\pi}|\theta_{mA}(u_j)-\theta_{mB}(u_j)|\}\end{array}\right\},\tag{5}$$

(vi) $\mathbb{S}_6(A,B)$

$$=1-\frac{1}{6}\sum_{j=1}^{n}\left\{\begin{array}{c}\{\max_j|s_A(u_j)-s_B(u_j)|\\+\frac{1}{2\pi}\max_j|\theta_{sA}(u_j)-\theta_{sB}(u_j)|\\+\max_j|t_A(u_j)-t_B(u_j)|\\+\frac{1}{2\pi}\max_j|\theta_{tA}(u_j)-\theta_{tB}(u_j)|\\+\max_j|m_A(u_j)-m_B(u_j)|\\+\frac{1}{2\pi}\max_j|\theta_{mA}(u_j)-\theta_{mB}(u_j)|\}\end{array}\right\},\tag{6}$$

(vii) $\mathbb{S}_7(A,B)$

$$=1-\frac{\sum_{j=1}^{n}\left\{\begin{array}{c}\{|s_A(u_j)-s_B(u_j)|+\frac{1}{2\pi}|\theta_{sA}(u_j)-\theta_{sB}(u_j)|\\+|t_A(u_j)-t_B(u_j)|+\frac{1}{2\pi}|\theta_{tA}(u_j)-\theta_{tB}(u_j)|\\+|m_A(u_j)-m_B(u_j)|+\frac{1}{2\pi}|\theta_{mA}(u_j)-\theta_{mB}(u_j)|\}\end{array}\right\}}{\sum_{j=1}^{n}\left\{\begin{array}{c}\{|s_A(u_j)+s_B(u_j)|+\frac{1}{2\pi}|\theta_{sA}(u_j)+\theta_{sB}(u_j))\\+|t_A(u_j)+t_B(u_j)|+\frac{1}{2\pi}|\theta_{tA}(u_j)+\theta_{tB}(u_j)|\\+|m_A(u_j)+m_B(u_j)|+\frac{1}{2\pi}|\theta_{mA}(u_j)+\theta_{mB}(u_j)|\}\end{array}\right\}},\tag{7}$$

Theorem 1: The proposed similarity measures $\mathbb{S}_t(t=1,2,\ldots,7)$ satisfies the following conditions:

(a) $0\le\mathbb{S}_t(A,B)\le 1$,
(b) $\mathbb{S}_t(A,B)=1$ if $A=B$,
(c) $\mathbb{S}_t(A,B)=\mathbb{S}_t(B,A)$,
(d) If $A\subseteq B\subseteq C$, then $\mathbb{S}_t(A,C)\le\mathbb{S}_{\approx}(A,B)$ and $\mathbb{S}_t(A,C)\le\mathbb{S}_t(B,C)$, where $A,B,\ C\in\Theta(U)$.

Proof: As per the definitions, the conditions "(a)-(c)" are simple to prove for the similarity measures $\mathbb{S}_t(t=1,2,\ldots,7)$. Hence, we will focus on proving the criteria given by "(d)" for the similarity measure $\mathbb{S}_5$ and $\mathbb{S}_7$. On the basis of these deliberations, similar proofs can be done for the other similarity measures. For the proof of the this, let $C=\{(u,(s_C(u),\theta_{sC}(u)),(t_C(u),\theta_{tC}(u)),(m_C(u),\theta_{mC}(u)))\,|\,u\in U\}$. If $A\subseteq B\subseteq C$. It implies that $s_A(u_j)\le s_B(u_j)\le s_C(u_j)$; $t_A(u_j)\le t_B(u_j)\le t_C(u_j)$; $m_A(u_j)\ge m_B(u_j)\ge m_C(u_j)$ and $\theta_{sA}(u_j)\le\theta_{sB}(u_j)\le\theta_{sC}(u_j)$; $\theta_{tA}(u_j)\le\theta_{tB}(u_j)\le\theta_{tC}(u_j)$; $\theta_{mA}(u_j)\ge\theta_{mB}(u_j)\ge\theta_{mC}(u_j)$.

(i)

$$\mathbb{S}_1(A,B)=\frac{1}{n}\sum_{j=1}^{n}\left\{\frac{\begin{array}{c}\{\min(s_A(u_j),s_B(u_j))\\+\frac{1}{2\pi}\min(\theta_{sA}(u_j),\theta_{sB}(u_j))\\+\min(t_A(u_j),t_B(u_j))\\+\frac{1}{2\pi}\min(\theta_{tA}(u_j),\theta_{tB}(u_j))\\+\min(m_A(u_j),m_B(u_j))\\+\frac{1}{2\pi}\min(\theta_{mA}(u_j),\theta_{mB}(u_j))\}\end{array}}{\begin{array}{c}\{\max(s_A(u_j),s_B(u_j))\\+\frac{1}{2\pi}\max(\theta_{sA}(u_j),\theta_{sB}(u_j))\\+\max(t_A(u_j),t_B(u_j))\\+\frac{1}{2\pi}\max(\theta_{tA}(u_j),\theta_{tB}(u_j))\\+\max(m_A(u_j),m_B(u_j))\\+\frac{1}{2\pi}\max(\theta_{mA}(u_j),\theta_{mB}(u_j))\}\end{array}}\right\}$$

$$=\frac{1}{n}\sum_{j=1}^{n}\left\{\frac{\begin{array}{c}\{s_A(u_j)+t_A(u_j)+m_B(u_j)\\+\frac{1}{2\pi}(\theta_{sA}(u_j)+\theta_{tA}(u_j)+\theta_{mB}(u_j))\end{array}}{\begin{array}{c}\{s_B(u_j)+t_B(u_j)+m_A(u_j)\\+\frac{1}{2\pi}(\theta_{sB}(u_j)+\theta_{tB}(u_j)+\theta_{mA}(u_j))\}\end{array}}\right\}$$

and $\mathbb{S}_1(A,C)$

$$=\frac{1}{n}\sum_{j=1}^{n}\left\{\frac{\begin{array}{c}\{\min(s_A(u_j),s_C(u_j))\\+\frac{1}{2\pi}\min(\theta_{sA}(u_j),\theta_{sC}(u_j))\\+\min(t_A(u_j),t_C(u_j))\\+\frac{1}{2\pi}\min(\theta_{tA}(u_j),\theta_{tC}(u_j))\\+\min(m_A(u_j),m_C(u_j))\\+\frac{1}{2\pi}\min(\theta_{mA}(u_j),\theta_{mC}(u_j))\}\end{array}}{\begin{array}{c}\{\max(s_A(u_j),s_C(u_j))\\+\frac{1}{2\pi}\max(\theta_{sA}(u_j),\theta_{sC}(u_j))\\+\max(t_A(u_j),t_C(u_j))\\+\frac{1}{2\pi}\max(\theta_{tA}(u_j),\theta_{tC}(u_j))\\+\max(m_A(u_j),m_C(u_j))\\+\frac{1}{2\pi}\max(\theta_{mA}(u_j),\theta_{mC}(u_j))\}\end{array}}\right\}$$

$$=\frac{1}{n}\sum_{j=1}^{n}\left\{\frac{\begin{array}{c}\{s_A(u_j)+t_A(u_j)+m_C(u_j)\\+\frac{1}{2\pi}(\theta_{sA}(u_j)+\theta_{tA}(u_j)+\theta_{mC}(u_j))\end{array}}{\begin{array}{c}\{s_C(u_j)+t_C(u_j)+m_A(u_j)\\+\frac{1}{2\pi}(\theta_{sC}(u_j)+\theta_{tC}(u_j)+\theta_{mA}(u_j))\}\end{array}}\right\}.$$

As $m_C(u_j)\le m_B(u_j$; $\theta_{mC}(u_j)\le\theta_{mB}(u_j$; $t_B(u_j)\le t_C(u_j$; $\theta_{tB}(u_j)\le\theta_{tC}(u_j$ and $s_B(u_j)\le s_C(u_j$; $\theta_{sB}(u_j)\le\theta_{sC}(u_j$. Therefore, $\mathbb{S}_1(A,C)\le\mathbb{S}_1(A,B)$. On the similar lines $\mathbb{S}_1(A,C)\le\mathbb{S}_1(B,C)$.

(ii) $A\subseteq B\subseteq C$ gives $|s_A(u_j)-s_B(u_j)|\le|s_A(u_j)-s_C(u_j)|$;$|t_A(u_j)-t_B(u_j)|\le|t_A(u_j)-t_C(u_j)|$; $|m_A(u_j)-m_B(u_j)|\le$

$|m_A(u_j) - m_C(u_j)|$ and $|\theta_{sA}(u_j) - \theta_{sB}(u_j)| \leq |\theta_{sA}(u_j) - \theta_{sC}(u_j)|$;$|\theta_{tA}(u_j) - \theta_{tB}(u_j)| \leq |\theta_{tA}(u_j) - \theta_{tC}(u_j)|$; $|\theta_{mA}(u_j) - \theta_{mB}(u_j)| \leq |\theta_{mA}(u_j) - \theta_{mC}(u_j)|$. Therefore,

$$\mathbb{S}_5(A, C)$$

$$= 1 - \frac{1}{6n}\sum_{j=1}^{n}\left\{\begin{array}{c} \{|s_A(u_j) - s_C(u_j)| \\ +\frac{1}{2\pi}|\theta_{sA}(u_j) - \theta_{sC}(u_j)| \\ +|t_A(u_j) - t_C(u_j)| \\ +\frac{1}{2\pi}|\theta_{tA}(u_j) - \theta_{tC}(u_j)| \\ +|m_A(u_j) - m_C(u_j)| \\ +\frac{1}{2\pi}|\theta_{mA}(u_j) - \theta_{mC}(u_j)|\} \end{array}\right\}$$

$$\leq 1 - \frac{1}{6n}$$

$$\times \sum_{j=1}^{n}\left\{\begin{array}{c} \{|s_A(u_j) - s_B(u_j)| + \frac{1}{2\pi}|\theta_{sA}(u_j) - \theta_{sB}(u_j)| \\ +|t_A(u_j) - t_B(u_j)| + \frac{1}{2\pi}|\theta_{tA}(u_j) - \theta_{tB}(u_j)| \\ +|m_A(u_j) - m_B(u_j)| + \frac{1}{2\pi}|\theta_{mA}(u_j) - \theta_{mB}(u_j)|\} \end{array}\right\}.$$

$$= \mathbb{S}_5(A, B)$$

Hence, $\mathbb{S}_5(A, C) \leq \mathbb{S}_5(A, B)$. On the similar lines, $\mathbb{S}_5(A, C) \leq \mathbb{S}_5(B, C)$ can be proved.

(iii) Now, we have

$$\left\{\begin{array}{c} \{|s_A(u_j) - s_B(u_j)| + \frac{1}{2\pi}|\theta_{sA}(u_j) - \theta_{sB}(u_j)| \\ +|t_A(u_j) - t_B(u_j)| + \frac{1}{2\pi}|\theta_{tA}(u_j) - \theta_{tB}(u_j)| \\ +|m_A(u_j) - m_B(u_j)| + \frac{1}{2\pi}|\theta_{mA}(u_j) - \theta_{mB}(u_j)|\} \end{array}\right\}$$

$$\leq \left\{\begin{array}{c} \{|s_A(u_j) - s_C(u_j)| + \frac{1}{2\pi}|\theta_{sA}(u_j) - \theta_{sC}(u_j)| \\ +|t_A(u_j) - t_C(u_j)| + \frac{1}{2\pi}|\theta_{tA}(u_j) - \theta_{tC}(u_j)| \\ +|m_A(u_j) - m_C(u_j)| + \frac{1}{2\pi}|\theta_{mA}(u_j) - \theta_{mC}(u_j)|\} \end{array}\right\}$$

$$\Rightarrow \frac{1}{\left\{\begin{array}{c} \{|s_A(u_j) - s_C(u_j)| + \frac{1}{2\pi}|\theta_{sA}(u_j) - \theta_{sC}(u_j)| \\ +|t_A(u_j) - t_C(u_j)| + \frac{1}{2\pi}|\theta_{tA}(u_j) - \theta_{tC}(u_j)| \\ +|m_A(u_j) - m_C(u_j)| + \frac{1}{2\pi}|\theta_{mA}(u_j) - \theta_{mC}(u_j)|\} \end{array}\right\}}$$

$$\leq \frac{1}{\left\{\begin{array}{c} \{|s_A(u_j) - s_B(u_j)| + \frac{1}{2\pi}|\theta_{sA}(u_j) - \theta_{sB}(u_j)| \\ +|t_A(u_j) - t_B(u_j)| + \frac{1}{2\pi}|\theta_{tA}(u_j) - \theta_{tB}(u_j)| \\ +|m_A(u_j) - m_B(u_j)| + \frac{1}{2\pi}|\theta_{mA}(u_j) - \theta_{mB}(u_j)|\} \end{array}\right\}}$$

$$\Rightarrow \frac{\begin{array}{c} 2 \times \{s_A(u_j) + t_C(u_j) + m_C(u_j) \\ +\frac{1}{2\pi}(\theta_{sA}(u_j) + \theta_{tC}(u_j) + \theta_{mC}(u_j))\} \end{array}}{\left\{\begin{array}{c} \{|s_A(u_j) - s_C(u_j)| + \frac{1}{2\pi}|\theta_{sA}(u_j) - \theta_{sC}(u_j)| \\ +|t_A(u_j) - t_C(u_j)| + \frac{1}{2\pi}|\theta_{tA}(u_j) - \theta_{tC}(u_j)| \\ +|m_A(u_j) - m_C(u_j)| + \frac{1}{2\pi}|\theta_{mA}(u_j) - \theta_{mC}(u_j)|\} \end{array}\right\}}$$

$$\leq \frac{\begin{array}{c} 2 \times \{s_A(u_j) + t_B(u_j) + m_B(u_j) \\ +\frac{1}{2\pi}(\theta_{sA}(u_j) + \theta_{tB}(u_j) + \theta_{mB}(u_j))\} \end{array}}{\left\{\begin{array}{c} \{|s_A(u_j) - s_B(u_j)| + \frac{1}{2\pi}|\theta_{sA}(u_j) - \theta_{sB}(u_j)| \\ +|t_A(u_j) - t_B(u_j)| + \frac{1}{2\pi}|\theta_{tA}(u_j) - \theta_{tB}(u_j)| \\ +|m_A(u_j) - m_B(u_j)| + \frac{1}{2\pi}|\theta_{mA}(u_j) - \theta_{mB}(u_j)|\} \end{array}\right\}}$$

$$\Rightarrow 1 + \frac{\begin{array}{c} 2 \times \{s_A(u_j) + t_C(u_j) + m_C(u_j) \\ +\frac{1}{2\pi}(\theta_{sA}(u_j) + \theta_{tC}(u_j) + \theta_{mC}(u_j))\} \end{array}}{\left\{\begin{array}{c} \{|s_A(u_j) - s_C(u_j)| + \frac{1}{2\pi}|\theta_{sA}(u_j) - \theta_{sC}(u_j)| \\ +|t_A(u_j) - t_C(u_j)| + \frac{1}{2\pi}|\theta_{tA}(u_j) - \theta_{tC}(u_j)| \\ +|m_A(u_j) - m_C(u_j)| + \frac{1}{2\pi}|\theta_{mA}(u_j) - \theta_{mC}(u_j)|\} \end{array}\right\}}$$

$$\leq 1 + \frac{\begin{array}{c} 2 \times \{s_A(u_j) + t_B(u_j) + m_B(u_j) \\ +\frac{1}{2\pi}(\theta_{sA}(u_j) + \theta_{tB}(u_j) + \theta_{mB}(u_j))\} \end{array}}{\left\{\begin{array}{c} \{|s_A(u_j) - s_B(u_j)| + \frac{1}{2\pi}|\theta_{sA}(u_j) - \theta_{sB}(u_j)| \\ +|t_A(u_j) - t_B(u_j)| + \frac{1}{2\pi}|\theta_{tA}(u_j) - \theta_{tB}(u_j)| \\ +|m_A(u_j) - m_B(u_j)| + \frac{1}{2\pi}|\theta_{mA}(u_j) - \theta_{mB}(u_j)|\} \end{array}\right\}}$$

$$\Rightarrow \frac{\left\{\begin{array}{c} \{|s_A(u_j) + s_C(u_j)| + \frac{1}{2\pi}|\theta_{sA}(u_j) + \theta_{sC}(u_j)| \\ +|t_A(u_j) + t_C(u_j)| + \frac{1}{2\pi}|\theta_{tA}(u_j) + \theta_{tC}(u_j)| \\ +|m_A(u_j) + m_C(u_j)| + \frac{1}{2\pi}|\theta_{mA}(u_j) + \theta_{mC}(u_j)|\} \end{array}\right\}}{\left\{\begin{array}{c} \{|s_A(u_j) - s_C(u_j)| + \frac{1}{2\pi}|\theta_{sA}(u_j) - \theta_{sC}(u_j)| \\ +|t_A(u_j) - t_C(u_j)| + \frac{1}{2\pi}|\theta_{tA}(u_j) - \theta_{tC}(u_j)| \\ +|m_A(u_j) - m_C(u_j)| + \frac{1}{2\pi}|\theta_{mA}(u_j) - \theta_{mC}(u_j)|\} \end{array}\right\}}$$

$$\leq \frac{\left\{\begin{array}{c} \{|s_A(u_j) + s_B(u_j)| + \frac{1}{2\pi}|\theta_{sA}(u_j) + \theta_{sB}(u_j)| \\ +|t_A(u_j) + t_B(u_j)| + \frac{1}{2\pi}|\theta_{tA}(u_j) + \theta_{tB}(u_j)| \\ +|m_A(u_j) + m_B(u_j)| + \frac{1}{2\pi}|\theta_{mA}(u_j) + \theta_{mB}(u_j)|\} \end{array}\right\}}{\left\{\begin{array}{c} \{|s_A(u_j) - s_B(u_j)| + \frac{1}{2\pi}|\theta_{sA}(u_j) - \theta_{sB}(u_j)| \\ +|t_A(u_j) - t_B(u_j)| + \frac{1}{2\pi}|\theta_{tA}(u_j) - \theta_{tB}(u_j)| \\ +|m_A(u_j) - m_B(u_j)| + \frac{1}{2\pi}|\theta_{mA}(u_j) - \theta_{mB}(u_j)|\} \end{array}\right\}}$$

$$\Rightarrow 1 - \frac{\left\{\begin{array}{c} \{|s_A(u_j) - s_C(u_j)| + \frac{1}{2\pi}|\theta_{sA}(u_j) - \theta_{sC}(u_j)| \\ +|t_A(u_j) - t_C(u_j)| + \frac{1}{2\pi}|\theta_{tA}(u_j) - \theta_{tC}(u_j)| \\ +|m_A(u_j) - m_C(u_j)| + \frac{1}{2\pi}|\theta_{mA}(u_j) - \theta_{mC}(u_j)|\} \end{array}\right\}}{\left\{\begin{array}{c} \{|s_A(u_j) + s_C(u_j)| + \frac{1}{2\pi}|\theta_{sA}(u_j) + \theta_{sC}(u_j)| \\ +|t_A(u_j) + t_C(u_j)| + \frac{1}{2\pi}|\theta_{tA}(u_j) + \theta_{tC}(u_j)| \\ +|m_A(u_j) + m_C(u_j)| + \frac{1}{2\pi}|\theta_{mA}(u_j) + \theta_{mC}(u_j)|\} \end{array}\right\}}$$

$$\leq 1 - \frac{\left\{\begin{array}{c} \{|s_A(u_j) - s_B(u_j)| + \frac{1}{2\pi}|\theta_{sA}(u_j) - \theta_{sB}(u_j)| \\ +|t_A(u_j) - t_B(u_j)| + \frac{1}{2\pi}|\theta_{tA}(u_j) - \theta_{tB}(u_j)| \\ +|m_A(u_j) - m_B(u_j)| + \frac{1}{2\pi}|\theta_{mA}(u_j) - \theta_{mB}(u_j)|\} \end{array}\right\}}{\left\{\begin{array}{c} \{|s_A(u_j) + s_B(u_j)| + \frac{1}{2\pi}|\theta_{sA}(u_j) + \theta_{sB}(u_j)| \\ +|t_A(u_j) + t_B(u_j)| + \frac{1}{2\pi}|\theta_{tA}(u_j) + \theta_{tB}(u_j)| \\ +|m_A(u_j) + m_B(u_j)| + \frac{1}{2\pi}|\theta_{mA}(u_j) + \theta_{mB}(u_j)|\} \end{array}\right\}}.$$

Therefore, $\mathbb{S}_7(A, C) \leq \mathbb{S}_7(A, B)$. and on the similar lines $\mathbb{S}_7(A, C) \leq \mathbb{S}_7(B, C)$.

Next, we discuss some important properties of these proposed similarity measures as given below:

Property 1: For A, B $\in \Theta(U)$ and $t = 1, 2, \ldots, 7$, we have

(i) $\mathbb{S}_t(A, B^c) = \mathbb{S}_t(A^c, B), t \neq 3, 4;$
(ii) $\mathbb{S}_t(A^c, B^c) = \mathbb{S}_t(A, B), t \neq 3, 4;$
(iii) $\mathbb{S}_t(A \cap B, A \cup B) = \mathbb{S}_t(A, B).$

Proof: Here, we will prove part (i) only, on similar lines other parts can be proved. Let us consider two *CPFSs* $A = \{(u, (s_A(u), \theta_{sA}(u)), (t_A(u), \theta_{tA}(u)), (m_A(u), \theta_{mA}(u))) \mid u \in U\}$ and

$$B = \{(u, (s_B(u), \theta_{sB}(u)), (t_B(u), \theta_{tB}(u)), (m_B(u), \theta_{mB}(u))) \mid u \in U\}.$$

(i) Now, we shall make use of similarity measure $\mathbb{S}_1$ given in equation (1),

$$\mathbb{S}_1(A, B^c) = \frac{1}{n}\sum_{j=1}^{n}\left\{\frac{\begin{Bmatrix}\min(s_A(u_j), m_B(u_j)) \\ +\frac{1}{2\pi}\min(\theta_{sA}(u_j), \theta_{mB}(u_j)) \\ +\min(t_A(u_j), t_B(u_j)) \\ +\frac{1}{2\pi}\min(\theta_{tA}(u_j), \theta_{tB}(u_j)) \\ +\min(m_A(u_j), s_B(u_j)) \\ +\frac{1}{2\pi}\min(\theta_{mA}(u_j), \theta_{sB}(u_j))\end{Bmatrix}}{\begin{Bmatrix}\max(s_A(u_j), m_B(u_j)) \\ +\frac{1}{2\pi}\max(\theta_{sA}(u_j), \theta_{mB}(u_j)) \\ +\max(t_A(u_j), t_B(u_j)) \\ +\frac{1}{2\pi}\max(\theta_{tA}(u_j), \theta_{tB}(u_j)) \\ +\max(m_A(u_j), s_B(u_j)) \\ +\frac{1}{2\pi}\max(\theta_{mA}(u_j), \theta_{sB}(u_j))\end{Bmatrix}}\right\}$$
$$= \mathbb{S}_1(A^c, B).$$

Therefore, $\mathbb{S}_1(A, B^c) = \mathbb{S}_1(A^c, B)$. Similar can be done for $\mathbb{S}_t(A, B^c) = \mathbb{S}_t(A^c, B)$ for $t = 2, 5, 6, 7$.

Property 2: For A, B $\in \Theta(U)$ and $t = 5, 6$, we have

(i) $\mathbb{S}_t(A, A \cup B) = \mathbb{S}_t(B, A \cap B);$
(ii) $\mathbb{S}_t(A, A \cap B) = \mathbb{S}_t(B, A \cup B);$
(iii) $\mathbb{S}_t(A, A + B) = \mathbb{S}_t(B, A.B);$
(iv) $\mathbb{S}_t(A, A.B) = \mathbb{S}_t(B, A + B).$

Proof: Here, we will prove part (i) and (iii) only, on the similar lines other parts can be proved.

(i) Let us consider two *CPFSs* $A = \{(u, (s_A(u), \theta_{sA}(u)), (t_A(u), \theta_{tA}(u)), (m_A(u), \theta_{mA}(u))) \mid u \in U\}$ and $B = \{(u, (s_B(u), \theta_{sB}(u)), (t_B(u), \theta_{tB}(u)), (m_B(u), \theta_{mB}(u))) \mid u \in U\}$. Now, we shall make use of similarity measure $\mathbb{S}_5$ given in equation (5),

(i) $\mathbb{S}_5(A, A \cup B)$

$$= 1 - \frac{1}{6n}\sum_{j=1}^{n}\begin{Bmatrix}\{|s_A(u_j) - \max(s_A(u_j), s_B(u_j))| \\ +\frac{1}{2\pi}|\theta_{sA}(u_j) - \max(\theta_{sA}(u_j), \theta_{sB}(u_j))| \\ +|t_A(u_j) - \min(t_A(u_j), t_B(u_j))| \\ +\frac{1}{2\pi}|\theta_{tA}(u_j) - \min(\theta_{tA}(u_j), \theta_{tB}(u_j))| \\ +|m_A(u_j) - \min(m_A(u_j), m_B(u_j))| \\ +\frac{1}{2\pi}|\theta_{mA}(u_j) - \min(\theta_{mA}(u_j), \theta_{mB}(u_j))|\}\end{Bmatrix}$$

$$= 1 - \frac{1}{6n}\sum_{j=1}^{n}\begin{Bmatrix}\{|\min(0, s_A(u_j) - s_B(u_j))| \\ +\frac{1}{2\pi}|\min(0, \theta_{sA}(u_j) - \theta_{sB}(u_j))| \\ +|\max(0, t_A(u_j) - t_B(u_j))| \\ +\frac{1}{2\pi}|\max(0, \theta_{tA}(u_j) - \theta_{tB}(u_j))| \\ +|\max(0, m_A(u_j) - m_B(u_j))| \\ +\frac{1}{2\pi}|\max(0, \theta_{mA}(u_j) - \theta_{mB}(u_j))|\}\end{Bmatrix}.$$

and $\mathbb{S}_5(B, A \cap B)$

$$= 1 - \frac{1}{6n}\sum_{j=1}^{n}\begin{Bmatrix}\{|s_B(u_j) - \min(s_A(u_j), s_B(u_j))| \\ +\frac{1}{2\pi}|\theta_{sB}(u_j) - \min(\theta_{sA}(u_j), \theta_{sB}(u_j))| \\ +|t_B(u_j) - \max(t_A(u_j), t_B(u_j))| \\ +\frac{1}{2\pi}|\theta_{tB}(u_j) - \max(\theta_{tA}(u_j), \theta_{tB}(u_j))| \\ +|m_B(u_j) - \max(m_A(u_j), m_B(u_j))| \\ +\frac{1}{2\pi}|\theta_{mB}(u_j) - \max(\theta_{mA}(u_j), \theta_{mB}(u_j))|\}\end{Bmatrix}$$

$$= 1 - \frac{1}{6n}\sum_{j=1}^{n}\begin{Bmatrix}\{|\max(0, s_B(u_j) - s_A(u_j))| \\ +\frac{1}{2\pi}|\max(0, \theta_{sB}(u_j) - \theta_{sA}(u_j))| \\ +|\min(0, t_B(u_j) - t_A(u_j))| \\ +\frac{1}{2\pi}|\min(0, \theta_{tB}(u_j) - \theta_{tA}(u_j))| \\ +|\min(0, m_B(u_j) - m_A(u_j))| \\ +\frac{1}{2\pi}|\min(0, \theta_{mB}(u_j) - \theta_{mA}(u_j))|\}\end{Bmatrix}$$

$$= 1 - \frac{1}{6n}\sum_{j=1}^{n}\begin{Bmatrix}\{|\min(0, s_A(u_j) - s_B(u_j))| \\ +\frac{1}{2\pi}|\min(0, \theta_{sA}(u_j) - \theta_{sB}(u_j))| \\ +|\max(0, t_A(u_j) - t_B(u_j))| \\ +\frac{1}{2\pi}|\max(0, \theta_{tA}(u_j) - \theta_{tB}(u_j))| \\ +|\max(0, m_A(u_j) - m_B(u_j))| \\ +\frac{1}{2\pi}|\max(0, \theta_{mA}(u_j) - \theta_{mB}(u_j))|\}\end{Bmatrix}.$$
$$= \mathbb{S}_5(A, A \cup B).$$

Therefore, $\mathbb{S}_5(A, A \cup B) = \mathbb{S}_1(A, A \cap B)$. Similar can be done for $\mathbb{S}_6(A, A \cup B) = \mathbb{S}_6(A, A \cap B)$.

(ii) Again, we shall make use of equation (5), we get $\mathbb{S}_5(A, A + B) = 1 - \frac{1}{6n}$, as shown at the bottom of the next page.

Hence, $\mathbb{S}_5(A, A + B) = \mathbb{S}_5(B, A.B)$. On the similar lines we can prove $\mathbb{S}_6(A, A + B) = \mathbb{S}_6(B, A.B)$.

IV. EMPIRICAL COMPARISON OF THE PROPOSED SIMILARITY MEASURES

In this section, we explain the rationality of the introduced similarity measures on the account of the comparisons with the existing similarity measures in the literature on the basis of numerical computations.

Example 4.1: For illustrating the effectiveness of the introduced similarity measure $\mathbb{S}_1$, a comparative analysis between the existing similarity measures in the literature and the introduced similarity measures has been done. Some of the existing similarity measures in the literature are listed in Table 1 with their proper references.

Let A_i and B_i are the distinct *CPFSs* as shown in Table 2. In order to evaluate the effectiveness and feasibility of the presented similarity measures with some of the existing measures, we first convert the complex picture fuzzy information into the simple picture fuzzy by taking the phase terms corresponding to every complex picture fuzzy set equal to zero. Table 2 provides an overview of the outcomes for both the proposed and the existing similarity measures.

Note: The values of A_i and B_i for the different cases in Table 2 are given as below:

(a) Case 1. $A_1 = \{(0.5, 0.3\pi), (0.1, 0.5\pi), (0.4, 0.1\pi)\}, \{(0.4, 0.3\pi), (0.2, 0.1\pi), (0.3, 0.2\pi)\}$
$B_1 = \{(0.4, 0.6\pi), (0.1, 0.5\pi), (0.4, 0.3\pi)\}, \{(0.5, 0.9\pi), (0.1, 0.4\pi), (0.4, 0.3\pi)\}$,

(b) Case 2. $A_2 = \{(0.5, 0.3\pi), (0.1, 0.5\pi), (0.4, 0.1\pi)\}, \{(0.4, 0.3\pi), (0.2, 0.1\pi), (0.3, 0.2\pi)\}$
$B_2 = \{(0.5, 0.9\pi), (0.1, 0.2\pi), (0.3, 0.2\pi)\}, \{(0.3, 0.9\pi), (0.3, 0.1\pi), (0.2, 0.2\pi)\}$,

(c) Case 3. $A_3 = \{(0.0, 1.1\pi), (1.0, 0.2\pi), (0.0, 0.1\pi)\}, \{(0.0, 0.4\pi), (0.0, 1.1\pi), (1.0, 0.1\pi)\}$
$B_3 = \{(0.5, 0.4\pi), (0.0, 1.2\pi), (0.2, 0.1\pi)\}, \{(0.6, 0.7\pi), (0.3, 0.2\pi), (0.0, 0.3\pi)\}$,

(d) Case 4. $A_4 = \{(0.0, 1.1\pi), (1.0, 0.2\pi), (0.0, 0.1\pi)\}, \{(0.0, 0.4\pi), (0.0, 1.1\pi), (1.0, 0.1\pi)\}$
$B_4 = \{(0.9, 0.6\pi), (0.0, 0.3\pi), (0.1, 0.5\pi)\}, \{(0.7, 0.4\pi), (0.1, 1.1\pi), (0.0, 0.1\pi)\}$,

From Table 2, we observe that for first pair of *CPFSs* $\{A_1, B_1\}$ and $\{A_2, B_2\}$, $\mathbb{S}_{SM_1}(A_1, B_1) = \mathbb{S}_{SM_1}(A_2, B_2)$ when $A_1 = A_2, B_1 \neq B_2$, which results in an unreasonable/counter-intuitive situation. This situation also exists for other existing similarity measures such as $\mathbb{S}_{SM_2}$, $\mathbb{S}_{SM_3}$, $\mathbb{S}_{SM_4}$, $\mathbb{S}_{SM_5}$, $\mathbb{S}_{W_2}$, $\mathbb{S}_{W_3}$, $\mathbb{S}_{W_4}$, $\mathbb{S}_{W_5}$, $\mathbb{S}_{DT_1}$, $\mathbb{S}_{DT_2}$. On the similar lines, by taking the second pair *CPFSs* $\{A_3, B_3\}$, $\{A_4, B_4\}$, we also get some irrelevant results for the similarity measures $\mathbb{S}_{SM_1}$, $\mathbb{S}_{SM_3}$, $\mathbb{S}_{SM_4}$, $\mathbb{S}_{W_1}$, $\mathbb{S}_{W_2}$, $\mathbb{S}_{W_3}$, $\mathbb{S}_{W_4}$, $\mathbb{S}_{W_5}$, $\mathbb{S}_{WG_1}$, $\mathbb{S}_{WG_2}$. Now, it is clear from Table 2, that the proposed similarity measure has overcome the limitations of the existing similarity measure and gives the most reasonable results.

V. UTILIZATION OF PROPOSED SIMILARITY MEASURES IN PATTERN RECOGNITION PROBLEMS

In this section, the introduced similarity measure has been applied especially in the problems of ''pattern recognition and medical diagnosis''. Also, the introduced similarity measures are compared with the existing measures in the literature.

A. STEPS OF PROCEDURE FOR PATTERN RECOGNITION

Let $U = \{u_1, u_2, u_3, \ldots, u_n\}$ be a universal set, suppose there are m distinct patterns $P_i = \{< \rho_{A_i}(u_j), \tau_{A_i}(u_j), \omega_{A_i}(u_j) >| u_j \in U\}(i = 1, 2, \ldots, m)$ and a completely unknown pattern $Q = \{< \rho_Q(u_j), \tau_Q(u_j), \omega_Q(u_j) >| u_j \in U\}$. The steps involved in this methodology are given as follows:

Step 1: Computing the values of the similarity measure $\mathbb{S}_1(P_i, Q)(i = 1, 2, \ldots, m)$ between P_i & Q.

Step 2: Select the maximum value $\mathbb{S}_1(P_{i0}, Q)$ from $\mathbb{S}_1(P_i, Q)$ i.e., $\mathbb{S}_1(P_{i0}, Q) = \max_{1\leq i\leq m} \mathbb{S}_1(P_i, Q)$. After doing the necessary computations, the unknown pattern Q is classified to the pattern P_{i0} by making use of the ''principle of the maximum of similarity measures''. Also, the procedural steps for pattern recognition problems are shown in Figure 1. Now, these steps are applied in the following examples.

Example 5.1.1: Consider three unknown patterns P_1, P_2, P_3 and the pattern to be tested is Q, are given in the Table 3 under complex picture fuzzy environment.

$$\sum_{j=1}^{n}\left\{\begin{array}{c}\{|s_A(u_j) - (s_A(u_j) + s_B(u_j) - s_A(u_j).s_B(u_j))| \\ +|t_A(u_j) - t_A(u_j).t_B(u_j)| + |m_A(u_j) - m_A(u_j).m_B(u_j)| \\ +\frac{1}{2\pi}(|\theta_{sA}(u_j) - (\theta_{sA}(u_j) + \theta_{sB}(u_j) - \frac{\theta_{sA}(u_j).\theta_{sB}(u_j)}{2\pi})| \\ + |\theta_{tA}(u_j) - \frac{\theta_{tA}(u_j).\theta_{tB}(u_j)}{2\pi})| + |\theta_{mA}(u_j) - \frac{\theta_{mA}(u_j).\theta_{mB}(u_j)}{2\pi})|)\}\end{array}\right\}$$

$$= 1 - \frac{1}{6n}\sum_{j=1}^{n}\left\{\begin{array}{c}\{|s_B(u_j)(1 - s_A(u_j))| + |t_A(u_j)(1 - t_B(u_j))| \\ +|m_A(u_j)(1 - m_B(u_j))| + \frac{1}{2\pi}(|\theta_{sB}(u_j)(1 - \frac{\theta_{sA}(u_j)}{2\pi})| \\ + |\theta_{tB}(u_j)(1 - \frac{\theta_{tA}(u_j)}{2\pi}) + |\theta_{mB}(u_j)(1 - \frac{\theta_{mA}(u_j)}{2\pi})|)\}\end{array}\right\}.$$

$$\text{and } \mathbb{S}_5(B, A.B) = 1 - \frac{1}{6n}$$

$$\sum_{j=1}^{n}\left\{\begin{array}{c}\{|s_B(u_j) - s_A(u_j).s_B(u_j))| \\ +|t_B(u_j) - (t_A(u_j) + t_B(u_j) - t_A(u_j).t_B(u_j))| \\ +|m_B(u_j) - (m_A(u_j) + m_B(u_j) - m_A(u_j).m_B(u_j))| \\ +\frac{1}{2\pi}(|\theta_{sB}(u_j) - \frac{\theta_{sA}(u_j).\theta_{sB}(u_j)}{2\pi}| \\ +|\theta_{tB}(u_j) - (\theta_{tA}(u_j) + \theta_{tB}(u_j) - \frac{\theta_{tA}(u_j).\theta_{tB}(u_j)}{2\pi})| \\ + |\theta_{mB}(u_j) - (\theta_{mA}(u_j) + \theta_{mB}(u_j) - \frac{\theta_{mA}(u_j).\theta_{sB}(u_j)}{2\pi})|)\}\end{array}\right\}$$

$$= 1 - \frac{1}{6n}\sum_{j=1}^{n}\left\{\begin{array}{c}\{|s_B(u_j)(1 - s_A(u_j))| + |t_A(u_j)(1 - t_B(u_j))| \\ +|m_A(u_j)(1 - m_B(u_j))| + \frac{1}{2\pi}(|\theta_{sB}(u_j)(1 - \frac{\theta_{sA}(u_j)}{2\pi})| \\ + |\theta_{tB}(u_j)(1 - \frac{\theta_{tA}(u_j)}{2\pi}) + |\theta_{mB}(u_j)(1 - \frac{\theta_{mA}(u_j)}{2\pi})|)\}\end{array}\right\}$$

$$= \mathbb{S}_5(A, A + B).$$

TABLE 1. Existing similarity measures for *PFSs*.

Authors	Existing similarity measures for PFSs $A = \{< u, \rho_A(u), \tau_A(u), \omega_A(u) >\mid u \in U\}$ and $B = \{< u, \rho_B(u), \tau_B(u), \omega_B(u) >\mid u \in U\}$
Singh and Mishra [21]	$\mathbb{S}_{SM_1}(A,B) = 1 - \frac{1}{4n}\sum_{j=1}^{n}(\lvert\rho_A(u_j)-\rho_B(u_j)\rvert + \lvert\tau_A(u_j)-\tau_B(u_j)\rvert + \lvert\omega_A(u_j)-\omega_B(u_j)\rvert + \lvert\pi_A(u_j)-\pi_B(u_j)\rvert)$. $\mathbb{S}_{SM_2}(A,B) = 1-(\frac{1}{4n}\sum_{j=1}^{n}(\rho_A(u_j)-\rho_B(u_j))^2 + (\tau_A(u_j)-\tau_B(u_j))^2 + (\omega_A(u_j)-\omega_B(u_j))^2 + (\pi_A(u_j)-\pi_B(u_j))^2)^{1/2}$. $\mathbb{S}_{SM_3}(A,B) = 1 - \frac{1}{4n}\sum_{j=1}^{n}\max(\lvert\rho_A(u_j)-\rho_B(u_j)\rvert, \lvert\tau_A(u_j)-\tau_B(u_j)\rvert, \lvert\omega_A(u_j)-\omega_B(u_j)\rvert, \lvert\pi_A(u_j)-\pi_B(u_j)\rvert)$. $\mathbb{S}_{SM_4}(A,B) = 1 - (\frac{1}{4n}\sum_{j=1}^{n}\max((\rho_A(u_j)-\rho_B(u_j))^2 + (\tau_A(u_j)-\tau_B(u_j))^2 + (\omega_A(u_j)-\omega_B(u_j))^2 + (\pi_A(u_j)-\pi_B(u_j))^2)^{1/2}$. $\mathbb{S}_{SM_5}(A,B) = \frac{1}{4n}\sum_{j=1}^{n}(\min(\lvert\rho_A(u_j)-\rho_B(u_j)\rvert, \lvert\tau_A(u_j)-\tau_B(u_j)\rvert, \lvert\omega_A(u_j)-\omega_B(u_j)\rvert, \lvert\pi_A(u_j)-\pi_B(u_j)\rvert)/\max(\lvert\rho_A(u_j)-\rho_B(u_j)\rvert, \lvert\tau_A(u_j)-\tau_B(u_j)\rvert, \lvert\omega_A(u_j)-\omega_B(u_j)\rvert, \lvert\pi_A(u_j)-\pi_B(u_j)\rvert))$.
Wei [29]	$\mathbb{S}_{W_1}(A,B) = \frac{1}{n}\sum_{j=1}^{n}(\rho_A(u_j)\rho_B(u_j) + \tau_A(u_j)\tau_B(u_j) + \omega_A(u_j)\omega_B(u_j)/(\rho_A(u_j)^2 + \tau_A(u_j)^2 + \omega_A(u_j)^2)^{1/2}(\rho_B(u_j)^2 + \tau_B(u_j)^2 + \omega_B(u_j)^2)^{1/2})$. $\mathbb{S}_{W_2}(A,B) = \frac{1}{n}\sum_{j=1}^{n} Cos\frac{\pi}{2}(\lvert\rho_A(u_j)-\rho_B(u_j)\rvert \vee \lvert\tau_A(u_j)-\tau_B(u_j)\rvert \vee \lvert\omega_A(u_j)-\omega_B(u_j)\rvert \vee \lvert\pi_A(u_j)-\pi_B(u_j)\rvert)$. $\mathbb{S}_{W_3}(A,B) = \frac{1}{n}\sum_{j=1}^{n} Cos\frac{\pi}{4}(\lvert\rho_A(u_j)-\rho_B(u_j)\rvert + \lvert\tau_A(u_j)-\tau_B(u_j)\rvert + \lvert\omega_A(u_j)-\omega_B(u_j)\rvert + \lvert\pi_A(u_j)-\pi_B(u_j)\rvert)$. $\mathbb{S}_{W_4}(A,B) = \frac{1}{n}\sum_{j=1}^{n} Cot[\frac{\pi}{4} + \frac{\pi}{4}(\lvert\rho_A(u_j)-\rho_B(u_j)\rvert \vee \lvert\tau_A(u_j)-\tau_B(u_j)\rvert \vee \lvert\omega_A(u_j)-\omega_B(u_j)\rvert)]$. $\mathbb{S}_{W_5}(A,B) = \frac{1}{n}\sum_{j=1}^{n} Cot[\frac{\pi}{4} + \frac{\pi}{4}(\lvert\rho_A(u_j)-\rho_B(u_j)\rvert \vee \lvert\tau_A(u_j)-\tau_B(u_j)\rvert \vee \lvert\omega_A(u_j)-\omega_B(u_j)\rvert \vee \lvert\pi_A(u_j)-\pi_B(u_j)\rvert)]$.
Wei and Gao [28]	$\mathbb{S}_{WG_1}(A,B) = \frac{1}{n}\sum_{j=1}^{n}(2(\rho_A(u_j)\rho_B(u_j) + \tau_A(u_j)\tau_B(u_j) + \omega_A(u_j)\omega_B(u_j) + \pi_A(u_j)\pi_B(u_j))/(\rho_A(u_j)^2 + \tau_A(u_j)^2 + \omega_A(u_j)^2 + \pi_A(u_j)^2) + (\rho_B(u_j)^2 + \tau_B(u_j)^2 + \omega_B(u_j)^2 + \pi_B(u_j)^2))$. $\mathbb{S}_{WG_2}(A,B) = \sum_{j=1}^{n} 2(\rho_A(u_j)\rho_B(u_j) + \tau_A(u_j)\tau_B(u_j) + \omega_A(u_j)\omega_B(u_j) + \pi_A(u_j)\pi_B(u_j))/\sum_{j=1}^{n}(\rho_A(u_j)^2 + \tau_A(u_j)^2 + \omega_A(u_j)^2 + \pi_A(u_j)^2) + \sum_{j=1}^{n}(\rho_B(u_j)^2 + \tau_B(u_j)^2 + \omega_B(u_j)^2 + \pi_B(u_j)^2)$.
Dinh and Thao [22]	$\mathbb{S}_{DT_1}(A,B) = 1 - \frac{1}{n}(\sum_{j=1}^{n}[(\rho_A(u_j)-\rho_B(u_j))^2 + (\tau_A(u_j)-\tau_B(u_j))^2 + (\omega_A(u_j)-\omega_B(u_j))^2])^{1/2}$. $\mathbb{S}_{DT_2}(A,B) = 1 - \frac{1}{3n}\sum_{j=1}^{n}[(\rho_A(u_j)-\rho_B(u_j)) + (\tau_A(u_j)-\tau_B(u_j)) + (\omega_A(u_j)-\omega_B(u_j))]$.

The results of the classification of various similarity measures are given in Table 4, and $\mathbb{S}_1(P_3, Q) < \mathbb{S}_1(P_1, Q) < \mathbb{S}_1(P_2, Q)$. From this, we conclude that the unknown pattern matches with the pattern P_2 and the proposed similarity measure $\mathbb{S}_1$ gives consistent results along with $\mathbb{S}_{WG_1}$, $\mathbb{S}_{WG_2}$. The other existing similarity measures in the literature are unable to classify the unknown pattern Q because either they are giving the same values or the values cannot be calculated. Therefore, our proposed similarity measures are able to overcome the limitations of the existing similarity in an effective and superior manner.

Example 5.1.2. [29]: Consider a company that devises a new product and is looking for the best approach to increase sales. Now, there are four distinct possible approaches: Q_1: develop a product which is more wealthy customers oriented, Q_2: develop a product which is more intended towards middle and lower middle-class customers, Q_3: develop a product

TABLE 2. Comparisons with different similarity measures(unreasonable results are in bold-type).

	Case1	Case2	Case3	Case4
$\mathbb{S}_{SM_1}$ [21]	**0.925**	**0.925**	**0.500**	**0.500**
$\mathbb{S}_{SM_2}$ [21]	**0.913**	**0.913**	0.421	0.352
$\mathbb{S}_{SM_3}$ [21]	**0.975**	**0.975**	**0.250**	**0.250**
$\mathbb{S}_{SM_4}$ [21]	**0.998**	**0.998**	**0.250**	**0.250**
$\mathbb{S}_{SM_5}$ [21]	**0.125**	**0.125**	0.038	0.013
$\mathbb{S}_{W_1}$ [29]	0.984	0.971	**0.000**	**0.000**
$\mathbb{S}_{W_2}$ [29]	**0.988**	**0.988**	**0.000**	**0.000**
$\mathbb{S}_{W_3}$ [29]	**0.969**	**0.969**	**0.000**	**0.000**
$\mathbb{S}_{W_4}$ [29]	**0.854**	**0.854**	**0.000**	**0.000**
$\mathbb{S}_{W_5}$ [29]	**0.854**	**0.854**	**0.000**	**0.000**
$\mathbb{S}_{WG_1}$ [28]	0.972	0.949	**0.000**	**0.000**
$\mathbb{S}_{WG_2}$ [28]	0.973	0.969	**0.000**	**0.000**
$\mathbb{S}_{DT_1}$ [22]	**0.900**	**0.900**	0.086	0.089
$\mathbb{S}_{DT_2}$ [22]	**0.916**	**0.916**	0.300	0.366
[50]	0.839	0.500	0.772	0.0.777
[51]	**0.991**	**0.991**	0.623	0.0.567
$\mathbb{S}_1$(Proposed)	0.687	0.646	0.239	0.230

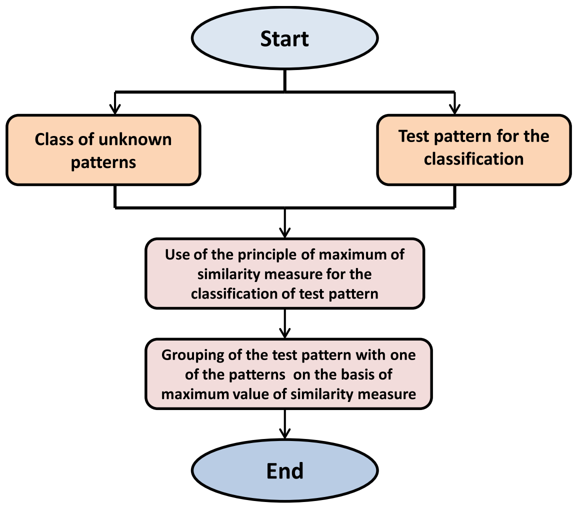

FIGURE 1. Procedural steps for solving a pattern recognition problem.

TABLE 3. Patterns and unknown pattern.

	u_1	u_2	u_3
P_1	$(0.5, 0.3\pi)$, $(0.1, 0.5\pi)$, $(0.1, 0.2\pi)$	$(0.3, 0.6\pi)$, $(0.1, 0.2\pi)$, $(0.3, 0.3\pi)$	$(0.3, 0.4\pi)$, $(0.1, 0.2\pi)$, $(0.4, 1.1\pi)$
P_2	$(0.4, 0.3\pi)$, $(0.3, 0.2\pi)$, $(0.2, 0.1\pi)$	$(0.3, 1.1\pi)$, $(0.2, 0.3\pi)$, $(0.5, 0.3\pi)$	$(0.4, 0.3\pi)$, $(0.1, 0.5\pi)$, $(0.3, 0.2\pi)$
P_3	$(0.2, 0.6\pi)$, $(0.4, 0.3\pi)$, $(0.3, 0.6\pi)$	$(0.1, 0.2\pi)$, $(0.4, 1.1\pi)$, $(0.4, 0.3\pi)$	$(0.0, 0.2\pi)$, $(0.0, 0.1\pi)$, $(0.0, 0.3\pi)$
Q	$(0.3, 0.4\pi)$, $(0.2, 0.3\pi)$, $(0.3, 0.6\pi)$	$(0.4, 0.6\pi)$, $(0.1, 0.5\pi)$, $(0.3, 0.2\pi)$	$(0.4, 0.2\pi)$, $(0.2, 0.3\pi)$, $(0.3, 0.5\pi)$

which is suitable to all customer types, Q_2: do not develop a product. After a thorough evaluation of the information, the approaches are condensed into the six characteristic features $C_i(i = 1, 2, \ldots, 6)$. Now, the four distinct approaches $Q_i(i = 1, 2, 3, 4)$ and a complete;y unknown approach D under the six characteristics are converted in the complex picture fuzzy

TABLE 4. Pattern recognition results for different similarity measures.

	$\mathbb{S}_1(P_1,Q)$	$\mathbb{S}_1(P_2,Q)$	$\mathbb{S}_1(P_3,Q)$	Classification results
$\mathbb{S}_{SM_1}$ [21]	0.900	0.900	0.750	Unable to classify
$\mathbb{S}_{SM_2}$ [21]	0.885	0.885	0.663	Unable to classify
$\mathbb{S}_{SM_3}$ [21]	0.967	0.967	0.883	Unable to classify
$\mathbb{S}_{SM_4}$ [21]	0.995	0.995	0.922	Unable to classify
$\mathbb{S}_{SM_5}$ [21]	0.875	0.875	0.046	Unable to classify
$\mathbb{S}_{W_1}$ [29]	0.919	0.954	NaN	Unable to classify
$\mathbb{S}_{W_2}$ [29]	0.976	0.976	0.666	Unable to classify
$\mathbb{S}_{W_3}$ [29]	0.943	0.943	0.639	Unable to classify
$\mathbb{S}_{W_4}$ [29]	0.812	0.812	0.617	Unable to classify
$\mathbb{S}_{W_5}$ [29]	0.812	0.812	0.473	Unable to classify
$\mathbb{S}_{WG_1}$ [28]	0.913	0.916	0.542	P_2
$\mathbb{S}_{WG_2}$ [28]	0.911	0.913	0.448	P_2
[50]	0.697	0.729	0.410	P_2
[51]	0.984	0.984	0.878	Unable to classify
$\mathbb{S}_1$(Proposed)	0.538	0.551	0.472	P_2

TABLE 5. Patterns and unknown pattern.

	Q_1	Q_2	Q_3	Q_4	D
C_1	$(0.53, 0.3\pi)$, $(0.33, 0.5\pi)$, $(0.09, 0.2\pi)$	$(1.0, 0.3\pi)$, $(0.0, 0.1\pi)$, $(0.0, 0.2\pi)$	$(0.91, 0.9\pi)$, $(0.03, 0.4\pi)$, $(0.02, 0.4\pi)$	$(0.85, 0.4\pi)$, $(0.09, 1.1\pi)$, $(0.05, 0.2\pi)$	$(0.90, 0.3\pi)$, $(0.05, 0.5\pi)$, $(0.02, 0.2\pi)$
C_2	$(0.89, 0.4\pi)$, $(0.08, 1.1\pi)$, $(0.03, 0.2\pi)$	$(0.13, 0.8\pi)$, $(0.64, 0.3\pi)$, $(0.21, 0.6\pi)$	$(0.07, 0.7\pi)$, $(0.09, 0.1\pi)$, $(0.05, 0.2\pi)$	$(0.74, 0.3\pi)$, $(0.16, 0.1\pi)$, $(0.10, 0.2\pi)$	$(0.68, 0.4\pi)$, $(0.08, 0.2\pi)$, $(0.21, 1.1\pi)$
C_3	$(0.42, 0.8\pi)$, $(0.35, 0.2\pi)$, $(0.18, 0.7\pi)$	$(0.03, 1.2\pi)$, $(0.82, 0.3\pi)$, $(0.13, 0.4\pi)$	$(0.04, 0.9\pi)$, $(0.85, 0.1\pi)$, $(0.10, 0.2\pi)$	$(0.02, 1.2\pi)$, $(0.89, 0.3\pi)$, $(0.05, 0.4\pi)$	$(0.05, 0.2\pi)$, $(0.87, 1.1\pi)$, $(0.06, 0.3\pi)$
C_4	$(0.08, 0.9\pi)$, $(0.89, 0.4\pi)$, $(0.02, 0.4\pi)$	$(0.73, 0.6\pi)$, $(0.15, 0.2\pi)$, $(0.08, 0.3\pi)$	$(0.68, 1.1\pi)$, $(0.26, 0.3\pi)$, $(0.06, 0.4\pi)$	$(0.08, 1.3\pi)$, $(0.84, 0.2\pi)$, $(0.06, 0.4\pi)$	$(0.13, 0.6\pi)$, $(0.75, 0.3\pi)$, $(0.09, 0.6\pi)$
C_5	$(0.33, 1.1\pi)$, $(0.51, 0.3\pi)$, $(0.12, 0.3\pi)$	$(0.52, 0.7\pi)$, $(0.31, 0.4\pi)$, $(0.16, 0.4\pi)$	$(0.15, 0.6\pi)$, $(0.76, 0.4\pi)$, $(0.07, 0.5\pi)$	$(0.16, 0.6\pi)$, $(0.71, 0.2\pi)$, $(0.05, 0.3\pi)$	$(0.15, 1.1\pi)$, $(0.73, 0.3\pi)$, $(0.08, 0.3\pi)$
C_6	$(0.17, 0.3\pi)$, $(0.53, 0.5\pi)$, $(0.13, 0.1\pi)$	$(0.51, 0.4\pi)$, $(0.24, 0.2\pi)$, $(0.21, 1.1\pi)$	$(0.31, 0.3\pi)$, $(0.39, 0.2\pi)$, $(0.25, 0.1\pi)$	$(1.00, 0.6\pi)$, $(0.0, 0.4\pi)$, $(0.0, 0.4\pi)$	$(0.91, 0.4\pi)$, $(0.03, 0.3\pi)$, $(0.00, 0.6\pi)$

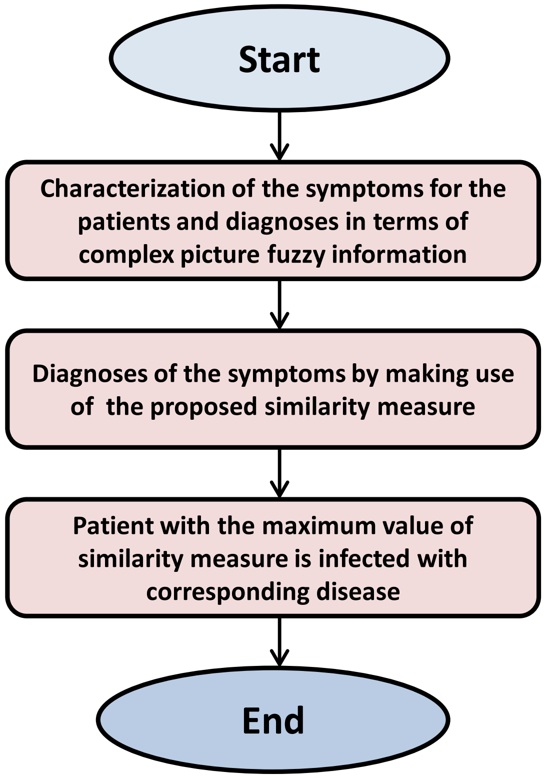

FIGURE 2. Procedural steps for solving a medical diagnosis problem.

TABLE 6. Comparison with existing measures.

	(Q_1,D)	(Q_2,D)	(Q_3,D)	(Q_1,D)
$\mathbb{S}_{W_1}$ [29]	0.813	0.656	0.787	0.994
$\mathbb{S}_{W_2}$ [29]	0.813	0.765	0.709	0.992
$\mathbb{S}_{W_3}$ [29]	0.813	0.757	0.707	0.989
$\mathbb{S}_{W_4}$ [29]	0.486	0.442	0.469	0.666
$\mathbb{S}_{W_5}$ [29]	0.486	0.442	0.440	0.665
$\mathbb{S}_1$(Proposed)	0.521	0.516	0.514	0.687

information given in Table 5. Further, by making use of the "principle of the maximum of similarity measures", the measure with the maximum value gives the optimal production approach.

From Table 6, we see that the results of the proposed similarity measure are equally consistent as those with the [29] and the unknown approach D matches with the approach Q_4.

B. APPLICATION IN MEDICAL DIAGNOSIS

Further, the proposed measure has been applied to the problems of medical diagnosis along with comparisons with the existing similarity measures in the literature. The steps involved in the methodology are given in Figure 2.

Example 5.2.1. [20]: Let us consider the set $P = \{P_1, P_2, P_3, P_4\}$ be the set of four patients and let S be the set of symptoms given as $S = \{$*Temperature*, *Headache*, *Stomach pain*, *Cough*, *Chest pain*$\}$. Let $D = \{$*Viral fever*, *Malaria*, *Typhoid*, *Stomach problem*, *Chest pain*$\}$ be the set of diagnoses. Now, the corresponding symptoms of the patients and diseases are given in Table 7 and Table 8 respectively. The information in the respective tables are stored in the form of complex picture fuzzy information. Now, we utilize the proposed similarity measure $\mathbb{S}_1$ for the diagnosis of the patients. Similarly, other proposed similarity measures can also be utilized.

Further, with the help of the "principle of the maximum of similarity measures", the maximum value of the similarity measure gives us the proper diagnosis. From Table 9, we conclude that the patient P_1 is suffering from *Viral fever*, P_2 is suffering from *Chest problem*, P_3 is suffering from *Typhoid*, P_4 is suffering from *stomach problem*.

Example 5.2.2: Let us consider the set $P = \{P_1, P_2, P_3\}$ be the set of four patients and let S be the set of symptoms given as $S = \{S_1, S_2, S_3, S_4, S_5\}$. Let $D = \{D_1, D_2, D_3, D_4\}$ be the set of diagnoses. Now, the corresponding symptoms of the patients and diseases are given in Table 10 and Table 11 respectively. The information in the respective tables is stored in the form of complex picture fuzzy information. Now, we again utilize the similarity measure $\mathbb{S}_1$ for the diagnosis of the patients.

Further, with the help of the "principle of the maximum of similarity measures", the maximum value of the similarity measure gives us the proper diagnosis.

TABLE 7. Characterization of symptoms for the patients.

	Temperature	Headache	Stomach pain	Cough	Chest pain
P_1	$(0.80, 0.3\pi)$, $(0.0, 0.5\pi)$, $(0.10, 0.2\pi)$	$(0.60, 0.3\pi)$, $(0.30, 0.1\pi)$, $(0.10, 0.2\pi)$	$(0.20, 0.9\pi)$, $(0.40, 0.4\pi)$, $(0.40, 0.4\pi)$	$(0.60, 0.4\pi)$, $(0.15, 1.1\pi)$, $(0.10, 0.2\pi)$	$(0.10, 0.3\pi)$, $(0.40, 0.5\pi)$, $(0.40, 0.2\pi)$
P_2	$(0.00, 0.4\pi)$, $(0.50, 1.1\pi)$, $(0.40, 0.2\pi)$	$(0.40, 0.8\pi)$, $(0.25, 0.3\pi)$, $(0.30, 0.6\pi)$	$(0.60, 0.7\pi)$, $(0.20, 0.1\pi)$, $(0.10, 0.2\pi)$	$(0.10, 0.3\pi)$, $(0.30, 0.1\pi)$, $(0.60, 0.2\pi)$	$(0.10, 0.4\pi)$, $(0.35, 0.2\pi)$, $(0.40, 1.1\pi)$
P_3	$(0.80, 0.8\pi)$, $(0.00, 0.2\pi)$, $(0.10, 0.7\pi)$	$(0.80, 1.2\pi)$, $(0.00, 0.3\pi)$, $(0.10, 0.4\pi)$	$(0.00, 0.9\pi)$, $(0.40, 0.1\pi)$, $(0.50, 0.2\pi)$	$(0.20, 1.2\pi)$, $(0.30, 0.3\pi)$, $(0.40, 0.4\pi)$	$(0.00, 0.2\pi)$, $(0.40, 1.1\pi)$, $(0.40, 0.3\pi)$
P_4	$(0.60, 0.9\pi)$, $(0.20, 0.4\pi)$, $(0.10, 0.4\pi)$	$(0.50, 0.6\pi)$, $(0.25, 0.2\pi)$, $(0.25, 0.3\pi)$	$(0.30, 1.1\pi)$, $(0.30, 0.3\pi)$, $(0.20, 0.4\pi)$	$(0.70, 1.3\pi)$, $(0.00, 0.2\pi)$, $(0.25, 0.4\pi)$	$(0.30, 0.6\pi)$, $(0.40, 0.3\pi)$, $(0.20, 0.6\pi)$

TABLE 8. Characterization of symptoms for the diagnoses.

	Temperature	Headache	Stomach pain	Cough	Chest pain
Viral fever	$(0.40, 0.2\pi)$, $(0.00, 0.6\pi)$, $(0.00, 0.2\pi)$	$(0.30, 0.5\pi)$, $(0.20, 0.3\pi)$, $(0.40, 0.2\pi)$	$(0.10, 0.9\pi)$, $(0.35, 0.4\pi)$, $(0.50, 0.4\pi)$	$(0.40, 0.4\pi)$, $(0.30, 1.1\pi)$, $(0.20, 0.2\pi)$	$(0.10, 0.3\pi)$, $(0.25, 0.5\pi)$, $(0.50, 0.2\pi)$
Malaria	$(0.70, 0.4\pi)$, $(0.00, 091\pi)$, $(0.00, 0.1\pi)$	$(0.20, 0.8\pi)$, $(0.40, 0.4\pi)$, $(0.35, 0.7\pi)$	$(0.00, 0.7\pi)$, $(0.40, 0.1\pi)$, $(0.50, 0.5\pi)$	$(0.70, 0.1\pi)$, $(0.10, 0.4\pi)$, $(0.00, 0.4\pi)$	$(0.10, 0.4\pi)$, $(0.30, 0.2\pi)$, $(0.50, 1.1\pi)$
Typhoid	$(0.30, 0.7\pi)$, $(0.40, 0.2\pi)$, $(0.30, 0.9\pi)$	$(0.60, 1.2\pi)$, $(0.20, 0.3\pi)$, $(0.10, 0.4\pi)$	$(0.20, 0.8\pi)$, $(0.30, 0.1\pi)$, $(0.40, 0.5\pi)$	$(0.20, 1.2\pi)$, $(0.35, 0.3\pi)$, $(0.30, 0.2\pi)$	$(0.10, 0.2\pi)$, $(0.20, 1.1\pi)$, $(0.60, 0.2\pi)$
Stomach problem	$(0.10, 0.8\pi)$, $(0.30, 0.4\pi)$, $(0.50, 0.6\pi)$	$(0.20, 0.6\pi)$, $(0.40, 0.2\pi)$, $(0.30, 0.2\pi)$	$(0.80, 1.1\pi)$, $(0.00, 0.3\pi)$, $(0.00, 0.5\pi)$	$(0.20, 1.1\pi)$, $(0.40, 0.2\pi)$, $(0.30, 0.4\pi)$	$(0.20, 0.6\pi)$, $(0.35, 0.2\pi)$, $(0.30, 0.6\pi)$
Chest problem	$(0.10, 0.7\pi)$, $(0.30, 0.4\pi)$, $(0.50, 0.3\pi)$	$(0.20, 0.5\pi)$, $(0.40, 0.2\pi)$, $(0.30, 0.4\pi)$	$(0.80, 1.1\pi)$, $(0.00, 0.3\pi)$, $(0.00, 0.3\pi)$	$(0.20, 1.3\pi)$, $(0.40, 0.2\pi)$, $(0.30, 0.3\pi)$	$(0.20, 0.5\pi)$, $(0.35, 0.3\pi)$, $(0.30, 0.5\pi)$

TABLE 9. Results obtained from the proposed similarity measure.

	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem
P_1	0.712	0.583	0.538	0.412	0.422
P_2	0.437	0.505	0.503	0.577	0.589
P_3	0.518	0.497	0.709	0.451	0.447
P_4	0.525	0.524	0.550	0.622	0.611

From Table 12, we conclude that the patient P_1 is suffering from D_2, P_2 is suffering from D_4, P_3 is suffering from D_3. From Table 13, we see that the results of the proposed similarity measure are equally consistent as those with the $\mathbb{S}_{W_1}$,$\mathbb{S}_{WG_1}$ and $\mathbb{S}_{WG_2}$, where other similarity measures are unable to diagnose the diseases.

TABLE 10. Characterization of symptoms for the patients.

	S_1	S_2	S_3	S_4	S_5
P_1	$(0.3, 0.3\pi)$, $(0.2, 0.5\pi)$, $(0.3, 0.2\pi)$	$(0.4, 0.3\pi)$, $(0.1, 0.1\pi)$, $(0.3, 0.2\pi)$	$(0.4, 0.9\pi)$, $(0.2, 0.4\pi)$, $(0.3, 0.4\pi)$	$(0.3, 0.4\pi)$, $(0.1, 1.1\pi)$, $(0.4, 0.2\pi)$	$(0.3, 0.3\pi)$, $(0.2, 0.5\pi)$, $(0.3, 0.2\pi)$
P_2	$(0.0, 0.4\pi)$, $(0.1, 1.1\pi)$, $(0.8, 0.2\pi)$	$(0.4, 0.8\pi)$, $(0.1, 0.3\pi)$, $(0.4, 0.6\pi)$	$(0.6, 0.7\pi)$, $(0.3, 0.1\pi)$, $(0.1, 0.2\pi)$	$(0.1, 0.3\pi)$, $(0.1, 0.1\pi)$, $(0.7, 0.2\pi)$	$(0.1, 0.4\pi)$, $(0.0, 0.2\pi)$, $(0.8, 1.1\pi)$
P_3	$(0.8, 0.8\pi)$, $(0.0, 0.2\pi)$, $(0.1, 0.7\pi)$	$(0.8, 1.2\pi)$, $(0.1, 0.3\pi)$, $(0.1, 0.4\pi)$	$(0.0, 0.9\pi)$, $(0.4, 0.1\pi)$, $(0.6, 0.2\pi)$	$(0.2, 1.2\pi)$, $(0.0, 0.3\pi)$, $(0.7, 0.4\pi)$	$(0.1, 0.2\pi)$, $(0.0, 1.1\pi)$, $(0.8, 0.3\pi)$

TABLE 11. Characterization of symptoms for the diagnoses.

	S_1	S_2	S_3	S_4	S_5
D_1	$(0.5, 0.2\pi)$, $(0.1, 0.6\pi)$, $(0.1, 0.2\pi)$	$(0.3, 0.5\pi)$, $(0.1, 0.3\pi)$, $(0.3, 0.2\pi)$	$(0.3, 0.9\pi)$, $(0.1, 0.4\pi)$, $(0.4, 0.4\pi)$	$(0.4, 0.4\pi)$, $(0.2, 1.1\pi)$, $(0.3, 0.2\pi)$	$(0.1, 0.3\pi)$, $(0.1, 0.5\pi)$, $(0.5, 0.2\pi)$
D_2	$(0.4, 0.4\pi)$, $(0.3, 091\pi)$, $(0.2, 0.1\pi)$	$(0.3, 0.8\pi)$, $(0.2, 0.4\pi)$, $(0.5, 0.7\pi)$	$(0.4, 0.7\pi)$, $(0.1, 0.1\pi)$, $(0.3, 0.5\pi)$	$(0.5, 0.1\pi)$, $(0.2, 0.4\pi)$, $(0.3, 0.4\pi)$	$(0.2, 0.4\pi)$, $(0.3, 0.2\pi)$, $(0.4, 1.1\pi)$
D_3	$(0.3, 0.7\pi)$, $(0.3, 0.2\pi)$, $(0.3, 0.9\pi)$	$(0.6, 1.2\pi)$, $(0.2, 0.3\pi)$, $(0.1, 0.4\pi)$	$(0.2, 0.8\pi)$, $(0.1, 0.1\pi)$, $(0.7, 0.5\pi)$	$(0.2, 1.2\pi)$, $(0.0, 0.3\pi)$, $(0.6, 0.2\pi)$	$(0.1, 0.2\pi)$, $(0.0, 1.1\pi)$, $(0.9, 0.2\pi)$
D_4	$(0.1, 0.8\pi)$, $(0.2, 0.4\pi)$, $(0.7, 0.6\pi)$	$(0.2, 0.6\pi)$, $(0.3, 0.2\pi)$, $(0.4, 0.2\pi)$	$(0.8, 1.1\pi)$, $(0.2, 0.3\pi)$, $(0.0, 0.5\pi)$	$(0.2, 1.1\pi)$, $(0.1, 0.2\pi)$, $(0.7, 0.4\pi)$	$(0.2, 0.6\pi)$, $(0.0, 0.2\pi)$, $(0.7, 0.6\pi)$

TABLE 12. Results obtained from the proposed similarity measure.

	D_1	D_2	D_3	D_4
P_1	0.572	0.583	0.490	0.506
P_2	0.462	0.528	0.469	0.628
P_3	0.468	0.393	0.756	0.494

TABLE 13. Medical diagnoses results for different similarity measures.

	P_1	P_2	P_3
$\mathbb{S}_{SM_1}$ [21]	Unable to diagnose	D_4	D_3
$\mathbb{S}_{SM_2}$ [21]	Unable to diagnose	D_4	D_3
$\mathbb{S}_{SM_3}$ [21]	Unable to diagnose	D_4	D_3
$\mathbb{S}_{SM_4}$ [21]	Unable to diagnose	D_4	D_3
$\mathbb{S}_{SM_5}$ [21]	Unable to diagnose	Unable to diagnose	Unable to diagnose
$\mathbb{S}_{W_1}$ [29]	D_2	D_4	D_3
$\mathbb{S}_{W_2}$ [29]	Unable to diagnose	D_4	Unable to diagnose
$\mathbb{S}_{W_3}$ [29]	Unable to diagnose	D_4	D_3
$\mathbb{S}_{W_4}$ [29]	Unable to diagnose	D_4	Unable to diagnose
$\mathbb{S}_{W_5}$ [29]	Unable to diagnose	D_4	Unable to diagnose
$\mathbb{S}_{WG_1}$ [28]	D_2	D_4	D_3
$\mathbb{S}_{WG_2}$ [28]	D_2	D_4	D_3
[50]	D_2	D_4	D_3
Proposed	D_2	D_4	D_3

VI. DISCUSSION ON ADVANTAGE OF SIMILARITY MEASURES FOR *CPFSS*

It has been duly explained in the introduction section regarding the advantages of utilizing complex picture fuzzy information in real-life applications such as fingerprint attendance management machines and other pattern recognition/medical diagnosis problems. Upon introducing the concept of similarity measures for complex picture fuzzy sets for the first time in literature, we have applied them in two different hypothetical problems of pattern recognition and medical diagnosis. The respective results of the problems under consideration have been already pointed out along with its computational part. In addition to this, we discuss and explicitly point out the important advantageous remarks as follows:

- The effectiveness of the similarity measures for *CPFSs* clearly gets reflected while dealing with the problems of time-periodicity and handling the 2-dimensional data set. Additionally, it incorporates all four components of uncertainty inherited in a picture-fuzzy environment.
- In the problem taken in Example 5.2.2, the superiority of the proposed measures has been illustrated with the help of a suitable comparative analysis.
- It may also be noted that the problem taken in Example 5.1.1 could not be addressed with the help of the existing similarity measures while this can be solved with the utilization of the proposed similarity measures.
- Overall, it has also been shown in Section IV that the values of the existing similarity measures are observed to be unreasonable in comparison with the values of introduced similarity measures.

Remark: The constraint on the sum of uncertainty components (i.e. one) is the limitation of the proposed model, the decision-makers have to choose the values so that the sum of all the uncertainty components remains one.

VII. CONCLUSION AND FUTURE WORK

The systematic notional deliberation of novel similarity measures under the complex picture fuzzy information setup has been successfully presented with proper proofs of validation and their properties. The empirical comparative analysis and comparison with similarity measures given by [50] and [51] depict that the existing similarity measures for complex picture fuzzy sets compute counter-intuitive values, which are not quite reasonable in real situations. However, the proposed measures show reasonably better acceptable values/ranges, illustrating their prominence. The implementation process of the proposed measures for different problems of pattern recognition is found to be computationally and effectively applicable. The other comparative analysis presented in the manuscript provides a better prospect with advantageous features for the problems under consideration. It may also be observed that the methodology containing the proposed similarity measures better handles problems of time-periodicity and manages 2-dimensional data sets more efficiently. In the literature, for the sake of further extension of the work, the notion of complex q-rung picture fuzzy sets can be introduced with their important algebraic operations, similarity measures, and various types of aggregation operators. Additionally, similar work can be done for complex q-rung picture fuzzy sets, and the measures can be applied to machine learning problems. Accordingly, the application field can be selected for some deterministic results and orientation.

DECLARATIONS AND COMPLIANCE WITH ETHICAL STANDARDS

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.
Conflict of interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this article.
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