IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received 19 May 2024, accepted 4 June 2024, date of publication 11 June 2024, date of current version 27 August 2024.

Digital Object Identifier 10.1109/ACCESS.2024.3412193

RESEARCH ARTICLE

Applications of q-Spherical Fuzzy Rough CODAS to the Assessment of a Problem Involving Renewable Energy Site Selection

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This work was supported by King Saud University, Riyadh, Saudi Arabia, under Grant RSPD2024R1060.

ABSTRACT Multi-criteria decision-making (MCDM) approaches prove to be effective and reliable in addressing problems under uncertain conditions. The q-spherical fuzzy rough set (q-SFRS) represents the latest advancement in fuzzy set theory. This article aims to introduce a novel approach, q-spherical fuzzy rough Combinative Distance-based Assessment (q-SFR-CODAS), by integrating CODAS with q-SFR set to address MCDM problems. The method utilizes the Hamming distance as the primary measure and the Euclidean distance as the secondary measure to assess the desirability of alternatives, calculated concerning the negative-ideal solution. Additionally, an illustrative example is presented to demonstrate the applicability of the proposed methodology. A comprehensive sensitivity analysis is conducted to validate the results of q-SFR-CODAS, comparing them with existing MCDM methods.

INDEX TERMS q-spherical fuzzy rough sets, combinative distance-based assessment, multiple criteria decision making involving renewable energy site selection.

I. INTRODUCTION

Energy plays a pivotal role in driving global economic growth and industrial activities. However, the escalating reliance on fossil fuels for energy generation has given rise to challenges, notably pollution and global warming. Consequently, there has been considerable discourse advocating for renewable energy as an eco-friendly, cost-effective, and sustainable alternative [1]. Renewable energy, derived from sources like solar, wind, water, heat, and biofuels, stands out due to its inexhaustibility, contribution to reducing dependence on finite resources like oil, enhancement of energy security, and preservation of the natural environment [2]. The evaluation and selection of suitable renewable energy portfolios constitute a multicriteria decision-making challenge, involving numerous and often conflicting criteria. The complexity is further compounded by the presence of uncertain data, necessitating a comprehensive assessment of alternative renewable energy portfolios. Multicriteria decision-making methods have proven successful in this context, contributing to the evaluation of renewable energy technologies, planning, policy development, and product selection. For instance, Yang et al. [3] propose the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) to assess renewable energy heating technology, determining the overall order of alternatives based on their proximity to the positive and negative ideal solutions. Garni et al. [4] employ the Analytical Hierarchy Process (AHP) to evaluate renewable power generation sources in Saudi Arabia, utilizing a four-level hierarchy to estimate the relative importance of decision elements. Kaya and Kahraman [5] introduce the Višekriterijumsko Kompromisno Rangiranje (VIKOR) and AHP for selecting the optimal renewable energy site alternative, considering criteria weights through pairwise comparisons in AHP and employing VIKOR for multicriteria selection.

The associate editor coordinating the review of this manuscript and approving it for publication was Asad Waqar Malik.

To address these challenges, this paper introduces a q-spherical fuzzy rough CODAS method for the evaluation and selection of renewable energy selection under uncertainty. The decision-making process incorporates risk analysis, and to aid subjective assessments by decision-makers, linguistic terms are represented using q-spherical fuzzy rough fuzzy numbers. A novel algorithm is proposed to generate a renewable energy site selection across all selection criteria. An illustrative example is provided to demonstrate the effectiveness of the proposed method in addressing the renewable energy site selection problem.

A. RENEWABLE ENERGY EVALUATION AND SITE SELECTION PROBLEM

Effectively utilizing renewable energy involves considering numerous factors, necessitating the simultaneous evaluation and selection of a specific renewable energy portfolio. This process requires taking into account diverse perspectives and accommodating the interests of various stakeholders in the decision-making process [7]. Extensive research has been conducted to identify relevant factors for evaluating and selecting suitable renewable energy portfolios. Kaya and Kahraman [5] emphasize the importance of considering factors such as capital cost, energy cost, operations and maintenance cost, safety of the energy system, land requirement, and emission reduction when evaluating renewable energy technologies. Wibowo and Grandhi [6] highlight technical capabilities, environmental friendliness, and cost-effectiveness as crucial factors in the evaluation of renewable energy portfolios. Scarpa and Willis [7] identify energy costs and the cost of new renewable energy projects as the top influential factors in decision-making. Mahapatra and Gustavsson [8] stress economic factors like annual and investment costs. Streimikiene et al. [9] underscore the significance of renewable heating technology, performance, and safety in the selection process. Amer and Daim [10] consider technical maturity, system efficiency, deployment time, and research and development costs in their assessment. Troldborg et al. [11] highlight network stability and ease of decentralization as crucial considerations. Stein [12] points out the importance of technology feasibility, compatibility with national energy policy, and environmental impact. Brand and Missaoui [13] mention reliability, resource reserves, safety in covering peak demand, network stability, and capital cost as key factors. Mourmouris and Potolias [14]stress technology feasibility, operating costs, and environmental impact. Pappas et al. [15] identify emission reduction, waste disposal needs, and social and political acceptance as crucial in evaluating renewable power generation sources. Chatzimouratidis and Pilavachi [16] add that national economic development should be considered. Montoya et al. [17] emphasize total cost, environmental impacts of new renewable energy power plants, and environmental costs of electricity generation as essential factors for consideration. The evaluation and selection of renewable energy selection, framed as a multicriteria decision-making problem, involves several steps: (a) identification of different portfolios, (b) selection of relevant evaluation criteria, (c) assessment of each portfolio, (d) calculation of overall performance index values using alternative performance ratings and criteria weights, and (e) selection of the most suitable renewable energy portfolio in each situation [6], [18]. To facilitate the subjective assessment by decision-makers, linguistic terms are employed, represented by q-spherical fuzzy rough numbers with approximate values ranging between 0 and 1.

B. LITERATURE REVIEW

In the realm of renewable energy site decision-making, the intersection of multi-criteria decision-making (MCDM) and fuzzy set theory forms a robust framework. MCDM, as a systematic approach, enables the assessment of alternatives against various criteria, crucial in the renewable energy landscape with its multifaceted considerations. Simultaneously, fuzzy set theory, designed to handle uncertainties, brings granularity to the decision process by accommodating imprecise information. The amalgamation of these methodologies, often referred to as fuzzy MCDM, is particularly pertinent to renewable energy scenarios. MCDM provides the structured backbone for decision problems, defining criteria and preferences, while fuzzy set theory adeptly manages uncertainties, especially in scenarios involving subjective assessments and imprecise data. The synergy of these approaches creates a nuanced decision-making tool tailored to the intricacies of renewable energy choices, capturing both the diversity of criteria and the uncertainties inherent in the evaluation process. This synthesis enriches the decision-making landscape, offering a comprehensive and adaptive methodology for stakeholders navigating the complexities of renewable energy adoption. In the discourse of decision-making methodologies, it is imperative to delineate between multi-attribute decision-making (MADM) and multi-criteria decision-making (MCDM) due to their nuanced approaches. MADM primarily deals with situations where a decision-maker evaluates alternatives based on a set of predefined attributes or characteristics. Each alternative is assessed independently concerning these attributes, and a final decision is derived from a collective analysis. On the other hand, MCDM extends this paradigm by incorporating a more intricate layer of multiple criteria, encompassing not only attributes but also subjective preferences, often conflicting in nature. MCDM allows for a more comprehensive evaluation, considering diverse criteria that might involve qualitative and quantitative factors, as well as the subjective judgments of decision-makers. In the context of renewable energy, MADM might assess alternatives based on individual attributes like cost, efficiency, and environmental impact, while MCDM would consider these factors holistically, recognizing the interplay and trade-offs between them. This distinction is crucial for researchers and practitioners navigating decision-making landscapes, especially in the complex and multifaceted domain of renewable energy adoption.

The origins of fuzzy set theory [19] may be traced back to the standard crisp set theory. In addition, Zadeh introduced the notion of a fuzzy set, which focuses solely on the membership function that received a positive evaluation. Membership functions in fuzzy sets define the degree of membership of an element in a set, ranging from 0 to 1. Essentially, they describe how much a given input value belongs to a particular fuzzy set. These functions map input values to membership values, indicating the degree of similarity between the input and the fuzzy set. For example, in a fuzzy set describing the concept of ''tallness'' in humans, the membership function might assign a membership value close to 1 for individuals with heights significantly above average, decreasing gradually as heights decrease towards the average. This gradual decrease captures the fuzziness inherent in human perception of tallness. Membership functions can take various shapes, such as triangular, trapezoidal, Gaussian, or sigmoidal, depending on the characteristics of the input variables and the problem domain. The choice of shape and parameters of the membership functions significantly influences the behavior and performance of a fuzzy logic system. In summary, membership functions in fuzzy sets play a crucial role in quantifying the degree of membership of elements in fuzzy sets, allowing for the representation of uncertainty and vagueness in real-world problems. Atanassov [20] introduced the intuitionistic fuzzy set (IFS) to expand upon the concept of fuzzy sets. The intuitionistic fuzzy set (IFS) incorporates both the positive and negative grades with the stipulation that their cumulative value should not exceed 1. In 2014, Coung expanded upon the concepts of fuzzy sets and IFS introduced a groundbreaking notion known as the picture fuzzy set (PFS) [21]. This innovation provided a fresh perspective within this field of study. Within the framework of PFS, the author delved into the categorization of grades into positive, neutral, and negative classifications. Gündoğdu and Kahraman [22] propose a spherical fuzzy set (SFS) as a possible way to address this challenge. Scholarly interest in the subject of SFS has grown in recent years. Kahraman and his research team [23] suggested the innovative conception of (q-SFS) in their determinations to concentrate the descendants of uncertainty. This innovative perception has provided evidence to be favorable in accompanying students in making well-informed varieties. The concept of rough sets (RS) was first introduced by Pawlak [24], [25] as a means of dealing with uncertainty. When examined from a mathematical perspective, this configuration demonstrates attributes that could be construed as vagueness and indeterminacy. Rough set theory (RST) is a modification of the traditional set theory, that uses the notion of connection to elucidate the operations of information systems. Researchers have acknowledged that the applicability of the equivalence relation in Pawlak's relational semantic theory is subject to notable constraints in a range of real-world situations, a point emphasized by multiple scholars. It is well acknowledged to initiate the concept of a ''q-spherical fuzzy set.'' Every single element in the q-SFS framework is classified as either positive, neutral, or negative. The notions of q-SFRS were first presented by Azim et al. [26] in their research paper published in 2023. This fuzzy set combines the advantages inherent in both the RS and the q-SFS. This research introduces a practical approach to decision-making within the framework of q-spherical fuzzy rough sets, thereby expanding the existing knowledge in this field. Within q-SFRS, three distinct parameters involve lower and upper approximations. Our main objective in this study is to advance future research by devising novel aggregation operators alongside defuzzification methods. After a comprehensive analysis, it becomes clear that the concept of q-SFRSs holds substantial potential as an innovative idea, thereby paving the way for numerous opportunities in future research endeavors. The CODAS algorithm, introduced by Ghorabaee et al. [27] in 2016, addresses complex Multi-Criteria Decision-Making (MCDM) problems. It has found application in various MCDM scenarios involving different fuzzy sets, including Pythagorean fuzzy sets [28], neutrosophic sets [29], intuitionistic fuzzy sets [20], picture fuzzy sets [21] spherical fuzzy sets [22], q-spherical fuzzy sets [23], q-spherical fuzzy rough sets [26] and among others. Livability indicators in Istanbul's suburban districts have been a recent research focus, with spherical fuzzy CODAS used for evaluating livability indices [30]. Additionally, Gündoğdu and Kahraman [31] explored the expansion of CODAS with spherical fuzzy sets in decision-making, a topic also covered in the INFUS 2019 Conference proceedings [32]. The q-spherical fuzzy rough sets are chosen for their applicability in MCDM systems, leading to the development of the q-spherical fuzzy rough CODAS technique, a novel contribution to this study. This article marks the first presentation, to the best of current knowledge, of q-spherical fuzzy rough CODAS, providing researchers with a method for scenarios where information is represented by q-spherical fuzzy rough numbers. In the context of industry 4.0, Azim and their team [33] proposed a project prioritization method using the q-SFR analytic hierarchy process in 2023. Similarly, Ali et al. [34] introduced the concept of averaging aggregation operators within the framework of q-ROPFStS in 2023, exploring their applications in multiple attribute decision making (MADM). Recent advancements in decision-making methodologies for sustainable energy and environmental management have led to significant contributions in the field. Narayanamoorthy et al. [40] introduce a novel augmented Fermatean multiple criteria decision making (MCDM) perspective for identifying optimal locations for renewable energy power plants. Parthasarathy et al. [41] propose an end-to-end categorizing strategy for green energy sources using the picture q-rung orthopair fuzzy EXPROM-II: MADA approach. Rouyendegh et al. [37] present an intuitionistic fuzzy TOPSIS method tailored for the green supplier selection problem, addressing key challenges in sustainable supply chain management. Parthasarathy et al. [42] develop an idiosyncratic interval valued picture q-rung orthopair fuzzy decision-making model for electric vehicle battery charging technology selection. Akram and Ashraf [43] delve into multi-criteria group decision-making based on spherical fuzzy rough numbers, offering insights into complex decision scenarios. Manirathinam et al. [44] propose a comprehensive approach for selecting sustainable renewable energy systems for self-sufficient households, integrating Fermatean neutrosophic fuzzy stratified AHP-MARCOS methodology. Menekse and Akdag [45] introduce a novel interval-valued spherical fuzzy CODAS method for reopening readiness evaluation of academic units amidst the COVID-19 pandemic. Akram et al. [46] extend the CODAS method for multiple attribute group decision-making (MAGDM) with 2-tuple linguistic T-spherical fuzzy sets. Gündoğdu and Kahraman [47] focus on the optimal site selection of electric vehicle charging stations using the spherical fuzzy TOPSIS method. Gül and Aydoğdu [48] propose novel entropy measure definitions and their applications in a modified combinative distance-based assessment method under a picture fuzzy environment. Furthermore, research extends to the sustainability prioritization of technologies for cleaning up soils polluted with oil and petroleum products [49], and a spherical fuzzy decision-making method for evaluating the Industrial Internet of Things (IIoT) industry [50]. These studies collectively contribute to the development of robust decision-support systems for sustainable energy and environmental management.

C. MOTIVATION

The motivation behind this article lies in the recognition of q-SFRS offering greater flexibility compared to PFS and SFS in studying decision-making (DM) problems. The article addresses the complexity of MADM problems influenced by imprecise factors within the q-SFRS environment. It highlights the limitations of existing operators and proposes a beyond-state-of-the-art method to overcome these limitations, providing excellent findings for various information categories represented by q-SFRS data. The simplicity and comprehensiveness of CODAS are acknowledged, and the article aims to define CODAS within the context of q-SFRS for addressing challenging decision-making problems, ensuring more accurate and precise results in real-life MADM situations. Azim et al. [26] pioneered the advancement of q-SFRS theory and its application in multi-criteria decision-making (MCDM) problems. The enriched q-SFRS framework, with its broader applicability and adaptability to diverse circumstances, opens avenues for further research and development. These groundbreaking assumptions contribute significantly to knowledge, providing the foundation for comprehensive and imaginative approaches to real-world problem-solving. The integration of q-SFRSs through the CODAS method presents an intriguing potential for analysis in the dynamic field of decision-making. This combination aims to enhance the flexibility and precision of decision-making across various domains. The utilization of q-spherical fuzzy rough CODAS to address uncertainty and hesitancy in real-world decision situations has contributed to its popularity. The flexible background offered by q-SFRS theory, capable of managing and interpreting vague information, facilitates a more representative categorization of complex decision environments. On the other hand, the CODAS ranking is based on the proximity of an alternative to the ideal solution and its isolation from the negative solution, making it a well-established multi-criteria decision analysis approach. The integration of q-SFRSs with CODAS seeks to overcome the limitations of conventional decision-making methods when handling rough and vague data. This study aims to advance decision knowledge by proposing a novel and practical approach for decision-makers to navigate complex decision scenarios. The amalgamation of CODAS with q-SFRSs is anticipated to provide a sophisticated and accurate decision support method, particularly in areas where imprecision, hesitation, and complexity prevail. This stems from the realization that precise and unambiguous data is seldom a characteristic of real-world decision-making circumstances. By leveraging the synergies between q-SFRSs and CODAS, this research endeavors to equip decision-makers with a tool capable of navigating the intricacies and nuances in complex decision-making scenarios. The ultimate objective is to foster a comprehensive and adaptable decision validation framework for the advancement of decision knowledge.

1. Using q-spherical fuzzy rough CODAS to demonstrate how these sets may deal with ambiguity and uncertainty in actual decision-making settings.
2. Describing the adaptable framework provided by q-spherical fuzzy rough set theory for dealing with and modeling misinformation. Emphasizing the need to provide a more genuine picture of the decision's complicated environment.

While CODAS and q-spherical fuzzy rough sets have been individually explored, their integration in this study is novel. This fusion is designed to harness the strengths of both methods to tackle the complex nature of decision-making under uncertainty. By combining CODAS, which is effective in ranking alternatives based on their distances to ideal solutions, with q-SFRS, which adeptly handles uncertainty and vagueness, this study offers a comprehensive decision-making tool not previously available. This integration provides a robust framework capable of more accurately modeling and managing the ambiguity and uncertainty inherent in real-world MCDM problems. The flexibility and adaptability of q-SFRS to represent linguistic terms and handle imprecise data enrich the decision-making process beyond traditional methods. The use of Hamming and Euclidean distances within the q-SFR-CODAS framework is a methodological advancement. By prioritizing Hamming distance as the primary measure and Euclidean distance as the secondary, the approach ensures a nuanced and precise assessment of alternatives, enhancing the robustness of the decision-making process. The development of a novel algorithm to generate renewable energy site selections across all criteria represents a significant contribution. This algorithm is tailored to leverage the q-SFRS framework, ensuring that the decision-making process is both comprehensive and precise in handling uncertainty. The extensive sensitivity analysis conducted to validate the results of the q-SFR-CODAS method underscores the reliability and robustness of the proposed approach. By comparing the results with existing MCDM methods, this study demonstrates the superiority and applicability of our approach in various scenarios. The inclusion of an illustrative example showcasing the practical application of the proposed methodology in renewable energy site selection provides concrete evidence of its effectiveness. This example highlights the method's ability to address real-world decision-making challenges, offering clear advantages over traditional methods. The q-spherical fuzzy rough set environment offers greater flexibility and capability in dealing with uncertainty and hesitancy compared to other fuzzy set theories like PFS and SFS. By leveraging this environment, the study addresses the complexities of decision-making in a more nuanced and representative manner. The study proposes new aggregation operators and defuzzification methods within the q-SFRS framework, paving the way for future research and development in handling diverse information categories. The proposed q-SFR-CODAS method is not a mere generalization but a significant advancement in the field of MCDM. It combines the strengths of q-SFRS and CODAS, introduces novel distance measures and algorithms, and provides a comprehensive framework for addressing uncertainty in decision-making. The extensive validation and practical applications underscore the method's robustness and effectiveness, contributing valuable insights and tools for researchers and practitioners alike.

The subsequent sections of the study are structured as follows:

Section II offers a comprehensive overview of various concepts, including FS, IFS PFS, SFS, q-SFS, RS, and q-SFRS, providing a foundational understanding for the subsequent sections. In section III, we delve into the operational laws governing the q-SFR framework, focusing on key aggregation operators, namely the q-SFR arithmetic mean and q-SFR geometric mean. Additionally, the discussion encompasses the presentation of the score and accuracy functions within the context of q-SFR, providing a comprehensive understanding of their roles in decision-making processes. Section IV goes over the q-SFR CODAS approach in great depth. It delves into the development and practical application of this strategy, particularly when evaluating renewable energy selection. In section V, we present the q-spherical fuzzy rough CODAS approach as a useful tool for evaluating and contrasting renewable energy selection. In Section VI, we look at the managerial implications of the q-SFR CODAS techniques. We stress the relevance of sensitivity analysis in determining the robustness of the methods, as well as the significance of comparative analysis for evaluating multi-criteria decision-making rankings. In Section VII, where we summarize the major findings and stress the study's overall significance, the research ends. The comprehensive structure of the paper is shown in Figure 1.

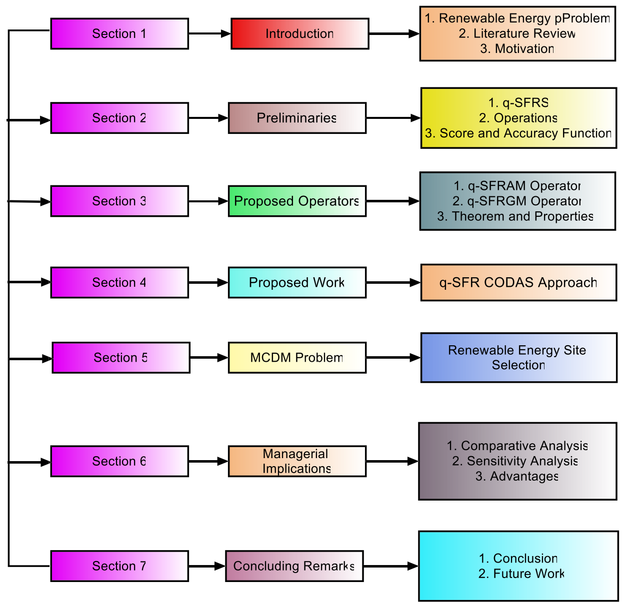

FIGURE 1. Structure of the research article.

II. PRELIMINARIES

In this section, we will look at a variety of mathematical ideas, beginning with an in-depth review of FS, IFS, PFS, SPS, q-SFS, and RS.

Definition 1: In 1965, Zadeh [19] proposed the idea of a fuzzy set as an extension of the conventional crisp set. The formal definition of a fuzzy set can be represented mathematically as follows:

$$\mathcal{A} = \{ \langle \mathfrak{x}, \zeta_{\mathcal{A}}(\mathfrak{x}) \rangle : \mathfrak{x} \in \mathcal{X} \} \tag{1}$$

where $0 \leq \zeta_{\mathcal{A}}(\mathfrak{x}) \leq 1$.

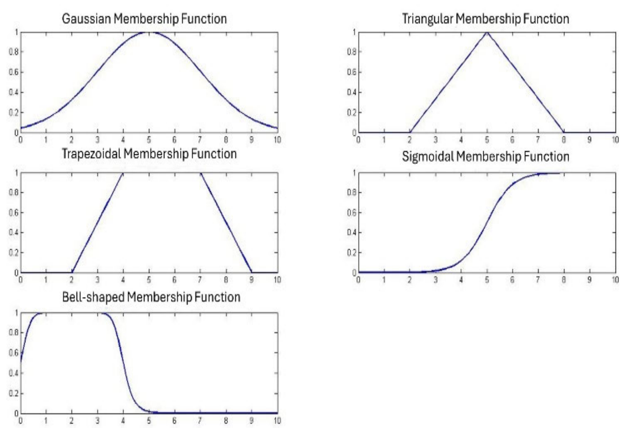

FIGURE 2. Some graphical representations of fuzzy spaces.

Definition 2: In 1986, Atanassov and Stoeva [20] proposed the intuitionistic fuzzy set (IFS) as an extension of the fuzzy set. The formal mathematical representation of an IFS is as follows:

$$\mathcal{A} = \{\langle \mathfrak{x}, \zeta_{\mathcal{A}}(\mathfrak{x}), \xi_{\mathcal{A}}(\mathfrak{x})\rangle : \mathfrak{x} \in \mathcal{X}\} \tag{2}$$

where $0 \leq \zeta_{\mathcal{A}}(\mathfrak{x}) + \xi_{\mathcal{A}}(\mathfrak{x}) \leq 1$.

Definition 3: [28] Let $\mathcal{X}$ be a non-empty finite set. A PyFS $\mathcal{A}$ over $\mathfrak{x} \in \mathcal{X}$ is defined as follows:

$$\mathcal{A} = \{\langle \mathfrak{x}, \zeta_{\mathcal{A}}(\mathfrak{x}), \xi_{\mathcal{A}}(\mathfrak{x})\rangle : \mathfrak{x} \in \mathcal{X}\} \tag{3}$$

where $\zeta_{\mathcal{A}}(\mathfrak{x})$ and $\xi_{\mathcal{A}}(\mathfrak{x})$ represent the MD and NMD of $\mathcal{A}$ respectively such that $\xi_{\mathcal{A}}(\mathfrak{x}), \xi_{\mathcal{A}}(\mathfrak{x}) \in [0, 1]$ and where $0 \leq (\zeta_{\mathcal{A}}(\mathfrak{x}))^2 + (\xi_{\mathcal{A}})^2 \leq 1$.

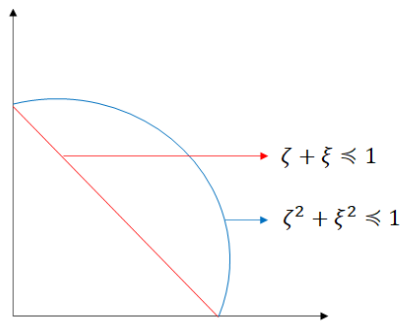

FIGURE 3. A comparison of the differences between Pythagorean and intuitionistic fuzzy spaces.

Definition 4: Building on the fundamental principles of FSs and IFSs, Cuong and his team [21] introduced the idea of a picture fuzzy set in 2014. Its definition can be expressed mathematically as follows:

$$\mathcal{A} = \{\langle \mathfrak{x}, \zeta_{\mathcal{A}}(\mathfrak{x}), \eta_{\mathcal{A}}(\mathfrak{x}), \xi_{\mathcal{A}}(\mathfrak{x})\rangle : \mathfrak{x} \in \mathcal{X}\}\ (4) \tag{4}$$

where $0 \leq \zeta_{\mathcal{A}}(\mathfrak{x}) + \eta_{\mathcal{A}}(\mathfrak{x}) + \xi_{\mathcal{A}}(\mathfrak{x}) \leq 1$.

The following symbols represent the representation of the membership functions for a fuzzy set in this situation, which includes positive, neutral, and negative aspects:

$\zeta_{\mathcal{A}}(\mathfrak{x})(\mathfrak{x}): X \to [0,1]$, $\eta_{\mathcal{A}}(\mathfrak{x}): \mathcal{X} \to [0,1]$ and $\xi_{\mathcal{A}}(\mathfrak{x}): \mathcal{X} \to [0,1]$ respectively.

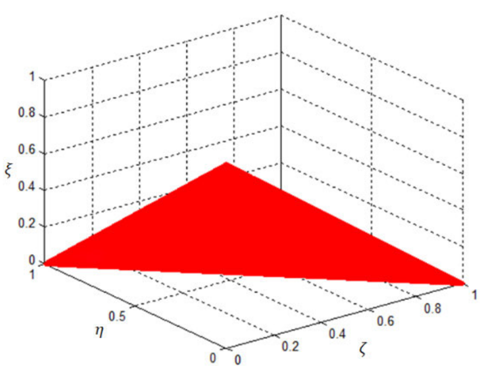

FIGURE 4. Picture membership grade space.

Definition 5: Gündoğdu and Kahraman [22] introduced the idea of a spherical fuzzy set in 2019, further advancing the picture fuzzy set framework. The concept can be expressed in the following way from a mathematical standpoint:

$$\mathcal{A} = \{\langle \mathfrak{x}, \zeta_{\mathcal{A}}(\mathfrak{x})(\mathfrak{x}), \eta_{\mathcal{A}}(\mathfrak{x}), \xi_{\mathcal{A}}(\mathfrak{x})\rangle : \mathfrak{x} \in \mathcal{X}\} \tag{5}$$

where $0 \leq (\zeta_{\mathcal{A}}(\mathfrak{x}))^2 + (\eta_{\mathcal{A}}(\mathfrak{x}))^2 + (\xi_{\mathcal{A}}(\mathfrak{x}))^2 \leq 1$.

Where the positive, neutral, and negative membership function for a fuzzy set is represented by $\zeta_{\mathcal{A}}(\mathfrak{x}): \mathcal{X} \to [0,1]$, $\eta_{\mathcal{A}}(\mathfrak{x}): \mathcal{X} \to [0,1]$ and $\xi_{\mathcal{A}}(\mathfrak{x}): \mathcal{X} \to [0,1]$ respectively.

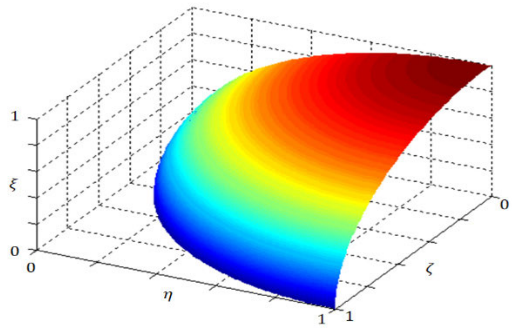

FIGURE 5. The condition $0 \leq (\zeta_{\mathcal{A}}(\mathfrak{x}))^2 + (\eta_{\mathcal{A}}(\mathfrak{x}))^2 + (\xi_{\mathcal{A}}(\mathfrak{x}))^2 \leq 1$ describes a spherical fuzzy set in three-dimensional space.

Definition 6: The idea of a q-SFS was introduced by Kahraman et al. [23] in the year 2020, as an extension of the existing notion of a spherical fuzzy set. Mathematically, the concept may be formally defined in the following manner.

$$\mathcal{A} = \{\langle \mathfrak{x}, \zeta_{\mathcal{A}}(\mathfrak{x})(\mathfrak{x}), \eta_{\mathcal{A}}(\mathfrak{x}), \xi_{\mathcal{A}}(\mathfrak{x})\rangle : \mathfrak{x} \in \mathcal{X}\} \tag{6}$$

Such that $0 \leq (\zeta_{\mathcal{A}}(\mathfrak{x}))^q + (\eta_{\mathcal{A}}(\mathfrak{x}))^q + (\xi_{\mathcal{A}}(\mathfrak{x}))^q \leq 1$ for all $q \geq 1$.

Where $\zeta_{\mathcal{A}}: \mathcal{X} \to [0,1]$, $\eta_{\mathcal{A}}: \mathcal{X} \to [0,1]$ and $\xi_{\mathcal{A}}: \mathcal{X} \to [0,1]$ correspond to the positive, neutral, and negative membership functions, respectively.

Definition 7: Pawlak [24] introduced the notion of RS in back 1982. The definition of rough set is as follows: The triplet $(\mathcal{G}_1, \mathcal{G}_2, \mathfrak{R})$ is referred to as an approximation space

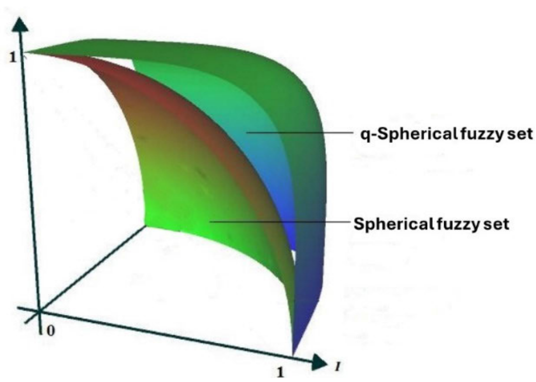

FIGURE 6. Graphical representation of the difference between spherical fuzzy set and q-spherical fuzzy set in three-dimensional space.

when considering an arbitrary binary relation $\mathfrak{R}$ on $\mathcal{G}_1 \times \mathcal{G}_2$. The $\underline{\mathfrak{R}}(\mathcal{A})$ and $\overline{\mathfrak{R}}(\mathcal{A})$ are defined for sets $\mathcal{X} \subseteq \mathcal{G}_1$ and $\boldsymbol{\mathcal{A}} \subseteq \mathcal{G}_2$.

$$\begin{pmatrix} \underline{\mathfrak{R}}(\mathcal{A}) = \{\mathfrak{x} \in \mathcal{G}_1 : [\mathfrak{x}]_{\mathcal{A}} \subseteq \mathcal{X}\} \\ \overline{\mathfrak{R}}(\mathcal{A}) = \left\{\mathfrak{x} \in \mathcal{G}_1 : [\mathfrak{x}]_{\mathcal{A}} \bigcap \mathcal{X} \neq \varphi\right\} \end{pmatrix} \tag{7}$$

where $[\mathfrak{x}]_{\mathcal{A}}$ represents the idea of indiscernibility.

The set $\left(\underline{\mathfrak{R}}(\mathcal{A}), \overline{\mathfrak{R}}(\mathcal{A})\right)$ is sometimes referred to as a rough set.

Definition 8: A q-spherical fuzzy relation $\mathfrak{R}$ in is a q-spherical fuzzy subset of $\mathcal{G}_1 \times \mathcal{G}_2$. and is given by

$$\begin{aligned} \mathfrak{R} = \{&\langle(\mathfrak{w}, x) : \zeta_{\mathfrak{R}}(\mathfrak{w}, x), \eta_{\mathfrak{R}}(\mathfrak{w}, x), \xi_{\mathfrak{R}}(\mathfrak{w}, x)\rangle : \\ &\left((\zeta_{\mathfrak{R}}(\mathfrak{w}, x))^q + (\eta_{\mathfrak{R}}(\mathfrak{w}, x))^q + (\xi_{\mathfrak{R}}(\mathfrak{w}, x))^q\right) \le 1 : \\ &\forall\, \mathfrak{w} \in \mathcal{G}_1, \S \in \mathcal{G}_2\}, \\ &\text{where } \zeta_R : \mathcal{G}_1 \times \mathcal{G}_2 \to [0,1],\ \eta_R : \mathcal{G}_1 \times \mathcal{G}_2 \to [0,1] \\ &\text{and } \xi_R : \mathcal{G}_1 \times \mathcal{G}_2 \to [0,1]. \end{aligned}$$

Definition 9: Assume that $\mathcal{G}_1$ and $\mathcal{G}_2$ are two no-empty sets. Let R be a q-SF relation from $\mathcal{G}_1$to $\mathcal{G}_2$. Then the triplet $(\mathcal{G}_1, \mathcal{G}_2, \mathfrak{R})$ is called q-SF approximation space. Now for any element $\mathfrak{w} \in \mathcal{G}_1$, the lower and upper approximation space of $\mathfrak{w}$ w.r.t approximation space $(\mathcal{G}_1, \mathcal{G}_2, \mathfrak{R})$ are presented and given as:

$$\mathcal{A} = (\underline{\mathcal{A}}, \overline{\mathcal{A}}) = \left\{\mathfrak{w}, \begin{pmatrix} \underline{\zeta}_{\mathcal{A}}(\mathfrak{w}), \underline{\zeta}_{\mathcal{A}}(\mathfrak{w}), \underline{\zeta}_{\mathcal{A}}(\mathfrak{w}), \\ \bar{\zeta}_{\mathcal{A}}(\mathfrak{w}), \bar{\zeta}_{\mathcal{A}}(\mathfrak{w}), \bar{\zeta}_{\mathcal{A}}(\mathfrak{w}) \end{pmatrix} : \mathfrak{w} \in \mathcal{G}_1\right\} \tag{8}$$

where,

$$\begin{aligned} \underline{\zeta}_{\mathcal{A}}(\mathfrak{w}) &= \bigwedge\nolimits_{x \in \mathcal{G}_2} \left\{\zeta_R(\mathfrak{w}, x) \bigwedge \zeta_A(x)\right\}, \\ \underline{\eta}_{\mathcal{A}}(\mathfrak{w}) &= \bigvee\nolimits_{x \in \mathcal{G}_2} \left\{\eta_R(\mathfrak{w}, x) \bigvee \eta_A(x)\right\}, \\ \underline{\xi}_{\mathcal{A}}(\mathfrak{w}) &= \bigvee\nolimits_{x \in \mathcal{G}_2} \left\{\xi_R(\mathfrak{w}, x) \bigvee \xi_A(x)\right\}, \\ \bar{\zeta}_{\mathcal{A}}(\mathfrak{w}) &= \bigvee\nolimits_{x \in \mathcal{G}_2} \left\{\zeta_R(\mathfrak{w}, x) \bigvee \zeta_A(x)\right\}, \\ \bar{\eta}_{\mathcal{A}}(\mathfrak{w}) &= \bigwedge\nolimits_{x \in U_2} \left\{\eta_R(\mathfrak{w}, x) \bigwedge \eta_A(x)\right\}, \\ \bar{\xi}_{\mathcal{A}}(\mathfrak{w}) &= \bigwedge\nolimits_{x \in \mathcal{G}_2} \left\{\xi_R(\mathfrak{w}, x) \bigwedge \xi_A(x)\right\}, \end{aligned}$$

with the condition that

$$\left(0 \le \underline{\zeta}_{\mathcal{A}}{}^q(\mathfrak{w}) + \underline{\eta}_{\mathcal{A}}{}^q(\mathfrak{w}) + \underline{\xi}_{\mathcal{A}}{}^q(\mathfrak{w}) \le 1\right)\Big) \text{ and } \left(0 \le \bar{\zeta}_{\mathcal{A}}{}^q(\mathfrak{w}) + \bar{\eta}_{\mathcal{A}}{}^q(\mathfrak{w}) + \bar{\xi}_{\mathcal{A}}{}^q(\mathfrak{w}) \le 1\right).$$

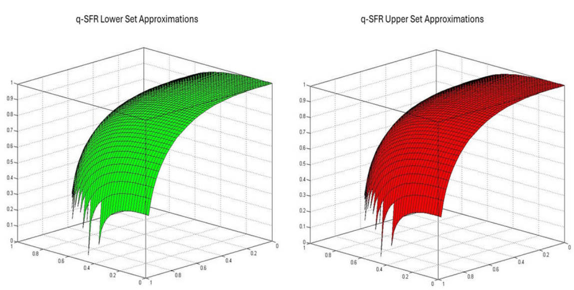

FIGURE 7. **Graphical representation of q-spherical fuzzy rough set-in three-dimensional space.**

The q-SFRS is defined as a pair of q-SFSs, where $\underline{A}$ is distinct from $\overline{A}$. To facilitate comprehension, we will denote the given concept as $\mathcal{A} = (\underline{\mathcal{A}}, \overline{\mathcal{A}})$, which is referred to as a q -spherical fuzzy rough number. The notation $\mathcal{A}_i$ represents the set that encompasses all q-SFR numbers.

III. OPERATIONAL LAWS AND AGGREGATION OPERATORS FOR $q - SFRNS$

Operational laws and aggregation operators play a crucial role in q-spherical fuzzy rough sets (q-SFRNs) by defining how different fuzzy rough sets interact and how information is aggregated or combined. Here's a brief overview of these concepts: The union of two q-SFRNs combines their elements, resulting in a set that contains all elements from both sets. In the context of q-SFRNs, this operation is essential for merging information from different sources or perspectives. The intersection of two q-SFRNs identifies the common elements between them. It helps in finding the overlapping information or shared characteristics between different sets. The algebraic sum operation combines the membership values of corresponding elements from two q-SFRNs. It provides a way to aggregate membership information from different sources while preserving their contributions. The algebraic product operation combines the membership values of corresponding elements from two q-SFRNs by taking their product. It emphasizes the joint membership of elements in both sets. The average operator calculates the mean membership value of the elements being aggregated. It provides a balanced view by considering the collective contribution of all sources. These operational laws and aggregation operators provide the foundational framework for performing operations and aggregating information within the context of q-spherical fuzzy rough sets. By understanding and utilizing these concepts effectively, researchers can analyze complex datasets and make informed decisions in various applications, such as decision-making, pattern recognition, and data analysis.

Definition 3.1: Let $\boldsymbol{\mathcal{A}}_1 = \left(\underline{\zeta}_1, \underline{\eta}_1, \underline{\xi}_1, \bar{\zeta}_1, \bar{\eta}_1, \bar{\xi}_1\right)$ and $\mathcal{A}_2 = \left(\underline{\zeta}_2, \underline{\eta}_2, \underline{\xi}_2, \bar{\zeta}_2, \bar{\eta}_2, \bar{\xi}_2\right)$ are two $\boldsymbol{q}$-SFRNs in $(\boldsymbol{\mathcal{G}}_1, \boldsymbol{\mathcal{G}}_2, \boldsymbol{\mathfrak{R}})$, then $\boldsymbol{\mathcal{A}}_1 \cap \boldsymbol{\mathcal{A}}_2 =$

$$\left[\left\langle \begin{matrix} \mathbf{min}\left(\underline{\zeta}_1, \underline{\zeta}_2\right), \mathbf{max}\left(\underline{\eta}_1, \underline{\eta}_2\right), \\ \mathbf{min}\left\{\begin{matrix} 1 - \left(\left(\mathbf{min}\left(\underline{\zeta}_1, \underline{\zeta}_2\right)\right)^q + \left(\mathbf{max}\left(\underline{\eta}_1, \underline{\eta}_2\right)\right)^q\right), \\ \mathbf{min}\left(\underline{\xi}_1, \underline{\xi}_2\right) \end{matrix}\right\} \\ \mathbf{min}\left(\bar{\zeta}_1, \underline{\zeta}_2\right), \mathbf{max}\left(\bar{\eta}_1, \bar{\eta}_2\right), \\ \mathbf{min}\left\{\begin{matrix} 1 - \left(\left(\mathbf{min}\left(\underline{\zeta}_1, \bar{\zeta}_2\right)\right)^q + \left(\mathbf{max}\left(\bar{\eta}_1, \bar{\eta}_2\right)\right)^q\right), \\ \mathbf{min}\left(\bar{\xi}_1, \bar{\xi}_2\right) \end{matrix}\right\} \end{matrix} \right\rangle\right] \tag{9}$$

Definition 3.2: Let $\mathcal{A}_1 = \left(\underline{\zeta}_1, \underline{\eta}_1, \underline{\xi}_1, \bar{\zeta}_1, \bar{\eta}_1, \bar{\xi}_1\right)$ and $\mathcal{A}_2 = \left(\underline{\zeta}_2, \underline{\eta}_2, \underline{\xi}_2, \bar{\zeta}_2, \bar{\eta}_2, \bar{\xi}_2\right)$ are two $\boldsymbol{q}$-SFRNs in

$(\mathcal{G}_1, \mathcal{G}, \mathcal{G})$, then $\overline{\mathcal{A}}_1 \cup \overline{\mathcal{A}}_2 =$

$$\left[\begin{array}{c} \left\langle \begin{array}{c} \mathbf{max}\left(\underline{\zeta}_1, \underline{\zeta}_2\right), \mathbf{min}\left(\underline{\eta}_1, \underline{\eta}_2\right), \\ \mathbf{max}\left\{\begin{array}{c} 1-\left(\left(\mathbf{max}\left(\underline{\zeta}_1, \underline{\zeta}_2\right)\right)^q + \left(\mathbf{min}\left(\underline{\eta}_1, \underline{\eta}_2\right)\right)^q\right), \\ \mathbf{max}\left(\underline{\xi}_1, \underline{\xi}_2\right) \end{array}\right\} \end{array}\right\rangle \\ \mathbf{max}\left(\bar{\zeta}_1, \underline{\zeta}_2\right), \mathbf{min}\left(\bar{\eta}_1, \bar{\eta}_2\right), \\ \mathbf{max}\left\{\begin{array}{c} 1-\left(\left(\mathbf{max}\left(\bar{\zeta}_1, \bar{\zeta}_2\right)\right)^q + \left(\mathbf{min}\left(\bar{\eta}_1, \bar{\eta}_2\right)\right)^q\right), \\ \mathbf{max}\left(\bar{\xi}_1, \bar{\xi}_2\right) \end{array}\right\} \end{array}\right] \tag{10}$$

Definition 3.3: Let $\mathcal{A}_1 = \left(\underline{\zeta}_1, \underline{\eta}_1, \underline{\xi}_1, \bar{\zeta}_1, \bar{\eta}_1, \bar{\xi}_1\right)$ and $\mathcal{A}_2 = \left(\underline{\zeta}_2, \underline{\eta}_2, \underline{\xi}_2, \bar{\zeta}_2, \bar{\eta}_2, \bar{\xi}_2\right)$ are two q-SFRNs in $(\mathcal{G}_1, \mathcal{G}_2, \mathfrak{R})$, then

$$\mathcal{A}_1 \oplus \mathcal{A}_2 = \left[\begin{array}{c} \sqrt[q]{\underline{\zeta}_1^q + \underline{\zeta}_2^q - \underline{\zeta}_1^q * \underline{\zeta}_2^q}, \underline{\eta}_1^q * \underline{\eta}_2^q, \\ \sqrt[q]{\left(1 - \underline{\zeta}_2^q * \underline{\xi}_1^q + \left(1 - \underline{\zeta}_1^q * \underline{\xi}_2^q\right) - \underline{\xi}_1^q * \underline{\xi}_2^q\right.}, \\ \sqrt[q]{\bar{\zeta}_1^q + \bar{\zeta}_2^q - \bar{\zeta}_1^q * \bar{\zeta}_2^q}, \bar{\eta}_1^q * \bar{\eta}_2^q, \end{array}\right] \tag{11}$$

Definition 3.4: Let $\mathcal{A}_1 = \left(\underline{\zeta}_1, \underline{\eta}_1, \underline{\xi}_1, \bar{\zeta}_1, \bar{\eta}_1, \bar{\xi}_1\right)$ and $\mathcal{A}_2 = \left(\underline{\zeta}_2, \underline{\eta}_2, \underline{\xi}_2, \bar{\zeta}_2, \bar{\eta}_2, \bar{\xi}_2\right)$ are two q-SFRNs in $(\mathcal{G}_1, \mathcal{G}_2, \mathfrak{R})$, then

$$\mathcal{A}_1 \otimes \mathcal{A}_2 = \left[\left\langle\begin{array}{c} \underline{\zeta}_1^q * \underline{\zeta}_2^q, \quad \sqrt[q]{\underline{\eta}_1^q + \underline{\eta}_2^q - \underline{\eta}_1^q * \underline{\eta}_2^q}, \\ \sqrt[q]{\left(1 - \underline{\eta}_2^q * \underline{\xi}_1^q + 1 - \underline{\eta}_1^q * \underline{\xi}_2^q\right) - \underline{\xi}_1^q * \underline{\xi}_2^q}, \\ \bar{\zeta}_1^q * \bar{\zeta}_2^q, \quad \sqrt[q]{\bar{\eta}_1^q + \bar{\eta}_2^q - \bar{\eta}_1^q * \bar{\eta}_2^q}, \\ \sqrt[q]{\left(1 - \bar{\eta}_2^q * \bar{\xi}_1^q + 1 - \bar{\eta}_1^q * \bar{\xi}_2^q\right) - \bar{\xi}_1^q * \bar{\xi}_2^q} \end{array}\right\rangle\right] \tag{12}$$

Definition 3.5: $\mathcal{A} = (\underline{\zeta}, \underline{\eta}, \underline{\xi}, \overline{\zeta}, \overline{\eta}, \overline{\xi})$ be any $q-$SFRNs in $(\mathcal{G}_1, \mathcal{G}_2, \mathfrak{R})$, and $\omega > 0$ be any scaler then,

$$\omega\mathcal{A} = \left[\left\langle\begin{array}{c} \sqrt[q]{1-(1-\underline{\zeta}^q)^\omega}, \ \underline{\eta}^{q\omega}, \ \sqrt[q]{(1-\underline{\zeta}^q)^\omega - \left(1-\underline{\zeta}^q-\underline{\xi}^q\right)^\omega}, \\ \sqrt[q]{1-(1-\bar{\zeta}^q)^\omega}, \ \bar{\eta}^{q\omega}, \ \sqrt[q]{(1-\bar{\zeta}^q)^\omega - \left(1-\bar{\zeta}^q-\bar{\xi}^q\right)^\omega} \end{array}\right\rangle\right] \tag{13}$$

Definition 3.6: $\mathcal{A} = (\underline{\zeta}, \underline{\eta}, \underline{\xi}, \overline{\zeta}, \overline{\eta}, \overline{\xi})$ be any $q-$SFRNs in $(\mathcal{G}_1, \mathcal{G}_2, \mathfrak{R})$, and $\omega > 0$ be any scaler then,

$$\mathcal{A}^\omega = \left[\left\langle\begin{array}{c} \underline{\zeta}^{q\omega}, \ \sqrt[q]{1-(1-\underline{\eta}^q)^\omega}, \ \sqrt[q]{(1-\underline{\eta}^q)^\omega - \left(1-\underline{\eta}^q-\underline{\xi}^q\right)^\omega}, \\ \bar{\zeta}^{q\omega}, \ \sqrt[q]{1-(1-\bar{\eta}^q)^\omega}, \ \sqrt[q]{(1-\bar{\eta}^q)^\omega - \left(1-\bar{\eta}^q-\bar{\xi}^q\right)^\omega} \end{array}\right\rangle\right]$$

Definition 3.7: Let $\mathcal{A} = (\underline{\zeta}, \underline{\eta}, \underline{\xi}, \overline{\zeta}, \overline{\eta}, \overline{\xi})$ be a $q-$SFRN. Then the score value which is denoted as A_Q can be determined by the following function.

$$Sco(\mathcal{A}) = \frac{2 + \left(\underline{\zeta}\right)^q + (\bar{\zeta})^q - \left(\underline{\eta}\right)^q - (\bar{\eta})^q - \left(\underline{\xi}\right)^q - (\bar{\xi})^q}{3} \tag{14}$$

where,

$$0 \le Sco\left(\mathcal{A}\right) \le 1.$$

Definition 3.8: Let $\mathcal{A} = (\underline{\zeta}, \underline{\eta}, \underline{\xi}, \overline{\zeta}, \overline{\eta}, \overline{\xi})$ be a $q-$SFRN. The accuracy of $\mathcal{A}$ is calculated by using the formula mentioned in Equation No. 10.

$$Acc(\mathcal{A}) = \frac{(\underline{\zeta})^q + (\bar{\zeta})^q - (\underline{\xi})^q - (\bar{\xi})^q}{2} \tag{15}$$

where $-1 \le Acc\left(\mathcal{A}\right) \le 1$.

Definition 3.9: Let $\mathcal{A}_1 = (\underline{\zeta}_1, \underline{\eta}_1, \underline{\xi}_1, \overline{\zeta}_1, \overline{\eta}_1, \overline{\xi}_1)$ and $\mathcal{A}_2 = (\underline{\zeta}_2, \underline{\eta}_2, \underline{\xi}_2, \overline{\zeta}_2, \overline{\eta}_2, \overline{\xi}_2)$ are two $q-$SFRNs, then

1. If $Sco(\mathcal{A}_1) < Sco(\mathcal{A}_2)$ then $\mathcal{A}_1 < \mathcal{A}_2$,
2. If $Sco(\mathcal{A}_1) > Sco(\mathcal{A}_2)$ then $\mathcal{A}_1 > \mathcal{A}_2$,
3. If $Sco\left(\mathcal{A}_1\right) = Sco(\mathcal{A}_2)$ then
   - If $Acc\left(\mathcal{A}_1\right) < Acc\left(\mathcal{A}_2\right)$ then $\mathcal{A}_1 < \mathcal{A}_2$,
   - If $Acc\left(\mathcal{A}_1\right) > Acc\left(\mathcal{A}_2\right)$ then $\mathcal{A}_1 > \mathcal{A}_2$,
   - If $Acc\left(\mathcal{A}_1\right) = Acc\left(\mathcal{A}_2\right)$ then $\mathcal{A}_1 = \mathcal{A}_2$.

Definition 3.10: Let $\mathcal{A}_1 = (\underline{\zeta}_1, \underline{\eta}_1, \underline{\xi}_1, \overline{\zeta}_1, \overline{\eta}_1, \overline{\xi}_1)$ and $\mathcal{A}_2 = (\underline{\zeta}_2, \underline{\eta}_2, \underline{\xi}_2, \overline{\zeta}_2, \overline{\eta}_2, \overline{\xi}_2)$ and $\mathcal{A} = (\underline{\zeta}, \underline{\eta}, \underline{\xi}, \overline{\zeta}, \overline{\eta}, \overline{\xi})$ be any three $q-$SFRNs, and ω, ω_1 and ω_2 are any positive integers then the following properties are held.

1. $\mathcal{A}_1 \oplus \mathcal{A}_2 = \mathcal{A}_2 \oplus \mathcal{A}_1$,
2. $\mathcal{A}_1 \otimes \mathcal{A}_2 = \mathcal{A}_2 \otimes \mathcal{A}_1$
3. $\omega\left(\mathcal{A}_1 \oplus \mathcal{A}_2\right) = \omega\mathcal{A}_1 \oplus \omega\mathcal{A}_2$,
4. $\omega_1\mathcal{A} \oplus \omega_2\mathcal{A} = \left(\omega_1 + \omega_2\right)\mathcal{A}$,
5. $\left(\mathcal{A}_1 \otimes \mathcal{A}_2\right)^\omega = \mathcal{A}_1^\omega \otimes \mathcal{A}_2^\omega$,
6. $\mathcal{A}^{\omega_1} \otimes \mathcal{A}^{\omega_2} = \mathcal{A}^{\omega_1 + \omega_2}$.

Definition 3.11: q-SFR Arithmetic Mean (q-SFRAM) operator concerning, $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)$; $\omega_i \in [0,1]$; $\sum_{i=1}^n \omega_i = 1$, q-SFRAM operator is mathematically defined as

$$\begin{aligned} &\text{q} - \text{SFRAM}_\omega\left(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_n\right) \\ &\quad = \omega_1\mathcal{A}_1 \oplus \omega_2\mathcal{A}_2 \oplus \omega_3\mathcal{A}_3 \oplus, \ldots, \oplus\omega_n\mathcal{A}_n \\ &\quad = \left[\left\langle\begin{array}{c} \sqrt[q]{\prod_{i=1}^n(1-(1-\underline{\zeta}_i^q)^{\omega_i})}, \ \prod_{i=1}^n \underline{\eta}_i^{q\omega_i}, \\ \sqrt[q]{\prod_{i=1}^n(1-\underline{\zeta}_i^q)^{\omega_i} - \prod_{i=1}^n\left(1-\underline{\zeta}_i^q-\underline{\xi}_i^q\right)^{\omega_i}}, \\ \sqrt[q]{\prod_{i=1}^n(1-(1-\bar{\zeta}_i^q)^{\omega_i})}, \ \prod_{i=1}^n \bar{\eta}_i^{q\omega_i}, \\ \sqrt[q]{\prod_{i=1}^n(1-\bar{\zeta}_i^q)^{\omega_i} - \prod_{i=1}^n\left(1-\bar{\zeta}_i^q-\bar{\xi}_i^q\right)^{\omega_i}} \end{array}\right\rangle\right] \end{aligned} \tag{16}$$

Definition 3.12: q-SFR Geometric Mean (q-SFRGM) operator concerning, $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)$; $\omega_i \in [0,1]$;

$\sum_{i=1}^{n}\omega_i = 1$, q-SFRGM operator is mathematically defined as

$$
\begin{aligned}
&q-\mathrm{SFRGM}_{\omega}\left(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_n\right)\\
&\quad = \omega_1\mathcal{A}_1 \otimes \omega_2\mathcal{A}_2 \otimes \omega_3\mathcal{A}_3 \otimes, \ldots, \otimes \omega_n\mathcal{A}_n\\
&\quad = \left[\left\langle\begin{array}{c}
\prod_{i=1}^{n}\underline{\zeta}_i^{q\omega_i}, \sqrt[q]{\prod_{i=1}^{n}(1-(1-\underline{\eta}_i^q)^{\omega_i})},\\
\sqrt[q]{\prod_{i=1}^{n}(1-\underline{\eta}_i^q)^{\omega_i}-\prod_{i=1}^{n}\left(1-\underline{\eta}_i^q-\underline{\xi}_i^q\right)^{\omega_i}},\\
\prod_{i=1}^{n}\overline{\zeta}_i^{q\omega_i}, \sqrt[q]{\prod_{i=1}^{n}(1-(1-\overline{\eta}_i^q)^{\omega_i})},\\
\sqrt[q]{\prod_{i=1}^{n}(1-\overline{\eta}_i^q)^{\omega_i}-\prod_{i=1}^{n}\left(1-\overline{\eta}_i^q-\overline{\xi}_i^q\right)^{\omega_i}}
\end{array}\right\rangle\right]
\end{aligned}
\tag{17}
$$

Theorem 1: Assuming $\mathcal{A}_1 = (\underline{\zeta}_1, \underline{\eta}_1, \underline{\xi}_1, \overline{\zeta}_1, \overline{\eta}_1, \overline{\xi}_1)$ and $\mathcal{A}_2 = (\underline{\zeta}_2, \underline{\eta}_2, \underline{\xi}_2, \overline{\zeta}_2, \overline{\eta}_2, \overline{\xi}_2)$ be any two $q-$SFRNs, then $\mathcal{A}_1 \oplus \mathcal{A}_2 = \mathcal{A}_2 \oplus \mathcal{A}_1$.

Proof: Let $\mathcal{A}_1 = (\underline{\zeta}_1, \underline{\eta}_1, \underline{\xi}_1, \overline{\zeta}_1, \overline{\eta}_1, \overline{\xi}_1)$ and $\mathcal{A}_2 = (\underline{\zeta}_2, \underline{\eta}_2, \underline{\xi}_2, \overline{\zeta}_2, \overline{\eta}_2, \overline{\xi}_2)$ be any two $q-$SFRNs. We are to prove $\mathcal{A}_1 \oplus \mathcal{A}_2 = \mathcal{A}_2 \oplus \mathcal{A}_1$.

For this, let

$$
\begin{aligned}
&\mathcal{A}_1 \oplus \mathcal{A}_2\\
&\quad = (\underline{\zeta}_1, \underline{\eta}_1, \underline{\xi}_1, \overline{\zeta}_1, \overline{\eta}_1, \overline{\xi}_1) \oplus (\underline{\zeta}_2, \underline{\eta}_2, \underline{\xi}_2, \overline{\zeta}_2, \overline{\eta}_2, \overline{\xi}_2)\\
&\quad = \left[\left\langle\begin{array}{c}
\sqrt[q]{\underline{\zeta}_1^q+\underline{\zeta}_2^q-\underline{\zeta}_1^q*\underline{\zeta}_2^q}, \underline{\eta}_1^q*\underline{\eta}_2^q,\\
\sqrt[q]{\left(1-\underline{\zeta}_2^q*\underline{\xi}_1^q+1-\underline{\zeta}_1^q*\underline{\xi}_2^q\right)-\underline{\xi}_1^q*\underline{\xi}_2^q},\\
\sqrt[q]{\overline{\zeta}_1^q+\overline{\zeta}_2^q-\overline{\zeta}_1^q*\overline{\zeta}_2^q}, \overline{\eta}_1^q*\overline{\eta}_2^q,\\
\sqrt[q]{\left(1-\overline{\zeta}_2^q*\overline{\xi}_1^q+1-\overline{\zeta}_1^q*\overline{\xi}_2^q\right)-\overline{\xi}_1^q*\overline{\xi}_2^q}
\end{array}\right\rangle\right]\\
&\quad = \left[\left\langle\begin{array}{c}
\sqrt[q]{\underline{\zeta}_2^q+\underline{\zeta}_1^q-\underline{\zeta}_2^q*\underline{\zeta}_1^q}, \underline{\eta}_2^q*\underline{\eta}_1^q,\\
\sqrt[q]{\left(1-\underline{\xi}_1^q*\underline{\zeta}_2^q+1-\underline{\xi}_2^q*\underline{\zeta}_1^q\right)-\underline{\xi}_2^q*\underline{\xi}_1^q},\\
\sqrt[q]{\overline{\zeta}_2^q+\overline{\zeta}_1^q-\overline{\zeta}_2^q*\overline{\zeta}_1^q}, \overline{\eta}_2^q*\overline{\eta}_1^q,\\
\sqrt[q]{\left(1-\overline{\xi}_1^q*\overline{\zeta}_2^q+1-\overline{\xi}_2^q*\overline{\zeta}_1^q\right)-\overline{\xi}_2^q*\overline{\xi}_1^q}
\end{array}\right\rangle\right]\\
&\quad = \mathcal{A}_2 \oplus \mathcal{A}_1.
\end{aligned}
$$

Theorem 2: Assuming $\mathcal{A}_1 = (\underline{\zeta}_1, \underline{\eta}_1, \underline{\xi}_1, \overline{\zeta}_1, \overline{\eta}_1, \overline{\xi}_1)$ and $\mathcal{A}_2 = (\underline{\zeta}_2, \underline{\eta}_2, \underline{\xi}_2, \overline{\zeta}_2, \overline{\eta}_2, \overline{\xi}_2)$ be any two $q-$SFRNs, then $\mathcal{A}_1 \otimes \mathcal{A}_2 = \mathcal{A}_2 \otimes \mathcal{A}_1$.

Proof: Let $\mathcal{A}_1 = (\underline{\zeta}_1, \underline{\eta}_1, \underline{\xi}_1, \overline{\zeta}_1, \overline{\eta}_1, \overline{\xi}_1)$ and $\mathcal{A}_2 = (\underline{\zeta}_2, \underline{\eta}_2, \underline{\xi}_2, \overline{\zeta}_2, \overline{\eta}_2, \overline{\xi}_2)$ be any two $q-$SFRNs. We are to prove $\mathcal{A}_1 \otimes \mathcal{A}_2 = \mathcal{A}_2 \otimes \mathcal{A}_1$.

For this, let

$$
\begin{aligned}
&\mathcal{A}_1 \oplus \mathcal{A}_2\\
&\quad = (\underline{\zeta}_1, \underline{\eta}_1, \underline{\xi}_1, \overline{\zeta}_1, \overline{\eta}_1, \overline{\xi}_1)\\
&\qquad \otimes (\underline{\zeta}_2, \underline{\eta}_2, \underline{\xi}_2, \overline{\zeta}_2, \overline{\eta}_2, \overline{\xi}_2)\\
&\quad = \left[\left\langle\begin{array}{c}
\underline{\zeta}_1^q*\underline{\zeta}_2^q, \sqrt[q]{\underline{\eta}_1^q+\underline{\eta}_2^q-\underline{\eta}_1^q*\underline{\eta}_2^q},\\
\sqrt[q]{\left(1-\underline{\eta}_2^q*\underline{\xi}_1^q+1-\underline{\eta}_1^q*\underline{\xi}_2^q\right)-\underline{\xi}_1^q*\underline{\xi}_2^q},\\
\overline{\zeta}_1^q*\overline{\zeta}_2^q, \sqrt[q]{\overline{\eta}_1^q+\overline{\eta}_2^q-\overline{\eta}_1^q*\overline{\eta}_2^q},\\
\sqrt[q]{\left(1-\overline{\eta}_2^q*\overline{\xi}_1^q+1-\overline{\eta}_1^q*\overline{\xi}_2^q\right)-\overline{\xi}_1^q*\overline{\xi}_2^q}
\end{array}\right\rangle\right]\\
&\quad = \left[\left\langle\begin{array}{c}
\underline{\zeta}_2^q*\underline{\zeta}_1^q, \sqrt[q]{\underline{\eta}_2^q+\underline{\eta}_1^q-\underline{\eta}_2^q*\underline{\eta}_1^q},\\
\sqrt[q]{\left(1-\underline{\xi}_1^q*\underline{\eta}_2^q+1-\underline{\xi}_2^q*\underline{\eta}_1^q\right)-\underline{\xi}_2^q*\underline{\xi}_1^q},\\
\overline{\zeta}_2^q*\overline{\zeta}_1^q, \sqrt[q]{\overline{\eta}_2^q+\overline{\eta}_1^q-\overline{\eta}_2^q*\overline{\eta}_1^q},\\
\sqrt[q]{\left(1-\overline{\xi}_1^q*\overline{\eta}_2^q+1-\overline{\xi}_2^q*\overline{\eta}_1^q\right)-\overline{\xi}_2^q*\overline{\xi}_1^q}
\end{array}\right\rangle\right]\\
&\quad = \mathcal{A}_2 \otimes \mathcal{A}_1.
\end{aligned}
$$

Theorem 3: Assuming $\mathcal{A}_1 = (\underline{\zeta}_1, \underline{\eta}_1, \underline{\xi}_1, \overline{\zeta}_1, \overline{\eta}_1, \overline{\xi}_1)$ and $\mathcal{A}_2 = (\underline{\zeta}_2, \underline{\eta}_2, \underline{\xi}_2, \overline{\zeta}_2, \overline{\eta}_2, \overline{\xi}_2)$ be any two $q-$SFRNs and ω be any positive integer then $\omega(\mathcal{A}_1 \oplus \mathcal{A}_2) = \omega\mathcal{A}_1 \oplus \omega\mathcal{A}_2$.

Proof: Let $\mathcal{A}_1 = (\underline{\zeta}_1, \underline{\eta}_1, \underline{\xi}_1, \overline{\zeta}_1, \overline{\eta}_1, \overline{\xi}_1)$ and $\mathcal{A}_2 = (\underline{\zeta}_2, \underline{\eta}_2, \underline{\xi}_2, \overline{\zeta}_2, \overline{\eta}_2, \overline{\xi}_2)$ be any two $q-$SFRNsand ω be any positive integer. We are to prove

$\omega(\mathcal{A}_1 \oplus \mathcal{A}_2) = \omega\mathcal{A}_1 \oplus \omega\mathcal{A}_2$. For this, let

$$
\begin{aligned}
&\omega\left(\mathcal{A}_1 \oplus \mathcal{A}_2\right)\\
&\quad = \omega\left(\underline{\zeta}_1, \underline{\eta}_1, \underline{\xi}_1, \overline{\zeta}_1, \overline{\eta}_1, \overline{\xi}_1\right) \oplus \omega\left(\underline{\zeta}_2, \underline{\eta}_2, \underline{\xi}_2, \overline{\zeta}_2, \overline{\eta}_2, \overline{\xi}_2\right).
\end{aligned}
$$

$$
\begin{aligned}
&\omega\left(\mathcal{A}_1 \oplus \mathcal{A}_2\right)\\
&\quad = \omega\left[\left\langle\begin{array}{c}
\sqrt[q]{\underline{\zeta}_1^q+\underline{\zeta}_2^q-\underline{\zeta}_1^q*\underline{\zeta}_2^q}, \underline{\eta}_1^q*\underline{\eta}_2^q,\\
\sqrt[q]{\left(1-\underline{\zeta}_2^q*\underline{\xi}_1^q+1-\underline{\zeta}_1^q*\underline{\xi}_2^q\right)-\underline{\xi}_1^q*\underline{\xi}_2^q},\\
\sqrt[q]{\overline{\zeta}_1^q+\overline{\zeta}_2^q-\overline{\zeta}_1^q*\overline{\zeta}_2^q}, \overline{\eta}_1^q*\overline{\eta}_2^q,\\
\sqrt[q]{\left(1-\overline{\zeta}_2^q*\overline{\xi}_1^q+1-\overline{\zeta}_1^q*\overline{\xi}_2^q\right)-\overline{\xi}_1^q*\overline{\xi}_2^q}
\end{array}\right\rangle\right]\\
&\quad = \left[\left\langle\begin{array}{c}
\sqrt[q]{1-(1-(\underline{\zeta}_1^q+\underline{\zeta}_2^q-\underline{\zeta}_1^q*\underline{\zeta}_2^q))^{\omega}}, \underline{\eta}_1^{q\omega} * \underline{\eta}_2^{q\omega},\\
\sqrt[q]{\left(\begin{array}{c}(1-(\underline{\zeta}_1^q+\underline{\zeta}_2^q-\underline{\zeta}_1^q*\underline{\zeta}_2^q)^{\omega}-\\ (1-(\underline{\zeta}_1^q+\underline{\zeta}_2^q-\underline{\zeta}_1^q*\underline{\zeta}_2^q))-\\ ((1-\underline{\zeta}_2^q*\underline{\xi}_1^q+1-\underline{\zeta}_1^q*\underline{\xi}_2^q)-\underline{\xi}_1^q*\underline{\xi}_1^q)\end{array}\right)^{\omega}},\\
\sqrt[q]{1-\overline{\zeta}_1^q+\overline{\zeta}_2^q-\overline{\zeta}_1^q*\overline{\zeta}_2^q)^{\omega}}, \overline{\eta}_1^{q\omega} * \overline{\eta}_2^{q\omega},\\
\sqrt[q]{\left(\begin{array}{c}(1-(\overline{\zeta}_1^q+\overline{\zeta}_2^q-\overline{\zeta}_1^q*\overline{\zeta}_2^q)^{\omega}-\\ (1-(\overline{\zeta}_1^q+\overline{\zeta}_2^q-\overline{\zeta}_1^q*\overline{\zeta}_2^q))-\\ ((1-\overline{\zeta}_2^q*\overline{\xi}_1^q+1-\overline{\zeta}_1^q*\overline{\xi}_2^q)-\overline{\xi}_1^q*\overline{\xi}_2^q)\end{array}\right)^{\omega}}
\end{array}\right\rangle\right].
\end{aligned}
$$

$$
\begin{aligned}
&\omega\mathcal{A}_1 \oplus \omega\mathcal{A}_2\\
&\quad = \left[\left\langle\begin{array}{c}
\sqrt[q]{1-(1-\underline{\zeta}_1^q)^{\omega}},\ \underline{\eta}_1^{q\omega},\ \sqrt[q]{(1-\underline{\zeta}_1^q)^{\omega}-\left(1-\underline{\zeta}_1^q-\underline{\xi}_1^q\right)^{\omega}},\\
\sqrt[q]{1-(1-\overline{\zeta}_1^q)^{\omega}},\ \overline{\eta}_1^{q\omega},\ \sqrt[q]{(1-\overline{\zeta}_1^q)^{\omega}-\left(1-\overline{\zeta}_1^q-\overline{\xi}_1^q\right)^{\omega}}
\end{array}\right\rangle\right]
\end{aligned}
$$

$$\oplus \left[\left\langle \begin{array}{c} \sqrt[q]{1-(1-\underline{\zeta}_2^q)^{\omega}},\ \underline{\eta}_2^{q\omega},\ \sqrt[q]{(1-\underline{\zeta}_2^q)^{\omega}-\left(1-\underline{\zeta}_2^q-\underline{\xi}_2^q\right)^{\omega}}, \\ \sqrt[q]{1-(1-\overline{\zeta}_2^q)^{\omega}},\ \overline{\eta}_2^{q\omega},\ \sqrt[q]{(1-\overline{\zeta}_2^q)^{\omega}-\left(1-\overline{\zeta}_2^q-\overline{\xi}_2^q\right)^{\omega}} \end{array}\right\rangle\right]$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\begin{array}{c}\left(1-\left(1-\underline{\zeta}_1^q\right)^{\omega}\right)+\left(1-\left(1-\underline{\zeta}_2^q\right)^{\omega}\right)- \\ \left(1-\left(1-\underline{\zeta}_1^q\right)^{\omega}\right)*\left(1-\left(1-\underline{\zeta}_2^q\right)^{\omega}\right)\end{array}}, \\ \underline{\eta}_1^{q\omega} * \underline{\eta}_2^{q\omega}, \\ \sqrt[q]{\begin{array}{c}\left(\left(1-\underline{\zeta}_1^q\right)^{\omega}-\left(1-\underline{\zeta}_1^q-\underline{\xi}_1^q\right)^{\omega}\right)+ \\ ((1-\underline{\zeta}_2^q)^{\omega}-(1-\underline{\zeta}_2^q-\underline{\xi}_2^q)^{\omega})- \\ \left(\left(1-\underline{\zeta}_1^q\right)^{\omega}-\left(1-\underline{\zeta}_1^q-\underline{\xi}_1^q\right)^{\omega}\right)* \\ ((1-\underline{\zeta}_2^q)^{\omega}-(1-\underline{\zeta}_2^q-\underline{\xi}_2^q)^{\omega})\end{array}}, \\ \sqrt[q]{\begin{array}{c}\left(1-\left(1-\overline{\zeta}_1^q\right)^{\omega}\right)+\left(1-\left(1-\overline{\zeta}_2^q\right)^{\omega}\right)- \\ \left(1-\left(1-\overline{\zeta}_1^q\right)^{\omega}\right)*\left(1-\left(1-\overline{\zeta}_2^q\right)^{\omega}\right)\end{array}}, \\ \overline{\eta}_1^{q\omega} * \overline{\eta}_2^{q\omega}, \\ \sqrt[q]{\begin{array}{c}\left(\left(1-\overline{\zeta}_1^q\right)^{\omega}-\left(1-\overline{\zeta}_1^q-\overline{\xi}_1^q\right)^{\omega}\right)+ \\ ((1-\overline{\zeta}_2^q)^{\omega}-(1-\overline{\zeta}_2^q-\overline{\xi}_2^q)^{\omega})- \\ \left(\left(1-\overline{\zeta}_1^q\right)^{\omega}-\left(1-\overline{\zeta}_1^q-\overline{\xi}_1^q\right)^{\omega}\right)* \\ ((1-\overline{\zeta}_2^q)^{\omega}-(1-\overline{\zeta}_2^q-\overline{\xi}_2^q)^{\omega})\end{array}} \end{array}\right\rangle\right]$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\left(1-\left(1-\underline{\zeta}_1^q\right)^{\omega}*\left(1-\underline{\zeta}_2^q\right)^{\omega}\right)},\ \underline{\eta}_1^{q\omega} * \underline{\eta}_2^{q\omega}, \\ \sqrt[q]{\begin{array}{c}(\left(1-\underline{\zeta}_1^q\right)^{\omega}*\left(1-\underline{\zeta}_2^q\right)^{\omega}- \\ (1-\underline{\zeta}_1^q-\underline{\xi}_1^q)^{\omega}*(1-\overline{\zeta}_2^q-\underline{\xi}_2^q)^{\omega})\end{array}} \\ \sqrt[q]{\left(1-\left(1-\overline{\zeta}_1^q\right)^{\omega}*\left(1-\overline{\zeta}_2^q\right)^{\omega}\right)},\ \overline{\eta}^{q\ \omega}_1 * \overline{\eta}_2^{q\ \omega}, \\ \sqrt[q]{\begin{array}{c}(\left(1-\overline{\zeta}_1^q\right)^{\omega}*\left(1-\overline{\zeta}_2^q\right)^{\omega}- \\ (1-\overline{\zeta}_1^q-\underline{\xi}_1^q)^{\omega}*(1-\overline{\zeta}_2^q-\underline{\xi}_2^q)^{\omega})\end{array}} \end{array}\right\rangle\right]$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{1-(1-(\underline{\zeta}_1^q+\underline{\zeta}_2^q-\underline{\zeta}_1^q*\underline{\zeta}_2^q))^{\omega}},\ \underline{\eta}_1^{q\omega} * \underline{\eta}_2^{q\omega}, \\ \sqrt[q]{\begin{array}{c}(1-(\underline{\zeta}_1^q+\underline{\zeta}_2^q-\underline{\zeta}_1^q*\underline{\zeta}_2^q)^{\omega}- \\ \left(\begin{array}{c}(1-(\underline{\zeta}_1^q+\underline{\zeta}_2^q-\underline{\zeta}_1^q*\underline{\zeta}_2^q))- \\ ((1-\underline{\zeta}_2^q*\underline{\xi}_1^q+1-\underline{\zeta}_1^q*\underline{\xi}_2^q)-\underline{\xi}_1^q*\underline{\xi}_1^q)\end{array}\right)^{\omega}\end{array}}, \\ \sqrt[q]{1-\overline{\zeta}_1^q+\overline{\zeta}_2^q-\overline{\zeta}_1^q*\overline{\zeta}_2^q)^{\omega}},\ \overline{\eta}_1^{q\omega} * \overline{\eta}_2^{q\omega}, \\ \sqrt[q]{\begin{array}{c}(1-(\overline{\zeta}_1^q+\overline{\zeta}_2^q-\overline{\zeta}_1^q*\overline{\zeta}_2^q)^{\omega}- \\ \left(\begin{array}{c}(1-(\overline{\zeta}_1^q+\overline{\zeta}_2^q-\overline{\zeta}_1^q*\overline{\zeta}_2^q))- \\ ((1-\overline{\zeta}_2^q*\overline{\xi}_1^q+1-\overline{\zeta}_1^q*\overline{\xi}_2^q)-\overline{\xi}_1^q*\overline{\xi}_2^q)\end{array}\right)^{\omega}\end{array}} \end{array}\right\rangle\right]$$

Theorem 4: Let $\mathcal{A} = (\underline{\zeta}, \underline{\eta}, \underline{\xi}, \overline{\zeta}, \overline{\eta}, \overline{\xi})$ be any $q-$SFRN, and ω_1 and ω_2 are any positive integers, then

$$\omega_1\mathcal{A} \oplus \omega_2\mathcal{A} = (\omega_1 + \omega_2)\,\mathcal{A}.$$

Proof: Let $\mathcal{A} = (\underline{\zeta}, \underline{\eta}, \underline{\xi}, \overline{\zeta}, \overline{\eta}, \overline{\xi})$ be any $q-$SFRN, and ω_1 and ω_2 are any positive integers, we are to show that $\omega_1\mathcal{A} \oplus \omega_2\mathcal{A} = (\omega_1 + \omega_2)\,\mathcal{A}$.

For this, let

$\omega_1\mathcal{A} \oplus \omega_2\mathcal{A}$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{1-(1-\underline{\zeta}^q)^{\omega_1}},\ \underline{\eta}^{q\omega_1},\ \sqrt[q]{(1-\underline{\zeta}^q)^{\omega_1}-\left(1-\underline{\zeta}^q-\underline{\xi}^q\right)^{\omega_1}}, \\ \sqrt[q]{1-(1-\overline{\zeta}^q)^{\omega_1}},\ \overline{\eta}^{q\omega_1},\ \sqrt[q]{(1-\underline{\zeta}^q)^{\omega_1}-\left(1-\underline{\zeta}^q-\underline{\xi}^q\right)^{\omega_1}} \end{array}\right\rangle\right]$$

$$\oplus \left[\left\langle \begin{array}{c} \sqrt[q]{1-(1-\underline{\zeta}^q)^{\omega_2}},\ \underline{\eta}^{q\omega_2},\ \sqrt[q]{(1-\underline{\zeta}^q)^{\omega_2}-\left(1-\underline{\zeta}^q-\underline{\xi}^q\right)^{\omega_2}}, \\ \sqrt[q]{1-(1-\overline{\zeta}^q)^{\omega_2}},\ \overline{\eta}^{q\omega_2},\ \sqrt[q]{(1-\underline{\zeta}^q)^{\omega_2}-\left(1-\overline{\zeta}^q-\overline{\xi}\right)^{\omega_2}} \end{array}\right\rangle\right]$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\begin{array}{c}\left(1-\left(1-\underline{\zeta}^q\right)^{\omega_1}\right)+\left(1-\left(1-\underline{\zeta}^q\right)^{\omega_2}\right)- \\ \left(1-\left(1-\underline{\zeta}^q\right)^{\omega_1}\right)*\left(1-\left(1-\underline{\zeta}^q\right)^{\omega_2}\right)\end{array}}, \\ \underline{\eta}^{q\omega_1+\omega_2}, \\ \sqrt[q]{\begin{array}{c}\left(\left(1-\underline{\zeta}^q\right)^{\omega_1}-\left(1-\underline{\zeta}^q-\underline{\xi}^q\right)^{\omega_1}\right)+ \\ ((1-\underline{\zeta}^q)^{\omega_2}-(1-\underline{\zeta}^q-\underline{\xi}^q)^{\omega_2})- \\ \left(\left(1-\underline{\zeta}^q\right)^{\omega_1}-\left(1-\underline{\zeta}^q-\underline{\xi}^q\right)^{\omega_1}\right)* \\ ((1-\underline{\zeta}^q)^{\omega_2}-(1-\underline{\zeta}^q-\underline{\xi}^q)^{\omega_2})\end{array}}, \\ \sqrt[q]{\begin{array}{c}\left(1-\left(1-\overline{\zeta}^q\right)^{\omega_1}\right)+\left(1-\left(1-\overline{\zeta}^q\right)^{\omega_2}\right)- \\ \left(1-\left(1-\overline{\zeta}^q\right)^{\omega_1}\right)*\left(1-\left(1-\overline{\zeta}^q\right)^{\omega_2}\right)\end{array}}, \\ \overline{\eta}^{q\omega_1+\omega_2}, \\ \sqrt[q]{\begin{array}{c}\left(\left(1-\overline{\zeta}^q\right)^{\omega_1}-\left(1-\overline{\zeta}^q-\overline{\xi}^q\right)^{\omega_1}\right)+ \\ ((1-\overline{\zeta}^q)^{\omega_2}-(1-\overline{\zeta}^q-\overline{\xi}^q)^{\omega_2})- \\ \left(\left(1-\overline{\zeta}^q\right)^{\omega_1}-\left(1-\overline{\zeta}^q-\overline{\xi}^q\right)^{\omega_1}\right)* \\ ((1-\overline{\zeta}^q)^{\omega_2}-(1-\overline{\zeta}^q-\overline{\xi}^q)^{\omega_2})\end{array}} \end{array}\right\rangle\right]$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{1-(1-\underline{\zeta}^q)^{\omega_1+\omega_2}}, \\ \underline{\eta}^{q\omega_1+\omega_2}, \\ \sqrt[q]{(1-\underline{\zeta}^q)^{\omega_1+\omega_2}-(1-\underline{\zeta}^q-\underline{\xi}^q)^{\omega_1+\omega_2}}, \\ \sqrt[q]{1-(1-\overline{\zeta}^q)^{\omega_1+\omega_2}}, \\ \overline{\eta}^{q\omega_1+\omega_2}, \\ \sqrt[q]{(1-\overline{\zeta}^q)^{\omega_1+\omega_2}-(1-\overline{\zeta}^q-\overline{\xi}^q)^{\omega_1+\omega_2}} \end{array}\right\rangle\right]$$

$= (\omega_1 + \omega_2)\,\mathcal{A}$.

Theorem 5: Let $\mathcal{A}_1 = (\underline{\zeta}_1, \underline{\eta}_1, \underline{\xi}_1, \overline{\zeta}_1, \overline{\eta}_1, \overline{\xi}_1)$ and $\mathcal{A}_2 = (\underline{\zeta}_2, \underline{\eta}_2, \underline{\xi}_2, \overline{\zeta}_2, \overline{\eta}_2, \overline{\xi}_2)$ and $\mathcal{A} = (\underline{\zeta}, \underline{\eta}, \underline{\xi}, \overline{\zeta}, \overline{\eta}, \overline{\xi})$ be any three $q-$SFRNs, and ω are any positive integer, then the $(\mathcal{A}_1 \otimes \mathcal{A}_2)^{\omega} = \mathcal{A}_1^{\omega} \otimes \mathcal{A}_2^{\omega}$.

Proof: Let $\mathcal{A} = (\underline{\zeta}, \underline{\eta}, \underline{\xi}, \overline{\zeta}, \overline{\eta}, \overline{\xi})$ be any $q-$SFRN, and ω_1 and ω_2 are any positive integers, we are to show that $(\mathcal{A}_1 \otimes \mathcal{A}_2)^{\omega} = \mathcal{A}_1^{\omega} \otimes \mathcal{A}_2^{\omega}$.

For this, let

$(\mathcal{A}_1 \otimes \mathcal{A}_2)^{\omega}$

$$= (\mathcal{A}_1 \otimes \mathcal{A}_2)^{\omega}$$

$$= \left[\left\langle \begin{array}{c} \underline{\zeta}_1^q * \underline{\zeta}_2^q, \sqrt[q]{\underline{\eta}_1^q + \underline{\eta}_2^q - \underline{\eta}_1^q * \underline{\eta}_2^q}, \\ \sqrt[q]{\left(1 - \underline{\eta}_2^q * \underline{\xi}_1^q + (1 - \underline{\eta}_1^q * \underline{\xi}_2^q\right) - \underline{\xi}_1^q * \underline{\xi}_2^q}, \\ \overline{\zeta}_1^q * \overline{\zeta}_2^q, \sqrt[q]{\overline{\eta}_1^q + \overline{\eta}_2^q - \overline{\eta}_1^q * \overline{\eta}_2^q}, \\ \sqrt[q]{\left(1 - \overline{\eta}_2^q * \overline{\xi}_1^q + 1 - \overline{\eta}_1^q * \overline{\xi}_2^q\right) - \overline{\xi}_1^q * \overline{\xi}_2^q} \end{array} \right\rangle \right]^{\omega}$$

$$= \left[\left\langle \begin{array}{c} \underline{\zeta}_1^{q\omega} * \underline{\zeta}_2^{q\omega}, \sqrt[q]{1 - \left(1 - \left(\underline{\eta}_1^q + \underline{\eta}_2^q - \underline{\eta}_1^q * \underline{\eta}_2^q\right)\right)^{\omega}}, \\ \sqrt[q]{\left\{ \begin{array}{c} (1 - (\underline{\eta}_1^q + \underline{\eta}_2^q - \underline{\eta}_1^q * \underline{\eta}_2^q)^{\omega} - \\ 1 - \left(\underline{\eta}_1^q + \underline{\eta}_2^q - \underline{\eta}_1^q * \underline{\eta}_2^q\right) - \\ (\left(1 - \underline{\eta}_2^q * \underline{\xi}_1^q + (1 - \underline{\eta}_1^q * \underline{\xi}_2^q\right) - \underline{\xi}_1^q * \underline{\xi}_2^q) \end{array} \right\}^{\omega}}, \\ \overline{\zeta}_1^{q\omega} * \overline{\zeta}_2^{q\omega}, \sqrt[q]{1 - (1 - (\overline{\eta}_1^q + \overline{\eta}_2^q - \overline{\eta}_1^q * \overline{\eta}_2^q))^{\omega}}, \\ \sqrt[q]{\left\{ \begin{array}{c} (1 - (\overline{\eta}_1^q + \overline{\eta}_2^q - \overline{\eta}_1^q * \overline{\eta}_2^q))^{\omega} - \\ 1 - (\overline{\eta}_1^q + \overline{\eta}_2^q - \overline{\eta}_1^q * \overline{\eta}_2^q) - \\ (\sqrt[q]{\left(1 - \overline{\eta}_2^q * \overline{\xi}_1^q + 1 - \overline{\eta}_1^q * \overline{\xi}_2^q\right) - \overline{\xi}_1^q * \overline{\xi}_2^q}) \end{array} \right\}^{\omega}} \end{array} \right\rangle \right]$$

$$= \left[\left\langle \begin{array}{c} \underline{\zeta}_1^{q\omega} * \underline{\zeta}_2^{q\omega}, \\ \sqrt[q]{(1 - (1 - \underline{\eta}_1^q))^{\omega} * (1 - (1 - \underline{\eta}_2^q))^{\omega}}, \\ \sqrt[q]{\left(1 - \underline{\eta}_1^q - \underline{\xi}_1^q\right)^{\omega} * \left(1 - \underline{\eta}_2^q - \underline{\xi}_2^q\right)^{\omega}}, \\ \overline{\zeta}_1^{q\omega} * \overline{\zeta}_2^{q\omega}, \\ \sqrt[q]{(1 - (1 - \overline{\eta}_1^q))^{\omega} * (1 - (1 - \overline{\eta}_2^q))^{\omega}}, \\ \sqrt[q]{\sqrt[q]{(1 - \overline{\eta}_1^q - \overline{\xi}_1^q)^{\omega} * \left(1 - \overline{\xi}_2^q - \underline{\xi}_2^q\right)^{\omega}}} \end{array} \right\rangle \right]$$

$$= \left[\left\langle \begin{array}{c} \underline{\zeta}_1^{q\omega} * \underline{\zeta}_2^{q\omega}, \\ \sqrt[q]{\begin{array}{c}(1 - (1 - \underline{\eta}_1^q)^{\omega}) + (1 - (1 - \underline{\eta}_2^q)^{\omega}) - \\ (1 - (1 - \underline{\eta}_1^q)^{\omega}) * (1 - (1 - \underline{\eta}_2^q)^{\omega})\end{array}}, \\ \sqrt[q]{\begin{array}{c}((1 - \underline{\eta}_1^q) - (1 - \underline{\eta}_1^q - \underline{\xi}_1^q)) + \\ ((1 - \underline{\eta}_2^q) - (1 - \underline{\eta}_2^q - \underline{\xi}_2^q)) - \\ ((1 - \underline{\eta}_1^q) - (1 - \underline{\eta}_1^q - \underline{\xi}_1^q)) * \\ ((1 - \underline{\eta}_2^q) - (1 - \underline{\eta}_2^q - \underline{\xi}_2^q))\end{array}}, \\ \overline{\zeta}_1^{q\omega} * \overline{\zeta}_2^{q\omega}, \\ \sqrt[q]{\begin{array}{c}(1 - (1 - \overline{\eta}_1^q)) + (1 - (1 - \overline{\eta}_2^q)) - \\ (1 - (1 - \overline{\eta}_1^q)) * (1 - (1 - \overline{\eta}_2^q))\end{array}}, \\ \sqrt[q]{\begin{array}{c}((1 - \overline{\eta}_1^q) - (1 - \overline{\eta}_1^q - \overline{\xi}_1^q)) + \\ ((1 - \overline{\eta}_2^q) - (1 - \overline{\eta}_2^q - \overline{\xi}_2^q)) - \\ ((1 - \overline{\eta}_1^q) - (1 - \overline{\eta}_1^q - \overline{\xi}_1^q)) * \\ ((1 - \overline{\eta}_2^q) - (1 - \overline{\eta}_2^q - \overline{\xi}_2^q))\end{array}} \end{array} \right\rangle \right]$$

$$= \left[\left\langle \begin{array}{c} \underline{\zeta}_1^{q\omega}, \sqrt[q]{1 - (1 - \underline{\eta}_1^q)^{\omega}}, \sqrt[q]{\begin{array}{c}\left(1 - \underline{\eta}_1^q\right)^{\omega} - \\ \left(1 - \underline{\eta}_1^q - \underline{\xi}_1^q\right)^{\omega}\end{array}}, \\ \overline{\zeta}_1^{q\omega}, \sqrt[q]{1 - (1 - \overline{\eta}_1^q)^{\omega}}, \sqrt[q]{\begin{array}{c}(1 - \overline{\eta}_1^q)^{\omega} - \\ \left(1 - \overline{\eta}_1^q - \overline{\xi}_1^q\right)^{\omega}\end{array}} \end{array} \right\rangle \right]$$

$$\otimes$$

$$\left[\left\langle \begin{array}{c} \underline{\zeta}_2^{q\omega}, \sqrt[q]{1 - (1 - \underline{\eta}_2^q)^{\omega}}, \sqrt[q]{\begin{array}{c}\left(1 - \underline{\eta}_2^q\right)^{\omega} - \\ \left(1 - \underline{\eta}_2^q - \underline{\xi}_2^q\right)^{\omega}\end{array}}, \\ \overline{\zeta}_2^{q\omega}, \sqrt[q]{1 - (1 - \overline{\eta}_2^q)^{\omega}}, \sqrt[q]{\begin{array}{c}(1 - \overline{\eta}_2^q)^{\omega} - \\ \left(1 - \overline{\eta}_2^q - \overline{\xi}_{12}^q\right)^{\omega}\end{array}} \end{array} \right\rangle \right]$$
$$= \mathcal{A}_1^{\omega} \otimes \mathcal{A}_2^{\omega}.$$

Theorem 6: Let $\mathcal{A} = (\underline{\zeta}, \underline{\eta}, \underline{\xi}, \overline{\zeta}, \overline{\eta}, \overline{\xi})$ be any three $q-$SFRN, ω_1 and ω_2 any positive integers, then $(\mathcal{A}_1 \otimes \mathcal{A}_2)^{\omega} = \mathcal{A}_1^{\omega} \otimes \mathcal{A}_2^{\omega}$.

Proof: Let $\mathcal{A} = (\underline{\zeta}, \underline{\eta}, \underline{\xi}, \overline{\zeta}, \overline{\eta}, \overline{\xi})$ be any $q-$SFRN, and ω_1 and ω_2 are any positive integers, we are to show that $\mathcal{A}^{\omega_1} \otimes \mathcal{A}^{\omega_2} = \mathcal{A}^{\omega_1 + \omega_2}$.

For this, let

$$\mathcal{A}^{\omega_1} \otimes \mathcal{A}^{\omega_2}$$
$$= \left[\left\langle \begin{array}{c} \underline{\zeta}^{q\omega_1}, \sqrt[q]{1 - (1 - \underline{\eta}^q)^{\omega_1}}, \sqrt[q]{(1 - \underline{\eta}^q)^{\omega_1} - \left(1 - \underline{\eta}^q - \underline{\xi}^q\right)^{\omega_1}}, \\ \overline{\zeta}^{q\omega_1}, \sqrt[q]{1 - (1 - \overline{\eta}^q)^{\omega_1}}, \sqrt[q]{(1 - \overline{\eta}^q)^{\omega_1} - \left(1 - \overline{\eta}^q - \overline{\xi}^q\right)^{\omega_1}} \end{array} \right\rangle \right]$$
$$\otimes$$
$$\left[\left\langle \begin{array}{c} \underline{\zeta}^{q\omega_2}, \sqrt[q]{1 - (1 - \underline{\eta}^q)^{\omega_2}}, \sqrt[q]{(1 - \underline{\eta}^q)^{\omega_2} - \left(1 - \underline{\eta}^q - \underline{\xi}^q\right)^{\omega_2}}, \\ \overline{\zeta}^{q\omega_2}, \sqrt[q]{1 - (1 - \overline{\eta}^q)^{\omega_2}}, \sqrt[q]{(1 - \overline{\eta}^q)^{\omega_2} - \left(1 - \overline{\eta}^q - \overline{\xi}^q\right)^{\omega_2}} \end{array} \right\rangle \right]$$

$$\left[\left\langle \begin{array}{c} \underline{\zeta}^{q\omega_1 + \omega_2}, \\ \sqrt[q]{\begin{array}{c}\left(1 - \left(1 - \underline{\eta}^q\right)^{\omega_1}\right) + \left(1 - \left(1 - \underline{\eta}^q\right)^{\omega_2}\right) - \\ \left(1 - \left(1 - \underline{\eta}^q\right)^{\omega_1}\right) * \left(1 - \left(1 - \underline{\eta}^q\right)^{\omega_2}\right)\end{array}}, \\ \sqrt[q]{\begin{array}{c}\left(\left(1 - \underline{\eta}^q\right)^{\omega_1} - \left(1 - \underline{\eta}^q - \underline{\xi}^q\right)^{\omega_1}\right) + \\ ((1 - \underline{\eta}^q)^{\omega_2} - (1 - \underline{\eta}^q - \underline{\xi}^q)^{\omega_2}) - \\ \left(\left(1 - \underline{\eta}^q\right)^{\omega_1} - \left(1 - \underline{\eta}^q - \underline{\xi}^q\right)^{\omega_1}\right) * \\ ((1 - \underline{\eta}^q)^{\omega_2} - (1 - \underline{\eta}^q - \underline{\xi}^q)^{\omega_2})\end{array}}, \\ \overline{\zeta}^{q\omega_1 + \omega_2}, \\ \sqrt[q]{\begin{array}{c}\left(1 - (1 - \overline{\eta}^q)^{\omega_1}\right) + \left(1 - (1 - \overline{\eta}^q)^{\omega_2}\right) - \\ \left(1 - (1 - \overline{\eta}^q)^{\omega_1}\right) * \left(1 - (1 - \overline{\eta}^q)^{\omega_2}\right)\end{array}}, \\ \sqrt[q]{\begin{array}{c}\left((1 - \overline{\eta}^q)^{\omega_1} - \left(1 - \overline{\eta}^q - \overline{\xi}^q\right)^{\omega_1}\right) + \\ ((1 - \overline{\eta}^q)^{\omega_2} - (1 - \overline{\eta}^q - \overline{\xi}^q)^{\omega_2}) - \\ \left((1 - \overline{\eta}^q)^{\omega_1} - \left(1 - \overline{\eta}^q - \overline{\xi}^q\right)^{\omega_1}\right) * \\ ((1 - \overline{\eta}^q)^{\omega_2} - (1 - \overline{\eta}^q - \overline{\xi}^q)^{\omega_2})\end{array}} \end{array} \right\rangle \right]$$

$$= \left[\left\langle \begin{array}{c} \underline{\zeta}^{q\omega_1 + \omega_2}, \\ \sqrt[q]{1 - (1 - \underline{\eta}^q)^{\omega_1 + \omega_2}}, \\ \sqrt[q]{(1 - \underline{\eta}^q)^{\omega_1 + \omega_2} - (1 - \underline{\eta}^q - \underline{\xi}^q)^{\omega_1 + \omega_2}}, \\ \overline{\zeta}^{q\omega_1 + \omega_2}, \\ \sqrt[q]{1 - (1 - \overline{\eta}^q)^{\omega_1 + \omega_2}}, \\ \sqrt[q]{(1 - \overline{\eta}^q)^{\omega_1 + \omega_2} - (1 - \overline{\eta}^q - \overline{\xi}^q)^{\omega_1 + \omega_2}} \end{array} \right\rangle \right]$$
$$= \mathcal{A}^{\omega_1 + \omega_2}.$$

Property 1: If all q-SFRNs $\mathcal{A}_i = \mathcal{A}$, then $\mathrm{q-SFRAM}_\omega(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_n) = \mathcal{A}$.

Property 2: If all q-SFRNs $\mathcal{A}_i = \mathcal{A}$, then $\mathrm{q-SFRGM}_\omega(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_n) = \mathcal{A}$.

Property 3: Let $\mathcal{A}_i = (\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i)$ $(i = 1, 2, 3, \ldots, n)$ be a collection of $q-$SFRNs in $(\mathcal{G}_1, \mathcal{G}_2, \mathfrak{R})$. For

$$\mathcal{A}^- = \left\langle \min \underline{\zeta}_i, \max \underline{\eta}_i, \max \underline{\xi}_i, \min \overline{\zeta}_i, \max \overline{\eta}_i, \max \overline{\xi}_i \right\rangle \text{ and}$$

$$\mathcal{A}^+ = \left\langle \max \underline{\zeta}_i, \min \underline{\eta}_i, \min \underline{\xi}_i, \max \overline{\zeta}_i, \min \overline{\eta}_i, \min \overline{\xi}_i \right\rangle, \text{ then}$$

$$\mathcal{A}^- \leq \mathrm{q-SFRAM}_\omega(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_n) \leq \mathcal{A}^+$$

Property 4: Let $\mathcal{A}_i = (\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i)$ $(i = 1, 2, 3, \ldots, n)$ be a collection of $q-$SFRNs in $(\mathcal{G}_1, \mathcal{G}_2, \mathfrak{R})$. For

$$\mathcal{A}^- = \left\langle \min \underline{\zeta}_i, \max \underline{\eta}_i, \max \underline{\xi}_i, \min \overline{\zeta}_i, \max \overline{\eta}_i, \max \overline{\xi}_i \right\rangle \text{ and}$$

$$\mathcal{A}^+ = \left\langle \max \underline{\zeta}_i, \min \underline{\eta}_i, \min \underline{\xi}_i, \max \overline{\zeta}_i, \min \overline{\eta}_i, \min \overline{\xi}_i \right\rangle, \text{ then}$$

$$\mathcal{A}^- \leq \mathrm{q-SFRGM}_\omega(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_n) \leq \mathcal{A}^+$$

Property 5: Assuming $\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right)$ $(i = 1, 2, \ldots, n)$ and$\mathcal{A}_i{}^* = \left(\underline{\zeta}_i{}^*, \underline{\eta}_i{}^*, \underline{\xi}_i{}^*, \overline{\zeta}_i{}^*, \overline{\eta}_i{}^*, \overline{\xi}_i{}^*\right)$ $(i = 1, 2, \ldots, n)$ be a collection of two $q-SFRNs$ such that $\mathcal{A}_i \leq \mathcal{A}_i{}^*$ for all i, then

$$\begin{aligned} &\mathrm{q-SFRAM}_\omega(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\ &\quad \leq \mathrm{q-SFRAM}_\omega\left(\mathcal{A}_1{}^*, \mathcal{A}_2{}^*, \ldots, \mathcal{A}_n{}^*\right). \end{aligned}$$

Property 6: Assuming $\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right)$ $(i = 1, 2, \ldots, n)$ and $\mathcal{A}_i{}^* = \left(\underline{\zeta}_i{}^*, \underline{\eta}_i{}^*, \underline{\xi}_i{}^*, \overline{\zeta}_i{}^*, \overline{\eta}_i{}^*, \overline{\xi}_i{}^*\right)$ $(i = 1, 2, \ldots, n)$ be a collection of two $q-SFRNs$ such that $\mathcal{A}_i \leq \mathcal{A}_i{}^*$ for all i, then

$$\begin{aligned} &\mathrm{q-SFRGM}_\omega(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\ &\quad \leq \mathrm{q-SFRGM}_\omega\left(\mathcal{A}_1{}^*, \mathcal{A}_2{}^*, \ldots, \mathcal{A}_n{}^*\right). \end{aligned}$$

Theorem 7: Assuming $\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right)$ $\left(i = 1, 2, \ldots, n\right)$ ne a collection of q-SFRNs concerning, $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)$; $\omega_i \in [0,1]$; $\sum_{i=1}^n \omega_i = 1$, the q-spherical fuzzy rough arithmetic mean operator $\mathrm{q-SFRAM}_\omega$ is defined as a mapping $\mathrm{q-SFRAM}_\omega : \mathcal{A}^n \longrightarrow \mathcal{A}$ characterized by

$$\mathrm{q-SFRAM}_\omega(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \oplus_{i=1}^n (\omega_i \mathcal{A}_i)$$

$$\begin{aligned} \mathrm{q-SFRAM}_\omega(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_n) &= \omega_1\mathcal{A}_1 \oplus \omega_2\mathcal{A}_2 \\ &\quad \oplus \omega_3\mathcal{A}_3 \oplus, \ldots, \oplus \omega_n\mathcal{A}_n \end{aligned}$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\prod_{i=1}^n (1-(1-\underline{\zeta}_i^q)^{\omega_i})},\ \prod_{i=1}^n \underline{\eta}_i^{q\omega_i}, \\ \sqrt[q]{\prod_{i=1}^n (1-\underline{\zeta}_i^q)^{\omega_i} - \prod_{i=1}^n \left(1-\underline{\zeta}_i^q-\underline{\xi}_i^q\right)^{\omega_i}}, \\ \sqrt[q]{\prod_{i=1}^n (1-(1-\overline{\zeta}_i^q)^{\omega_i})},\ \prod_{i=1}^n \overline{\eta}_i^{q\omega_i}, \\ \sqrt[q]{\prod_{i=1}^n (1-\overline{\zeta}_i^q)^{\omega_i} - \prod_{i=1}^n \left(1-\overline{\zeta}_i^q-\overline{\xi}_i^q\right)^{\omega_i}} \end{array} \right\rangle \right]$$

Proof: We will prove Theorem (7) by using mathematical induction.

Step 1: For $n = 2$, we have

For this, let

$$\omega_1\mathcal{A}_1 \oplus \omega_2\mathcal{A}_2$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{1-(1-\underline{\zeta}_1^q)^{\omega_1}},\ \underline{\eta}_1^{q\omega_1}, \sqrt[q]{(1-\underline{\zeta}_1^q)^{\omega_1} - \left(1-\underline{\zeta}_1^q-\underline{\xi}_1^q\right)^{\omega_1}}, \\ \sqrt[q]{1-(1-\overline{\zeta}_1^q)^{\omega_1}}, \overline{\eta}_1^{q\omega_1}, \sqrt[q]{(1-\overline{\zeta}_1^q)^{\omega_1} - \left(1-\underline{\zeta}_1^q-\overline{\xi}_1^q\right)^{\omega_1}} \end{array} \right\rangle \right]$$

$$\oplus$$

$$\left[\left\langle \begin{array}{c} \sqrt[q]{1-(1-\underline{\zeta}_2^q)^{\omega_2}}, \underline{\eta}_2^{q\omega_2}, \sqrt[q]{(1-\underline{\zeta}_2^q)^{\omega_2} - \left(1-\underline{\zeta}_2^q-\underline{\xi}_2^q\right)^{\omega_2}}, \\ \sqrt[q]{1-(1-\overline{\zeta}_2^q)^{\omega_2}},\ \overline{\eta}_2^{q\omega_2},\ \sqrt[q]{(1-\overline{\zeta}_2^q)^{\omega_2} - \left(1-\underline{\zeta}_2^q-\overline{\xi}_2^q\right)^{\omega_2}} \end{array} \right\rangle \right]$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\begin{array}{c} \left(1-\left(1-\underline{\zeta}_1^q\right)^{\omega_1}\right) + \left(1-\left(1-\underline{\zeta}_2^q\right)^{\omega_2}\right) - \\ \left(1-\left(1-\underline{\zeta}_1^q\right)^{\omega_1}\right) * \left(1-\left(1-\underline{\zeta}_2^q\right)^{\omega_2}\right) \end{array}}, \\ \underline{\eta}_1^{q\omega_1} * \underline{\eta}_2^{q\omega_2}, \\ \sqrt[q]{\begin{array}{c} \left(\left(1-\underline{\zeta}_1^q\right)^{\omega_1} - \left(1-\underline{\zeta}_1^q-\underline{\xi}_1^q\right)^{\omega_1}\right) + \\ ((1-\underline{\zeta}_2^q)^{\omega_2} - (1-\underline{\zeta}_2^q-\underline{\xi}_2^q)^{\omega_2}) - \\ \left(\left(1-\underline{\zeta}_1^q\right)^{\omega_1} - \left(1-\underline{\zeta}_1^q-\underline{\xi}_1^q\right)^{\omega_1}\right) * \\ ((1-\underline{\zeta}_2^q)^{\omega_2} - (1-\underline{\zeta}_2^q-\underline{\xi}_2^q)^{\omega_2}) \end{array}}, \\ \sqrt[q]{\begin{array}{c} \left(1-\left(1-\overline{\zeta}_1^q\right)^{\omega_1}\right) + \left(1-\left(1-\overline{\zeta}_2^q\right)^{\omega_2}\right) - \\ \left(1-\left(1-\overline{\zeta}_1^q\right)^{\omega_1}\right) * \left(1-\left(1-\overline{\zeta}_2^q\right)^{\omega_2}\right) \end{array}}, \\ \overline{\eta}_1^{q\omega_1} * \overline{\eta}_2^{q\omega_2}, \\ \sqrt[q]{\begin{array}{c} \left(\left(1-\overline{\zeta}_1^q\right)^{\omega_1} - \left(1-\overline{\zeta}_1^q-\overline{\xi}_1^q\right)^{\omega_1}\right) + \\ ((1-\overline{\zeta}_2^q)^{\omega_2} - (1-\overline{\zeta}_2^q-\overline{\xi}_2^q)^{\omega_2}) - \\ \left(\left(1-\overline{\zeta}_1^q\right)^{\omega_1} - \left(1-\overline{\zeta}_1^q-\overline{\xi}_1^q\right)^{\omega_1}\right) * \\ ((1-\overline{\zeta}_2^q)^{\omega_2} - (1-\overline{\zeta}_2^q-\overline{\xi}_2^q)^{\omega_2}) \end{array}} \end{array} \right\rangle \right]$$

$$= \left[\begin{array}{c} \left\langle \begin{array}{c} \sqrt[q]{1-\left(1-\underline{\zeta}_1^q\right)^{\omega_1} * \left(1-\underline{\zeta}_2^q\right)^{\omega_2}}, \\ \underline{\eta}_1^{q\omega_1} * \underline{\eta}_2^{q\omega_2}, \\ \sqrt[q]{\begin{array}{c}\left(1-\underline{\zeta}_1^q\right)^{\omega_1} * \left(1-\underline{\zeta}_2^q\right)^{\omega_2} - \\ (1-\underline{\zeta}_1^q-\underline{\xi}_1^q)^{\omega_1} * (1-\underline{\zeta}_2^q-\underline{\xi}_2^q)^{\omega_2}\end{array}}, \\ \sqrt[q]{1-(1-\overline{\zeta}_1^q)^{\omega_1} * (1-\overline{\zeta}_2^q)^{\omega_2}}, \\ \overline{\eta}_1^{q\omega_1} * \overline{\eta}_2^{q\omega_2}, \\ \sqrt[q]{\begin{array}{c}\left(1-\overline{\zeta}_1^q\right)^{\omega_1} * \left(1-\overline{\zeta}_2^q\right)^{\omega_2} - \\ (1-\overline{\zeta}_1^q-\overline{\xi}_1^q)^{\omega_1} * (1-\overline{\zeta}_2^q-\overline{\xi}_2^q)^{\omega_2}\end{array}} \end{array} \right\rangle \end{array}\right]$$

$$= \omega_1\mathcal{A}_1 \oplus \omega_2\mathcal{A}_2.$$

Hence the theorem is true for $n = 2$.

Step 2: Suppose that the theorem is true for $n = k$, we have:

$$\text{q} - \text{SFRAM}_\omega(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k) = \oplus_{i=1}^{k}(\omega_i\mathcal{A}_i)$$

$$\text{q} - \text{SFRAM}_\omega(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_k) = \omega_1\mathcal{A}_1\oplus\omega_2\mathcal{A}_2\oplus\omega_3\mathcal{A}_3\oplus, \ldots, \oplus\omega_k\mathcal{A}_k$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\prod_{i=1}^k(1-(1-\underline{\zeta}_i^q)^{\omega_i})},\ \prod_{i=1}^k \underline{\eta}_i^{q\omega_i}, \\ \sqrt[q]{\prod_{i=1}^k(1-\underline{\zeta}_i^q)^{\omega_i} - \prod_{i=1}^k\left(1-\underline{\zeta}_i^q-\underline{\xi}_i^q\right)^{\omega_i}}, \\ \sqrt[q]{\prod_{i=1}^k(1-(1-\overline{\zeta}_i^q)^{\omega_i})},\ \prod_{i=1}^k \overline{\eta}_i^{q\omega_i}, \\ \sqrt[q]{\prod_{i=1}^k(1-\overline{\zeta}_i^q)^{\omega_i} - \prod_{i=1}^k\left(1-\overline{\zeta}_i^q-\overline{\xi}_i^q\right)^{\omega_i}} \end{array}\right\rangle\right]$$

Step 3: We will prove that the theorem is true for $n = k + 1$. For this, we have:

$$\text{q}-\text{SFRAM}_\omega(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k, \mathcal{A}_{k+1}) = \oplus_{i=1}^{k}(\omega_i\mathcal{A}_i)\oplus\omega_{k+1}\mathcal{A}_{k+1}$$

$$\text{q}-\text{SFRAM}_\omega(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_k) = \omega_1\mathcal{A}_1\oplus\omega_2\mathcal{A}_2\oplus\omega_3\mathcal{A}_3\oplus, \ldots, \oplus\omega_k\mathcal{A}_k$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\prod_{i=1}^k(1-(1-\underline{\zeta}_i^q)^{\omega_i})},\ \prod_{i=1}^k \underline{\eta}_i^{q\omega_i}, \\ \sqrt[q]{\prod_{i=1}^k\left(1-\underline{\zeta}_i^q\right)^{\omega_i} - \prod_{i=1}^k\left(1-\underline{\zeta}_i^q-\underline{\xi}_i^q\right)^{\omega_i}}, \\ \sqrt[q]{\prod_{i=1}^k(1-(1-\overline{\zeta}_i^q)^{\omega_i})},\ \prod_{i=1}^k \overline{\eta}_i^{q\omega_i}, \\ \sqrt[q]{\prod_{i=1}^k\left(1-\overline{\zeta}_i^q\right)^{\omega_i} - \prod_{i=1}^k\left(1-\overline{\zeta}_i^q-\overline{\xi}_i^q\right)^{\omega_i}} \end{array}\right\rangle\right]$$

$$\oplus \left[\left\langle \begin{array}{c} \sqrt[q]{(1-(1-\underline{\zeta}_{k+1}^q)^{\omega_{k+1}})},\ \underline{\eta}_{k+1}^{q\ \omega_{k+1}}, \\ \sqrt[q]{(1-\underline{\zeta}_{k+1}^q)^{\omega_{k+1}} - \left(1-\underline{\zeta}_{k+1}^q-\underline{\xi}_{k+1}^q\right)^{\omega_{k+1}}}, \\ \sqrt[q]{(1-(1-\overline{\zeta}_{k+1}^q)^{\omega_{k+1}})},\ \overline{\eta}_{k+1}^{q\ \omega_{k+1}}, \\ \sqrt[q]{(1-\overline{\zeta}_{k+1}^q)^{\omega_{k+1}} - \left(1-\overline{\zeta}_{k+1}^q-\overline{\xi}_{k+1}^q\right)^{\omega_{k+1}}} \end{array}\right\rangle\right]$$

$$\text{q} - \text{SFRAM}_\omega(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{k+1}) = \oplus_{i=1}^{k+1}(\omega_i\mathcal{A}_i)$$

$$\text{q} - \text{SFRAM}_\omega(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_{k+1}) = \omega_1\mathcal{A}_1\oplus\omega_2\mathcal{A}_2\oplus\omega_3\mathcal{A}_3\oplus, \ldots, \oplus\omega_{k+1}\mathcal{A}_{k+1}$$

$$= \left[\left\langle \begin{array}{c} \sqrt[q]{\prod_{i=1}^{k+1}(1-(1-\underline{\zeta}_i^q)^{\omega_i})},\ \prod_{i=1}^{k+1} \underline{\eta}_i^{q\omega_i}, \\ \sqrt[q]{\prod_{i=1}^{k+1}(1-\underline{\zeta}_i^q)^{\omega_i} - \prod_{i=1}^{k+1}\left(1-\underline{\zeta}_i^q-\underline{\xi}_i^q\right)^{\omega_i}}, \\ \sqrt[q]{\prod_{i=1}^{k+1}(1-(1-\overline{\zeta}_i^q)^{\omega_i})},\ \prod_{i=1}^{k+1} \overline{\eta}_i^{q\omega_i}, \\ \sqrt[q]{\prod_{i=1}^{k+1}(1-\overline{\zeta}_i^q)^{\omega_i} - \prod_{i=1}^{k+1}\left(1-\overline{\zeta}_i^q-\overline{\xi}_i^q\right)^{\omega_i}} \end{array}\right\rangle\right]$$

Hence the theorem is true for $n = k + 1$.

Theorem 8: Assuming $\mathcal{A}_i = \left(\underline{\zeta}_i, \underline{\eta}_i, \underline{\xi}_i, \overline{\zeta}_i, \overline{\eta}_i, \overline{\xi}_i\right)$ $(i = 1, 2, \ldots, n)$ ne a collection of q-SFRNs concerning, $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)$; $\omega_i \in [0,1]$; $\sum_{i=1}^n \omega_i = 1$, the q-spherical fuzzy rough geometric mean operator $\text{q} - \text{SFRGM}_\omega$ is defined as a mapping $\text{q} - \text{SFRGM}_\omega : \mathcal{A}^n \longrightarrow \mathcal{A}$ characterized by

$$\text{q} - \text{SFRGM}_\omega(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \otimes_{i=1}^{n}(\omega_i\mathcal{A}_i)$$

$$\text{q} - \text{SFRGM}_\omega(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_n) = \omega_1\mathcal{A}_1\otimes\omega_2\mathcal{A}_2\otimes\omega_3\mathcal{A}_3\otimes, \ldots, \otimes\omega_n\mathcal{A}_n$$

$$= \left[\left\langle \begin{array}{c} \prod_{i=1}^n \underline{\zeta}_i^{q\omega_i},\ \sqrt[q]{\prod_{i=1}^n(1-(1-\underline{\eta}_i^q)^{\omega_i})}, \\ \sqrt[q]{\prod_{i=1}^n(1-\underline{\eta}_i^q)^{\omega_i} - \prod_{i=1}^n\left(1-\underline{\eta}_i^q-\underline{\xi}_i^q\right)^{\omega_i}}, \\ \prod_{i=1}^n \overline{\zeta}_i^{q\omega_i},\ \sqrt[q]{\prod_{i=1}^n(1-(1-\overline{\eta}_i^q)^{\omega_i})}, \\ \sqrt[q]{\prod_{i=1}^n(1-\overline{\eta}_i^q)^{\omega_i} - \prod_{i=1}^n\left(1-\overline{\eta}_i^q-\overline{\xi}_i^q\right)^{\omega_i}} \end{array}\right\rangle\right]$$

Proof: We will prove Theorem (8) by using mathematical induction.

Step 1: For $n = 2$, we have

For this, let

$$\mathcal{A}_1^{\omega_1}\otimes\mathcal{A}_2^{\omega_2}$$

$$= \left[\left\langle \begin{array}{c} \underline{\zeta}_1^{q\omega_1},\ \sqrt[q]{1-(1-\underline{\eta}_1^q)^{\omega_1}},\ \sqrt[q]{(1-\underline{\eta}_1^q)^{\omega_1}-\left(1-\underline{\eta}_1^q-\underline{\xi}_1^q\right)^{\omega_1}}, \\ \overline{\zeta}_1^{q\omega_1},\ \sqrt[q]{1-(1-\overline{\eta}_1^q)^{\omega_1}},\ \sqrt[q]{(1-\overline{\eta}_1^q)^{\omega_1}-\left(1-\overline{\eta}_1^q-\overline{\xi}_1^q\right)^{\omega_1}} \end{array}\right\rangle\right]$$

$$\otimes$$

$$\left[\left\langle \begin{array}{c} \underline{\zeta}_2^{q\omega_2},\ \sqrt[q]{1-(1-\underline{\eta}_2^q)^{\omega_2}},\ \sqrt[q]{(1-\underline{\eta}_2^q)^{\omega_2}-\left(1-\underline{\eta}_2^q-\underline{\xi}_2^q\right)^{\omega_2}}, \\ \overline{\zeta}_2^{q\omega_2},\ \sqrt[q]{1-(1-\overline{\eta}_2^q)^{\omega_2}},\ \sqrt[q]{(1-\overline{\eta}_2^q)^{\omega_2}-\left(1-\overline{\eta}_2^q-\overline{\xi}_2^q\right)^{\omega_2}} \end{array}\right\rangle\right]$$

$$
= \left[\begin{array}{c} \left\langle \begin{array}{c} \underline{\zeta}_1^{q\omega_1} * \underline{\zeta}_2^{q\omega_2}, \\ \sqrt[q]{\begin{array}{c}\left(1-\left(1-\underline{\eta}_1^q\right)^{\omega_1}\right)+\left(1-\left(1-\underline{\eta}_2^q\right)^{\omega_2}\right)- \\ \left(1-\left(1-\underline{\eta}_1^q\right)^{\omega_1}\right)*\left(1-\left(1-\underline{\eta}_2^q\right)^{\omega_2}\right)\end{array}}, \\ \sqrt[q]{\begin{array}{c}\left(\left(1-\underline{\eta}_1^q\right)^{\omega_1}-\left(1-\underline{\eta}_1^q-\underline{\xi}_1^q\right)^{\omega_1}\right)+ \\ ((1-\underline{\eta}_2^q)^{\omega_2}-(1-\underline{\eta}_2^q-\underline{\xi}_2^q)^{\omega_2})- \\ \left(\left(1-\underline{\eta}_1^q\right)^{\omega_1}-\left(1-\underline{\eta}_1^q-\underline{\xi}_1^q\right)^{\omega_1}\right)* \\ ((1-\underline{\eta}_2^q)^{\omega_2}-(1-\underline{\eta}_2^q-\underline{\xi}_2^q)^{\omega_2})\end{array}}, \\ \overline{\zeta}_1^{q\omega_1} * \overline{\zeta}_2^{q\omega_2}, \\ \sqrt[q]{\begin{array}{c}(1-(1-\overline{\eta}_1^q)^{\omega_1})+(1-(1-\overline{\eta}_2^q)^{\omega_2})- \\ (1-(1-\overline{\eta}_1^q)^{\omega_1})*(1-(1-\overline{\eta}_2^q)^{\omega_2})\end{array}}, \\ \sqrt[q]{\begin{array}{c}\left((1-\overline{\eta}_1^q)^{\omega_1}-\left(1-\overline{\eta}_1^q-\overline{\xi}_1^q\right)^{\omega_1}\right)+ \\ ((1-\overline{\eta}_2^q)^{\omega_2}-(1-\overline{\eta}_2^q-\overline{\xi}_2^q)^{\omega_2})- \\ \left((1-\overline{\eta}_1^q)^{\omega_1}-\left(1-\overline{\eta}_1^q-\overline{\xi}_1^q\right)^{\omega_1}\right)* \\ ((1-\overline{\eta}_2^q)^{\omega_2}-(1-\overline{\eta}_2^q-\overline{\xi}_2^q)^{\omega_2})\end{array}} \end{array} \right\rangle \end{array} \right]
$$

$$
= \left[\begin{array}{c} \left\langle \begin{array}{c} \underline{\zeta}_1^{q\omega_1} * \underline{\zeta}_2^{q\omega_2}, \\ \sqrt[q]{1-\left(1-\underline{\eta}_1^q\right)^{\omega_1}*\left(1-\underline{\eta}_2^q\right)^{\omega_2}}, \\ \sqrt[q]{\begin{array}{c}\left(1-\underline{\eta}_1^q\right)^{\omega_1}*\left(1-\underline{\eta}_2^q\right)^{\omega_2}- \\ (1-\underline{\eta}_1^q-\underline{\xi}_1^q)^{\omega_1}*(1-\underline{\eta}_2^q-\underline{\xi}_2^q)^{\omega_2}\end{array}}, \\ \overline{\zeta}_1^{q\omega_1} * \overline{\zeta}_2^{q\omega_2}, \\ \sqrt[q]{1-(1-\overline{\eta}_1^q)^{\omega_1}*(1-\overline{\eta}_2^q)^{\omega_2}}, \\ \sqrt[q]{\begin{array}{c}\left(1-\overline{\eta}_1^q\right)^{\omega_1}*\left(1-\overline{\eta}_2^q\right)^{\omega_2}- \\ (1-\overline{\eta}_1^q-\overline{\xi}_1^q)^{\omega_1}*(1-\overline{\eta}_2^q-\overline{\xi}_2^q)^{\omega_2}\end{array}} \end{array} \right\rangle \end{array} \right]
$$

$$
= \mathcal{A}_1^{\omega_1} \otimes \mathcal{A}_2^{\omega_2}.
$$

Hence the theorem is true for $n = 2$.

Step 2: Suppose that the theorem is true for $n = k$, we have:

$$
\mathrm{q-SFRGM}_\omega(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k) = \otimes_{i=1}^k \mathcal{A}_i^{\omega_i}
$$

$$
\mathrm{q-SFRGM}_\omega(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_k) = \mathcal{A}_1^{\omega_1} \otimes \mathcal{A}_2^{\omega_2} \otimes \mathcal{A}_3^{\omega_3} \otimes, \ldots, \otimes \mathcal{A}_k^{\omega_k}
$$

$$
= \left[\left\langle \begin{array}{c} \prod_{i=1}^k \underline{\zeta}_i^{q\omega_i}, \sqrt[q]{\prod_{i=1}^k (1-(1-\underline{\eta}_i^q)^{\omega_i})}, \\ \sqrt[q]{\prod_{i=1}^k (1-\underline{\eta}_i^q)^{\omega_i} - \prod_{i=1}^k \left(1-\underline{\eta}_i^q-\underline{\xi}_i^q\right)^{\omega_i}}, \\ \prod_{i=1}^k \overline{\zeta}_i^{q\omega_i}, \sqrt[q]{\prod_{i=1}^k (1-(1-\overline{\eta}_i^q)^{\omega_i})}, \\ \sqrt[q]{\prod_{i=1}^k (1-\overline{\eta}_i^q)^{\omega_i} - \prod_{i=1}^k \left(1-\overline{\eta}_i^q-\overline{\xi}_i^q\right)^{\omega_i}} \end{array} \right\rangle \right]
$$

Step 3: We will prove that the theorem is true for $n = k + 1$. For this, we have:

$$
\mathrm{q-SFRGM}_\omega(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k, \mathcal{A}_{k+1})
$$
$$
= \otimes_{i=1}^k \mathcal{A}_i^{\omega_i} \otimes \mathcal{A}_{k+1}^{\omega_{k+1}}
$$
$$
= \mathcal{A}_1^{\omega_1} \otimes \mathcal{A}_2^{\omega_2} \otimes \mathcal{A}_3^{\omega_3} \otimes, \ldots, \otimes \mathcal{A}_k^{\omega_k} \otimes \mathcal{A}_{k+1}^{\omega_{k+1}}
$$

$$
= \left[\left\langle \begin{array}{c} \prod_{i=1}^k \underline{\zeta}_i^{q\omega_i}, \sqrt[q]{\prod_{i=1}^k (1-(1-\underline{\eta}_i^q)^{\omega_i})}, \\ \sqrt[q]{\prod_{i=1}^k \left(1-\underline{\eta}_i^q\right)^{\omega_i} - \prod_{i=1}^k \left(1-\underline{\eta}_i^q-\underline{\xi}_i^q\right)^{\omega_i}}, \\ \prod_{i=1}^k \overline{\zeta}_i^{q\omega_i}, \sqrt[q]{\prod_{i=1}^k (1-(1-\overline{\eta}_i^q)^{\omega_i})}, \\ \sqrt[q]{\prod_{i=1}^k (1-\overline{\eta}_i^q)^{\omega_i} - \prod_{i=1}^k \left(1-\overline{\eta}_i^q-\overline{\xi}_i^q\right)^{\omega_i}} \end{array} \right\rangle \right]
$$
$$
\otimes \left[\left\langle \begin{array}{c} \underline{\zeta}_{k+1}^{q\ \omega_{k+1}}, \sqrt[q]{(1-(1-\underline{\eta}_{k+1}^q)^{\omega_{k+1}})}, \\ \sqrt[q]{(1-\underline{\eta}_{k+1}^q)^{\omega_{k+1}} - \left(1-\underline{\eta}_{k+1}^q-\underline{\xi}_{k+1}^q\right)^{\omega_{k+1}}}, \\ \overline{\zeta}_{k+1}^{q\ \omega_{k+1}}, \sqrt[q]{(1-(1-\overline{\eta}_{k+1}^q)^{\omega_{k+1}})}, \\ \sqrt[q]{(1-\overline{\eta}_{k+1}^q)^{\omega_{k+1}} - \left(1-\overline{\eta}_{k+1}^q-\overline{\xi}_{k+1}^q\right)^{\omega_{k+1}}} \end{array} \right\rangle \right]
$$

$$
\mathrm{q-SFRGM}_\omega(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k, \mathcal{A}_{k+1}) = \otimes_{i=1}^k \mathcal{A}_i^{\omega_i} \otimes \mathcal{A}_{k+1}^{\omega_{k+1}}
$$
$$
= \mathcal{A}_1^{\omega_1} \otimes \mathcal{A}_2^{\omega_2} \otimes \mathcal{A}_3^{\omega_3} \otimes, \ldots, \otimes \mathcal{A}_k^{\omega_k} \otimes \mathcal{A}_{k+1}^{\omega_{k+1}}
$$
$$
= \left[\left\langle \begin{array}{c} \prod_{i=1}^{k+1} \underline{\zeta}_i^{q\omega_i}, \sqrt[q]{\prod_{i=1}^{k+1} (1-(1-\underline{\eta}_i^q)^{\omega_i})}, \\ \sqrt[q]{\prod_{i=1}^{k+1} (1-\underline{\eta}_i^q)^{\omega_i} - \prod_{i=1}^{k+1} \left(1-\underline{\eta}_i^q-\underline{\xi}_i^q\right)^{\omega_i}}, \\ \prod_{i=1}^{k+1} \overline{\zeta}_i^{q\omega_i}, \sqrt[q]{\prod_{i=1}^{k+1} (1-(1-\overline{\eta}_i^q)^{\omega_i})}, \\ \sqrt[q]{\prod_{i=1}^{k+1} (1-\overline{\eta}_i^q)^{\omega_i} - \prod_{i=1}^{k+1} \left(1-\overline{\eta}_i^q-\overline{\xi}_i^q\right)^{\omega_i}} \end{array} \right\rangle \right]
$$

Hence the theorem is true for $n = k + 1$.

IV. $q-$SPHERICAL FUZZY ROUGH CODAS APPROACH

One approach that may be used to illustrate a (MCDM) problem is to use a decision matrix. When a system is functioning in a q-spherical fuzzy rough environment, the components of a decision matrix represent the evaluation values of all conceivable alternatives with relation to each criterion. Let the expression $X = \{x_1, x_2, x_3, \ldots, x_m\}\,(m \geq 2)$ denotes a discrete collection of m possible alternatives and the values of all alternatives about each criterion. This will be done inside the confines of a q-spherical fuzzy rough environment. When functioning in a rough q-spherical fuzzy environment, the collection of criteria should be indicated by $C = \{C_1, C_2, C_3, \ldots, C_n\}$. Let's name the weight vector that contains all the requirements of w and describe it as something that must meet the constraints $0 \leq \omega_i \leq 1$ and $\sum_{i=1}^n \omega_i = 1$ to be considered valid. by marking it as $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)$.

Step 1: The person in charge of making choices completes the assessment matrix using the linguistic terms listed in Table 1.

Step 2: The decision matrix is converted into a weighted q-spherical fuzzy rough decision matrix using the q-spherical fuzzy rough values from Table 1. The weighted q-spherical

TABLE 1. q-SFRNs with linguistic terms.

Importance Level	$(\zeta, \underline{\eta}, \underline{\xi}, \overline{\zeta}, \overline{\eta}, \overline{\xi})$
Very High Importance (VHI)	(0.85, 0.25, 0.25, 0.80, 0.20, 0.20)
High Importance (HI)	(0.75, 0.35, 0.35, 0.70, 0.30, 0.30)
Slightly More Importance (SMI)	(0.65, 0.45, 0.45, 0.60, 0.40, 0.40)
Equally Importance (EI)	(0.55, 0.55, 0.55, 0.54, 0.54, 0.54)
Low Importance (LI)	(0.50, 0.75, 0.50, 0.51, 0.70, 0.51)

fuzzy rough decision matrix is obtained by multiplying the weights of the criteria by the evaluations and may be produced using the q-SFRGM operator.

Step 3: Utilizing the score function that is given to defuzzification of the weighted decision matrix. This can be done by using the formula below.

$$Sco\left(C_j\left(X_{i\omega}\right)\right) = \frac{2+\left(\underline{\zeta}\right)^q+\left(\overline{\zeta}\right)^q-\left(\underline{\eta}\right)^q-(\overline{\eta})^q-\left(\underline{\xi}\right)^q-\left(\overline{\xi}\right)^q}{3}.$$

Step 4: The Step 3 score values are used to construct a q-SFRNIS.

For the q-SFRNIS:

$$X^- = \left\{C_j,\ min_i < Score\left(C_j\left(X_{iw}\right)\right) > j = 1, 2, \ldots, n\right\}$$

Step 5: Evaluate the normalized Hamming distance formula from q-SFRNIS for alternatives by formula.

$$D_H\left(X_i, X^-\right) = \sqrt[q]{\frac{1}{4n}\sum\nolimits_{i=1}^{n}\left(\begin{array}{l}\left|\underline{\zeta}^q_{X_i}-\underline{\zeta}^q_{X^-}\right|+\left|\overline{\zeta}^q_{X_i}-\overline{\zeta}^q_{X^-}\right|+\\ \left|\underline{\eta}^q_{X_i}-\underline{\eta}^q_{X^-}\right|+\left|\overline{\eta}^q_{X_i}-\overline{\eta}^q_{X^-}\right|+\\ \left|\underline{\xi}^q_{X_i}-\underline{\xi}^q_{X^-}\right|+\left|\overline{\xi}^q_{X_i}-\overline{\xi}^q_{X^-}\right|+\\ \left|\underline{\pi}^q_{X_i}-\underline{\pi}^q_{X^-}\right|+\left|\overline{\pi}^q_{Xi}-\overline{\pi}^q_{X^-}\right|\end{array}\right)}$$

Step 6: Evaluate the normalized Euclidean distance formula from q-SFRNIS for alternatives by formula.

$$D_E(X_i, X^-) = \sqrt[q]{\frac{1}{4n}\sum\nolimits_{i=1}^{n}\left(\begin{array}{l}\left(\underline{\zeta}^q_{X_i}-\underline{\zeta}^q_{X^-}\right)^2+\left(\overline{\zeta}^q_{X_i}-\overline{\zeta}^q_{X^-}\right)^2+\\ \left(\underline{\eta}^q_{X_i}-\underline{\eta}^q_{X^-}\right)^2+\left(\overline{\eta}^q_{X_i}-\overline{\eta}^q_{X^-}\right)^2+\\ \left(\underline{\xi}^q_{X_i}-\underline{\xi}^q_{X^-}\right)^2+\left(\overline{\xi}^q_{X_i}-\overline{\xi}^q_{X^-}\right)^2+\\ \left(\underline{\pi}^q_{X_i}-\underline{\pi}^q_{X^-}\right)^2+\left(\overline{\pi}^q_{Xi}-\overline{\pi}^q_{X^-}\right)^2\end{array}\right)}$$

Step 7: Find the relative assessment matrix by using the formula as follows:

$$r_{ik} = \left(D_{E_i}-D_{E_j}\right)+\omega\right)\left(D_{E_i}-D_{E_j}\right)\cdot\left(D_{H_i}-D_{H_j}\right),$$

where $k\in\{1, 2, 3, \ldots, n\}$ and ω is a threshold function that is defined in eq. (11). The decision maker can define the threshold value of this function. In this study, we practice $\omega = 0.03$.

$$\omega(x) = \begin{cases}1 & if\,|x| \geq \Omega\\ 0 & if\,|x| \leq \Omega\end{cases}$$

Step 8: Estimate how each alternative would do on an appraisal score. The highest AS_i the score represents the optimal solution.

$$AS_i = \sum_{k=1}^{n} r_{ik}$$

Figure 8 shows a widely used framework for multi-criteria decision-making (MCDM) methodology. The configuration presented in Figure 8 typically incorporates linguistic terms, alternatives, criteria weights, distances, and evaluation metrics, culminating in a ranking of alternatives.

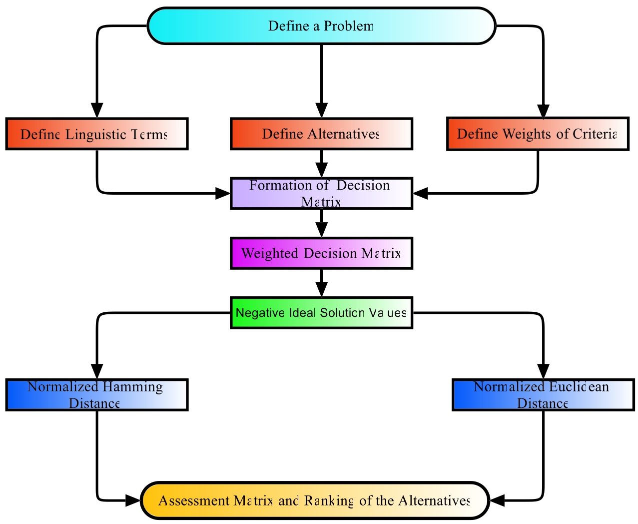

FIGURE 8. Illustrates the normative framework used in the multi-criteria decision-making process.

V. DECISION MODEL AND ITS APPLICATIONS WITH THE HELP OF NUMERICAL EXAMPLE

The research findings of many scientists indicate that the region of the United States is the finest site in the world to generate renewable energy due to the natural conditions. The selection of a site location to create a wind power farm is an application of the methodology that we have proposed. Regarding this objective, most of the preference was given to the following four cities: Evaluations are being done in four cities (A_1: California A_2: North Carolina, A_3: New Jersey, A_4: Colorado). Following an in-depth analysis of the relevant prior research, four criteria have been established. The environmental conditions (C_1), the economic situation (C_2), the technical opportunities (C_3), and the site attributes (C_4) are all considered to be criteria. To begin, the evaluations of the criteria are solicited from a group of individuals responsible for making decisions about the objective, making use of the linguistic terms given in Table 1. The person who oversaw making the choice did not consider the evaluation, which is shown in Table 2. Figure 9 illustrates a decision tree used for renewable energy site selection.

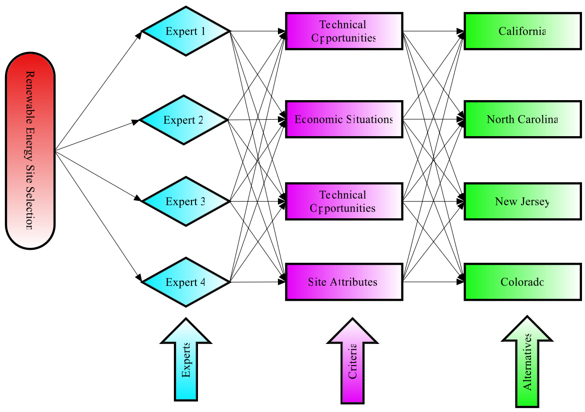

FIGURE 9. Decision tree for renewable energy site selection.

TABLE 2. Evaluation of the alternatives for renewable energy site selection.

Alternatives	C_1	C_2	C_3	C_4
A_1	SMI	HI	LI	SMI
A_2	EI	HI	HI	HI
A_3	HI	SMI	SMI	SMI
A_4	SMI	VHI	LI	VHI

These assessments are turned into q-spherical fuzzy rough sets using the methodological stages outlined in Table 3.

TABLE 3. q-spherical fuzzy rough evaluations.

Alternatives	C_1	C_2
A_1	(0.65,0.45,0.45 0.60,0.40,0.40)	(0.75,0.35,0.35, 0.70,0.30,0.30)
A_2	(0.55,0.55,055, 0.54,0.54,0.54)	(0.75,0.35,0.35, 0.70,0.30,0.30)
A_3	(0.75,0.35,0.35, 0.70,0.30,0.30)	(0.65,0.45,0.45, 0.60,0.40,0.40)
A_4	(0.65,0.45,0.45, 0.60,0.40,0.40)	(0.85,0.25,0.25, 0.80,0.20,0.20)
Alternatives	C_3	C_4
A_1	(0.50,0.75,0.50, 0.51,0.70,0.51)	(0.65,0.45,0.45, 0.60,0.40,0.40)
A_2	(0.75,0.35,0.35, 0.70,0.30,0.30)	(0.75,0.35,0.35, 0.70,0.30,0.30)
A_3	(0.65,0.45,0.45, 0.60,0.40,0.40)	(0.65,0.45,0.45, 0.60,0.40,0.40)
A_4	(0.50,0.75,0.50, 0.57,0.70,0.51)	(0.85,0.25,0.25, 0.80,0.20,0.20)

The next step is to figure out how to make a weighted decision matrix. The person making the decision picks the weights of the criteria, which are given in Table 4.

TABLE 4. Weights of the criteria.

Weight of the criteria	C_1	C_2
$(\underline{\zeta},\underline{\eta},\underline{\xi},\overline{\zeta},\overline{\eta},\overline{\xi})$	(0.3,0.1,0.1 0.3,0.4,0.1)	(0.3,0.2,0.2, 0.3,0.3,0.2)
Weight of the criteria	C_3	C_4
$(\underline{\zeta},\underline{\eta},\underline{\xi},\overline{\zeta},\overline{\eta},\overline{\xi})$	(0.2,0.3,0.3, 0.2,0.1,0.3)	(0.2,0.4,0.4, 0.2,0.2,0.4)

The q-SFRWGM operator is used to aggregate the perspectives of decision-makers while also accounting for their relative importance. The weights are multiplied by the decision matrix to make the weighted q-spherical fuzzy rough decision matrix. Table 5 shows this.

TABLE 5. Weighted q-spherical fuzzy rough decision matrix.

A_i	C_1	C_2
A_1	(0.0074,0.4515,0.4512, 0.0058,0.4985,0.3932)	(0.0114,0.3697,0.3680, 0.0093,0.3763,0.3234)
A_2	(0.0045,0.5509,0.5506, 0.0043,0.5957,0.5290)	(0.0114,0.3697,0.3680, 0.0093,0.3763,0.3234)
A_3	(0.0022,0.4872,0.4802, 0.0017,0.4019,0.4439)	(0.0022, ,0.4872,0.4802, 0.0017,0.4019,0.4439)
A_4	(0.0074,0.4515,0.4512, 0.0058,0.4985,0.3932)	(0.0166,0.2864,0.2854, 0.0138,0.3264,0.2502)
A_i	C_3	C_4
A_1	(0.0010,0.7591,0.5115, 0.0011,0.7004,0.5274)	(0.0022,0.5305,0.5163, 0.0017,0.4150,0.4923)
A_2	(0.0034,0.4096,0.4049, 0.0027,0.3036,0.3745)	(0.0034,0.4705,0.4620, 0.0027,0.3264,0.4437)
A_3	(0.0022,0.4872,0.4802, 0.0017,0.4019,0.4439)	(0.0022,0.5305,0.5163, 0.0017,0.4150,0.4923)
A_4	(0.0010,0.7591,0.5115, 0.0011,0.7004,0.5274)	(0.0049,0.4284,0.4247, 0.0041,0.2516,0.4139)

The next step is to calculate the negative ideal q-SFR values. To achieve this goal, defuzzification of the q-SFR values is conducted using the score function. The outcomes are given in Table 6, which is shown below.

TABLE 6. Defuzzification of the q-SFR values.

Alternatives	C_1	C_2	C_3	C_4
A_1	0.5438	0.6042	0.3128	0.5074
A_2	0.4355	0.6042	0.5948	0.5584
A_3	0.5990	0.5487	0.5404	0.5074
A_4	0.5438	0.6343	0.3128	0.5860

The values of the q-SFR Decision Matrix are displayed in Table 6 after they have been defuzzified.

The NIS values are calculated with the help of the information supplied in Table 6, which is followed by the information displayed in Table 7, which concludes the process.

According to the data shown in Table 8, which represents the normalized hamming distance and normalized Euclidean distance. In the last step of the procedure, the relative

TABLE 7. NIS values.

	C_1	C_2
NIS	$\begin{pmatrix}0.0045,0.5509,0.5506\\0.0043,0.5957,0.5290\end{pmatrix}$	$\begin{pmatrix}0.0074,0.4617,0.4594\\0.0058,0.4469,0.4106\end{pmatrix}$
	C_3	C_4
NIS	$\begin{pmatrix}0.0010,0.7591,0.5115\\0.0011,0.7004,0.5274\end{pmatrix}$	$\begin{pmatrix}0.0022,0.5305,0.5163\\0.0017,0.4150,0.4923\end{pmatrix}$

TABLE 8. The distance of each alternative from NIS values.

Alternatives	NH Distance	NE Distance
A_1	0.3124	0.1273
A_2	0.4180	0.2539
A_3	0.4182	0.2482
A_4	0.3705	0.1527

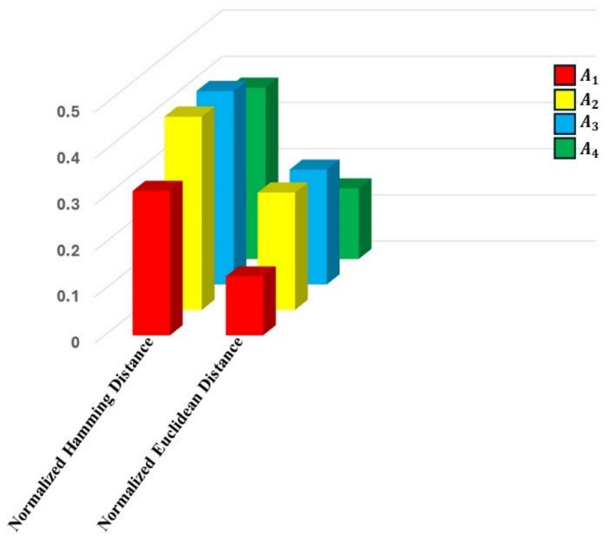

FIGURE 10. Graphical representation of normalized hamming distance and normalized Euclidean distance.

assessment matrix is computed making use of step 7, which can be found in Table 9.

TABLE 9. The relative assessment matrix of the alternatives.

	C_1	C_2	C_3	C_4
A_1	0.0000	−0.2322	−0.2269	−0.0835
A_2	0.2322	0.0000	0.0055	0.1487
A_3	0.2267	−0.0055	0.0000	0.1432
A_4	0.0835	−0.1487	−0.1432	0.0000

Following is the graphical representation of Normalized Hamming Distance and Normalized Euclidean Distance.

Following is the graphical representation of the ranking of the alternatives based on their appraisal scores.

We have concluded that alternative A_2(North Carolina), which has the highest appraisal ratio, is the most convenient alternative because of the results of the study, and

TABLE 10. Ranking of the alternatives.

Alternatives	Appraisal Score	Ranking
A_1	−0.5426	4
A_2	0.3864	1
A_3	0.3644	2
A_4	−0.2084	3

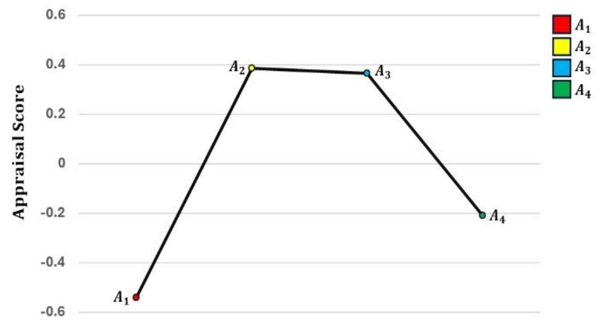

FIGURE 11. Graphical representation of ranking of the alternatives based on their appraisal score.

we have come to this conclusion based on the data. This alternative is followed by A_3(New Jersey), A_4(Colorado), and A_1(California).

A. EFFECT OF q ON RANKING ORDER AND SCORE VALUES

To fulfill the constraint requirement $(0\leq \underline{\zeta}_{\mathcal{A}}{}^{q}(\mathfrak{w})+\underline{\eta}_{\mathcal{A}}{}^{q}(\mathfrak{w})+\underline{\xi}_{\mathcal{A}}{}^{q}(\mathfrak{w})\leq 1)$ and $(0\leq \overline{\zeta}_{\mathcal{A}}{}^{q}(\mathfrak{w})+\overline{\eta}_{\mathcal{A}}{}^{q}(\mathfrak{w})+\overline{\xi}_{\mathcal{A}}{}^{q}(\mathfrak{w})\leq 1)$, and then by examining the attribute values, the decision maker is capable of identifying a minimum numerical parameter q. For example, while evaluating an alternative, if the attribute values are (0.8,0.7,0.9,0.9,0.8,0.7), one should choose q as 3 or q as 4, as both configurations meet the criterion. However, we employed several values of q in Step 4 of the novel approach to solve the case to fully evaluate the effect of parameter q on the experimental results. Table 9 presents the results of these modifications and indicates that A_2 is at the top, followed by A_3, A_4, and finally, A_1. Notable is the relevance of the best alternative and the unchanging ranking. Table 10 illustrates this point. Specifically, when q equals 1. The alternatives and ratings offered do not adhere to the requirements of either 1 (i.e., under PFRS environment $\left(0\leq \underline{\zeta}_{\mathcal{A}}(\mathfrak{w})+\underline{\eta}_{\mathcal{A}}(\mathfrak{w})+\underline{\xi}_{\mathcal{A}}(\mathfrak{w})\leq 1\right)$ and $\left(0\leq \overline{\zeta}_{\mathcal{A}}(\mathfrak{w})+\overline{\eta}_{\mathcal{A}}(\mathfrak{w})+\overline{\xi}_{\mathcal{A}}(\mathfrak{w})\leq 1\right)$) or 2 (i.e., under SFRS environment $(0\leq \underline{\zeta}_{\mathcal{A}}{}^{2}(\mathfrak{w})+\underline{\eta}_{\mathcal{A}}{}^{2}(\mathfrak{w})+\underline{\xi}_{\mathcal{A}}{}^{2}(\mathfrak{w})\leq 1)$ and $(0\leq \overline{\zeta}_{\mathcal{A}}{}^{2}(\mathfrak{w})+\overline{\eta}_{\mathcal{A}}{}^{2}(\mathfrak{w})+\overline{\xi}_{\mathcal{A}}{}^{2}(\mathfrak{w})\leq 1)$.

Table 11 illustrates the consistent consistency in the ranking order of alternatives at different q-parameter values. This enduring stability of the hierarchy offers decision-makers a reliable framework for evaluating test alternatives within a limited set. It establishes a safe and flexible environment for careful examination and informed decision-making based on defined parameters.

TABLE 11. Sorting alternatives according to their respective parameter q values.

Parameter q	Ranking order	Best alternative
$q = 1$	$A_2 > A_3 > A_4 > A_1$	A_2
$q = 2$	$A_2 > A_3 > A_4 > A_1$	A_2
$q = 3$	$A_2 > A_3 > A_4 > A_1$	A_2
$q = 4$	$A_2 > A_3 > A_4 > A_1$	A_2
$q = 5$	$A_2 > A_3 > A_4 > A_1$	A_2
$q = 6$	$A_2 > A_3 > A_4 > A_1$	A_2
$q = 7$	$A_2 > A_3 > A_4 > A_1$	A_2
$q = 8$	$A_2 > A_3 > A_4 > A_1$	A_2
$q = 9$	$A_2 > A_3 > A_4 > A_1$	A_2
$q = 10$	$A_2 > A_3 > A_4 > A_1$	A_2
$q = 11$	$A_2 > A_3 > A_4 > A_1$	A_2

B. TEST OF VALIDITY

To demonstrate the versatility of the proposed technique in diverse contexts, we use the evaluation protocol introduced by Wang and Trianafilo [35] as follows:

Step 1: Changing the ranking values of sub-optimal alternatives that indicate inferior quality is not expected to affect the identification of optimal alternatives. It preserves the highest-ranked choice, assuming a constant relative weight for the criteria.

Step 2: Transitivity should be followed in the procedure.

Step 3: When using the same decision-making process for a given problem that has been broken into smaller ones, the initial ranking of the alternatives should be preserved.

1) TEST OF VALIDITY UTILIZING CRITERIA 1

The alternatives ranked by using our suggested method are $A_2 > A_3 > A_4 > A_1$. Based on test criteria 1, we replaced the non-optimal alternative A_1 with the lowest alternative A_1^* to evaluate the stability of the suggested method. (0.58,0.45,0.89,0.65,0.74,0.63), (0.58,0.85,0.74,0.63,0.25, 0.87), and (0.85,0.25,0.85,0.85,0.78,0.36) were used as the rating values of A_1^*. The aggregated score values for the alternatives were as follows after we used our suggested methodology: $Sco(A_2) = 0.8523$, $Sco(A_3) = 0.78561$, $Sco(A_4) = 0.6589$, and $Sco(A_1^*) = 0.4523$. As a result, $A_2 > A_3 > A_4 > A_1$ is the new ranking order, and the best alternative still adheres to the first suggested strategy. Consequently, our method meets test requirement 1 by producing a consistent result.

2) TEST OF VALIDITY EMPLOYING CRITERIA 2 AND 3

The fragmented decision-making subcases are regarded as $\{A_1, A_2, A_3\}$, $\{A_2, A_3, A_4\}$ and $\{A_1, A_3, A_4\}$ to assess the validity based on criteria 2 and 3. They rank in the following sequence via the procedures mentioned: $A_2 > A_3 > A_1$, $A_2 > A_3 > A_4$ and $A_3 > A_4 > A_1$. After combining all the findings, the overall ranking appears as $A_2 > A_3 > A_4 > A_1$, This is perfectly consistent with the results of the initial decision-making process. Consequently, our proposed strategy meets the criteria stated in requirements 2 and 3.

VI. MANAGERIAL IMPLICATIONS

The framework shows remarkable industry flexibility and efficacy in a variety of decision-making contexts. Executives in a variety of fields can effectively use the q-SFR CODAS model's capacity for a range of objectives. For example, it demonstrates its value in the renewable energy site selection procedure by subsidiary the estimation of various considerations of the procedure to establish the best beneficial energy supply. Moreover, it might assist in the choice of conservation performances, granting supervisors to select the beneficial approach to preserve their tools or approach. The additional region where the demonstration may be worked is in the assessment of machines in developed situations, which can assist supervisors in deciding the effectiveness and pertinence of numerous automated organizations. It may also be consumed in the collection of substantial operating tackle, and supplementary executives in making sophisticated findings observing the best and highest beneficial tools for their requirements. It is significant, nonetheless, to identify that the executive construction procedure within this structure is manipulated by the inclinations of specialists and participants. While the template suggests a systematic and methodical methodology for decision-making, the assumptions and standings are determined by the decision-makers' conclusions and inclinations. As an outcome, involving professionals and participants is significant in confirming the legitimacy and reputation of the conclusions. The two most important assessments are conceded to enhance the trustworthiness and sturdiness of the attained outcomes. The two most important assessments are conceded to enhance the trustworthiness and sturdiness of the attained outcomes.

This conclusion is an effective tool for decision-makers to rank, consider, and judge conclusions from several alternatives, each explored using separate conditions. It promotes a better knowledge of trade agreements and allows for more informed decision-making by highlighting the advantages and disadvantages of each possibility. By performing a sensitivity analysis, important insights are gained about the stability and sensitivity of the results. Decision-makers can examine how various factors influence their choices, enhancing their ability to make adaptive decisions in a dynamic environment. Incorporating this analysis into the decision-making process enables managers to increase reliability and confidence in their strategic decisions.

The q-SFR CODAS model, combined with comparative and sensitivity analysis, offers a comprehensive framework that equips managers in diverse industries and applications with the tools needed to make informed and flexible decisions.

A. COMPARATIVE ANALYSIS

A comparative study is conducted to validate the robustness and effectiveness of this research against other contemporary multiple criteria decision-making (MCDM) methods. To achieve this goal, the problem is solved using six different MCDM models that operate within the framework of spherical fuzzy and fuzzy methods. The models selected for comparison include SF TOPSIS [22], SF CODAS [32], SF EDAS [36], IF TOPSIS [37], IF CODAS [38], and IF EDAS [39]. Table 12 presents the evaluation rankings derived from both the proposed model and the existing six MCDM models, highlighting comparative evaluations in different decision-making frameworks. The references for the other models selected for comparison should indeed be provided for transparency and academic integrity. These references would typically include the sources or publications where the SF TOPSIS [22], SF CODAS [32], SF EDAS [36], IF TOPSIS [37], IF CODAS [38], and IF EDAS [39] models were introduced or described. As for why these specific models were chosen for comparison, several factors might have influenced their selection.

TABLE 12. **Appraisal scores of different approaches.**

Approaches	Appraisal scores			
	A_1	A_2	A_3	A_4
SF TOPSIS [22]	0.0000	1.7960	3.9770	3.9740
SF CODAS [32]	−0.2190	−0.1030	0.0101	0.3270
SF EDAS [36]	0.2600	0.0000	−0.0400	−0.2700
IF TOPSIS [37]	0.5527	0.7701	0.2307	0.7688
IF CODAS [38]	−18.7610	−3.3600	6.5100	15.611
IF EDAS [39]	0.4446	0.7025	0.4780	0.6138
[This Paper]	−0.5426	0.3864	0.3644	−0.2084

Amidst the propositions, calculations, and practical applications outlined above, the distinct advantages of embracing q-spherical fuzzy rough sets come to light, illuminating a path toward enhanced decision-making:

1. Traditional fuzzy sets and intuitionistic fuzzy sets, while valuable, often falter in capturing comprehensive information in certain contexts. The constraints imposed by membership and non-membership degrees can stifle the expression of nuanced opinions by decision-makers.
2. In response to these limitations, Yager's introduction of Pythagorean fuzzy sets broadened the scope of representation, allowing for a more diverse array of applications.
3. However, within the realm of uncertain information, such as in voting systems, the rigidity of picture fuzzy sets may prove restrictive, particularly in accommodating decision-maker flexibility.
4. Enter spherical fuzzy numbers, offering a solution that gracefully navigates diverse information sets without exceeding the bounds of unity. This adaptability empowers decision-makers to allocate membership values according to their unique preferences.
5. The incorporation of q-spherical fuzzy rough sets, along with associated algorithms, presents a versatile framework with far-reaching implications across various decision-making processes.
6. Furthermore, the proposed aggregation operators excel in handling imprecise information, offering a level of reliability that surpasses existing methodologies.
7. The applicability of q-spherical fuzzy rough sets spans a multitude of domains, including stock investment analysis, airline service quality evaluation, investment banking authority selection, and electronic learning factor assessment, underscoring their broad utility and relevance.
8. By embracing the advantages inherent in q-spherical fuzzy rough sets, decision-makers are better equipped to traverse the intricate landscapes of decision-making with heightened confidence and precision.
9. Regarding the specific concerns about the limitations of picture fuzzy rough sets (PFSRS) and spherical fuzzy rough sets, it's important to acknowledge that they are constrained by specific numerical bounds within their approximations. In contrast, q-spherical fuzzy rough sets offer a broader scope of representation, allowing for a more nuanced handling of information sets. This distinction underscores the versatility and potential superiority of q-SFRS in handling complex decision-making scenarios.
10. q-spherical fuzzy rough sets are more general than other algebraic structures because they incorporate lower and upper approximations with membership, neutral, and non-membership degrees. The inclusion of a q-parameter further enhances their robustness compared to picture fuzzy sets and spherical fuzzy sets. Additionally, the merging of CODAS methods strengthens their robustness.

Table 13 represents the pros and cons of the proposed operators and existing operators along with their year of publications.

TABLE 13. **The pros and cons of both the proposed and existing operators.**

Operators	Approximations Set	Parameter	Year
		q	
SF TOPSIS [22]	×	×	2019
SF CODAS [32]	×	×	2020
SF EDAS [36]	×	×	2022
IF TOPSIS [37]	×	×	2020
IF CODAS [38]	×	×	2020
IF EDAS [39]	×	×	2020
[This Paper]	✓	✓	2024

It is important to recognize that each approach comes with its own set of limitations. For example, the FFRS technique allows decision-makers to rank alternatives within the constraints of $(0 \leq \underline{\zeta}_{\mathcal{A}}^{3}(\mathfrak{w}) + \underline{\eta}_{\mathcal{A}}^{3}(\mathfrak{w}) + \underline{\xi}_{\mathcal{A}}^{3}(\mathfrak{w}) \leq 1)$ and $(0 \leq \overline{\zeta}_{\mathcal{A}}^{3}(\mathfrak{w}) + \overline{\eta}_{\mathcal{A}}^{3}(\mathfrak{w}) + \overline{\xi}_{\mathcal{A}}^{3}(\mathfrak{w}) \leq 1)$. In contrast, the spherical fuzzy rough approach directs decision makers to rank the alternatives obeying the condition that $(0 \leq \underline{\zeta}_{\mathcal{A}}^{2}(\mathfrak{w}) + \underline{\eta}_{\mathcal{A}}^{2}(\mathfrak{w}) + \underline{\xi}_{\mathcal{A}}^{2}(\mathfrak{w}) \leq 1)$ and $(0 \leq \overline{\zeta}_{\mathcal{A}}^{2}(\mathfrak{w}) + \overline{\eta}_{\mathcal{A}}^{2}(\mathfrak{w}) + \overline{\xi}_{\mathcal{A}}^{2}(\mathfrak{w}) \leq 1)$. To overcome these limitations, the proposed approach provides a more flexible environment for decision-makers. By reducing these constraints, decision-makers can provide more accurate classifications and make well-informed decisions. The unique features of different techniques including the comparison characteristics of the proposed approach are presented in Table 14.

TABLE 14. Comparison of characteristics between different methods.

Approaches	MD	Neu − MD	Non − MD	q
PFS	Yes	Yes	Yes	No
SFS	Yes	Yes	Yes	No
$q - SFS$	Yes	Yes	Yes	Yes
$q - SFRS$	Yes	Yes	Yes	Yes

Figure 12 represents some particular cases of q-spherical fuzzy rough sets.

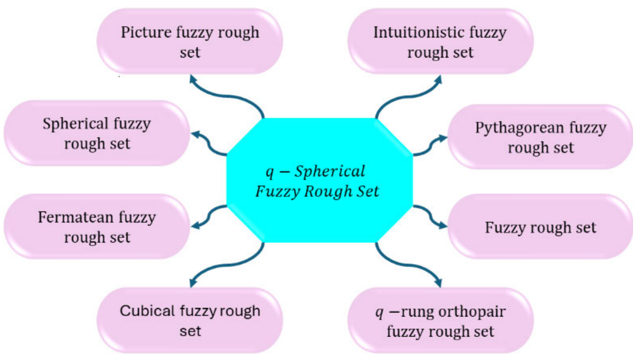

FIGURE 12. Some particular cases of different algebraic structures compared to q-spherical fuzzy rough sets.

B. SENSITIVITY ANALYSIS

In this study, to validate the developed model, two separate sensitivity analyses concerning changes in criteria and decision-making weights on the final ranking are presented. In this first study, a temporal sensitivity analysis is performed based on each criterion. For this purpose, the weight values of the reference criteria, i.e., high importance, equal importance, and low importance, are determined to see the effect of changing the criteria weight on the final ranking. Then, assigning these reference values to each criterion one by one, the model is run, and the alternatives are ranked. The results obtained according to the total 12 scenarios thus obtained are presented in Figure 13.

In all scenarios, alternative A_2 ranks first and alternative A_1 ranks last. Even with extreme values, altering the criterion weights has little effect on model output. In the second analysis, the weights of the decision makers are significantly changed, and 15 different scenarios are obtained based on different values of the weights. Figure 14 presents the final ranking of the decision makers' weight distribution. Alternative A_2 is the best choice in all scenarios, while alternative A_1 is the worst choice. Although the ranking order of the two alternatives may vary depending on the combination of

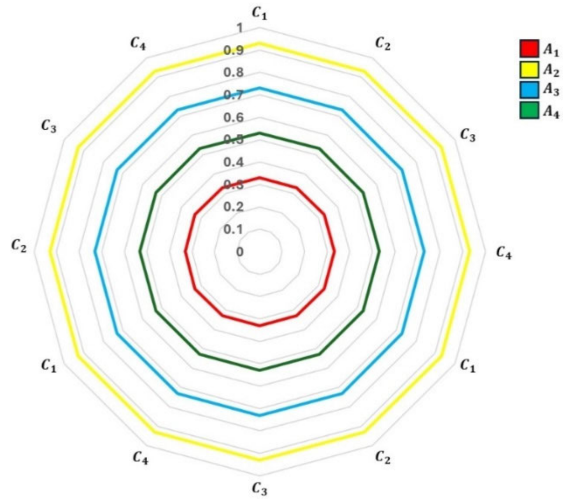

FIGURE 13. Alternative classification considering variations in criteria weights.

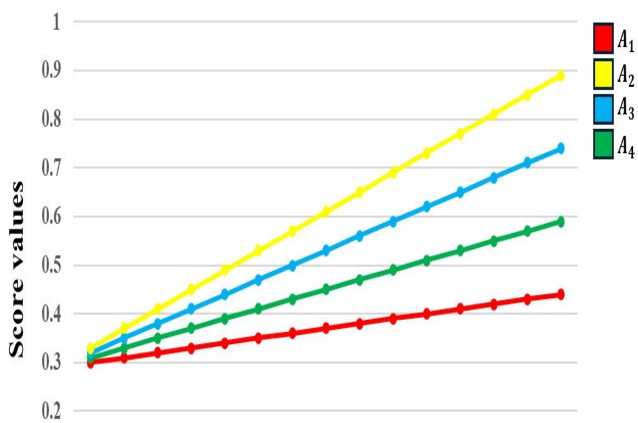

FIGURE 14. Alternative rankings in response to adjustments in decision-making weights.

weights used, the proposed approach generally produces reliable results and has reasonable consistency across different decision-weighting scenarios.

C. ADVANTAGES

The proposed technique has various benefits:

1. The addition of parameter q to the aggregation operators gives decision-makers a great deal of freedom. This versatility allows them to tailor the settings to the individual needs and preferences of the decision-making scenarios. The decision process's versatility allows for varying degrees of membership and non-membership, making it appropriate for a broad range of real-world scenarios.
2. The parametric character of the suggested operators enables decision-makers to fine-tune the impact of membership and non-membership degrees. This degree of control enables decision-makers to accurately tailor the aggregation process to their preferences and the unique aspects of the situation at hand.

3. The symmetry of the suggested aggregation operators concerning the parameter ensures that the ranking orders of alternatives stay generally consistent across parameter values. This stability is critical in decision-making because it prevents the outcomes from being impacted by the decision-makers' pessimism or optimism.

D. LIMITATIONS

Every research endeavor inherently has limitations, and the methodology proposed in this study is no exception. Below is a discussion of these constraints:

1. The applicability of the proposed technique may be limited to specific domains or decision contexts. Understanding these limitations is critical to determining the optimal use of the recommended strategy.
2. As with any research approach, the proposed method relies on certain assumptions and simplifications to facilitate analysis. It is important to recognize that these assumptions may not align perfectly with real-world scenarios, potentially limiting the broad or practical applicability of the results.
3. The accomplishment of the suggested framework is established through a case study including four alternatives and four characteristics. It is critical to identify that the pattern may be expanded to integrate more possibilities and abilities in future efforts.
4. For several values of the parameter q, alternative ranking orders are calculated. It is important to note that more investigations might be conducted to investigate the hierarchical order for other values of these considerations.

VII. CONCLUSION

The execution of the q-spherical fuzzy rough CODAS knowledge signifies substantial progress in the decision-making method. Its flexibility offers a vast scope of relevance, making it a valued apparatus for decision-makers in a variety of fields. Future revisions should concentrate on evaluation investigations, evaluating q-SFR CODAS to other CODAS additions, and measuring similarities. This comparison analysis will focus on the advantages and difficulties of various decision-making approaches, which will aid in the expansion of the q-spherical fuzzy rough CODAS methodology. Additionally, utilizing this investigation involves investigating other decisive factors and studying the combination of alternative pieces of knowledge. Such adaptations would improve the performance's flexibility, acknowledging it to manage challenging decision-making circumstances in a variability of restraints. A multidimensional methodology for decision-making is also needed. Discovering numerous methods and methods to the same difficulty grants researchers a better identification of the elements that influence decision conclusions. The full expertise of the strong suit and boundaries of each approach may be achieved by comparing the outcomes produced from these diverse methods. To sum up, the q-spherical fuzzy rough CODAS methodology has substantial assurance and impacts substantially on the field of decision-making. Its continuing expansion, association with alternative procedures, and examination of different views will strengthen its relevance and provide a more open and capable decision-making method. In the future, we can apply this approach using different algebraic structures like soft sets, etc.

CONFLICT OF INTEREST

The authors confirm that they do not possess any discernible conflicting financial interests or personal relationships that could appear to impact the research detailed in this paper.

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