Received 16 May 2024, accepted 29 May 2024, date of publication 4 June 2024, date of current version 11 June 2024.
Digital Object Identifier 10.1109/ACCESS.2024.3409420

RESEARCH ARTICLE

Biswapped Networks and Their Maximal Twin-Preserving Subgraphs With a Motivation on Connection Number in Network-Based Approaches

KHAWLAH ALHULWAH[1], ALI N. A. KOAM[2], NASREEN ALMOHANNA[1], ALI AHMAD[3], AND MUHAMMAD AZEEM[4]
[1]Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11623, Saudi Arabia
[2]Department of Mathematics, College of Science, Jazan University, Jazan 45142, Saudi Arabia
[3]Department of Computer Science, College of Engineering and Computer Science, Jazan University, Jazan 45142, Saudi Arabia
[4]Department of Mathematics, Riphah International University, Lahore 54000, Pakistan
Corresponding author: Nasreen Almohanna (nimohanna@imamu.edu.sa)

This work was supported by the Deanship of Scientific Research, Imam Mohammad Ibn Saud Islamic University (IMSIU), under Grant IMSIU-RG23037.

ABSTRACT The use of optoelectronic technology in Optical Transposition Interconnection Systems (OTIS) offers an effective solution to the ongoing problem of storing and sending data with comprehensive information. This is due to the reduced power requirements and broad bandwidth capabilities of optoelectronic systems, which make them well-suited for this task. The integration of radio communication and electrical technology has transformed OTIS into a highly valued network, enhancing the efficiency of existing optoelectronic computers. OTIS is characterized by the swapped network that is formed with the help of path graph $\mathfrak{P}_m$ and denoted as $B(\mathfrak{P}_m)$. This research work focused on certain topological invariants (TIs) about the number of connections between nodes of the graph $B(\mathfrak{P}_m)$ and its largest subgraph that preserves twin nodes ($M(B(\mathfrak{P}_m))$). First Zagreb connection index, geometric-arithmetic connection index, Randić connection index, reduced reciprocal Randić connection index, sum-connectivity connection index, first, second and third redefined Zagreb connection indices are considered in this work two types of biswapped and OTIS networks.

INDEX TERMS Biswapped network, maximal twin-preserving subgraph, connection number, topological indices.

I. INTRODUCTION

An important class of parameters related to graphs is the set of TIs. Discovering various entities such as the characteristics of algebraic structures [1], [2], the analysis of chemical graphs without tentative input [3], [4], [5], and knowing networks through TIs [6], [7], represents a current focus in graph theory, in [8]. Over the past decade, the application of TIs to study several networks, including the biswapped network [9], derived network [10], butterfly network [11], and benes network [12], [13] has grew important attraction. The last 50 years have observed the numerous TIs introduced [14], [15], [16].

Researchers of [17] and [18] proposed the concept of atom bond connectivity index that has been working to examine the stability of linear alkanes and cycloalkanes, and also the strain energy. Authors of [19] and [20], introduced the new TIs with connection-based and determined Zagreb indices

The associate editor coordinating the review of this manuscript and approving it for publication was Yu-Da Lin.

for wheel-related graphs and dendrimers. Another TI namely, the harmonic index that was introduced in 2012 by [21], is associated with the eigenvalues of graphs and remains a current interest. In 2017 [22], introduced new TIs, named ev-valency and ve-valency, and later worked by [23]. Some TIs for algebraic structures [24], frameworks [25], [26], [27] and M-polynomials are determined in [28]. In 2022 [29], simplicial networks are studied with the Sombor index, and general exponential multiplicative ZIs in terms of a unified approach for its extremal values [30], and the Gaussian-based indices named by the Estrada, of graphs [31]. For further progress on the learning of TIs, refer to [32], [33], [34], and [35].

The concept of system analysis is highly valuable in various technical sectors today, including the application of artificial intelligence in the development of sustainable supply chains [36], community strategy plans via social networks [38], postoperative health monitoring [37], and phase change in a storage container [39]. A new direction of research work in [40] starts a valuable link between system investigation and TIs using arithmetical apparatus. In OTIS, processors are interconnected continuously, forming a graph that represents their connections. From a topological perspective, an OTIS must possess several graph-theoretic characteristics. These include a least diameter to facilitate efficient text routing, a minimum valency to manage communication costs, a high linked concentration to handle responsibilities, and the inclusion of paths and cycles of different orders to streamline simulations and text routing. Searching connectivity networks requires a multidisciplinary integration of discrete mathematics, engineering, and, computer science.

Optical transposition interconnection networks (OTISs), which combine optical and electronic technologies for data transport, have been designed and developed [41]. The swapped OTIS is a particular sort of OTIS that is highly successful in transmitting data via networks. It is constructed by swapping the components of a base network and consists of m copies of this network. The biswapped network discussed in [42] is an alternative version of OTIS, where the original OTIS has been replaced with a different one. Consists of a total of $2m$ replicas of a fundamental network, each having a unique connectivity pattern as described in the following section. The idea to use Biswapped networks is preferable to using swapped networks because of their modular architecture.

A significant benefit is that if the basic networking is a Cayley visualization, the swapped structure created from that base graph is also a Cayley graph. This property is lacking in the swapped network. Furthermore, multi-swapped networks exhibit specific attributes that render them well-suited for application as optoelectronic interconnection networks as demonstrated in [43] and [44]. Dynamic graph convolutional network-based prediction of the urban grid-level taxi demand-supply imbalance using GPS trajectories [45], [46], [47]. Robust tube-based model predictive control with Koopman operators [48], [49], [50]. Unified spatial temporal neighbor attention network for dynamic traffic prediction [51], [52], [53]. For further study on this topic see [54], [55], and [56]. On resolvability and domination related parameters of complete multipartite graphs [57], [58], [59]. On the fault-tolerant metric dimension of certain interconnection networks [60], [61].

This article focuses on examining bi-swapped networks and their subgraphs using connection-based TIs. The result involves the preservation of twin subgraphs that recall specific parameters of the original graph. This study establishes certain connection-based topological invariants (TIs) of biswapped networks and finds that their Spanning subgraph that preserves the maximum number of twin vertices exhibits a high valency of similarity. Hence, the examination of twin-preserving subgraphs proves to be a helpful approach in situations where complete data from an intricate network is unattainable. The sole limitation pertains to the existence of Twin-Preserving Spanning Subgraphs (TPSS). The structure of this article is given as In section I, the introduction and background are discussed. Section II provides definitions and required formulas for TIs, as well as the structure of the biswapped OTIS ($B(\mathfrak{P}_m)$) for the reader's understanding. In section III, we introduce novel findings: firstly, we establish the construction of $B(\mathfrak{P}_m)$ and its maximal subgraph that preserves twins ($M(B(\mathfrak{P}_m))$); Furthermore, we assessed the outcomes of TIs using connectivity metrics for both families. In addition, we conducted a comparison of the formulas for TIs in $B(\mathfrak{P}_m)$ and $M(B(\mathfrak{P}_m))$ using both numerical and graphical methods. Section IV is dedicated to the conclusion, which includes a description of the supply and an appraisal of the latest findings.

II. DEFINITIONS AND TERMINOLOGIES

Before presenting the research findings, we will first offer precise definitions and formulas derived from the domain of graph theory. Additionally, we will elucidate the notions of biswapped and swapped OTIS. In addition, we describe the methodology used to produce the results and the tools used to enable a comparison of the outcomes. We employ conventional nomenclature derived from the field of graph theory. For instance, $(V(G), E(G))$ or (V, E) represents the collection of vertices and edges of a graph, respectively. The distance between α and β is denoted by $d(\alpha, \beta)$. Two vertices are considered adjacent if the distance between them is exactly one unit. The valency of a vertex refers to the number of nodes that are directly connected to it. The connection number of a vertex refers to the number of nodes that are at a distance of two from it. In addition, the literature often employs the abbreviations of subgraphs and spanning subgraphs, as evidenced by references such as [62] and [63].

Let a graph G and a particular subgraph which is called a spanning subgraph, and we used the notation S_G are said to be twin-preserving if the twin vertices of G are also present in S_G. The statement can be reformulated in the following

TABLE 1. Some TIs.

Sr. No.	Name of TIs	Abbreviation	Formula
1	First Zagreb connection index	FZCI	$\sum_{uv\in E(G)} (C_u + C_v)$
2	Second Zagreb connection index	SZCI	$\sum_{uv\in E(G)} (C_u \times C_v)$
3	Geometric arithmetic connection index	GACI	$\sum_{uv\in E(G)} \frac{2\sqrt{C_u \times C_v}}{C_u + C_v}$
4	Atom bond connectivity connection index	ABCCI	$\sum_{uv\in E(G)} \sqrt{\frac{C_u + C_v - 2}{C_u \times C_v}}$
5	Symmetric Division Connection Index	SDCI	$\sum_{uv\in E(G)} \left(\frac{\min(C_u,C_v)}{\max(C_u,C_v)} + \frac{\max(C_u,C_v)}{\min(C_u,C_v)}\right)$
6	Harmonic connection index	HCI	$\sum_{uv\in E(G)} \frac{2}{C_u + C_v}$
7	Augmented Zagreb connection index	ACI	$\sum_{uv\in E(G)} \left(\frac{C_u \times C_v}{C_u + C_v - 2}\right)^3$
8	Hyper Zagreb connection index	HZCI	$\sum_{uv\in E(G)} \left(C_u + C_v\right)^2$
9	Randic connection index	RCI	$\sum_{uv\in E(G)} \sqrt{\frac{1}{C_u \times C_v}}$
10	Reciprocal Randić connection index	RRCI	$\sum_{uv\in E(G)} \sqrt{C_u \times C_v}$
11	Reduced Reciprocal Randić connection index	RRRCI	$\sum_{uv\in E(G)} \sqrt{(C_u - 1)(C_v - 1)}$
12	variation of the Randić connection index	VRCI	$\frac{1}{\max\{C_u,C_v\}}$
13	Sum Connectivity connection index	SCCI	$\sum_{uv\in E(G)} \sqrt{\frac{1}{C_u + C_v}}$
14	Forgotten connection index	FCI	$\sum_{uv\in E(G)} \left((C_u)^2 + (C_v)^2\right)$
15	Albertson connection Index	ACI	$\sum_{uv\in E(G)} \lvert C_u - C_v \rvert$
16	First Redefined Zagreb Connection Index	FRZC	$\sum_{uv\in E(G)} \frac{C_u + C_v}{C_u \times C_v}$
17	Second Redefined Zagreb Connection Index	SRZC	$\sum_{uv\in E(G)} \frac{C_u \times C_v}{C_u + C_v}$
18	Third Redefined Zagreb Connection Index	TRZC	$\sum_{uv\in E(G)} (C_u \times C_v)(C_u + C_v)$

manner: in a subgraph S_G that preserves twins, the vertices that are twins in the graph G are likewise twins in S_G [64]. Now, we will review the formulas of TIs listed in Table 1.

Recently, researchers introduced indices that depend on the vertices' Connection Numbers (CN). The cardinality of the vertices that are two distances distant from a vertex's u value is known as its CN.

Definition 1: For a graph G, the first Zagreb connection index ($ZCI_1(G)$) and second Zagreb connection index ($ZCI_2(G)$) are defined as:

$$ZCI_1(G) = \sum_{\alpha\in V(G)} (C_\alpha)^2 = \sum_{\alpha\beta\in E(G)} \left(C_\alpha + C_\beta\right) \quad (1)$$

$$ZCI_2(G) = \sum_{\alpha\beta\in E(G)} \left(C_\alpha \times C_\beta\right) \quad (2)$$

Consider the introduction of swapped and biswapped OTISs in the network context. The swapped OTIS denoted as SO_G, is derived from a base graph G. Its vertex set $V(SO_G)$ and edge set $E(SO_G)$ are defined as follows: $V(SO_G) = \{x_{t,s} : x_{t,s} \in V(G), t, s \in \mathbb{N}\}$ and $E(SO_G) = \{(x_{t,s}, x_{u,v}) : (x_{t,s}, x_{u,v}) \in E(G)\} \cup \{(x_{t,s}, x_{s,t}) : x_{t,s}, x_{s,t} \in V(G), t \neq s\}$. If $|V(G)| = m^2$, then SO_G is formed by m copies of G, with each copy called as a cluster in SO_G.

Vertices in SO_G are denoted as $x_{t,s}$, where ts signifies the address of the vertex at position s in cluster t. In the swapped OTIS(G), edges between clusters exist between $x_{t,s}$ and $x_{s,t}$ when $t \neq s$. The vertex $x_{t,t}$ points to the processor t in cluster t, and no cluster other than t has an edge incident to $x_{t,t}$ [65].

The ev-valency and the ve-valency-based TIs and entropies of swapped OTIS for the base graph $\mathfrak{P}_m$ are computed in [66] and [67]. On the other hand, the biswapped OTIS, denoted as BO_G, is also an OTIS with the following vertex and edge definitions:

$V(BO_G) = \{(0, x, y), (1, x, y) : x, y \in V(G)\}$ and $E(BO_G) = \{((0, x_1, y)(0, x_2, y)), ((1, x_1, y)(1, x_2, y)) : (x_2, x_1) \in E(G), y \in V(G)\} \cup \{((0, x, y)(1, x, y)) : x, y \in V(G)\}$.

In [68], authors established some TIs related to the path graph and the complete graph as the basis graph for the biswapped network. The biswapped network $BO_{\mathfrak{P}_m}$ comprises $2m$ copies of $\mathfrak{P}_m$, with $2m^2$ vertices denoted as $x_{t,s}, y_{t,s}$, where $1 \leq t, s \leq m$. The vertices $x_{t,s}$ and $y_{t,s}$ represent the upper and lower layers, respectively, in $BO_{\mathfrak{P}_m}$. Edges among $x_{t,s}$'s follow the adjacency pattern in the base graph $\mathfrak{P}_m$. Similarly, $y_{t,s}$'s are connected by the adjacency pattern of $\mathfrak{P}_m$.

Edges between $x_{t,s}$'s and $y_{t,s}$'s follow the rule that $x_{t,s}$ is adjacent to $y_{s,t}$ for all $t \neq s$, and $x_{t,t}$ is adjacent to $y_{t,t}$ for all $1 \leq t \leq m$. For a more comprehensive understanding of the construction, refer to [67] and [68].

III. RESULTS AND DISCUSSIONS

In this section, we construct a Spanning subgraph that preserves the maximum number of twin vertices $M(B(\mathfrak{P}_m))$ of a biswapped OTIS over a base graph $\mathfrak{P}_m$. The construction of $M(B(\mathfrak{P}_m))$ involves a direct approach, where an edge is removed, and all the properties of a TPSS are verified. Afterward, through an examination of the composition of $B(\mathfrak{P}_m)$ and the subgraph that was created, specific quantitative metrics, known as connection number-based TIs, are calculated. These TIs provide complete structural specifications of the network through numerical values. The computation of TIs involves inspecting the subgraph for the number of edges, vertices, and connectivity patterns among vertices. The connection numbers-based TIs obtained from the analysis of the network $B(\mathfrak{P}_m)$ and $M(B(\mathfrak{P}_m))$ are listed in Table 8.

A. BISWAPPED NETWORKS $B(\mathfrak{P}_M)$ AND THEIR SPANNING SUBGRAPH THAT PRESERVES THE MAXIMUM NUMBER OF TWIN VERTICES

In this subsection, we choose a spanning subgraph $M(B(\mathfrak{P}_m))$ of $B(\mathfrak{P}_m)$ and prove that this subgraph is twin-preserving and maximal, which is evident from the construction.

The set $V(B(\mathfrak{P}_m))$ can be partitioned as: $\{x_{1,j} : 1 \leq j \leq m\} \cup \{x_{2,j} : 1 \leq j \leq m\} \cup \cdots \cup \{x_{m,j} : 1 \leq j \leq m\} \cup \{y_{1,j} : 1 \leq j \leq m\} \cup \{y_{2,j} : 1 \leq j \leq m\} \cup \cdots \cup \{y_{m,j} : 1 \leq j \leq m\}$.

The neighborhoods of all vertices of $B(\mathfrak{P}_m)$ are: $N(x_{i,1}) = \{y_{1,i}, x_{i,2} : 1 \leq i \leq m\}$, $N(x_{i,j}) = \{y_{j,i}, x_{i,j-1}, x_{i,j+1} : 2 \leq j \leq m-1, 1 \leq i \leq m\}$, $N(x_{i,m}) = \{y_{m,i}, x_{i,m-1} : 1 \leq i \leq m\}$, $N(y_{i,1}) = \{x_{1,i}, y_{i,2} : 1 \leq i \leq m\}$, $N(y_{i,j}) = \{x_{j,i}, y_{i,j-1}, y_{i,j+1} : 2 \leq j \leq m-1, 1 \leq i \leq m\}$, $N(y_{i,m}) = \{x_{m,i}, y_{i,m-1} : 1 \leq i \leq m\}$.

A keen observation reveals that all vertices of $B(\mathfrak{P}_m)$ possess unique neighborhoods, resulting in the existence of singleton twins in $B(\mathfrak{P}_m)$. By removing the edge $x_{2,1}y_{1,2}$, a maximal spanning subgraph $M(B(\mathfrak{P}_m))$ of $B(\mathfrak{P}_m)$ is obtained. The neighborhoods of vertices in $M(B(\mathfrak{P}_m))$ are unaltered, except for $x_{2,1}$ and $y_{1,2}$, whose neighborhoods in $M(B(\mathfrak{P}_m))$ are $N(x_{2,1}) = \{x_{2,2}\}$ and $N(y_{1,2}) = \{y_{1,1}, y_{1,3}\}$. The neighborhoods of $x_{2,1}$ and $y_{1,2}$ in $B(\mathfrak{P}_m)$ are $N(x_{2,1}) = \{x_{2,2}, y_{1,2}\}$ and $N(y_{1,2}) = \{x_{2,1}, y_{1,1}, y_{1,3}\}$. From these neighborhoods, it is clear that $M(B(\mathfrak{P}_m))$ also has singleton twins. Therefore, $M(B(\mathfrak{P}_m))$ is a TPSS of $B(\mathfrak{P}_m)$ and is maximal. Similarly, the removal of any other edge of $B(\mathfrak{P}_m)$ yields a family of twin-preserving subgraphs.

The following equation is a generalized description or formula of the topological indices. In this expression $\phi(G)$ can be any topological descriptor or index and $\Gamma(u, v)$ is the edge type formula of any topological descriptor or index.

$$\phi(G) = \sum_{uv \in E(G)} \Gamma(u, v).$$

By making a small adjustment to the process of constructing the spanning subgraph (specifically, by eliminating two or more particular edges), we can generate a diverse range of subgraphs, which opens up intriguing possibilities for further investigation.

B. BISWAPPED GRAPH $B(\mathfrak{P}_M)$

In this section, we computed main results of the related to the biswapped graph $B(\mathfrak{P}_m)$.

Lemma 1: Let $B(\mathfrak{P}_m)$ be a biswapped network with $m \geq 6$. Then $\phi(B(\mathfrak{P}_m)) = 3m^2\Big(\Gamma(6,6)\Big) + 2m\Big(4\Gamma(4,5) + 2\Gamma(5,5) + 4\Gamma(5,6) - 11\Gamma(6,6)\Big) + 4\Big(\Gamma(3,3) + 2\Gamma(3,4) + 2\Gamma(4,4) - 4\Gamma(4,5) - 4\Gamma(5,5) - 6\Gamma(5,6) + 9\Gamma(6,6)\Big)$.

Proof: Let $B(\mathfrak{P}_m)$ be a biswapped network has the maximum valency 3 and maximum connection number 6. The total number of vertices and edges are: $2m^2$ and $3m^2 - 2m$, respectively. In the graph of $B(\mathfrak{P}_m)$, connection numbers of vertices are 3, 4, 5 or 6.

Let $E_{u,v}$ be the edge partition with end vertices have connection number u and v. The edge partition function for the vertices of $B(\mathfrak{P}_m)$ based on their connection numbers will be:

$$E_{3,3} = \{uv \in E(\mathfrak{P}_m) : u = 3, v = 3\} \quad (3)$$
$$E_{3,4} = \{uv \in E(\mathfrak{P}_m) : u = 3, v = 4\} \quad (4)$$
$$E_{4,4} = \{uv \in E(\mathfrak{P}_m) : u = 4, v = 4\} \quad (5)$$
$$E_{4,5} = \{uv \in E(\mathfrak{P}_m) : u = 4, v = 5\} \quad (6)$$
$$E_{5,5} = \{uv \in E(\mathfrak{P}_m) : u = 5, v = 5\} \quad (7)$$
$$E_{5,6} = \{uv \in E(\mathfrak{P}_m) : u = 5, v = 6\} \quad (8)$$
$$E_{6,6} = \{uv \in E(\mathfrak{P}_m) : u = 6, v = 6\} \quad (9)$$

Note that $E(B(\mathfrak{P}_m)) = E_{3,3} \cup E_{3,4} \cup E_{4,4} \cup E_{4,5} \cup E_{5,5} \cup E_{5,6} \cup E_{6,6}$. The number of edges incident to two vertices of connection number 3 are 4, so $|E_{3,3}| = 4$. The number of edges incident to one vertex of connection number 3 and other vertex of connection number 4 are 8. So $|E_{3,4}| = 8$. The number of edges incident to one vertex of connection number 4 and other vertex of connection number 4, 5 are 4, $8m-16$, respectively. So $|E_{4,4}| = 8$ and $|E_{4,5}| = 8m-16$. The number of edges incident to one vertex of connection number 5 and other vertex of connection number 5, 6 are $4m-16$, $8m-24$, respectively. So $|E_{5,5}| = 4m-16$, $|E_{5,6}| = 8m-24$. The number of edges incident to two vertices of connection number 6 are $3m^2 - 22m + 36$, so $|E_{6,6}| = 3m^2 - 22m + 36$.

Hence, $\phi(B(\mathfrak{P}_m)) =$

$$\sum_{uv \in E(\mathfrak{P}_m)} \Gamma(u, v)$$
$$= \sum_{uv \in E_{3,3}} \Gamma(3,3) + \sum_{uv \in E_{3,4}} \Gamma(3,4) + \sum_{uv \in E_{4,4}} \Gamma(4,4)$$
$$+ \sum_{uv \in E_{4,5}} \Gamma(4,5) + \sum_{uv \in E_{5,5}} \Gamma(5,5) + \sum_{uv \in E_{5,6}} \Gamma(5,6)$$

$$+\sum_{uv\in E_{6,6}}\Gamma(6,6)$$
$$=(4)\,\Gamma(3,3)+(8)\,\Gamma(3,4)+(8)\,\Gamma(4,4)+(8m-16)\,\Gamma(4,5)+(4m-16)\,\Gamma(5,5)+(8m-24)\,\Gamma(5,6)+\left(3m^2-22m+36\right)\Gamma(6,6)$$

After simplification, we get

$$\phi(B(\mathfrak{P}_m))=3m^2\Big(\Gamma(6,6)\Big)+2m\Big(4\Gamma(4,5)+2\Gamma(5,5)+4\Gamma(5,6)-11\Gamma(6,6)\Big)+4\Big(\Gamma(3,3)+2\Gamma(3,4)+2\Gamma(4,4)-4\Gamma(4,5)-4\Gamma(5,5)-6\Gamma(5,6)+9\Gamma(6,6)\Big).$$

□

So, by using the Lemma 1 and putting different functions instead of $\Gamma(u,v)$ one can produce different types of connection-based TIs.

In the following theorems, we determined the connection-based TIs of Biswapped networks.

Theorem 1: Let $B(\mathfrak{P}_m)$ be a biswapped network with $m \geq 6$, then the first Zagreb connection index

$$FZCI(B(\mathfrak{P}_m))=36m^2-64m+8$$

the second Zagreb connection index

$$SZCI(B(\mathfrak{P}_m))=108m^2-292m+116.$$

Proof: From the Definition 1, the first Zagreb connection index $FZCI(B(\mathfrak{P}_m))$ of $B(\mathfrak{P}_m)$, we obtain $\Gamma(u,v)=(u+v)$. So $\Gamma(3,3)=6$, $\Gamma(3,4)=7$, $\Gamma(4,4)=8$, $\Gamma(4,5)=9$, $\Gamma(5,5)=10$, $\Gamma(5,6)=11$, and $\Gamma(6,6)=12$. Thus by Lemma 1,

$$FZCI(B(\mathfrak{P}_m))=36m^2-64m+8.$$

From the Definition 1, the first Zagreb connection index $SZCI(B(\mathfrak{P}_m))$ of $B(\mathfrak{P}_m)$, we obtain $\Gamma(u,v)=(u\times v)$. So $\Gamma(3,3)=9$, $\Gamma(3,4)=12$, $\Gamma(4,4)=16$, $\Gamma(4,5)=20$, $\Gamma(5,5)=25$, $\Gamma(5,6)=30$, and $\Gamma(6,6)=36$. Thus by Lemma 1,

$$SZCI(B(\mathfrak{P}_m))=108m^2-292m+116.$$

□

Theorem 2: Let $B(\mathfrak{P}_m)$ be a biswapped network with $m \geq 6$, then the geometric arithmetic connection index

$$GACI(B(\mathfrak{P}_m))=3\,m^2+\left(\frac{32\sqrt{5}}{9}-18+\frac{16\sqrt{30}}{11}\right)m+32+\frac{32\sqrt{3}}{7}-\frac{64\sqrt{5}}{9}-\frac{48\sqrt{30}}{11}$$

the atom bond connectivity connection index

$$ABCCI(B(\mathfrak{P}_m))=\frac{\sqrt{10}\,m^2}{2}+\left(\frac{4\sqrt{35}}{5}+\frac{8\sqrt{2}}{5}+\frac{4\sqrt{30}}{5}-\frac{11\sqrt{10}}{3}\right)m+\frac{8}{3}+\frac{4\sqrt{15}}{3}+2\sqrt{6}-\frac{8\sqrt{35}}{5}-\frac{32\sqrt{2}}{5}-\frac{12\sqrt{30}}{5}+6\sqrt{10}$$

the symmetric division connection index

$$SDCI(B(\mathfrak{P}_m))=6m^2-\frac{10}{3}m-\frac{14}{15}$$

the harmonic connection index

$$HCI(B(\mathfrak{P}_m))=\frac{1}{2}m^2+\frac{181}{495}m+\frac{1732}{3465}$$

the augmented Zagreb connection index

$$AZCI(B(\mathfrak{P}_m))=\frac{17496}{125}m^2-\frac{62452697407}{148176000}m+\frac{8784103777}{37044000}$$

the hyper Zagreb connection index

$$HZCI(B(\mathfrak{P}_m))=432m^2-1152m+432.$$

Proof: By using the Lemma 1 and Table 2 in the formula that are given in Table 1, we get the required results.

□

Theorem 3: Let $B(\mathfrak{P}_m)$ be a biswapped network with $m \geq 6$, then the Randić connection index

$$RCI(B(\mathfrak{P}_m))=\frac{m^2}{2}+\left(\frac{4\sqrt{5}}{5}-\frac{43}{15}+\frac{4\sqrt{30}}{15}\right)m+\frac{92}{15}+\frac{4\sqrt{3}}{3}-\frac{8\sqrt{5}}{5}-\frac{4\sqrt{30}}{5}$$

the Reciprocal Randić connection index

$$RRCI(B(\mathfrak{P}_m))=18m^2+\left(16\sqrt{5}-112+8\sqrt{30}\right)m+180+16\sqrt{3}-32\sqrt{5}-24\sqrt{30}$$

the Reduced Reciprocal Randić connection index

$$RRRCI(B(\mathfrak{P}_m))=15m^2+\left(16\sqrt{3}+16\sqrt{5}-94\right)m+148+8\sqrt{6}-32\sqrt{3}-48\sqrt{5}$$

variation of the Randić connection index

$$VRCI(B(\mathfrak{P}_m))=\frac{1}{2}m^2+\frac{1}{15}m+\frac{14}{15}.$$

Proof: By using the Lemma 1 and Table 3 in the formula that are given in Table 1, we get the required results.

□

Theorem 4: Let $B(\mathfrak{P}_m)$ be a biswapped network with $m \geq 6$, then the Sum Connectivity connection index

$$SCCI(B(\mathfrak{P}_m))=\frac{\sqrt{3}\,m^2}{2}+\left(\frac{8}{3}+\frac{2\sqrt{10}}{5}+\frac{8\sqrt{11}}{11}-\frac{11\sqrt{3}}{3}\right)m+\frac{2\sqrt{6}}{3}+\frac{8\sqrt{7}}{7}+2\sqrt{2}-\frac{16}{3}-\frac{8\sqrt{10}}{5}-\frac{24\sqrt{11}}{11}+6\sqrt{3}$$

the Forgotten connection index

$$FCI(B(\mathfrak{P}_m))=216m^2-568m+200$$

TABLE 2. Values of $\Gamma(u, v)$ with respect to Connection numbers for Theorem 2.

$\Gamma(u,v)$	(3,3)	(3,4)	(4,4)	(4,5)	(5,5)	(5,6)	(6,6)
$\frac{2\sqrt{u\times v}}{u+v}$	1	$\frac{4\sqrt{3}}{7}$	1	$\frac{4\sqrt{5}}{9}$	1	$\frac{2\sqrt{30}}{11}$	1
$\sqrt{\frac{u+v-2}{u\times v}}$	$\frac{2}{3}$	$\frac{\sqrt{15}}{6}$	$\frac{\sqrt{6}}{4}$	$\frac{\sqrt{35}}{10}$	$\frac{2\sqrt{2}}{5}$	$\frac{\sqrt{30}}{10}$	$\frac{\sqrt{10}}{6}$
$\frac{\min(u,v)}{\max(u,v)}+\frac{\max(u,v)}{\min(u,v)}$	2	$\frac{25}{12}$	2	$\frac{41}{20}$	2	$\frac{61}{31}$	2
$\frac{2}{u+v}$	$\frac{1}{3}$	$\frac{2}{7}$	$\frac{1}{4}$	$\frac{2}{9}$	$\frac{1}{5}$	$\frac{2}{11}$	$\frac{1}{6}$
$\left(\frac{u\times v}{u+v-2}\right)^3$	$\frac{729}{64}$	$\frac{1728}{125}$	$\frac{512}{27}$	$\frac{8000}{343}$	$\frac{15625}{512}$	$\frac{1000}{27}$	$\frac{5832}{125}$
$\left(u+v\right)^2$	36	49	64	81	100	121	144

TABLE 3. Values of $\Gamma(u, v)$ with respect to Connection numbers for Theorem 3.

$\Gamma(u,v)$	(3,3)	(3,4)	(4,4)	(4,5)	(5,5)	(5,6)	(6,6)
$\sqrt{\frac{1}{u\times v}}$	$\frac{1}{3}$	$\frac{1}{\sqrt{12}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{20}}$	$\frac{1}{5}$	$\frac{1}{\sqrt{30}}$	$\frac{1}{6}$
$\sqrt{u\times v}$	3	$\sqrt{12}$	4	$\sqrt{20}$	5	$\sqrt{30}$	6
$\sqrt{(u-1)\times(v-1)}$	2	$\sqrt{6}$	3	$\sqrt{12}$	4	$\sqrt{20}$	5
$\frac{1}{\max\{u,v\}}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{6}$

the Albertson connection Index

$$ACI(B(\mathfrak{P}_m)) = 16m - 32$$

the first redefined Zagreb connection index

$$FRZC(B(\mathfrak{P}_m)) = m^2 + \frac{4}{5}m + \frac{14}{15}$$

the second redefined Zagreb connection index

$$SRZC(B(\mathfrak{P}_m)) = 9m^2 - \frac{1624}{99}m + \frac{1874}{693}$$

the third redefined Zagreb connection index

$$TRZC(B(\mathfrak{P}_m)) = 1296m^2 - 4424m + 2664.$$

Proof: By using the Lemma 1 and Table 4 in the formula that are given in Table 1, we get the required results. □

C. SPANNING SUBGRAPH THAT PRESERVES THE MAXIMUM NUMBER OF TWIN VERTICES ($M(B(\mathfrak{P}_M))$) OF $B(\mathfrak{P}_M)$

Lemma 2: Let $M(B(\mathfrak{P}_m))$ be a Spanning subgraph that preserves the maximum number of twin vertices of $B(\mathfrak{P}_m)$ with $m \geq 6$. Then $\mathcal{T}(M(B(\mathfrak{P}_m))) = 3m^2\Big(\Gamma(6,6)\Big) + 2m\Big(4\Gamma(4,5)+2\Gamma(5,5)+4\Gamma(5,6)-11\Gamma(6,6)\Big)+\Big(\Gamma(2,5)+5\Gamma(3,3)+7\Gamma(3,4)+\Gamma(3,5)+7\Gamma(4,4)-18\Gamma(4,5)-16\Gamma(5,5)-24\Gamma(5,6)+36\Gamma(6,6)\Big)$.

Proof: Let $M(B(\mathfrak{P}_m))$ be a Spanning subgraph that preserves the maximum number of twin vertices of $B(\mathfrak{P}_m)$ has the minimum valency 1 and maximum connection number 6. The total number of vertices and edges are: $2m^2$ and $3m^2-2m-1$, respectively. In the graph of $M(B(\mathfrak{P}_m))$, connection numbers of vertices are 2, 3, 4, 5 and 6 and their cardinalities are: 1, 9, $4m-2$, $8m-24$ and $2m^2-12m+16$, respectively. Let $E_{u,v}$ be the edge partition with end vertices have connection number u and v. The edge partition function for the vertices of $M(B(\mathfrak{P}_m))$ based on their connection numbers will be:

$$E_{2,5} = \{uv \in E(\mathfrak{P}_m) : u = 2, v = 5\} \tag{10}$$
$$E_{3,3} = \{uv \in E(\mathfrak{P}_m) : u = 3, v = 3\} \tag{11}$$
$$E_{3,4} = \{uv \in E(\mathfrak{P}_m) : u = 3, v = 4\} \tag{12}$$
$$E_{3,5} = \{uv \in E(\mathfrak{P}_m) : u = 3, v = 5\} \tag{13}$$
$$E_{4,4} = \{uv \in E(\mathfrak{P}_m) : u = 4, v = 4\} \tag{14}$$
$$E_{4,5} = \{uv \in E(\mathfrak{P}_m) : u = 4, v = 5\} \tag{15}$$
$$E_{5,5} = \{uv \in E(\mathfrak{P}_m) : u = 5, v = 5\} \tag{16}$$
$$E_{5,6} = \{uv \in E(\mathfrak{P}_m) : u = 5, v = 6\} \tag{17}$$
$$E_{6,6} = \{uv \in E(\mathfrak{P}_m) : u = 6, v = 6\} \tag{18}$$

Note that $E(M(B(\mathfrak{P}_m))) = E_{2,5} \cup E_{3,3} \cup E_{3,4} \cup E_{3,5} \cup E_{4,4} \cup E_{4,5} \cup E_{5,5} \cup E_{5,6} \cup E_{6,6}$. The number of edges incident to one vertex of connection number 2 and other vertex of connection number 5 are 1. So $|E_{2,5}| = 1$. The number of edges incident to two vertices of connection number 3 are 5, so $|E_{3,3}| = 5$. The number of edges incident to one vertex of connection number 3 and other vertex of connection number 4 and 5 are 7 and 1, respectively. So $|E_{3,4}| = 7$ and $|E_{3,5}| = 1$. The number of edges incident to one vertex of connection number 4 and other vertex of connection number 4, 5 are 7, $8m-18$, respectively. So $|E_{4,4}| = 7$ and $|E_{4,5}| = 8m-18$. The number of edges incident to one vertex of connection number 5 and other vertex of connection number 5, 6 are $4m-16$, $8m-24$, respectively. So $|E_{5,5}| = 4m-16$, $|E_{5,6}| = 8m-24$. The number of edges incident to two vertices of connection number 6 are $3m^2-22m+36$, so $|E_{6,6}| = 3m^2-22m+36$.

Hence, $\psi(M(B(\mathfrak{P}_m))) =$

$$\sum_{uv\in E(\mathfrak{P}_m)} \Gamma(u,v)$$
$$= \sum_{uv\in E_{2,5}} \Gamma(2,5) + \sum_{uv\in E_{3,3}} \Gamma(3,3) + \sum_{uv\in E_{3,4}} \Gamma(3,4)$$

TABLE 4. Values of $\Gamma(u, v)$ with respect to Connection numbers for Theorem 4.

$\Gamma(u,v)$	(3,3)	(3,4)	(4,4)	(4,5)	(5,5)	(5,6)	(6,6)
$\frac{1}{\sqrt{u+v}}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{7}}$	$\frac{1}{\sqrt{8}}$	$\frac{1}{3}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{11}}$	$\frac{1}{\sqrt{12}}$
$\lvert u-v\rvert$	0	1	0	1	0	1	0
u^2+v^2	18	25	32	41	50	61	72
$\frac{u+v}{u\times v}$	$\frac{2}{3}$	$\frac{7}{12}$	2	$\frac{9}{20}$	$\frac{2}{5}$	$\frac{11}{30}$	$\frac{1}{3}$
$\frac{u\times v}{u+v}$	$\frac{3}{2}$	$\frac{12}{7}$	2	$\frac{20}{9}$	$\frac{5}{2}$	$\frac{30}{11}$	3
$(u+v)(u\times v)$	54	84	128	180	250	330	432

$$
\begin{aligned}
&+\sum_{uv\in E_{3,5}}\Gamma(3,5)+\sum_{uv\in E_{4,4}}\Gamma(4,4)\\
&+\sum_{uv\in E_{4,5}}\Gamma(4,5)\\
&+\sum_{uv\in E_{5,5}}\Gamma(5,5)+\sum_{uv\in E_{5,6}}\Gamma(5,6)+\sum_{uv\in E_{6,6}}\Gamma(6,6)\\
&=(1)\,\Gamma(2,5)+(5)\,\Gamma(3,3)+(7)\,\Gamma(3,4)\\
&+(1)\,\Gamma(3,5)+(7)\,\Gamma(4,4)\\
&+(8m-18)\,\Gamma(4,5)+(4m-16)\,\Gamma(5,5)\\
&+(8m-24)\,\Gamma(5,6)+\Big(3m^2-22m+36\Big)\,\Gamma(6,6)
\end{aligned}
$$

After simplification, we get

$$
\begin{aligned}
&\psi(M(B(\mathfrak{P}_m)))\\
&=3m^2\Big(\Gamma(6,6)\Big)+2m\Big(4\Gamma(4,5)+2\Gamma(5,5)\\
&+4\Gamma(5,6)-11\Gamma(6,6)\Big)+\Big(\Gamma(2,5)+5\Gamma(3,3)+7\Gamma(3,4)\\
&+\Gamma(3,5)+7\Gamma(4,4)-18\Gamma(4,5)\\
&-16\Gamma(5,5)-24\Gamma(5,6)+36\Gamma(6,6)\Big).
\end{aligned}
$$

□

In the following theorems, we determined the connection-based TIs of Spanning subgraph that preserves the maximum number of twin vertices of $B(\mathfrak{P}_m)$.

Theorem 5: Let $M(B(\mathfrak{P}_m))$ be a Spanning subgraph that preserves the maximum number of twin vertices of $B(\mathfrak{P}_m)$ with $m \geq 6$. Then

the first Zagreb connection index

$$FZCI(M(B(\mathfrak{P}_m))) = 36m^2 - 64m - 4$$

the second Zagreb connection index

$$SZCI(M(B(\mathfrak{P}_m))) = 108m^2 - 292m + 82.$$

Proof: From the Definition 1, the first Zagreb connection index $FZCI(M(B(\mathfrak{P}_m)))$ of $M(B(\mathfrak{P}_m))$, we obtain $\Gamma(u,v) = (u+v)$. So $\Gamma(2,5) = 7$, $\Gamma(3,3) = 6$, $\Gamma(3,4) = 7$, $\Gamma(3,5) = 8$, $\Gamma(4,4) = 8$, $\Gamma(4,5) = 9$, $\Gamma(5,5) = 10$, $\Gamma(5,6) = 11$, and $\Gamma(6,6) = 12$. Thus by Lemma 1,

$$FZCI(M(B(\mathfrak{P}_m))) = 36m^2 - 64m - 4.$$

From the Definition 1, the first Zagreb connection index $SZCI(M(B(\mathfrak{P}_m)))$ of $M(B(\mathfrak{P}_m))$, we obtain $\Gamma(u,v) = (u \times v)$. So $\Gamma(2,5) = 10$, $\Gamma(3,3) = 9$, $\Gamma(3,4) = 12$, $\Gamma(3,5) = 15$, $\Gamma(4,4) = 16$, $\Gamma(4,5) = 20$, $\Gamma(5,5) = 25$, $\Gamma(5,6) = 30$, and $\Gamma(6,6) = 36$. Thus by Lemma 1,

$$SZCI(M(B(\mathfrak{P}_m))) = 108m^2 - 292m + 82.$$

□

Theorem 6: Let $M(B(\mathfrak{P}_m))$ be a Spanning subgraph that preserves the maximum number of twin vertices of $B(\mathfrak{P}_m)$ with $m \geq 6$. Then

the geometric arithmetic connection index

$$
\begin{aligned}
GACI(M(B(\mathfrak{P}_m))) &= 3m^2 + \left(\frac{32\sqrt{5}}{9} - 18 + \frac{16\sqrt{30}}{11}\right)m\\
&+ \frac{2\sqrt{10}}{7} + 32 + 4\sqrt{3} + \frac{\sqrt{15}}{4} - 8\sqrt{5} - \frac{48\sqrt{30}}{11}
\end{aligned}
$$

the atom bond connectivity connection index

$$
\begin{aligned}
&ABCCI(M(B(\mathfrak{P}_m))) =\\
&\frac{\sqrt{10}\,m^2}{2} + \left(\frac{4\sqrt{35}}{5} + \frac{8\sqrt{2}}{5} + \frac{4\sqrt{30}}{5} - \frac{11\sqrt{10}}{3}\right)m\\
&- \frac{59\sqrt{2}}{10} + \frac{10}{3} + \frac{7\sqrt{15}}{6} + \frac{31\sqrt{10}}{5}\\
&+ \frac{7\sqrt{6}}{4} - \frac{9\sqrt{35}}{5} - \frac{12\sqrt{30}}{5}
\end{aligned}
$$

the symmetric division connection index

$$SDCI(M(B(\mathfrak{P}_m))) = 6m^2 - \frac{10}{3}m - \frac{39}{20}$$

the harmonic connection index

$$HCI(M(B(\mathfrak{P}_m))) = \frac{1}{2}m^2 + \frac{181}{495}m + \frac{449}{1155}$$

the augmented Zagreb connection index

$$
\begin{aligned}
AZCI(M(B(\mathfrak{P}_m))) &= \frac{17496}{125}m^2 - \frac{62452697407}{148176000}m\\
&+ \frac{14277324667}{74088000}
\end{aligned}
$$

the hyper Zagreb connection index

$$HZCI(M(B(\mathfrak{P}_m))) = 432m^2 - 1152m + 306.$$

Proof: By using the Lemma 2 and Table 5 in the formula that are given in Table 1, we get the required results. □

Theorem 7: Let $M(B(\mathfrak{P}_m))$ be a Spanning subgraph that preserves the maximum number of twin vertices of $B(\mathfrak{P}_m)$ with $m \geq 6$. Then the Randić connection index

$$RCI(M(B(\mathfrak{P}_m)))$$

TABLE 5. Values of $\Gamma(u,v)$ with respect to Connection numbers for Theorem 6.

$\Gamma(u,v)$	(2,5)	(3,3)	(3,4)	(3,5)	(4,4)	(4,5)	(5,5)	(5,6)	(6,6)
$\frac{2\sqrt{u\times v}}{u+v}$	$\frac{2\sqrt{10}}{7}$	1	$\frac{4\sqrt{3}}{7}$	$\frac{\sqrt{15}}{4}$	1	$\frac{4\sqrt{5}}{9}$	1	$\frac{2\sqrt{30}}{11}$	1
$\sqrt{\frac{u+v-2}{u\times v}}$	$\frac{\sqrt{2}}{2}$	$\frac{2}{3}$	$\frac{\sqrt{15}}{6}$	$\frac{\sqrt{10}}{5}$	$\frac{\sqrt{6}}{4}$	$\frac{\sqrt{35}}{10}$	$\frac{2\sqrt{2}}{5}$	$\frac{\sqrt{30}}{10}$	$\frac{\sqrt{10}}{6}$
$\frac{\min(u,v)}{\max(u,v)}+\frac{\max(u,v)}{\min(u,v)}$	$\frac{29}{10}$	2	$\frac{25}{12}$	$\frac{34}{15}$	2	$\frac{41}{20}$	2	$\frac{61}{31}$	2
$\frac{2}{u+v}$	$\frac{2}{7}$	$\frac{1}{3}$	$\frac{2}{7}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{2}{9}$	$\frac{1}{5}$	$\frac{2}{11}$	$\frac{1}{6}$
$\left(\frac{u\times v}{u+v-2}\right)^3$	8	$\frac{729}{64}$	$\frac{1728}{125}$	$\frac{125}{8}$	$\frac{512}{27}$	$\frac{8000}{343}$	$\frac{15625}{512}$	$\frac{1000}{27}$	$\frac{5832}{125}$
$(u+v)^2$	49	36	49	64	64	81	100	121	144

TABLE 6. Values of $\Gamma(u,v)$ with respect to Connection numbers for Theorem 7.

$\Gamma(u,v)$	(2,5)	(3,3)	(3,4)	(3,5)	(4,4)	(4,5)	(5,5)	(5,6)	(6,6)
$\sqrt{\frac{1}{u\times v}}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{3}$	$\frac{1}{\sqrt{12}}$	$\frac{1}{\sqrt{15}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{20}}$	$\frac{1}{5}$	$\frac{1}{\sqrt{30}}$	$\frac{1}{6}$
$\sqrt{u\times v}$	$\sqrt{10}$	3	$\sqrt{12}$	$\sqrt{15}$	4	$\sqrt{20}$	5	$\sqrt{30}$	6
$\sqrt{(u-1)\times(v-1)}$	2	2	$\sqrt{6}$	$\sqrt{8}$	3	$\sqrt{12}$	4	$\sqrt{20}$	5
$\frac{1}{\max\{u,v\}}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{6}$

TABLE 7. Values of $\Gamma(u,v)$ with respect to Connection numbers for Theorem 8.

$\Gamma(u,v)$	(2,5)	(3,3)	(3,4)	(3,5)	(4,4)	(4,5)	(5,5)	(5,6)	(6,6)
$\frac{1}{\sqrt{u+v}}$	$\frac{1}{\sqrt{7}}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{7}}$	$\frac{1}{\sqrt{8}}$	$\frac{1}{\sqrt{8}}$	$\frac{1}{3}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{11}}$	$\frac{1}{\sqrt{12}}$
$\lvert u-v\rvert$	3	0	1	2	0	1	0	1	0
u^2+v^2	29	18	25	34	32	41	50	61	72
$\frac{u+v}{u\times v}$	$\frac{7}{10}$	$\frac{2}{3}$	$\frac{7}{12}$	$\frac{8}{15}$	2	$\frac{9}{20}$	$\frac{2}{5}$	$\frac{11}{30}$	$\frac{1}{3}$
$\frac{u\times v}{u+v}$	$\frac{10}{7}$	$\frac{3}{2}$	$\frac{12}{7}$	$\frac{15}{8}$	2	$\frac{20}{9}$	$\frac{5}{2}$	$\frac{30}{11}$	3
$(u+v)(u\times v)$	70	54	84	120	128	180	250	330	432

$$= \frac{m^2}{2} + \left(\frac{4\sqrt{5}}{5} - \frac{43}{15} + \frac{4\sqrt{30}}{15}\right)m + \frac{\sqrt{10}}{10} + \frac{373}{60} + \frac{7\sqrt{3}}{6} + \frac{\sqrt{15}}{15} - \frac{9\sqrt{5}}{5} - \frac{4\sqrt{30}}{5}$$

the Reciprocal Randić connection index

$$RRCI(M(B(\mathfrak{P}_m))) = 18m^2 + \left(16\sqrt{5} - 112 + 8\sqrt{30}\right)m + \sqrt{10} + 179 + 14\sqrt{3} + \sqrt{15} - 36\sqrt{5} - 24\sqrt{30}$$

the Reduced Reciprocal Randić connection index

$$RRRCI(M(B(\mathfrak{P}_m))) = 15m^2 + \left(16\sqrt{3} + 16\sqrt{5} - 94\right)m + 149 + 7\sqrt{6} + 2\sqrt{2} - 36\sqrt{3} - 48\sqrt{5}$$

variation of the Randić connection index

$$VRCI(M(B(\mathfrak{P}_m))) = \frac{1}{2}m^2 + \frac{1}{15}m + \frac{23}{30}.$$

Proof: By using the Lemma 2 and Table 6 in the formula that are given in Table 1, we get the required results. □

Theorem 8: Let $M(B(\mathfrak{P}_m))$ be a Spanning subgraph that preserves the maximum number of twin vertices of $B(\mathfrak{P}_m)$ with $m \geq 6$. Then the Sum Connectivity connection index

$$SCCI(M(B(\mathfrak{P}_m))) = \frac{\sqrt{3}\,m^2}{2} + \left(\frac{8}{3} + \frac{2\sqrt{10}}{5} + \frac{8\sqrt{11}}{11} - \frac{11\sqrt{3}}{3}\right)m + \frac{8\sqrt{7}}{7} + \frac{5\sqrt{6}}{6} + 2\sqrt{2} - 6 - \frac{8\sqrt{10}}{5} - \frac{24\sqrt{11}}{11} + 6\sqrt{3}$$

the Albertson connection Index

$$ACI(M(B(\mathfrak{P}_m))) = 16m - 30$$

the Forgotten connection index

$$FCI(M(B(\mathfrak{P}_m))) = 216m^2 - 568m + 142$$

the first redefined Zagreb connection index

$$FRZC(M(B(\mathfrak{P}_m))) = m^2 + \frac{4}{5}m + \frac{17}{20}$$

the second redefined Zagreb connection index

$$SRZC(M(B(\mathfrak{P}_m))) = 9m^2 - \frac{1624}{99}m - \frac{401}{616}$$

the third redefined Zagreb connection index

$$TRZC(M(B(\mathfrak{P}_m))) = 1296m^2 - 4424m + 2336.$$

Proof: By using the Lemma 2 and Table 7 in the formula that are given in Table 1, we get the required results. □

TABLE 8. Comparison of TIs of biswapped graph $B(\mathfrak{P}_m)$ and its TPSS $M(B(\mathfrak{P}_m))$ with respect to Connection numbers.

Abbreviation of TIs	TIs values of $B(\mathfrak{P}_m)$	TIs values of $M(B(\mathfrak{P}_m))$	Difference
FZCI	$36m^2-64m+8$	$36m^2-64m-4$	12
SZCI	$108m^2-292m+116$	$108m^2-292m+82$	34
GACI	$3m^2+(\frac{32\sqrt{5}}{9}-18+\frac{16\sqrt{30}}{11})m+32+\frac{32\sqrt{3}}{7}-\frac{64\sqrt{5}}{9}-\frac{48\sqrt{30}}{11}$	$3m^2+(\frac{32\sqrt{5}}{9}-18+\frac{16\sqrt{30}}{11})m+\frac{2\sqrt{10}}{7}+32+4\sqrt{3}+\frac{\sqrt{15}}{4}-8\sqrt{5}-\frac{48\sqrt{30}}{11}$	$\frac{4\sqrt{3}}{7}+\frac{8\sqrt{5}}{9}-\frac{2\sqrt{10}}{7}-\frac{\sqrt{15}}{4}$
ABCCI	$\frac{\sqrt{10}\,m^2}{2}+(\frac{4\sqrt{35}}{5}+\frac{8\sqrt{2}}{5}+\frac{4\sqrt{30}}{5}-\frac{11\sqrt{10}}{3})m+\frac{8}{3}+\frac{4\sqrt{15}}{3}+2\sqrt{6}-\frac{8\sqrt{35}}{5}-\frac{32\sqrt{2}}{5}-\frac{12\sqrt{30}}{5}+6\sqrt{10}$	$\frac{\sqrt{10}\,m^2}{2}+(\frac{4\sqrt{35}}{5}+\frac{8\sqrt{2}}{5}+\frac{4\sqrt{30}}{5}-\frac{11\sqrt{10}}{3})m-\frac{59\sqrt{2}}{10}+\frac{10}{3}+\frac{7\sqrt{15}}{6}+\frac{31\sqrt{10}}{5}+\frac{7\sqrt{6}}{4}-\frac{9\sqrt{35}}{5}-\frac{12\sqrt{30}}{5}$	$-\frac{2}{3}+\frac{\sqrt{15}}{6}+\frac{\sqrt{6}}{4}+\frac{\sqrt{35}}{5}-\frac{\sqrt{2}}{2}-\frac{31\sqrt{10}}{5}$
SDCI	$6m^2-\frac{10}{3}m-\frac{14}{15}$	$6m^2-\frac{10}{3}m-\frac{39}{20}$	$\frac{61}{60}$
HCI	$\frac{1}{2}m^2+\frac{181}{495}m+\frac{1732}{3465}$	$\frac{1}{2}m^2+\frac{181}{495}m+\frac{449}{1155}$	$\frac{1}{9}$
ACI	$\frac{17496}{125}m^2-\frac{62452697407}{148176000}m+\frac{8784103777}{37044000}$	$\frac{17496}{125}m^2-\frac{62452697407}{148176000}m+\frac{14277324667}{74088000}$	$\frac{3290882887}{74088000}$
HZCI	$432m^2-1152m+432$	$432m^2-1152m+306$	126
RCI	$\frac{m^2}{2}+(\frac{4\sqrt{5}}{5}-\frac{43}{15}+\frac{4\sqrt{30}}{15})m+\frac{92}{15}+\frac{4\sqrt{3}}{3}-\frac{8\sqrt{5}}{5}-\frac{4\sqrt{30}}{5}$	$\frac{m^2}{2}+(\frac{4\sqrt{5}}{5}-\frac{43}{15}+\frac{4\sqrt{30}}{15})m+\frac{\sqrt{10}}{10}+\frac{373}{60}+\frac{7\sqrt{3}}{6}+\frac{\sqrt{15}}{15}-\frac{9\sqrt{5}}{5}-\frac{4\sqrt{30}}{5}$	$-\frac{1}{12}+\frac{\sqrt{3}}{6}+\frac{\sqrt{5}}{5}-\frac{\sqrt{10}}{10}-\frac{\sqrt{15}}{15}$
RRCI	$18m^2+(16\sqrt{5}-112+8\sqrt{30})m+180+16\sqrt{3}-32\sqrt{5}-24\sqrt{30}$	$18m^2+(16\sqrt{5}-112+8\sqrt{30})m+\sqrt{10}+179+14\sqrt{3}+\sqrt{15}-36\sqrt{5}-24\sqrt{30}$	$1+2\sqrt{3}+4\sqrt{5}-\sqrt{10}-\sqrt{15}$
RRRCI	$15m^2+(16\sqrt{3}+16\sqrt{5}-94)m+148+8\sqrt{6}-32\sqrt{3}-48\sqrt{5}$	$15m^2+(16\sqrt{3}+16\sqrt{5}-94)m+149+7\sqrt{6}+2\sqrt{2}-36\sqrt{3}-48\sqrt{5}$	$-1+\sqrt{6}+4\sqrt{3}-2\sqrt{2}$
VRCI	$\frac{1}{2}m^2+\frac{1}{15}m+\frac{14}{15}$	$\frac{1}{2}m^2+\frac{1}{15}m+\frac{23}{30}$	$\frac{1}{6}$
SCCI	$\frac{\sqrt{3}\,m^2}{2}+(\frac{8}{3}+\frac{2\sqrt{10}}{5}+\frac{8\sqrt{11}}{11}-\frac{11\sqrt{3}}{3})m+\frac{2\sqrt{6}}{3}+\frac{8\sqrt{7}}{7}+2\sqrt{2}-\frac{16}{3}-\frac{8\sqrt{10}}{5}-\frac{24\sqrt{11}}{11}+6\sqrt{3}$	$\frac{\sqrt{3}\,m^2}{2}+(\frac{8}{3}+\frac{2\sqrt{10}}{5}+\frac{8\sqrt{11}}{11}-\frac{11\sqrt{3}}{3})m+\frac{8\sqrt{7}}{7}+\frac{5\sqrt{6}}{6}+2\sqrt{2}-6-\frac{8\sqrt{10}}{5}-\frac{24\sqrt{11}}{11}+6\sqrt{3}$	$-\frac{\sqrt{6}}{6}+\frac{2}{3}$
ACI	$16m-32$	$16m-30$	-2
FCI	$216m^2-568m+200$	$216m^2-568m+142$	58
FRZC	$m^2+\frac{4}{5}m+\frac{14}{15}$	$m^2+\frac{4}{5}m+\frac{17}{20}$	$\frac{1}{12}$
SRZC	$9m^2-\frac{1624}{99}m+\frac{1874}{693}$	$9m^2-\frac{1624}{99}m-\frac{401}{616}$	$\frac{1691}{504}$
TRZC	$1296m^2-4424m+2664$	$1296m^2-4424m+2336$	328

TABLE 9. Numerical values of TIS for biswapped graph $B(\mathfrak{P}_m)$.

m	FZCI	SZCI	GACI	ABCCI	SDCI	HCI	AZCI	HZCI
6	920	2252	95.620	55.657	195.07	20.694	2747.1	9072
7	1324	3364	132.54	75.993	269.73	27.559	4145.2	13536
8	1800	4692	175.45	99.494	356.40	35.425	5823.3	18864
9	2348	6236	224.37	126.16	455.07	44.291	7781.2	25056
10	2968	7996	279.29	155.98	565.73	54.156	10019	32112
11	3660	9972	340.21	188.96	688.40	65.022	12537	40032
12	4424	12164	407.13	225.12	823.07	76.888	15335	48816
13	5260	14572	480.05	264.42	969.73	89.753	18413	58464

TABLE 10. Numerical values of TIS for biswapped graph $B(\mathfrak{P}_m)$.

m	RCI	RRCI	RRRCI	VRCI	SCCI	FCI	ACI	FRZC	SRZC	TRZC
6	20.780	458.27	361.78	19.333	31.382	4568	64	41.733	228.28	22776
7	27.663	659.87	526.28	25.900	42.633	6808	80	55.533	328.88	35200
8	35.545	897.47	720.77	33.467	55.617	9480	96	71.333	447.47	50216
9	44.427	1171.1	945.27	42.033	70.332	12584	112	89.133	584.07	67824
10	54.311	1480.6	1199.8	51.600	86.779	16120	128	108.93	738.66	88024
11	65.194	1826.2	1484.2	62.167	104.96	20088	144	130.73	911.26	110820
12	77.076	2207.9	1798.7	73.733	124.87	24488	160	154.53	1101.9	136200
13	89.960	2625.5	2143.2	86.300	146.50	29320	176	180.33	1310.5	164180

D. COMPARISON OF TIS OF $B(\mathfrak{P}_M)$ AND $M(B(\mathfrak{P}_M))$

This section provides a comparison between the computed topological invariants (TIs) for the maximal twin-preserving subgraph $M(B(\mathfrak{P}_m))$ and $B(\mathfrak{P}_m)$ with connection number. The comparisons are presented in Table 8. The values corresponding to the various TIs for $B(\mathfrak{P}_m)$ are displayed in the first column of Table 8. The values for $M(B(\mathfrak{P}_m))$ and the discrepancy are presented in the second and third columns, respectively. The third column of Table 8 demonstrates that the TIs only vary by a constant.

TABLE 11. Numerical values of TIS for Spanning subgraph that preserves the maximum number of twin vertices $M(B(\mathfrak{P}_m))$.

m	FZCI	SZCI	GACI	ABCCI	SDCI	HCI	AZCI	HZCI
6	908	2218	94.514	55.222	20.583	20.583	2702.7	8946
7	1312	3330	131.43	75.559	27.448	27.448	4100.8	13410
8	1788	4658	174.35	99.059	35.314	35.314	5778.8	18738
9	2336	6202	223.27	125.72	44.180	44.180	7736.8	24930
10	2956	7962	278.19	155.54	54.045	54.045	9974.7	31986
11	3648	9938	339.10	188.53	64.911	64.911	12493	39906
12	4412	12130	406.02	224.69	76.777	76.777	15290	48690
13	5248	14538	478.94	263.99	89.642	89.642	18368	58338

TABLE 12. Numerical values of TIS for Spanning subgraph that preserves the maximum number of twin vertices $M(B(\mathfrak{P}_m))$.

m	RCI	RRCI	RRRCI	VRCI	SCCI	ACI	FCI	FRZC	SRZC	TRZC
6	20.702	451.9	356.24	19.167	31.123	66	4510	41.65	224.92	22448
7	27.585	653.49	520.73	25.733	42.375	82	6750	55.45	325.52	34872
8	35.467	891.09	715.22	33.3	55.358	98	9422	71.25	444.12	49888
9	44.351	1164.7	939.72	41.867	70.073	114	12526	89.05	580.71	67496
10	54.233	1474.3	1194.1	51.433	86.521	130	16062	108.85	735.31	87696
11	65.116	1819.8	1478.6	62	104.7	146	20030	130.65	907.9	110490
12	76.999	2201.5	1793.1	73.567	124.61	162	24430	154.45	1098.5	135870
13	89.881	2619.1	2137.6	86.133	146.25	178	29262	180.25	1307.1	163850

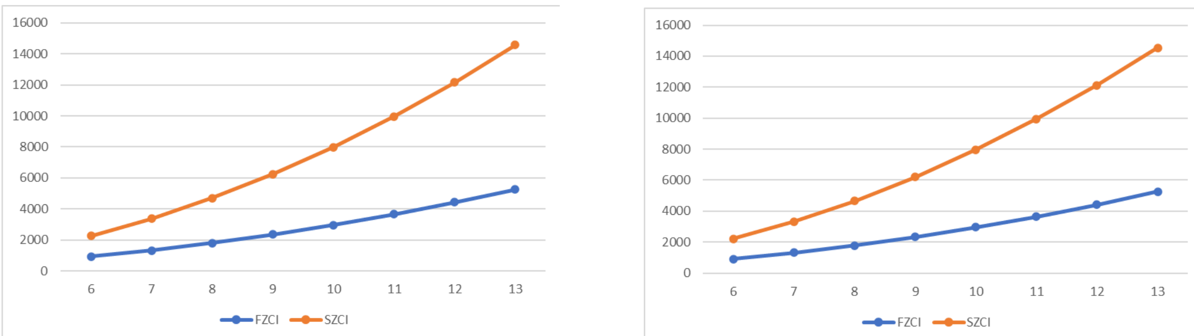

FIGURE 1. **Graphical representation of Theorem 1 and Theorem 5.**

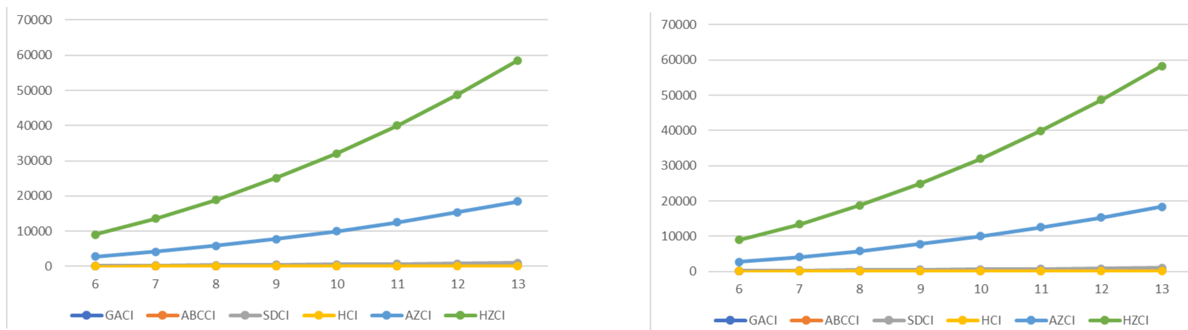

FIGURE 2. **Graphical representation of Theorem 2 and Theorem 6.**

Furthermore, we have opted to include graphical trends to illustrate the distinction across the calculated TIs of the $B(\mathfrak{P}_m)$ and $M(B(\mathfrak{P}_m))$. The numerical values of TIS for the biswapped graph $B(\mathfrak{P}_m)$ are shown in Table 9 and Table 10, while numerical values of TIS for the spanning subgraph that preserves the maximum number of twin vertices $M(B(\mathfrak{P}_m))$ are shown in Table 11 and Table 12. The left side figure of Figure 1 gives values of TIs in Theorem 2 for $B(\mathfrak{P}_m)$, and the right side figure of Figure 1 gives values of TIs in Theorem 6 for $M(B(\mathfrak{P}_m))$ by graphical way. Both graphs exhibit comparable patterns, except one graph is positioned above/below the other.

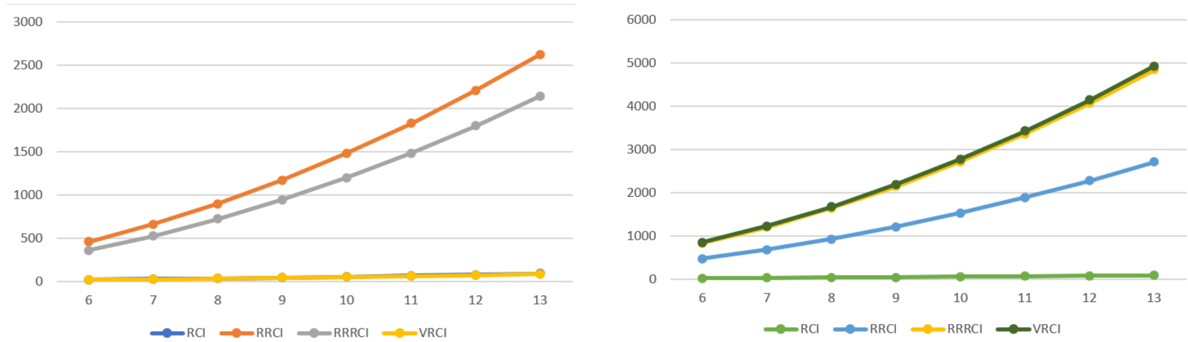

FIGURE 3. Graphical representation of Theorem 3 and Theorem 7.

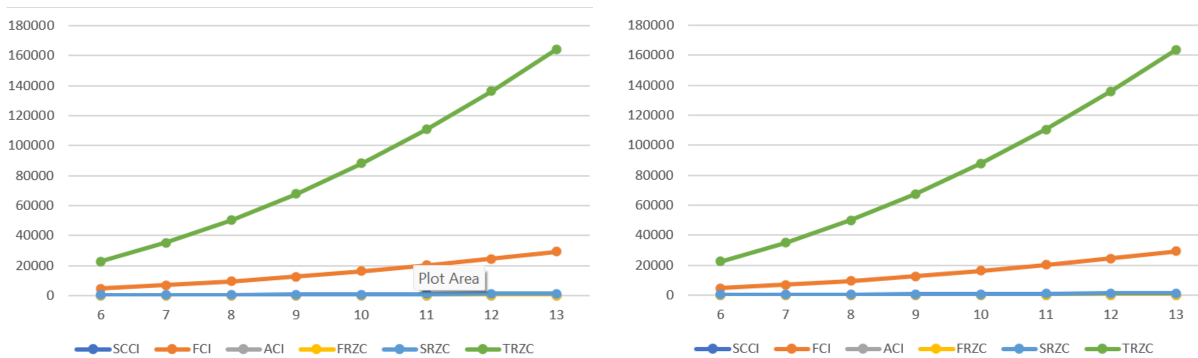

FIGURE 4. Graphical representation of Theorem 4 and Theorem 8.

Similarly, Figures 2, 3, and 4 are used to show the trends and comparisons of the other TIs.

IV. CONCLUSION

Various techniques are available for streamlining complex networks, and one significant strategy involves employing twin nodes that replicate the connectivity pattern of the entire network. This concept has been expanded by selecting a TPSS from a fundamental graph, allowing for the examination of specific graph-related properties that are maintained by the subgraph. Twin nodes are essential for calculating various TIs. One motivation for constructing a TPSS of a simple graph is to reduce the intricacy of a network. More precisely, twin nodes can remain unaltered when the size of the network is decreased. The calculation of TIs based on connection counts is also contingent upon the presence of twin nodes. Hence, investigating the correlation between these topological invariants (TIs) for graphs and their TPSS is intriguing.

One further advantage of studying TPSS is their capacity to preserve specific graph parameters. Examining this particular subgraph is advantageous in situations where analysing the full graph or network is impractical, or when there is incomplete data, such as lacking information about specific nodes or edges. The TPSS are valued in engineering for molecular analysis in such situations [69].

This work examines a subgraph of $B(\mathfrak{P}_m)$ that preserves twins to the maximum extent possible. We then determine its topological invariants along with their corresponding connection numbers. By comparing the TIs of $B(\mathfrak{P}_m)$ and its maximal twin-preserving subgraphs, we find that the TIs differ only by a constant term. A comparitive study presented in this work are derived using analytical equations, graphs and tables. Our study distinguishes itself from most current works on TIs by incorporating two key elements: the objective is to construct a subgraph that preserves the maximum number of twin vertices, and then compare the TIs between the original graph and this subgraph.

REFERENCES

[1] K. Selvakumar, P. Gangaeswari, and G. Arunkumar, "The Wiener index of the zero-divisor graph of a finite commutative ring with unity," *Discrete Appl. Math.*, vol. 311, pp. 72–84, Apr. 2022.

[2] T. Asir and V. Rabikka, "The Wiener index of the zero-divisor graph of $\mathbb{Z}_n$," *Discret. Appl. Math.*, vol. 319, pp. 461–471, Oct. 2022.

[3] S. A. U. H. Bokhary, Adnan, M. K. Siddiqui, and M. Cancan, "On topological indices and QSPR analysis of drugs used for the treatment of breast cancer," *Polycyclic Aromatic Compounds*, vol. 42, no. 9, pp. 6233–6253, Oct. 2022.

[4] M. Azeem, "Cycle-super magic labeling of polyomino linear and zig-zag chains," *J. Oper. Intell.*, vol. 1, no. 1, pp. 67–81, Nov. 2023, doi: 10.31181/jopi1120235.

[5] M. Azeem and M. F. Nadeem, "Metric-based resolvability of polycyclic aromatic hydrocarbons," *Eur. Phys. J. Plus*, vol. 136, no. 4, p. 395, Apr. 2021, doi: 10.1140/epjp/s13360-021-01399-8.

[6] M. Imran, S. Hayat, and M. Y. H. Mailk, "On topological indices of certain interconnection networks," *Appl. Math. Comput.*, vol. 244, pp. 936–951, Oct. 2014.
[7] J. M. Xu, *Topological Structure and Analysis of Interconnection Networks*. Boston, MA, USA: Kluwer, 2001, pp. 13–52.
[8] R. J. Wilson and L. W. Beineke, *Applications of Graph Theory*. New York, NY, USA: Academic, 1979.
[9] M. F. Nadeem, W. Ali, and H. M. A. Siddiqui, "Locating number of biswapped networks," *Int. J. Found. Comput. Sci.*, vol. 33, nos. 6–7, pp. 667–690, Sep. 2022.
[10] M. Imran, A. Q. Baig, and H. Ali, "On topological properties of dominating David derived networks," *Can. J. Chem.*, vol. 94, no. 2, pp. 137–148, Feb. 2016.
[11] X. Liu and Q. P. Gu, "Multicasts on WDM all-optical butterfly networks," *J. Inf. Sci. Eng.*, vol. 18, pp. 1049–1058, Apr. 2002.
[12] V. E. Benes, *Mathematical Theory of Connecting Networks and Telephone*. New York, NY, USA: Murray Hill, 1965, pp. 19–65.
[13] P. D. Manuel, M. I. Abd-El-Barr, I. Rajasingh, and B. Rajan, "An efficient representation of Benes networks and its applications," *J. Discrete Algorithms*, vol. 6, no. 1, pp. 11–19, Mar. 2008.
[14] M. Dai, L. Luo, J. Ren, H. Yu, and G. Sun, "PSACCF: Prioritized online slice admission control considering fairness in 5G/B5G networks," *IEEE Trans. Netw. Sci. Eng.*, vol. 9, no. 6, pp. 4101–4114, Nov. 2022, doi: 10.1109/TNSE.2022.3195862.
[15] G. Sun, Z. Xu, H. Yu, and V. Chang, "Dynamic network function provisioning to enable network in box for industrial applications," *IEEE Trans. Ind. Informat.*, vol. 17, no. 10, pp. 7155–7164, Oct. 2021, doi: 10.1109/TII.2020.3042872.
[16] T. Li, H. Shi, X. Bai, K. Zhang, and G. Bin, "Early performance degradation of ceramic bearings by a twin-driven model," *Mech. Syst. Signal Process.*, vol. 204, Dec. 2023, Art. no. 110826, doi: 10.1016/j.ymssp.2023.110826.
[17] E. Estrada, L. A. Torres, L. Rodriguez, and I. Gutman, "An atom-bond connectivity index: Modelling the enthalpy of formation of alkanes," *Indian J. Chem. A*, vol. 37, pp. 849–855, Jan. 1998.
[18] E. Estrada, "Generalization of topological indices," *Chem. Phys. Lett.*, vol. 336, nos. 3–4, pp. 248–252, Mar. 2001.
[19] M. Javaid, U. Ali, and K. Siddiqui, "Novel connection based Zagreb indices of several wheel-related graphs," *Comput. J. Comb. Math.*, vol. 1, pp. 1–28, Jan. 2021.
[20] A. Sattar, M. Javaid, and M. N. Alam, "On the studies of dendrimers via connection-based molecular descriptors," *Math. Problems Eng.*, vol. 2022, pp. 1–13, Mar. 2022.
[21] L. Zhong, "The harmonic index for graphs," *Appl. Math. Lett.*, vol. 25, no. 3, pp. 561–566, Mar. 2012.
[22] M. Chellali, T. W. Haynes, S. T. Hedetniemi, and T. M. Lewis, "On *ve*-degrees and *ev*-degrees in graphs," *Discrete Math.*, vol. 340, pp. 31–38, Feb. 2017.
[23] B. Horoldagva, K. C. Das, and T. A. Selenge, "On *ve*-degree and *ev*-degree of graphs," *Disc. Optim.*, vol. 31, no. 2, pp. 1–7, 2019.
[24] A. S. Alali, S. Ali, N. Hassan, A. M. Mahnashi, Y. Shang, and A. Assiry, "Algebraic structure graphs over the commutative ring zm: Exploring topological indices and entropies using M-polynomials," *Mathematics*, vol. 11, no. 18, p. 3833, Sep. 2023, doi: 10.3390/math11183833.
[25] S. Zaman, K. H. Hakami, S. Rasheed, and F. T. Agama, "Reduced reverse degree-based topological indices of graphyne and graphdiyne nanoribbons with applications in chemical analysis," *Sci. Rep.*, vol. 14, no. 1, p. 547, Jan. 2024, doi: 10.1038/s41598-023-51112-1.
[26] M. Azeem, M. K. Jamil, and Y. Shang, "Notes on the localization of generalized hexagonal cellular networks," *Mathematics*, vol. 11, no. 4, p. 844, Feb. 2023, doi: 10.3390/math11040844.
[27] M. Azeem, S. Anwar, M. K. Jamil, M. Saeed, and M. Deveci, "Topological numbers of fuzzy soft graphs and their application," *Inf. Sci.*, vol. 667, May 2024, Art. no. 120468, doi: 10.1016/j.ins.2024.120468.
[28] M. Irfan, H. U. Rehman, H. Almusawa, S. Rasheed, and I. A. Baloch, "M-polynomials and topological indices for line graphs of chain silicate network and H-naphtalenic nanotubes," *J. Math.*, vol. 2021, pp. 1–11, Mar. 2021, doi: 10.1155/2021/5551825.
[29] Y. Shang, "Sombor index and degree-related properties of simplicial networks," *Appl. Math. Comput.*, vol. 419, Apr. 2022, Art. no. 126881.
[30] R. Ismail, M. Azeem, Y. Shang, M. Imran, and A. Ahmad, "A unified approach for extremal general exponential multiplicative zagreb indices," *Axioms*, vol. 12, no. 7, p. 675, Jul. 2023.
[31] Y. Shang, "Lower bounds for Gaussian Estrada index of graphs," *Symmetry*, vol. 10, no. 8, p. 325, Aug. 2018.
[32] S. Ahmad, H. M. A. Siddiqui, A. Ali, M. R. Farahani, M. Imran, and I. N. Cangul, "On Wiener index and Wiener polarity index of some polyomino chains," *J. Discrete Math. Sci. Cryptogr.*, vol. 22, no. 7, pp. 1151–1164, Oct. 2019.
[33] A. Ahmad, A. N. A. Koam, I. Masmali, M. Azeem, and H. Ghazwani, "Connection number topological aspect for backbone DNA networks," *Eur. Phys. J. E*, vol. 46, no. 12, p. 120, Dec. 2023.
[34] M. Azeem, M. K. Jamil, A. Javed, and A. Ahmad, "Verification of some topological indices of Y-Junction based nanostructures by M-polynomials," *J. Math.*, vol. 2022, pp. 1–18, Jul. 2022, doi: 10.1155/2022/8238651.
[35] S. Anwar, M. Azeem, and M. K. Jamil, "Topological numbers of fuzzy soft graphs and their applications in globalizing the world by mutual trade," *Appl. Soft Comput.*, vol. 159, Jul. 2024, Art. no. 111642, doi: 10.1016/j.asoc.2024.111642.
[36] M. Maghsoudi, S. Shokouhyar, A. Ataei, S. Ahmadi, and S. Shokoohyar, "Co-authorship network analysis of AI applications in sustainable supply chains: Key players and themes," *J. Cleaner Prod.*, vol. 422, Oct. 2023, Art. no. 138472.
[37] K. A. McLean, S. R. Knight, T. M. Diehl, C. Varghese, N. Ng, M. A. Potter, S. N. Zafar, M.-M. Bouamrane, and E. M. Harrison, "Readiness for implementation of novel digital health interventions for postoperative monitoring: A systematic review and clinical innovation network analysis," *Lancet Digit. Health*, vol. 5, no. 5, pp. e295–e315, May 2023.
[38] W. Zhang, M. Zhang, L. Yuan, and F. Fan, "Social network analysis and public policy: What's new?" *J. Asian Public Policy*, vol. 16, no. 2, pp. 115–145, May 2023.
[39] T. Xiao, Z. Liu, L. Lu, H. Han, X. Huang, X. Song, X. Yang, and X. Meng, "LSTM-BP neural network analysis on solid-liquid phase change in a multi-channel thermal storage tank," *Eng. Anal. with Boundary Elements*, vol. 146, pp. 226–240, Jan. 2023.
[40] J.-B. Liu, Y.-Q. Zheng, and X.-B. Peng, "The statistical analysis for sombor indices in a random polygonal chain networks," *Discrete Appl. Math.*, vol. 338, pp. 218–233, Oct. 2023.
[41] G. C. Marsden, P. J. Marchand, P. Harvey, and S. C. Esener, "Optical transpose interconnection system architectures," *Opt. Lett.*, vol. 18, no. 13, p. 1083, 1993.
[42] W. Xiao, W. Chen, M. He, W. Wei, and B. Parhami, "Biswapped networks and their topological properties," in *Proc. 8th ACIS Int. Conf. Softw. Eng., Artif. Intell., Netw., Parallel/Distributed Comput. (SNPD)*, Jul. 2007, pp. 19–38.
[43] I. A. Stewart, "Multiswapped networks and their topological and algorithmic properties," *J. Comput. Syst. Sci.*, vol. 79, no. 8, pp. 1269–1286, Dec. 2013.
[44] Y. Yin, Y. Guo, Q. Su, and Z. Wang, "Task allocation of multiple unmanned aerial vehicles based on deep transfer reinforcement learning," *Drones*, vol. 6, no. 8, p. 215, Aug. 2022, doi: 10.3390/drones6080215.
[45] X. Xu, W. Liu, and L. Yu, "Trajectory prediction for heterogeneous traffic-agents using knowledge correction data-driven model," *Inf. Sci.*, vol. 608, pp. 375–391, Aug. 2022, doi: 10.1016/j.ins.2022.06.073.
[46] H. Yang, Z. Li, and Y. Qi, "Predicting traffic propagation flow in urban road network with multi-graph convolutional network," *Complex Intell. Syst.*, vol. 10, no. 1, pp. 23–35, Feb. 2024, doi: 10.1007/s40747-023-01099-z.
[47] H. Yang and Z. Li, "Dynamic graph convolutional network-based prediction of the urban grid-level taxi demand–supply imbalance using GPS trajectories," *ISPRS Int. J. Geo-Inf.*, vol. 13, no. 2, p. 34, Jan. 2024, doi: 10.3390/ijgi13020034.
[48] X. Zhang, W. Pan, R. Scattolini, S. Yu, and X. Xu, "Robust tube-based model predictive control with Koopman operators," *Automatica*, vol. 137, Mar. 2022, Art. no. 110114, doi: 10.1016/j.automatica.2021.110114.
[49] X. Li and Y. Sun, "Application of RBF neural network optimal segmentation algorithm in credit rating," *Neural Comput. Appl.*, vol. 33, no. 14, pp. 8227–8235, Jul. 2021, doi: 10.1007/s00521-020-04958-9.
[50] X. Xi, J. Li, Z. Wang, H. Tian, and R. Yang, "The effect of high-order interactions on the functional brain networks of boys with ADHD," *Eur. Phys. J. Special Topics*, Apr. 2024, doi: 10.1140/epjs/s11734-024-01161-y.
[51] Z. Wang, M. Chen, X. Xi, H. Tian, and R. Yang, "Multi-chimera states in a higher order network of FitzHugh–Nagumo oscillators," *Eur. Phys. J. Special Topics*, Mar. 2024, doi: 10.1140/epjs/s11734-024-01143-0.

[52] W. Long, Z. Xiao, D. Wang, H. Jiang, J. Chen, Y. Li, and M. Alazab, ''Unified spatial–temporal neighbor attention network for dynamic traffic prediction,'' *IEEE Trans. Veh. Technol.*, vol. 72, no. 2, pp. 1515–1529, Feb. 2023, doi: 10.1109/TVT.2022.3209242.
[53] Z. Xiao, H. Fang, H. Jiang, J. Bai, V. Havyarimana, H. Chen, and L. Jiao, ''Understanding private car aggregation effect via spatio-temporal analysis of trajectory data,'' *IEEE Trans. Cybern.*, vol. 53, no. 4, pp. 2346–2357, Apr. 2023, doi: 10.1109/TCYB.2021.3117705.
[54] G. Sun, Z. Xu, H. Yu, X. Chen, V. Chang, and A. V. Vasilakos, ''Low-latency and resource-efficient service function chaining orchestration in network function virtualization,'' *IEEE Internet Things J.*, vol. 7, no. 7, pp. 5760–5772, Jul. 2020, doi: 10.1109/JIOT.2019.2937110.
[55] G. Sun, G. Zhu, D. Liao, H. Yu, X. Du, and M. Guizani, ''Cost-efficient service function chain orchestration for low-latency applications in NFV networks,'' *IEEE Syst. J.*, vol. 13, no. 4, pp. 3877–3888, Dec. 2019, doi: 10.1109/JSYST.2018.2879883.
[56] G. Sun, Y. Li, D. Liao, and V. Chang, ''Service function chain orchestration across multiple domains: A full mesh aggregation approach,'' *IEEE Trans. Netw. Service Manage.*, vol. 15, no. 3, pp. 1175–1191, Sep. 2018, doi: 10.1109/TNSM.2018.2861717.
[57] S. Hayat, A. Khan, and Y. Zhong, ''On Resolvability- and domination-related parameters of complete multipartite graphs,'' *Mathematics*, vol. 10, no. 11, p. 1815, May 2022.
[58] H. Raza, S. Hayat, and X.-F. Pan, ''On the fault-tolerant metric dimension of convex polytopes,'' *Appl. Math. Comput.*, vol. 339, pp. 172–185, Dec. 2018.
[59] H. M. A. Siddiqui, S. Hayat, A. Khan, M. Imran, A. Razzaq, and J.-B. Liu, ''Resolvability and fault-tolerant resolvability structures of convex polytopes,'' *Theor. Comput. Sci.*, vol. 796, pp. 114–128, Dec. 2019.
[60] H. Raza, S. Hayat, and X.-F. Pan, ''On the fault-tolerant metric dimension of certain interconnection networks,'' *J. Appl. Math. Comput.*, vol. 60, nos. 1–2, pp. 517–535, Jun. 2019.
[61] S. Hayat, A. Khan, M. Y. H. Malik, M. Imran, and M. K. Siddiqui, ''Fault-tolerant metric dimension of interconnection networks,'' *IEEE Access*, vol. 8, pp. 145435–145445, 2020.
[62] R. J. Wilson, *Introduction to Graph Theory*. Upper Saddle River, NJ, USA: Prentice-Hall, 1996, p. 118.
[63] S. R. Nayaka and S. P. Puttaswamy, ''The open neighborhood number of a graph,'' *Int. J. Eng. Sci.*, vol. 1, pp. 52–54, Jan. 2017.
[64] K. Wahid, A. Das, A. Rani, S. Amanat, M. Imran, and U. Ali, ''On twin preserving spanning subgraph,'' *J. Intell. Fuzzy Syst.*, vol. 40, no. 5, pp. 9505–9513, Apr. 2021.
[65] I. F. Hussain, S. Afridi, A. M. Qureshi, G. Ali, and U. Ali, ''Fault-tolerant resolvability of swapped optical transpose interconnection system,'' *J. Math.*, vol. 2022, pp. 1–6, May 2022.
[66] N. Zahra, M. Ibrahim, and M. K. Siddiqui, ''On topological indices for swapped networks modeled by optical transpose interconnection system,'' *IEEE Access*, vol. 8, pp. 200091–200099, 2020.
[67] A. Rani, A. Fahad, I. F. Hussain, U. Ali, and M. Ashraf, ''A comparison between certain graph parameters of biswapped networks and their maximal twin-preserving subgraphs with applications in determining complexity,'' *Meas. Sci. Technol.*, vol. 35, no. 1, Jan. 2024, Art. no. 015603, doi: 10.1088/1361-6501/acffe7.
[68] N. Zahra and M. Ibrahim, ''On *ve*-degree and *ev*-degree based topological indices of certain otis biswapped network,'' *Adv. Math., Sci. J.*, vol. 10, no. 6, pp. 2887–2908, Jun. 2021.
[69] Y. Shang, ''Subgraph robustness of complex networks under attacks,'' *IEEE Trans. Syst. Man, Cybern. Syst.*, vol. 49, no. 4, pp. 821–832, Apr. 2019.

KHAWLAH ALHULWAH received the M.Sc. degree in mathematics from Princess Nourah Bint Abdulrahman University, Riyadh, Saudi Arabia, in 2010, and the Ph.D. degree in mathematics from Western Michigan University, MI, USA, in 2017. She is currently an Assistant Professor with the College of Science, Imam Mohammad Ibn Saud Islamic University, Saudi Arabia. Her research interests include graph theory, graph coloring, graph labeling, metric dimension, minimal doubly resolving sets, distances in graphs, and topological indices of graphs.

ALI N. A. KOAM received the B.Sc. degree in mathematics from King Abdulaziz University, Jeddah, Saudi Arabia, in 2007, and the Ph.D. degree in mathematics from Leicester University, Leicester, U.K., in 2016. He is currently the Vice-Dean of Academic Development of Quality with Jazan University. His research interests include homological algebra, associative algebra, logical algebra, decision-making, and fuzzy set theory. He has authored several journal articles on these topics.

NASREEN ALMOHANNA received the M.Sc. degree in mathematics from Princess Nourah Bint Abdulrahman University, Riyadh, Saudi Arabia, in 2009, and the M.Sc. and Ph.D. degrees in mathematics from Western Michigan University, MI, USA, in 2018 and 2019, respectively. She is currently an Assistant Professor with the College of Science, Imam Mohammad Ibn Saud Islamic University, Saudi Arabia. Her research interests include graph theory, graph coloring, graph labeling, connectivity, metric dimension, minimal doubly resolving sets, distances in graphs, and topological indices of graphs.

ALI AHMAD received the M.Sc. degree in mathematics from Punjab University, Lahore, Pakistan, in 2000, the M.Phil. degree in mathematics from Bahauddin Zakariya University, Multan, Pakistan, in 2005, and the Ph.D. degree in mathematics from the Abdus Salam School of Mathematical Sciences, GC University, Lahore, in 2010. He is currently an Assistant Professor with the College of Computer Science and Information Technology, Jazan University, Saudi Arabia. His research interests include graph labeling, metric dimension, minimal doubly resolving sets, distances in graphs, and topological indices of graphs.

MUHAMMAD AZEEM received the B.S. degree from COMSATS University Islamabad, Lahore Campus, in 2018, and the M.Phil. degree in mathematics from Riphah International University, Lahore Campus, in 2023. He is currently a Research Associate with the Department of Mathematics, Riphah International University, Lahore Campus. He has 100 published research articles in reputed international journals of mathematics and informatics. His research interests include control theory, fuzzy set theory, fuzzy graph theory, decision making, metric graph theory, graph labeling, spectral graph theory, and soliton theory. He is a referee of several international mathematical journals. He is an editor of several international mathematical journals.