IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received 9 May 2024, accepted 28 May 2024, date of publication 3 June 2024, date of current version 14 June 2024.

Digital Object Identifier 10.1109/ACCESS.2024.3408679

RESEARCH ARTICLE

Selection of Energy Trading Platform for Peer-to-Peer (P2P) Energy Trading by Using a Multi-Attribute Decision-Making Approach Based on Bipolar Fuzzy Aczel-Alsina Prioritized Aggregation Operators

UBAID UR REHMAN[1], FATEN LABASSI[2], TURKI ALSURAIHEED[3], TAHIR MAHMOOD[1], AND MERAJ ALI KHAN[2]

[1]Department of Mathematics and Statistics, International Islamic University Islamabad, Islamabad 44000, Pakistan
[2]Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11566, Saudi Arabia
[3]Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Saudi Arabia

Corresponding author: Ubaid Ur Rehman (ubaid5@outlook.com)

This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-RG23089).

ABSTRACT The selection of the proper P2P energy trading platform is a complicated multi-attribute decision-making (MADM) dilemma that involves evaluating different alternatives against various attribute. Traditional MADM techniques often fail to capture the bipolarity of certain attribute, where positive and negative aspects are simultaneously present. This duality of attribute therefore requires a more advance method of modeling and decision making (DM). The bipolar fuzzy set (BFS) framework presents a possible research gap-filling solution by enabling the consideration of both positive and negative information associated with each attribute at the same time. This article aims to use BFS to create a MADM technique that will be able to model the bipolarity of the criteria, and provide a structured approach for the selection of the most suitable P2P energy trading platform. Further, this article contains various aggregation operators within BFS based on Aczel-Alsina (AA) t-norm and t-conorm, which play a critical role in the proposed MADM approach. After that, the case study, ‘‘Selection of Energy Trading Platform for P2P Energy Trading’’ is investigated by employing the invented approach. In the end, the advantages and dominance of the inaugurated work over some of the prevailing literature are demonstrated through comparative analysis.

INDEX TERMS Energy trading, Aczel-Alsina t-norm and t-conorm, MADM, bipolar fuzzy information.

I. INTRODUCTION

Energy Trading Platorms for P2P Energy Trading as pioneering innovation within the energy sector are designed to meet the growing need for decentralized and environmentally friendly energy solutions. Today, with climate change and energy security issues at the top of the agenda, these platforms open up a great opportunity to make energy trading available to the masses and to let the individuals and communities become part of the energy market. The main claim of P2P energy trading platforms is their ability to hold direct transactions between energy producers and consumers, avoiding the centralized systems. Through the application of innovative technologies, including block chain, smart contracts, and the Internet of Things (IoT), these platforms can usher in a new era of hassle-free and secure energy trading, with transparency, efficiency, and cost-effectiveness as key features. The prosumers (producers

The associate editor coordinating the review of this manuscript and approving it for publication was F. R. Islam.

© 2024 The Authors. This work is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 License.
For more information, see https://creativecommons.org/licenses/by-nc-nd/4.0/

and consumers) can now buy and sell locally generated renewable energy, which facilitates a system that is not only sustainable but also resilient. Besides this, these platforms advocate for the integration of distributed energy resources, such as rooftop solar panels, wind turbines, and energy storage systems, allowing individuals and communities to operate at a lower carbon footprint while supporting the worldwide effort to reduce CO2 emissions and tackle climate change.

The development of P2P energy trading is seen as one of the most innovative concepts in the energy market and allows direct and decentralized transactions between energy producers and consumers. Different researchers examine different aspects of the P2P energy exchange, like market design, challenges, opportunities, and realizations. Parag and Sovacool [1] argued that the design of electricity markets is a key factor for the prosumer era, where consumers also become producers (prosumers) of energy. They stress the need to involve the local energy markets and the prosumers who should actively participate in the making of the energy system. Zhang et al. [2] devised the existing P2P energy trading projects, trying to learn their features, technologies, and business models. The main issues include regulatory obstacles, privacy issues, and scalability. Andoni et al. [3] presented a systematic review of blockchain technology's possible impacts on the energy sector, including its applications to P2P energy trading. They talked about what problems and opportunities exist in using blockchain in the energy system. Zhou et al. [4] simulated a multi-agent framework to investigate the different P2P energy-sharing mechanisms, which include factors like energy pricing, fairness, and prosumer satisfaction. Guerrero et al. [5] suggested a P2P energy trading mechanism based on decentralization that considers the constraints of the network and the low-voltage network levels, to facilitate secure and efficient energy transactions. Park and Yong [6] discussed P2P electricity trading, addressing trading mechanisms, pricing strategies, and regulatory issues. Zhang et al. [7] interpreted a P2P energy trading mechanism in a microgrid with a grid connection whereby prosumers can buy or sell energy through a market-based system. Sousa et al. [8] through their systematic review, illustrated the P2P and community-based energy markets' potential benefits, disadvantages, and future research directions. Paudel et al. [9] offered a game-theoretic model of P2P energy trading in a prosumer-oriented community-based microgrid which takes into account factors like prosumer preference, energy prices, and fairness. Moret and Pinson [10] first defined energy collectives, as a new form of market participation for communities, which in turn promotes fairness and sustainability. Wang et al. [11] created a P2P energy trading model within microgrids by taking into account the various willingness factors, such as energy demand, energy price, and ecology. Roy et al. [12] assessed the possible worth of P2P energy trading in the Australian National Electricity Market, where they identified the potential benefits and challenges of such a system. The graphical interpretation of the P2P model devised by Roy et al. [12] is interpreted in Fig 1.

The mathematical concept of a fuzzy set (FS) [13] enables the depiction of ambiguity or uncertainty in information. FS assigns each element a term of support, ranging from 0 to 1, reflecting the extent to which that element exhibits the features of the set, as opposed to classical sets, where an element is either placed or not placed into the set. Due to the inherent uncertainty and fuzziness of many real-world occurrences, which cannot be precisely characterized by means of classical set theory, FS is required. For instance, it is challenging to describe ideas like "beauty" or "tallness" using conventional set theory because they are arbitrary and susceptible to context. In areas like artificial intelligence, control systems, and decision-making (DM), FS has found extensive use as a potent tool for modeling and studying such hazy and imprecise events. Tang et al. [14] investigated electronic marketing strategies with the assistance of the fuzzy multi-criteria DM (MCDM) approach. Mohaghar et al. [15] discussed the selection of marketing strategy by fusing VIKOR and AHP techniques under a fuzzy environment. Lin et al. [16] investigated competitive marketing strategy with fuzzy group DM. Gholami and Seyyed-Esfahani [17] did the competitive market strategy selection by employing fuzzy DEMATEL and AHP techniques. Lee et al. [18] discussed marketing strategy in community colleges by utilizing fuzzy assessment. With the help of fuzzy ANP, Oztaysi et al. [19] analyzed the marketing strategy for shopping malls. A bipolar FS (BFS) investigated by Zhang [20] is a type of FS that assigns not only the positive term of support but also assign negative term of support to the element placed in $[-1, 0]$. BFS is helpful in circumstances, where it is not enough to simply categorize elements as either place or not place in a set. For instance, in DM dilemmas, it may be critical to keep in view both negative and positive aspects of elements instead of just positive aspects. More, BFS is a critical tool for modeling uncertain and complex circumstances, where the traditional crisp sets or FS are not sufficient. Riaz et al. [21] initiated sine trigonometric AOs for bipolar fuzzy (BF) numbers (BFNs). Jana et al. [22] analyzed Dombi AOs for BFNs and Jana et al. [23] initiated prioritized AOs relying on the Dombi norm and t-conorm for BFNs. The Hamacher AOs for BFNs were discussed by Wei et al. [24]. Numerous researchers studied various DM techniques under BF information such as Karabasevic et al. [25] initiated MULTIMOORA, Alsolame and Alshehri [26] studied VIKOR, Akram et al. [27] investigated TOPSIS and ELECTRE-I, Deva and Felix [28] initiated DEMATEL, Jana and Pal [29] developed EDAS and Liu et al. [30] discussed SWARA-MABAC.

A. MOTIVATION AND CONTRIBUTION

The energy trading platforms for P2P energy trading are one of the key issues with regard to BFS that are the extension of traditional FS. BFSs are appropriate for the representation of

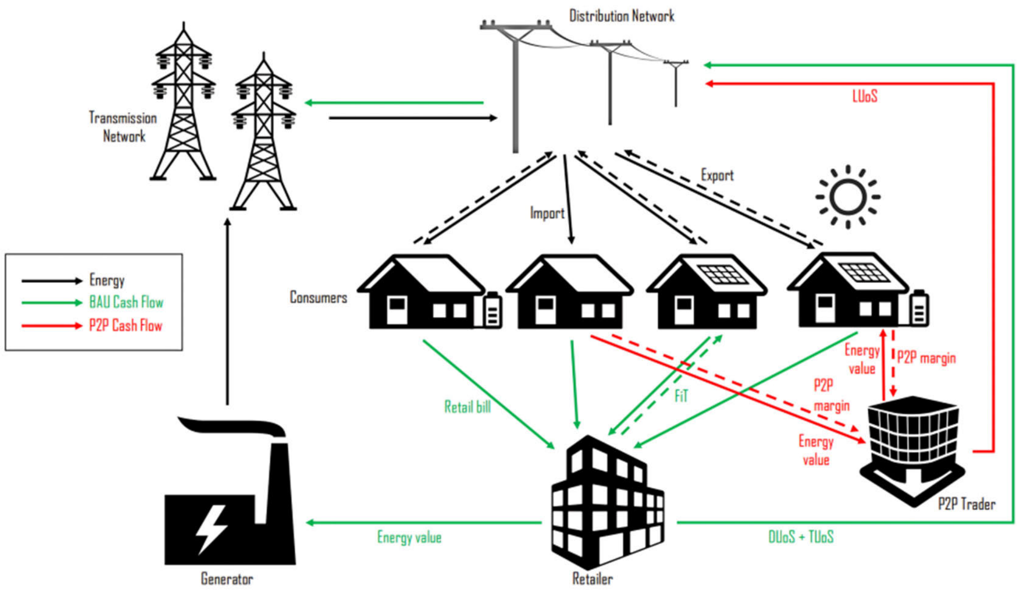

FIGURE 1. The interpretation of P2P trading model.

both positive and negative information at the same time and can therefore be used for modeling complex and multifaceted systems such as energy trading platforms. In the context of P2P energy trading, BFSs can be utilized to represent both the negative and positive aspects of each criterion. Consequently, BFSs can be applied to energy trading to assess the positive and negative influences on factors such as carbon emission, energy security, and grid stability. The utilization of BFSs in the design as well as the operation of P2P energy trading platforms enables the development of more sophisticated and strong decision-making systems that consider the multi-faceted nature of energy systems. This may result in an increase in the efficiency and sustainability of energy trading, better congruence with prosumers' preferences, and more effective management of uncertainties and risks related to decentralized energy production and distribution.

Further, in various MADM dilemmas, one attribute can be before the other existing attributes. Similarly, in various dilemmas, the experts or decision analysts have the prioritizations. To cope with such sort of information where the attributes contain prioritization or experts contain prioritization, Yager [31] interpreted the notion of prioritized AOs. Afterward, the prioritized ordered weighted averaging operators were deduced by Yager [33]. Yu and Xu [33] initiated prioritized AOs for intuitionistic fuzzy numbers (IFNs). Khan et al. [34] diagnosed prioritized AOs for Pythagorean fuzzy numbers. Riaz et al. [35] investigated prioritized AOs for q-rung orthopair fuzzy numbers. Aczél and Alsina [36] interpreted Aczel-'Alsina t-norm and t-conorm, which are recent additions to the notion of FS and have the benefits of changeability by adapting a parameter. Because of their importance, they got more attention from various researchers who employed them in numerous areas such as Sarfraz et al. [37] investigated prioritized AOs relying on Aczel-Alsina t-norm and conorm under IFNs, Senapati et al. [38] and diagnosed Aczel-Alsina (AA) AOs for IFNs, Senapati [39] developed AA AOs for picture fuzzy numbers. After a lot of investigation and study, we observed that no research can provide the advantages and benefits of prioritization, Aczel-Alsina t-norm, and t-conorm under bipolar fuzzy information at the same time and there is no MADM problem by using such operators to handle energy trading platform for P2P energy trading. To fill this gap in the literature and also keep in mind the real-life importance of these theories, in this manuscript, we inaugurate prioritized AOs under bipolar fuzzy information relying on the Aczel-Alsina t-norm and t-conorm. These AOs are BFAAPRA, BFAAPRWA, BFAAPRG, and BFAAPRWA. Further, this manuscript contains an approach of MADM with the assistance of diagnosed AOs to cope with complicated genuine-life MADM dilemmas (in particular, the selection of an energy trading platform for P2P energy trading).

B. ARRANGEMENT OF THE ARTICLE

The rest of this script is developed as: Section II, contains a few prevailing concepts and correct primary operations for the notion of BFNs. Section III, includes bipolar fuzzy Aczel-Alsina (BFAA) prioritized averaging (BFAAPRA), BFAA prioritized weighted averaging (BFAAPRWA), BFAA prioritized geometric (BFAAPRG), and BFAA prioritized weighted averaging (BFAAPRWA) and AA operations for BFS. Section IV, includes an approach of MADM for tackling MADM dilemmas and a case study ''selection of energy trading platform for P2P energy trading''. Section V, includes the comparative analysis, and Section VI, includes the concluding remarks.

II. PRELIMINARIES

In this section, we overview a few prevailing concepts and initiated correct primary operations for the notion of BFNs.

Definition 1 [36]: AA t-norm and t-conorm are symbolized as

$$\mathbb{C}^{\mathfrak{f}}_{\mathcal{A}}\left(\mathfrak{z}_1,\mathfrak{z}_2\right)=\begin{cases}\acute{\varsigma}_{\mathbb{D}}\left(\mathfrak{z}_1,\mathfrak{z}_2\right) & \text{if } \mathfrak{f}=0\\ \min\left(\mathfrak{z}_1,\mathfrak{z}_2\right) & \text{if } \mathfrak{f}=\infty\\ e^{-\left(\left(-\log\mathfrak{z}_1\right)^{\mathfrak{f}}+\left(-\log\mathfrak{z}_2\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}}, & \text{otherwise}\end{cases}\tag{1}$$

$$\mathbb{C}^{**\mathfrak{f}}_{\mathcal{A}}\left(\mathfrak{z}_1,\mathfrak{z}_2\right)=\begin{cases}\acute{\varsigma}^{**}_{\mathbb{D}}\left(\mathfrak{z}_1,\mathfrak{z}_2\right) & \text{if } \mathfrak{f}=0\\ \max\left(\mathfrak{z}_1,\mathfrak{z}_2\right) & \text{if } \mathfrak{f}=\infty\\ 1-e^{-\left(\left(-\log\left(1-\mathfrak{z}_1\right)\right)^{\mathfrak{f}}+\left(-\log\left(1-\mathfrak{z}_2\right)\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}}, & \text{otherwise}\end{cases}\tag{2}$$

where $\mathfrak{f}\in[0,\ \infty]$.

Definition 2 [20]: A BFS is a type $\Upsilon=\left\{\left(\mathfrak{z},\ \mathring{U}_{\mathcal{P}-\Upsilon}\left(\mathfrak{z}\right),\ \mathring{U}_{\mathcal{N}-\Upsilon}\left(\mathfrak{z}\right)\right)\ |\ \mathfrak{z}\in\mathfrak{Z}\right\}$, where the $\mathring{U}_{\mathcal{P}-\Upsilon}\left(\mathfrak{z}\right)$ would utilize as a positive term of support that is placed in $[0,\ 1]$ and $\mathring{U}_{\mathcal{N}-\Upsilon}\left(\mathfrak{z}\right)$ would utilize as a negative term of support that placed in $[-1,\ 0]$. Further, the set $\Upsilon=\left(\mathring{U}_{\mathcal{P}-\Upsilon},\mathring{U}_{\mathcal{N}-\Upsilon}\right)$ is identified as BFN in this script.

Definition 3 [24]: The following Eq. interprets the score value of a BFN $\Upsilon=\left(\mathring{U}_{\mathcal{P}-\Upsilon},\mathring{U}_{\mathcal{N}-\Upsilon}\right)$

$$\dot{\mathcal{S}}_F\left(\Upsilon\right)=\frac{1}{2}\left(1+\mathring{U}_{\mathcal{P}-\Upsilon}\left(\mathfrak{z}\right)+\mathring{U}_{\mathcal{N}-\Upsilon}\left(\mathfrak{z}\right)\right),\quad \dot{\mathcal{S}}_F\in[0,\ 1]\tag{3}$$

Definition 4 [24]: The following Eq. interprets the accuracy value of a BFN $\Upsilon=\left(\mathring{U}_{\mathcal{P}-\Upsilon},\mathring{U}_{\mathcal{N}-\Upsilon}\right)$

$$\mathcal{H}_F\left(\Upsilon\right)=\frac{\mathring{U}_{\mathcal{P}-\Upsilon}\left(\mathfrak{z}\right)-\mathring{U}_{\mathcal{N}-\Upsilon}\left(\mathfrak{z}\right)}{2},\ \mathcal{H}_F\in[0,\ 1]\tag{4}$$

With the assistance of Eq. (3) and (4) the underneath given

1. If $\dot{\mathcal{S}}_F\left(\Upsilon_1\right)<\dot{\mathcal{S}}_F\left(\Upsilon_2\right)$, then $\Upsilon_1<\Upsilon_2$;
2. If $\dot{\mathcal{S}}_F\left(\Upsilon_1\right)>\dot{\mathcal{S}}_F\left(\Upsilon_2\right)$, then $\Upsilon_1>\Upsilon_2$;
3. If $\dot{\mathcal{S}}_F\left(\Upsilon_1\right)=\dot{\mathcal{S}}_F\left(\Upsilon_2\right)$, then
   - if $\mathcal{H}_F\left(\Upsilon_1\right)<\mathcal{H}_F\left(\Upsilon_2\right)$, then $\Upsilon_1<\Upsilon_2$;
   - if $\mathcal{H}_F\left(\Upsilon_1\right)>\mathcal{H}_F\left(\Upsilon_2\right)$, then $\Upsilon_1>\Upsilon_2$;
   - if $\mathcal{H}_F\left(\Upsilon_1\right)=\mathcal{H}_F\left(\Upsilon_2\right)$, then $\Upsilon_1=\Upsilon_2$.

Definition 5 [31]: Let a gathering of attributes i.e., $\mathfrak{A}_{\mathring{A}}=\left\{\mathfrak{A}_{\mathring{A}-1},\ \mathfrak{A}_{\mathring{A}-2},\ \ldots,\mathfrak{A}_{\mathring{A}-\bar{\eta}}\right\}$, and let a prioritization among the interpreted attributes by a linear order that is $\mathfrak{A}_{\mathring{A}-1}\succ\mathfrak{A}_{\mathring{A}-2}\succ\ldots>\mathfrak{A}_{\mathring{A}-\bar{\eta}}$. Further, if $\dot{\upsilon}<\alpha$, then $\mathfrak{A}_{\mathring{A}-\dot{\upsilon}}$ is prior than $\mathfrak{A}_{\mathring{A}-\alpha}$. $\mathfrak{A}_{\mathring{A}-\dot{\upsilon}}\left(\mathfrak{z}\right)$ is the assessment outcome of the performance of any alternative under the attribute $\mathfrak{A}_{\mathring{A}-\dot{\upsilon}}$ and $\mathfrak{A}_{\mathring{A}-\dot{\upsilon}}\in[0,\ 1]$. If

$$PRA\left(\mathfrak{A}_{\mathring{A}-1}\left(\mathfrak{z}\right),\ \mathfrak{A}_{\mathring{A}-2}\left(\mathfrak{z}\right)\ \ldots,\ \mathfrak{A}_{\mathring{A}-\dot{\upsilon}}\left(\mathfrak{z}\right)\right)=\underset{\dot{\upsilon}=1}{\overset{\bar{\eta}}{\oplus}}\frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\bar{\eta}}\dot{T}_{\dot{\upsilon}}}\mathfrak{A}_{\mathring{A}-\dot{\upsilon}}\left(\mathfrak{z}\right)\tag{5}$$

III. ACZEL-ALSINA PRIORITIZED AOS FOR BF NUMBERS

In this part of the script, we inaugurate BFAAPRA, BFAAPRWA, BFAAPRG, and BFAAPRWG. But first, we would inaugurate AA operations for BFS.

Definition 6: The underneath statements interpret the AA operations for any two BFNs, identified as $\Upsilon_1=\left(\mathring{U}_{\mathcal{P}-\Upsilon_1},\ \mathring{U}_{\mathcal{N}-\Upsilon_1}\right)$ and $\Upsilon_2=\left(\mathring{U}_{\mathcal{P}-\Upsilon_2},\ \mathring{U}_{\mathcal{N}-\Upsilon_2}\right)$, which use the AA t-norm and t-conorm with $\mathfrak{f}\geq 1$, and $\alpha>0$,

1. $\Upsilon_1\oplus\Upsilon_2$

$$=\left(\begin{array}{c}1-e^{-\left(\left(-\log\left(1-\mathring{U}_{\mathcal{P}-\Upsilon_1}\right)\right)^{\mathfrak{f}}+\left(-\log\left(1-\mathring{U}_{\mathcal{P}-\Upsilon_2}\right)\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}},\\ -\left(e^{-\left(\left(-\log\left|\mathring{U}_{\mathcal{N}-\Upsilon_1}\right|\right)^{\mathfrak{f}}+\left(-\log\left|\mathring{U}_{\mathcal{N}-\Upsilon_2}\right|\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}}\right)\end{array}\right)$$

2. $\Upsilon_1\otimes\Upsilon_2$

$$=\left(\begin{array}{c}e^{-\left(\left(-\log\left(\mathring{U}_{\mathcal{P}-\Upsilon_1}\right)\right)^{\mathfrak{f}}+\left(-\log\left(\mathring{U}_{\mathcal{P}-\Upsilon_2}\right)\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}},\\ -1+e^{-\left(\left(-\log\left(1+\mathring{U}_{\mathcal{N}-\Upsilon_1}\right)\right)^{\mathfrak{f}}+\left(-\log\left(1+\mathring{U}_{\mathcal{N}-\Upsilon_2}\right)\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}}\end{array}\right)$$

3. $$\alpha\Upsilon_1=\left(\begin{array}{c}1-e^{-\left(\alpha\left(-\log\left(1-\mathring{U}_{\mathcal{P}-\Upsilon_1}\right)\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}},\\ -\left(e^{-\left(\alpha\left(-\log\left|\mathring{U}_{\mathcal{N}-\Upsilon_1}\right|\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}}\right)\end{array}\right)$$

4. $$\Upsilon_1^{\alpha}=\left(\begin{array}{c}e^{-\left(\alpha\left(-\log\mathring{U}_{\mathcal{P}-\Upsilon_1}\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}},\\ -1+e^{-\left(\alpha\left(-\log\left(1+\mathring{U}_{\mathcal{N}-\Upsilon_1}\right)\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}}\end{array}\right)$$

Definition 7: The BFAAPRA operator for the class of BFNs, $\Upsilon_{\dot{\upsilon}}=\left(\mathring{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}},\ \mathring{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right)$ $(\dot{\upsilon}=1,\ 2,\ ..,\ \bar{\eta})$ is inaugurated as

$$BFAAPRA\left(\Upsilon_1,\ \Upsilon_2,\ldots,\ \Upsilon_{\bar{\eta}}\right)=\underset{\dot{\upsilon}=1}{\overset{\bar{\eta}}{\oplus}}\frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\bar{\eta}}\dot{T}_{\dot{\upsilon}}}\Upsilon_{\dot{\upsilon}}\tag{6}$$

Noted that $\dot{T}_1=1$, $\dot{T}_{\dot{\upsilon}}=\prod_{\dot{\upsilon}=1}^{\bar{\eta}-1}\dot{\mathcal{S}}_F\left(\Upsilon_{\dot{\upsilon}}\right)$, $\dot{\upsilon}=1,\ 2,\ \ldots,\bar{\eta}$ and $\dot{\mathcal{S}}_F\left(\Upsilon_{\dot{\upsilon}}\right)$ is the score value of BFN $\Upsilon_{\dot{\upsilon}}$.

Theorem 1: If $\Upsilon_{\dot{\upsilon}}=\left(\mathring{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}},\ \mathring{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right)$ $(\dot{\upsilon}=1,\ 2,\ ..,\ \bar{\eta})$ describe the class of BFNs, then utilizing the BFAAPRA operator to this class results in an aggregated value in the form of BFN

$$BFAAPRA\left(\Upsilon_1,\ \Upsilon_2,\ldots,\ \Upsilon_{\bar{\eta}}\right)=\left(\begin{array}{c}1-e^{-\left(\sum_{\dot{\upsilon}=1}^{\bar{\eta}}\frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\bar{\eta}}\dot{T}_{\dot{\upsilon}}}\left(-\log\left(1-\mathring{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}\right)\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}},\\ -e^{-\left(\sum_{\dot{\upsilon}=1}^{\bar{\eta}}\frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\bar{\eta}}\dot{T}_{\dot{\upsilon}}}\left(-\log\left|\mathring{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right|\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}}\end{array}\right)\tag{7}$$

Proof: This proof would be done by mathematical induction. Suppose $\tilde{\eta}=2$

$$\frac{\dot{T}_1}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\grave{\Upsilon}_1=\left(\begin{array}{c}1-e^{-\left(\frac{\dot{T}_1}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\left(-\log\left(1-\mathring{U}_{\mathcal{P}-\grave{\Upsilon}_1}\right)\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}},\\ -e^{-\left(\frac{\dot{T}_1}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\left(-\log\left|\mathring{U}_{\mathcal{N}-\grave{\Upsilon}_1}\right|\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}}\end{array}\right)$$

$$\frac{\dot{T}_2}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\grave{\Upsilon}_2=\left(\begin{array}{c}1-e^{-\left(\frac{\dot{T}_2}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\left(-\log\left(1-\mathring{U}_{\mathcal{P}-\grave{\Upsilon}_2}\right)\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}},\\ -e^{-\left(\frac{\dot{T}_2}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\left(-\log\left|\mathring{U}_{\mathcal{N}-\grave{\Upsilon}_2}\right|\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}}\end{array}\right)$$

$$\frac{\dot{T}_1}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\grave{\Upsilon}_1\oplus\frac{\dot{T}_2}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\grave{\Upsilon}_2=\left(\begin{array}{c}1-e^{-\left(\frac{\dot{T}_1}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\left(-\log\left(1-\mathring{U}_{\mathcal{P}-\grave{\Upsilon}_1}\right)\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}},\\ -e^{-\left(\frac{\dot{T}_1}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\left(-\log\left|\mathring{U}_{\mathcal{N}-\grave{\Upsilon}_1}\right|\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}}\end{array}\right)\oplus\left(\begin{array}{c}1-e^{-\left(\frac{\dot{T}_2}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\left(-\log\left(1-\mathring{U}_{\mathcal{P}-\grave{\Upsilon}_2}\right)\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}},\\ -e^{-\left(\frac{\dot{T}_2}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\left(-\log\left|\mathring{U}_{\mathcal{N}-\grave{\Upsilon}_2}\right|\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}}\end{array}\right)$$

$$=\left(\begin{array}{c}1-e^{-\left(\sum_{\dot{\upsilon}=1}^{2}\frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\left(-\log\left(1-\mathring{U}_{\mathcal{P}-\grave{\Upsilon}_{\dot{\upsilon}}}\right)\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}},\\ -e^{-\left(\sum_{\dot{\upsilon}=1}^{2}\frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\left(-\log\left|\mathring{U}_{\mathcal{N}-\grave{\Upsilon}_{\dot{\upsilon}}}\right|\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}}\end{array}\right)$$

From above, note that Eq. (7) is held for $\tilde{\eta}=2$. Next, suppose that Eq. (7) is held for $\tilde{\eta}=\mathfrak{R}$

$$BFAAPRA\left(\grave{\Upsilon}_1,\grave{\Upsilon}_2,\ldots,\grave{\Upsilon}_{\mathfrak{R}}\right)=\left(\begin{array}{c}1-e^{-\left(\sum_{\dot{\upsilon}=1}^{\mathfrak{R}}\frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\left(-\log\left(1-\mathring{U}_{\mathcal{P}-\grave{\Upsilon}_{\dot{\upsilon}}}\right)\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}},\\ -e^{-\left(\sum_{\dot{\upsilon}=1}^{\mathfrak{R}}\frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\left(-\log\left|\mathring{U}_{\mathcal{N}-\grave{\Upsilon}_{\dot{\upsilon}}}\right|\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}}\end{array}\right)$$

Now suppose that $\tilde{\eta}=\mathfrak{R}+1$,

$$BFAAPRA\left(\grave{\Upsilon}_1,\grave{\Upsilon}_2,\ldots,\grave{\Upsilon}_{\mathfrak{R}+1}\right)=BFAAPRA\left(\grave{\Upsilon}_1,\grave{\Upsilon}_2,\ldots,\grave{\Upsilon}_{\mathfrak{R}}\right)\oplus\left(\frac{\dot{T}_{\mathfrak{R}+1}}{\sum_{\dot{\upsilon}=1}^{\mathfrak{R}+1}\dot{T}_{\dot{\upsilon}}}\grave{\Upsilon}_{\mathfrak{R}+1}\right)$$

$$=\left(\begin{array}{c}1-e^{-\left(\sum_{\dot{\upsilon}=1}^{\mathfrak{R}}\frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\left(-\log\left(1-\mathring{U}_{\mathcal{P}-\grave{\Upsilon}_{\dot{\upsilon}}}\right)\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}},\\ -e^{-\left(\sum_{\dot{\upsilon}=1}^{\mathfrak{R}}\frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\left(-\log\left|\mathring{U}_{\mathcal{N}-\grave{\Upsilon}_{\dot{\upsilon}}}\right|\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}}\end{array}\right)\oplus\left(\begin{array}{c}1-e^{-\left(\frac{\dot{T}_{\mathfrak{R}+1}}{\sum_{\dot{\upsilon}=1}^{\mathfrak{R}+1}\dot{T}_{\dot{\upsilon}}}\left(-\log\left(1-\mathring{U}_{\mathcal{P}-\grave{\Upsilon}_{\mathfrak{R}+1}}\right)\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}},\\ -e^{-\left(\frac{\dot{T}_{\mathfrak{R}+1}}{\sum_{\dot{\upsilon}=1}^{\mathfrak{R}+1}\dot{T}_{\dot{\upsilon}}}\left(-\log\left|\mathring{U}_{\mathcal{N}-\grave{\Upsilon}_{\mathfrak{R}+1}}\right|\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}}\end{array}\right)$$

$$=\left(\begin{array}{c}1-e^{-\left(\sum_{\dot{\upsilon}=1}^{\mathfrak{R}+1}\frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\left(-\log\left(1-\mathring{U}_{\mathcal{P}-\grave{\Upsilon}_{\dot{\upsilon}}}\right)\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}},\\ -e^{-\left(\sum_{\dot{\upsilon}=1}^{\mathfrak{R}+1}\frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\left(-\log\left|\mathring{U}_{\mathcal{N}-\grave{\Upsilon}_{\dot{\upsilon}}}\right|\right)^{\mathfrak{f}}\right)^{\frac{1}{\mathfrak{f}}}}\end{array}\right)$$

From above, it is clear that Eq. (7) is held for $\tilde{\eta}=\mathfrak{R}+1$. Consequently, Eq. (7) is held for all $\tilde{\eta}$.

The BFAAPRA operator satisfies the properties that are idempotency, monotonicity, and boundedness.

Idempotency: If $\grave{\Upsilon}_{\dot{\upsilon}}=\left(\mathring{U}_{\mathcal{P}-\grave{\Upsilon}_{\dot{\upsilon}}},\mathring{U}_{\mathcal{N}-\grave{\Upsilon}_{\dot{\upsilon}}}\right)$ $(\dot{\upsilon}=1,2,..,\tilde{\eta})$ describe the class of BFNs, and $\grave{\Upsilon}_{\dot{\upsilon}}=\grave{\Upsilon}\ \forall\ \dot{\upsilon}$, then we get

$$BFAAPRA\left(\grave{\Upsilon}_1,\grave{\Upsilon}_2,\ldots,\grave{\Upsilon}_{\tilde{\eta}}\right)=\grave{\Upsilon}$$

Monotonicity: If $\grave{\Upsilon}_{\dot{\upsilon}}=\left(\mathring{U}_{\mathcal{P}-\grave{\Upsilon}_{\dot{\upsilon}}},\mathring{U}_{\mathcal{N}-\grave{\Upsilon}_{\dot{\upsilon}}}\right)$ and $\grave{\Upsilon}'_{\dot{\upsilon}}=\left(\mathring{U}_{\mathcal{P}-\grave{\Upsilon}'_{\dot{\upsilon}}},\mathring{U}_{\mathcal{N}-\grave{\Upsilon}'_{\dot{\upsilon}}}\right)$ $(\dot{\upsilon}=1,2,..,\tilde{\eta})$ describe two classes of BFNs, and $\grave{\Upsilon}_{\dot{\upsilon}}\leq\grave{\Upsilon}'_{\dot{\upsilon}}\ \forall\ \dot{\upsilon}$ i.e. $\mathring{U}_{\mathcal{P}-\grave{\Upsilon}_{\dot{\upsilon}}}\leq\mathring{U}_{\mathcal{P}-\grave{\Upsilon}'_{\dot{\upsilon}}}$, $\mathring{U}_{\mathcal{N}-\grave{\Upsilon}_{\dot{\upsilon}}}\leq\mathring{U}_{\mathcal{N}-\grave{\Upsilon}'_{\dot{\upsilon}}}$, then we get

$$BFAAPRA\left(\grave{\Upsilon}_1,\grave{\Upsilon}_2,\ldots,\grave{\Upsilon}_{\tilde{\eta}}\right)\leq BFAAPRA\left(\grave{\Upsilon}'_1,\grave{\Upsilon}'_2,\ldots,\grave{\Upsilon}'_{\tilde{\eta}}\right)$$

Boundedness: If $\grave{\Upsilon}_{\dot{\upsilon}}=\left(\mathring{U}_{\mathcal{P}-\grave{\Upsilon}_{\dot{\upsilon}}},\mathring{U}_{\mathcal{N}-\grave{\Upsilon}_{\dot{\upsilon}}}\right)$ $(\dot{\upsilon}=1,2,..,\tilde{\eta})$ describe the class of BFNs, and if $\grave{\Upsilon}^-=\left(\min_{\dot{\upsilon}}\left\{\mathring{U}_{\mathcal{P}-\grave{\Upsilon}_{\dot{\upsilon}}}\right\},\max_{\dot{\upsilon}}\left\{\mathring{U}_{\mathcal{N}-\grave{\Upsilon}_{\dot{\upsilon}}}\right\}\right)$, and $\grave{\Upsilon}^+=\left(\max_{\dot{\upsilon}}\left\{\mathring{U}_{\mathcal{P}-\grave{\Upsilon}_{\dot{\upsilon}}}\right\},\min_{\dot{\upsilon}}\left\{\mathring{U}_{\mathcal{N}-\grave{\Upsilon}_{\dot{\upsilon}}}\right\}\right)$, then we get

$$\grave{\Upsilon}^-\leq BFAAPRA\left(\grave{\Upsilon}_1,\grave{\Upsilon}_2,\ldots,\grave{\Upsilon}_{\tilde{\eta}}\right)\leq\grave{\Upsilon}^+$$

Definition 8: The BFAAPRWA operator for the class of BFNs, $\grave{\Upsilon}_{\dot{\upsilon}}=\left(\mathring{U}_{\mathcal{P}-\grave{\Upsilon}_{\dot{\upsilon}}},\mathring{U}_{\mathcal{N}-\grave{\Upsilon}_{\dot{\upsilon}}}\right)$ $(\dot{\upsilon}=1,2,..,\tilde{\eta})$ is inaugurated as

$$BFAAPRWA\left(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{\tilde{\eta}}\right) = \bigoplus_{\dot{\upsilon}=1}^{\tilde{\eta}} \frac{\omega_{\omega\nu-\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \omega_{\omega\nu-\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\Upsilon_{\dot{\upsilon}} \quad (8)$$

Noted that $\omega_{\omega\nu} = \left(\omega_{\omega\nu-1}, \omega_{\omega\nu-2}, \ldots, \omega_{\omega\nu-\tilde{\eta}}\right)^T$ as a weight vector and $0 \le \omega_{\omega\nu-\dot{\upsilon}} \le 1$ for all $\dot{\upsilon}$ and $\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \omega_{\omega\nu-\dot{\upsilon}} = 1$. Further, $\dot{T}_1 = 1, \dot{T}_{\dot{\upsilon}} = \prod_{\dot{\upsilon}=1}^{\tilde{\eta}-1} \dot{S}_F\left(\Upsilon_{\dot{\upsilon}}\right), \dot{\upsilon} = 1, 2, \ldots, \tilde{\eta}$ and $\dot{S}_F\left(\Upsilon_{\dot{\upsilon}}\right)$ is the score value of BFN $\Upsilon_{\dot{\upsilon}}$.

Theorem 2: If $\Upsilon_{\dot{\upsilon}} = \left(\dot{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}, \dot{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right)$ $(\dot{\upsilon} = 1, 2, .., \tilde{\eta})$ describe the class of BFNs, then utilizing the BFAAPRWA operator to this class results in an aggregated value in the form of BFN

$$BFAAPRWA\left(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{\tilde{\eta}}\right) = \left(\begin{array}{c} 1 - e^{-\left(\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \frac{\omega_{\omega\nu-\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \omega_{\omega\nu-\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\left(-\log\left(1-\dot{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}\right)\right)^{f}\right)^{\frac{1}{f}}}, \\ -e^{-\left(\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \frac{\omega_{\omega\nu-\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \omega_{\omega\nu-\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\left(-\log\left|\dot{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right|\right)^{f}\right)^{\frac{1}{f}}} \end{array}\right) \quad (9)$$

The BFAAPRWA operator satisfies the properties that are idempotency, monotonicity, and boundedness.

Idempotency: If $\Upsilon_{\dot{\upsilon}} = \left(\dot{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}, \dot{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right)$ $(\dot{\upsilon} = 1, 2, .., \tilde{\eta})$ describe the class of BFNs, and $\Upsilon_{\dot{\upsilon}} = \Upsilon\ \forall\ \dot{\upsilon}$, then we get

$$BFAAPRWA\left(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{\tilde{\eta}}\right) = \Upsilon$$

Monotonicity: If $\Upsilon_{\dot{\upsilon}} = \left(\dot{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}, \dot{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right)$ and $\Upsilon'_{\dot{\upsilon}} = \left(\dot{U}_{\mathcal{P}-\Upsilon'_{\dot{\upsilon}}}, \dot{U}_{\mathcal{N}-\Upsilon'_{\dot{\upsilon}}}\right)$ $(\dot{\upsilon} = 1, 2, .., \tilde{\eta})$ describe two classes of BFNs, and $\Upsilon_{\dot{\upsilon}} \le \Upsilon'_{\dot{\upsilon}}\ \forall\ \dot{\upsilon}$ i.e. $\dot{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}} \le \dot{U}_{\mathcal{P}-\Upsilon'_{\dot{\upsilon}}}, \dot{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}} \le \dot{U}_{\mathcal{N}-\Upsilon'_{\dot{\upsilon}}}$, then we get

$$BFAAPRWA\left(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{\tilde{\eta}}\right) \le BFAAPRWA\left(\Upsilon'_1, \Upsilon'_2, \ldots, \Upsilon'_{\tilde{\eta}}\right)$$

Boundedness: If $\Upsilon_{\dot{\upsilon}} = \left(\dot{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}, \dot{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right)$ $(\dot{\upsilon} = 1, 2, .., \tilde{\eta})$ describe the class of BFNs, and if $\Upsilon^- = \left(\min_{\dot{\upsilon}}\left\{\dot{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}\right\}, \max_{\dot{\upsilon}}\left\{\dot{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right\}\right)$, and $\Upsilon^+ = \left(\max_{\dot{\upsilon}}\left\{\dot{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}\right\}, \min_{\dot{\upsilon}}\left\{\dot{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right\}\right)$, then we get

$$\Upsilon^- \le BFAAPRWA\left(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{\tilde{\eta}}\right) \le \Upsilon^+$$

Definition 9: The BFAAPRG operator for the class of BFNs, $\Upsilon_{\dot{\upsilon}} = \left(\dot{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}, \dot{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right)$ $(\dot{\upsilon} = 1, 2, .., \tilde{\eta})$ is inaugurated as

$$BFAAPRG\left(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{\tilde{\eta}}\right) = \bigotimes_{\dot{\upsilon}=1}^{\tilde{\eta}} \left(\Upsilon_{\dot{\upsilon}}\right)^{\frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}} \quad (10)$$

Noted that $\dot{T}_1 = 1, \dot{T}_{\dot{\upsilon}} = \prod_{\dot{\upsilon}=1}^{\tilde{\eta}-1} \dot{S}_F\left(\Upsilon_{\dot{\upsilon}}\right), \dot{\upsilon} = 1, 2, \ldots, \tilde{\eta}$ and $\dot{S}_F\left(\Upsilon_{\dot{\upsilon}}\right)$ is the score value of BFN $\Upsilon_{\dot{\upsilon}}$.

Theorem 3: If $\Upsilon_{\dot{\upsilon}} = \left(\dot{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}, \dot{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right)$ $(\dot{\upsilon} = 1, 2, .., \tilde{\eta})$ describe the class of BFNs, then utilizing the BFAAPRG operator to this class results in an aggregated value in the form of BFN

$$BFAAPRG\left(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{\tilde{\eta}}\right) = \left(\begin{array}{c} e^{-\left(\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}\left(-\log\dot{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}\right)^{f}\right)^{\frac{1}{f}}}, \\ -1 + e^{-\left(\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}\left(-\log\left(1+\dot{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right)\right)^{f}\right)^{\frac{1}{f}}} \end{array}\right) \quad (11)$$

Proof: This proof would be done by mathematical induction. Suppose $\tilde{\eta} = 2$

$$\left(\Upsilon_1\right)^{\frac{\dot{T}_1}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}} = \left(\begin{array}{c} e^{-\left(\frac{\dot{T}_1}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}\left(-\log\dot{U}_{\mathcal{P}-\Upsilon_1}\right)^{f}\right)^{\frac{1}{f}}}, \\ -1 + e^{-\left(\frac{\dot{T}_1}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}\left(-\log\left(1+\dot{U}_{\mathcal{N}-\Upsilon_1}\right)\right)^{f}\right)^{\frac{1}{f}}} \end{array}\right)$$

$$\left(\Upsilon_2\right)^{\frac{\dot{T}_2}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}} = \left(\begin{array}{c} e^{-\left(\frac{\dot{T}_2}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}\left(-\log\dot{U}_{\mathcal{P}-\Upsilon_2}\right)^{f}\right)^{\frac{1}{f}}}, \\ -1 + e^{-\left(\frac{\dot{T}_2}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}\left(-\log\left(1+\dot{U}_{\mathcal{N}-\Upsilon_2}\right)\right)^{f}\right)^{\frac{1}{f}}} \end{array}\right)$$

$$\left(\Upsilon_1\right)^{\frac{\dot{T}_1}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}} \otimes \left(\Upsilon_2\right)^{\frac{\dot{T}_2}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}} = \left(\begin{array}{c} e^{-\left(\frac{\dot{T}_1}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}\left(-\log\dot{U}_{\mathcal{P}-\Upsilon_1}\right)^{f}\right)^{\frac{1}{f}}}, \\ -1 + e^{-\left(\frac{\dot{T}_1}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}\left(-\log\left(1+\dot{U}_{\mathcal{N}-\Upsilon_1}\right)\right)^{f}\right)^{\frac{1}{f}}} \end{array}\right) \otimes \left(\begin{array}{c} e^{-\left(\frac{\dot{T}_2}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}\left(-\log\dot{U}_{\mathcal{P}-\Upsilon_2}\right)^{f}\right)^{\frac{1}{f}}}, \\ -1 + e^{-\left(\frac{\dot{T}_2}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}\left(-\log\left(1+\dot{U}_{\mathcal{N}-\Upsilon_2}\right)\right)^{f}\right)^{\frac{1}{f}}} \end{array}\right)$$

$$= \left(\begin{array}{c} e^{-\left(\sum_{\dot{\upsilon}=1}^{2} \frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}\left(-\log\dot{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}\right)^{f}\right)^{\frac{1}{f}}}, \\ -1 + e^{-\left(\sum_{\dot{\upsilon}=1}^{2} \frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}\left(-\log\left(1+\dot{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right)\right)^{f}\right)^{\frac{1}{f}}} \end{array}\right)$$

From above, note that Eq. (11) is held for $\tilde{\eta} = 2$. Next, suppose that Eq. (11) is held for $\tilde{\eta} = \mathfrak{R}$

$$BFAAPRG\left(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{\mathfrak{R}}\right) = \left(e^{-\left(\sum_{\dot{\upsilon}=1}^{\mathfrak{R}} \frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}\left(-\log \mathring{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}\right)^{f}\right)^{\frac{1}{f}}}, -1 + e^{-\left(\sum_{\dot{\upsilon}=1}^{\mathfrak{R}} \frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}\left(-\log\left(1+\mathring{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right)\right)^{f}\right)^{\frac{1}{f}}} \right)$$

Now suppose that $\tilde{\eta} = \mathfrak{R}+1$,

$$\begin{aligned} &BFAAPRG\left(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{\mathfrak{R}+1}\right) \\ &= BFAAPRG\left(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{\mathfrak{R}}\right) \otimes \left(\Upsilon_{\mathfrak{R}+1}\right)^{\frac{\dot{T}_{\mathfrak{R}+1}}{\sum_{\dot{\upsilon}=1}^{\mathfrak{R}+1} \dot{T}_{\dot{\upsilon}}}} \\ &= \left(e^{-\left(\sum_{\dot{\upsilon}=1}^{\mathfrak{R}} \frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}\left(-\log \mathring{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}\right)^{f}\right)^{\frac{1}{f}}}, -1 + e^{-\left(\sum_{\dot{\upsilon}=1}^{\mathfrak{R}} \frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}\left(-\log\left(1+\mathring{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right)\right)^{f}\right)^{\frac{1}{f}}} \right) \\ &\otimes \left(e^{-\left(\frac{\dot{T}_{\mathfrak{R}+1}}{\sum_{\dot{\upsilon}=1}^{\mathfrak{R}+1} \dot{T}_{\dot{\upsilon}}}\left(-\log \mathring{U}_{\mathcal{P}-\Upsilon_{\mathfrak{R}+1}}\right)^{f}\right)^{\frac{1}{f}}}, -1 + e^{-\left(\frac{\dot{T}_{\mathfrak{R}+1}}{\sum_{\dot{\upsilon}=1}^{\mathfrak{R}+1} \dot{T}_{\dot{\upsilon}}}\left(-\log\left(1+\mathring{U}_{\mathcal{N}-\Upsilon_{\mathfrak{R}+1}}\right)\right)^{f}\right)^{\frac{1}{f}}} \right) \\ &= \left(e^{-\left(\sum_{\dot{\upsilon}=1}^{\mathfrak{R}+1} \frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}\left(-\log \mathring{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}\right)^{f}\right)^{\frac{1}{f}}}, -1 + e^{-\left(\sum_{\dot{\upsilon}=1}^{\mathfrak{R}+1} \frac{\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \dot{T}_{\dot{\upsilon}}}\left(-\log\left(1+\mathring{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right)\right)^{f}\right)^{\frac{1}{f}}} \right) \end{aligned}$$

From above, it is clear that Eq. (11) is held for $\tilde{\eta} = \mathfrak{R}+1$. Consequently, Eq. (11) is held for all $\tilde{\eta}$.

The BFAAPRG operator satisfies the properties that are idempotency, monotonicity, and boundedness.

Idempotency: If $\Upsilon_{\dot{\upsilon}} = \left(\mathring{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}, \mathring{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right)$ $(\dot{\upsilon} = 1, 2, .., \tilde{\eta})$ describe the class of BFNs, and $\Upsilon_{\dot{\upsilon}} = \Upsilon\ \forall\ \dot{\upsilon}$, then we get

$$BFAAPRG\left(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{\tilde{\eta}}\right) = \Upsilon$$

Monotonicity: If $\Upsilon_{\dot{\upsilon}} = \left(\mathring{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}, \mathring{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right)$ and $\Upsilon'_{\dot{\upsilon}} = \left(\mathring{U}_{\mathcal{P}-\Upsilon'_{\dot{\upsilon}}}, \mathring{U}_{\mathcal{N}-\Upsilon'_{\dot{\upsilon}}}\right)$ $(\dot{\upsilon} = 1, 2, .., \tilde{\eta})$ describe two classes of BFNs, and $\Upsilon_{\dot{\upsilon}} \leq \Upsilon'_{\dot{\upsilon}}\ \forall\ \dot{\upsilon}$ i.e. $\mathring{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}} \leq \mathring{U}_{\mathcal{P}-\Upsilon'_{\dot{\upsilon}}}, \mathring{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}} \leq \mathring{U}_{\mathcal{N}-\Upsilon'_{\dot{\upsilon}}}$, then we get

$$\begin{aligned} &BFAAPRG\left(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{\tilde{\eta}}\right) \\ &\qquad \leq BFAAPRG\left(\Upsilon'_1, \Upsilon'_2, \ldots, \Upsilon'_{\tilde{\eta}}\right) \end{aligned}$$

Boundedness: If $\Upsilon_{\dot{\upsilon}} = \left(\mathring{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}, \mathring{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right)$ $(\dot{\upsilon} = 1, 2, .., \tilde{\eta})$ describe the class of BFNs, and if $\Upsilon^- = \left(\min_{\dot{\upsilon}}\left\{\mathring{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}\right\}, \max_{\dot{\upsilon}}\left\{\mathring{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right\}\right)$, and $\Upsilon^+ = \left(\max_{\dot{\upsilon}}\left\{\mathring{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}\right\}, \min_{\dot{\upsilon}}\left\{\mathring{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right\}\right)$, then we get

$$\Upsilon^- \leq BFAAPRG\left(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{\tilde{\eta}}\right) \leq \Upsilon^+$$

Definition 10: The BFAAPRWG operator for the class of BFNs, $\Upsilon_{\dot{\upsilon}} = \left(\mathring{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}, \mathring{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right)$ $(\dot{\upsilon} = 1, 2, .., \tilde{\eta})$ is inaugurated as

$$BFAAPRWG\left(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{\tilde{\eta}}\right) = \bigotimes_{\dot{\upsilon}=1}^{\tilde{\eta}} \left(\Upsilon_{\dot{\upsilon}}\right)^{\frac{\omega_{\omega\nu-\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \omega_{\omega\nu-\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}} \tag{12}$$

Noted that $\omega_{\omega\nu} = \left(\omega_{\omega\nu-1}, \omega_{\omega\nu-2}, \ldots, \omega_{\omega\nu-\tilde{\eta}}\right)^T$ as a weight vector and $0 \leq \omega_{\omega\nu-\dot{\upsilon}} \leq 1$ for all $\dot{\upsilon}$ and $\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \omega_{\omega\nu-\dot{\upsilon}} = 1$. Further, $\dot{T}_1 = 1, \dot{T}_{\dot{\upsilon}} = \prod_{\dot{\upsilon}=1}^{\tilde{\eta}-1} \dot{S}_F\left(\Upsilon_{\dot{\upsilon}}\right)$, $\dot{\upsilon} = 1, 2, \ldots, \tilde{\eta}$ and $\dot{S}_F\left(\Upsilon_{\dot{\upsilon}}\right)$ is the score value of BFN $\Upsilon_{\dot{\upsilon}}$.

Theorem 4: If $\Upsilon_{\dot{\upsilon}} = \left(\mathring{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}, \mathring{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right)$ $(\dot{\upsilon} = 1, 2, .., \tilde{\eta})$ describe the class of BFNs, then utilizing the BFAAPRWG operator to this class results in an aggregated value in the form of BFN

$$BFAAPRWG\left(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{\tilde{\eta}}\right) = \left(e^{-\left(\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \frac{\omega_{\omega\nu-\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \omega_{\omega\nu-\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\left(-\log \mathring{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}\right)^{f}\right)^{\frac{1}{f}}}, -1 + e^{-\left(\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \frac{\omega_{\omega\nu-\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}{\sum_{\dot{\upsilon}=1}^{\tilde{\eta}} \omega_{\omega\nu-\tilde{\eta}}\dot{T}_{\dot{\upsilon}}}\left(-\log\left(1+\mathring{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right)\right)^{f}\right)^{\frac{1}{f}}} \right) \tag{13}$$

The BFAAPRWG operator satisfies the properties that are idempotency, monotonicity, and boundedness.

Idempotency: If $\Upsilon_{\dot{\upsilon}} = \left(\mathring{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}, \mathring{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right)$ $(\dot{\upsilon} = 1, 2, .., \tilde{\eta})$ describe the class of BFNs, and $\Upsilon_{\dot{\upsilon}} = \Upsilon\ \forall\ \dot{\upsilon}$, then we get

$$BFAAPRWG\left(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{\tilde{\eta}}\right) = \Upsilon$$

Monotonicity: If $\Upsilon_{\dot{\upsilon}} = \left(\mathring{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}, \mathring{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right)$ and $\Upsilon'_{\dot{\upsilon}} = \left(\mathring{U}_{\mathcal{P}-\Upsilon'_{\dot{\upsilon}}}, \mathring{U}_{\mathcal{N}-\Upsilon'_{\dot{\upsilon}}}\right)$ $(\dot{\upsilon} = 1, 2, .., \tilde{\eta})$ describe two classes of BFNs, and $\Upsilon_{\dot{\upsilon}} \leq \Upsilon'_{\dot{\upsilon}}\ \forall\ \dot{\upsilon}$ i.e. $\mathring{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}} \leq \mathring{U}_{\mathcal{P}-\Upsilon'_{\dot{\upsilon}}}, \mathring{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}} \leq \mathring{U}_{\mathcal{N}-\Upsilon'_{\dot{\upsilon}}}$, then we get

$$\begin{aligned} &BFAAPRWG\left(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{\tilde{\eta}}\right) \\ &\qquad \leq BFAAPRWG\left(\Upsilon'_1, \Upsilon'_2, \ldots, \Upsilon'_{\tilde{\eta}}\right) \end{aligned}$$

Boundedness: If $\Upsilon_{\dot{\upsilon}} = \left(\mathring{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}, \mathring{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right)$ $(\dot{\upsilon} = 1, 2, .., \tilde{\eta})$ describe the class of BFNs, and if $\Upsilon^- = \left(\min_{\dot{\upsilon}}\left\{\mathring{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}\right\}, \max_{\dot{\upsilon}}\left\{\mathring{U}_{\mathcal{N}-\Upsilon_{\dot{\upsilon}}}\right\}\right)$, and $\Upsilon^+ = \left(\max_{\dot{\upsilon}}\left\{\mathring{U}_{\mathcal{P}-\Upsilon_{\dot{\upsilon}}}\right\},\right.$

$\min_{\mathring{\upsilon}} \left\{ \mathring{U}_{\mathcal{N}-\Upsilon_{\mathring{\upsilon}}} \right\} \Big)$, then we get

$$\Upsilon^{-} \leq BFAAPRWG\left(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{\tilde{\eta}}\right) \leq \Upsilon^{+}$$

IV. MADM RELIES ON DEVELOPED OPERATORS UNDER BF INFORMATION

In this part, we induce an approach of MADM by employing the inaugurated operators under BF information and then investigate a study case related to the marketing strategies for business growth.

Consider a class of alternatives, identified as $\Upsilon_{\mathring{A}} = \left\{\Upsilon_{\mathring{A}-1}, \Upsilon_{\mathring{A}-2}, \ldots, \Upsilon_{\mathring{A}-\tilde{\eta}}\right\}$, which contains $\tilde{\eta}$ various alternatives. Additionally, there are Ω attributes, identified as $\mathfrak{A}_{\mathring{A}} = \left\{\mathfrak{A}_{\mathring{A}-1}, \mathfrak{A}_{\mathring{A}-2}, \ldots, \mathfrak{A}_{\mathring{A}-\Omega}\right\}$, that are utilized to assess these alternatives and there is prioritization among the attributes interpreted with the assistance of linear order $\mathfrak{A}_{\mathring{A}-1} > \mathfrak{A}_{\mathring{A}-2} > \ldots > \mathfrak{A}_{\mathring{A}-\Omega}$, show that the priority of $\mathfrak{A}_{\mathring{A}-\mathring{\upsilon}}$ is higher than $\mathfrak{A}_{\mathring{A}-\alpha}$ if $\mathring{\upsilon} < \alpha$. Further, in any circumstance, if the decision analyst wants to deliver the weight to each attribute that is $\omega_{\omega\upsilon} = \left(\omega_{\omega\upsilon-1}, \omega_{\omega\upsilon-2}, \ldots, \omega_{\omega\upsilon-\alpha}\right)^T$ such that $0 \leq \omega_{\omega\upsilon-\alpha} \leq 1$, and $\sum_{\alpha=1}^{\Omega} \omega_{\omega\upsilon-\alpha} = 1$. The decision analyst assessment values for these alternatives based on the attribute will be in the model of BF information i.e., $\Upsilon = \left(\mathring{U}_{\mathcal{P}-\Upsilon}, \mathring{U}_{\mathcal{N}-\Upsilon}\right)$, where $\mathring{U}_{\mathcal{P}-\Upsilon} \in [0, 1]$ and $\mathring{U}_{\mathcal{N}-\Upsilon} \in [-1, 0]$, which will develop a BF decision matrix. To address this MADM dilemma, we propose the below steps.

Step 1: In every DM dilemma there may be two types of information (cost and benefit type). In the case of cost type, the normalization of the BF decision matrix would be required which would be done by the below formula

$$N = \begin{cases} \left(\mathring{U}_{\mathcal{P}-\Upsilon}, \mathring{U}_{\mathcal{N}-\Upsilon}\right) & \textit{for benefit type} \\ \left(1 - \mathring{U}_{\mathcal{P}-\Upsilon}, -1 - \mathring{U}_{\mathcal{N}-\Upsilon}\right) & \textit{for cost type} \end{cases} \tag{14}$$

In the case of benefit type, normalization wouldn't be required.

Step 2: Evaluate $T_{\mathring{\upsilon}\alpha}$ as

$$T_{\mathring{\upsilon}\alpha} = \prod_{\rho=1}^{\alpha-1} \mathring{S}_F\left(\Upsilon_{\mathring{\upsilon}\rho}\right) \tag{15}$$

where, $\mathring{\upsilon} = 1, 2, \ldots, \tilde{\eta}$, $\alpha = 1, 2, \ldots, \Omega$ and $T_{\mathring{\upsilon}1} = 1$ for $\mathring{\upsilon} = 1, 2, \ldots, \tilde{\eta}$.

Step 3: Here, the inaugurated operators that are BFAAPRA, and BFAAPRG operators would be utilized to achieve the aggregative result. If the decision-analyst considered the weight of each attribute then BFAAPRWA or BFAAPRWG operator would be utilized to achieve the required result.

Step 4: Eq. (3) would be utilized to achieve the score values of the aggregated result. However, if the score function encounters any issues, then the accuracy values would be calculated instead.

Step 5: Here, the ranking would be determined based on the accuracy or score values to get the best option.

For showing the applicability of the proposed method we interpret the underneath numerical example.

A. STUDY CASE

With the world moving towards a more sustainable and decentralized energy system, the idea of the P2P energy trade has been greatly publicized. P2P energy trading platforms provide individuals and communities with the opportunity to buy and sell renewable energy that is locally generated, which leads to energy democracy, reduction of carbon footprint, and self-sufficiency. In this regard, Greenville Renewable Energy Solutions (GRES), an innovative company that is at the forefront of renewable energy technologies, has chosen Oakwood Community as a perfect place for deployment of a new 2-way energy trading platform. Within, the framework of the Oakwood Community, a futuristic residential neighborhood, a self-sufficient energy ecosystem is the goal. Moved by a passion for sustainable development and energy independence, the community plans to take advantage of the possibility of using distributed renewable energy by installing rooftop solar photovoltaic systems and small-scale wind turbines. The residents of Oakwood Community ambition to take part actively in the energy market using the P2P energy trading platform, exchanging their surplus energy among themselves and cutting down their dependency on the centralized energy systems.

Keeping in mind that this is a major undertaking, the GRES recognizes the critical role played by the P2P energy trading platform in tailoring it to the Oakwood Community's specific requirements. To design a thorough and informed process for the selection, GRES has identified four energy platforms that are $\Upsilon_{\mathring{A}-1}$, $\Upsilon_{\mathring{A}-2}$, $\Upsilon_{\mathring{A}-3}$, and $\Upsilon_{\mathring{A}-4}$ which would be asses by the underneath four attribute

$\mathfrak{A}_{\mathring{A}-1}$**: Technology Compatibility:** It is the criterion that determines the compatibility of the platform with the existing energy infrastructure and communication protocols to integrate with the community. It looks at, for example, how well it works with smart meters, advanced metering infrastructure (AMI), communication standards, and its ability to work with renewable energy systems. The platform can be developed with high technology compatibility, which will enable it to seamlessly interact with the community's energy ecosystem thereby lowering implementation barriers and making energy transactions efficient.

$\mathfrak{A}_{\mathring{A}-2}$**: Transaction Costs:** The transaction costs are related to the charges, fees, as well as commissions that are involved in energy trades that are done on the platform. This criterion critically examines the cost-effectiveness and feasibility of the platform for P2P energy trading. Reduction of transaction costs encourages trading and makes it more financially reliable for participants which in turn increases the adoption of renewable energy generation and consumption. Forums with clear fee structures and rates competitive for participants to reduce the costs are better than the rest.

$\mathfrak{A}_{\mathring{A}-3}$**: User Interface and Experience:** The user interface and experience criterion is all about the ease of

TABLE 1. The expert assessment values in the form of BF decision-matrix.

	$\mathfrak{A}_{Ḁ-1}$	$\mathfrak{A}_{Ḁ-2}$	$\mathfrak{A}_{Ḁ-3}$	$\mathfrak{A}_{Ḁ-4}$
$\Upsilon_{Ḁ-1}$	$(0.67, -0.34)$	$(0.53, -0.43)$	$(0.19, -0.44)$	$(0.76, -0.61)$
$\Upsilon_{Ḁ-2}$	$(0.78, -0.45)$	$(0.64, -0.32)$	$(0.45, -0.54)$	$(0.43, -0.57)$
$\Upsilon_{Ḁ-3}$	$(0.89, -0.23)$	$(0.75, -014.)$	$(0.27, -0.65)$	$(0.85, -0.37)$
$\Upsilon_{Ḁ-4}$	$(0.56, -0.55)$	$(0.24, -0.62)$	$(0.54, -0.14)$	$(0.74, -0.64)$

TABLE 2. The aggregated result of table 1 by employing invented operators with $\mathfrak{f} = 5$.

Operators	$\Upsilon_{Ḁ-1}$	$\Upsilon_{Ḁ-2}$	$\Upsilon_{Ḁ-3}$	$\Upsilon_{Ḁ-4}$
BFAAPRA	$(0.364, -0.987)$	$(0.419, -0.989)$	$(0.556, -0.835)$	$(0.316, -0.96)$
BFAAPRWA	$(0.353, -0.988)$	$(0.418, -0.988)$	$(0.503, -0.752)$	$(0.425, -0.99998)$
BFAAPRG	$(0.9989, -0.0012)$	$(0.9992, -0.0015)$	$(0.9999, -0.0002)$	$(0.9987, -0.001)$
BFAAPRWG	$(0.9989, -0.0013)$	$(0.9992, -0.0016)$	$(0.9999, -0.0002)$	$(0.9986, -0.0011)$

TABLE 3. The score values of table 2.

Operators	$\dot{S}_F(\Upsilon_{Ḁ-1})$	$\dot{S}_F(\Upsilon_{Ḁ-2})$	$\dot{S}_F(\Upsilon_{Ḁ-3})$	$\dot{S}_F(\Upsilon_{Ḁ-4})$
BFAAPRA	0.1887	0.2149	0.3603	0.1779
BFAAPRWA	0.1821	0.2149	0.3756	0.2131
BFAAPRG	0.99882	0.99889	0.99986	0.99881
BFAAPRWG	0.9988	0.99882	0.99987	0.99874

TABLE 4. The ranking of alternatives.

Operators	Ranking
BFAAPRA	$\Upsilon_{Ḁ-3} > \Upsilon_{Ḁ-2} > \Upsilon_{Ḁ-1} > \Upsilon_{Ḁ-4}$
BFAAPRWA	$\Upsilon_{Ḁ-3} > \Upsilon_{Ḁ-2} > \Upsilon_{Ḁ-4} > \Upsilon_{Ḁ-1}$
BFAAPRG	$\Upsilon_{Ḁ-3} > \Upsilon_{Ḁ-2} > \Upsilon_{Ḁ-1} > \Upsilon_{Ḁ-4}$
BFAAPRWG	$\Upsilon_{Ḁ-3} > \Upsilon_{Ḁ-2} > \Upsilon_{Ḁ-1} > \Upsilon_{Ḁ-4}$

use, accessibility, and functionality of the energy trading platform. It involves considering aspects like user-friendly design, intuitive navigation, real-time data visualization, and interactive elements. A platform with a good UX in the sense of a well-designed and intuitive user interface increases user engagement, user satisfaction, and user adoption. Transparent and easy-to-understand presentation of power data and trading choices lets the traders make sound decisions and actively take part in P2P energy trading.

$\mathfrak{A}_{Ḁ-4}$**: Data Security and Privacy:** Data security and privacy are the critical questions to be answered while choosing an energy trading platform since the energy consumption data is so sensitive. This criterion measures the system's safeguards for user data, ensuring confidentiality, integrity, and availability as well as observance of the privacy laws. Robust encryption protocols, safe data storage, access controls, and public privacy policies are the key components of a trusted energy trading platform. Platforms that ensure the security and privacy of data are the ones that inspire confidence among participants and therefore establish the trustworthiness of the P2P energy transactions.

The prioritization between these attributes is $\mathfrak{A}_{Ḁ-1} \succ \mathfrak{A}_{Ḁ-2} \succ \mathfrak{A}_{Ḁ-3} \succ \mathfrak{A}_{Ḁ-4}$. The expert GRES also wants to interpret his weightage to the attribute thus the weight of attributes is (0.18, 0.29, 0.32, 0.31). The assessment values of the expert would form the BF decision matrix which is demonstrated in Table 1.

Step 1: The taken information is benefit type so no need for the process of normalization.

Step 2: Evaluated $\mathrm{T}_{ʋα}$ is

$$\mathrm{T}_{ʋα} = \begin{bmatrix} 1 & 0.655 & 0.366 & 0.137 \\ 1 & 0.665 & 0.439 & 0.2 \\ 1 & 0.83 & 0.668 & 0.207 \\ 1 & 0.505 & 0.157 & 0.11 \end{bmatrix}$$

Step 3: By the utilization of inaugurated operators that are BFAAPRA, BFAAPRG, BFAAPRWA, and BFAAPRWG operators, the achieved result is demonstrated in Table 2.

TABLE 5. The score values of invented and current work.

Source	$\dot{S}_F(\Upsilon_{\underset{\circ}{A}-1})$	$\dot{S}_F(\Upsilon_{\underset{\circ}{A}-2})$	$\dot{S}_F(\Upsilon_{\underset{\circ}{A}-3})$	$\dot{S}_F(\Upsilon_{\underset{\circ}{A}-4})$
Sarfraz et al. [37]	×≃×≃×	×≃×≃×	×≃×≃×	×≃×≃×
Senapati et al. [38]	×≃×≃×	×≃×≃×	×≃×≃×	×≃×≃×
Senapati [39]	×≃×≃×	×≃×≃×	×≃×≃×	×≃×≃×
Jana et al. [22] (BFDPA operator)	0.675	0.657	0.67	0.674
Jana et al. [22] (BFDPG operator)	0.223	0.246	0.236	0.232
Wei et al. [24] (BFHWA operator)	0.633	0.698	0.621	0.576
Wei et al. [24] (BFHWG operator)	0.343	0.401	0.439	0.462
Inaugurated operator (BFAAPRA)	0.1887	0.2149	0.3603	0.1779
Inaugurated operator (BFAAPRWA)	0.1821	0.2149	0.3756	0.2131
Inaugurated operator (BFAAPRG)	0.99882	0.99889	0.99986	0.99881
Inaugurated operator (BFAAPRWG)	0.9988	0.99882	0.99987	0.99874

TABLE 6. The ranking of current and invented work.

Source	Ranking
Sarfraz et al. [37]	×≃×≃×
Senapati et al. [38]	×≃×≃×
Senapati [39]	×≃×≃×
Jana et al. [22] (BFDPA operator)	$\Upsilon_{\underset{\circ}{A}-1} > \Upsilon_{\underset{\circ}{A}-4} > \Upsilon_{\underset{\circ}{A}-3} > \Upsilon_{\underset{\circ}{A}-2}$
Jana et al. [22] (BFDPG operator)	$\Upsilon_{\underset{\circ}{A}-2} > \Upsilon_{\underset{\circ}{A}-3} > \Upsilon_{\underset{\circ}{A}-4} > \Upsilon_{\underset{\circ}{A}-1}$
Wei et al. [24] (BFHWA operator)	$\Upsilon_{\underset{\circ}{A}-3} > \Upsilon_{\underset{\circ}{A}-2} > \Upsilon_{\underset{\circ}{A}-1} > \Upsilon_{\underset{\circ}{A}-4}$
Wei et al. [24] (BFHWG operator)	$\Upsilon_{\underset{\circ}{A}-2} > \Upsilon_{\underset{\circ}{A}-1} > \Upsilon_{\underset{\circ}{A}-3} > \Upsilon_{\underset{\circ}{A}-4}$
Inaugurated operator (BFAAPRA)	$\Upsilon_{\underset{\circ}{A}-3} > \Upsilon_{\underset{\circ}{A}-2} > \Upsilon_{\underset{\circ}{A}-1} > \Upsilon_{\underset{\circ}{A}-4}$
Inaugurated operator (BFAAPRWA)	$\Upsilon_{\underset{\circ}{A}-3} > \Upsilon_{\underset{\circ}{A}-2} > \Upsilon_{\underset{\circ}{A}-4} > \Upsilon_{\underset{\circ}{A}-1}$
Inaugurated operator (BFAAPRG)	$\Upsilon_{\underset{\circ}{A}-3} > \Upsilon_{\underset{\circ}{A}-2} > \Upsilon_{\underset{\circ}{A}-1} > \Upsilon_{\underset{\circ}{A}-4}$
Inaugurated operator (BFAAPRWG)	$\Upsilon_{\underset{\circ}{A}-3} > \Upsilon_{\underset{\circ}{A}-2} > \Upsilon_{\underset{\circ}{A}-1} > \Upsilon_{\underset{\circ}{A}-4}$

Step 4: By Eq. (3) the achieved score values of the aggregated result are displayed in Table 3.

Step 5: The determined ranking based on score values to get the best option is displayed in Table 4.

This result shows that $\Upsilon_{\underset{\circ}{A}-3}$ is the best and finest energy platform for P2P energy trading that GRES will adopt.

V. COMPARATIVE ANALYSIS

To interpret the advantages and supremacy of the invented theory, it is compulsory to compare the invented theory with current theories, like prioritized AA AOs under intuitionistic fuzzy information demonstrated by Sarfraz et al. [37], AA AOs for intuitionistic fuzzy information interpreted by Senapati et al. [38], picture fuzzy AA AOs developed by Senapati [39], Dombi prioritized AOs under BF information deduced by Jana et al. [22] and Hamacher AOs for BF information described by Wei et al. [24]. To facilitate this comparison, the data of Table 1 is revisited and analyzed employing the inaugurated and current work and the results are part of Tables 5 and 6.

The AOs inaugurated by Sarfraz et al. [37], and Senapati et al. [38] are merely able to cope with fuzzy and intuitionistic fuzzy information however, the information involving the negative side of the opinion can't manage with the structure of intuitionistic FS. As a result, the AOs inaugurated by Sarfraz et al. [37], and Senapati et al. [38] are unable to manage the BF information. Likewise, the AA AOs for picture FS investigated by Senapati [39] are merely able to cope with picture fuzzy information but can't manage the negative side of the opinion. As a result, the AOs inaugurated Senapati [39] are unable to tackle BF information. More, Jana et al. [22] inaugurated BF Dombi prioritized averaging (BFDPRA) and BF Dombi prioritized geometric (BFDPRG) and Wei et al. [24] investigated BF Hamacher weighted averaging (BFHWA) and BF Hamacher weighted geometric (BFHWG) AOs under the structure of BFS. Consequently, the data of Table 1 can solve through the AOs investigated by Jana et al. [22] and Wei et al. [24] and the results are part of Tables 5 and 6. We can notice from Tables 5 and 6 that the both averaging and geometric AOs investigated by Jana et al. [22] and Wei et al. [24] demonstrate various results. However, our inaugurated averaging and geometric AOs based on AA operations demonstrate that only $\Upsilon_{\underset{\circ}{A}-3}$ is the finest choice. Furthermore, the proposed work is the perfect tool for managing two (positive and negative side) sided information and managing genuine-life MADM dilemmas where two-sided information is involved. The inauguration can also reduce the environment of fuzzy information.

VI. CONCLUSION

In the time of a decentralized energy system and the rising need for sustainable approaches, a suitable P2P energy trading platform is one of the most challenging decisions to be made. The study has pioneered a new MADM framework that leverages BFSs to deal with the complexity of bipolar criteria involved in this field. The proposed MADM framework, strengthened with the potential of BFSs, is the breakthrough answer to the long-time puzzle of modeling and evaluating criteria that have both positive and negative aspects at the same time. Through this approach, we will be able to overcome the limitations of the traditional methods by taking into account the duality of the criteria, therefore, leading to more comprehensive and fruitful decision-making processes. Furthermore, this article contained a set of new AOs, which are based on AA t-norm and t-conorm within BFS. Such operators play the role of a super-efficient aggregator, providing aggregated results of DM data and enabling the achievement of resilient and sustainable decisions for the energy trading platform selection. The usefulness of our methodology is demonstrated by a case study: "Selection of Energy Trading Platforms for P2P Energy Trading", where it is shown how our method is effective and advantageous. The findings of this study not only reinforce the model but also highlight its power in facilitating energy democracy, cutting carbon footprints, and allowing communities to be self-sufficient. Moreover, comparative analyses with the present literature revealed the incontrovertible superiority and dominance of the introduced MADM approach, which, in turn, places it in a category of a significant contribution to the sphere of sustainable energy systems and decentralized energy trading.

As in the existing literature, various researchers utilized the AA t-norm and t-conorm to get various AOs in different generalization FS. In the future, we aim to expand this work in the setting of bipolar complex fuzzy set [40], bipolar complex fuzzy soft set [41], bipolar complex spherical fuzzy set [42], complex hesitant FS [43], and aggregation theory [44], [45].

ACKNOWLEDGMENT

This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-RG23089).

REFERENCES

[1] Y. Parag and B. K. Sovacool, "Electricity market design for the prosumer era," *Nature Energy*, vol. 1, no. 4, pp. 1–6, Mar. 2016.

[2] C. Zhang, J. Wu, C. Long, and M. Cheng, "Review of existing peer-to-peer energy trading projects," *Energy Proc.*, vol. 105, pp. 2563–2568, May 2017.

[3] M. Andoni, V. Robu, D. Flynn, S. Abram, D. Geach, D. Jenkins, P. McCallum, and A. Peacock, "Blockchain technology in the energy sector: A systematic review of challenges and opportunities," *Renew. Sustain. Energy Rev.*, vol. 100, pp. 143–174, Feb. 2019.

[4] Y. Zhou, J. Wu, and C. Long, "Evaluation of peer-to-peer energy sharing mechanisms based on a multiagent simulation framework," *Appl. Energy*, vol. 222, pp. 993–1022, Jul. 2018.

[5] J. Guerrero, A. C. Chapman, and G. Verbic, "Decentralized P2P energy trading under network constraints in a low-voltage network," *IEEE Trans. Smart Grid*, vol. 10, no. 5, pp. 5163–5173, Sep. 2019.

[6] C. Park and T. Yong, "Comparative review and discussion on P2P electricity trading," *Energy Proc.*, vol. 128, pp. 3–9, Sep. 2017.

[7] C. Zhang, J. Wu, M. Cheng, Y. Zhou, and C. Long, "A bidding system for peer-to-peer energy trading in a grid-connected microgrid," *Energy Proc.*, vol. 103, pp. 147–152, Dec. 2016.

[8] T. Sousa, T. Soares, P. Pinson, F. Moret, T. Baroche, and E. Sorin, "Peer-to-peer and community-based markets: A comprehensive review," *Renew. Sustain. Energy Rev.*, vol. 104, pp. 367–378, Apr. 2019.

[9] A. Paudel, K. Chaudhari, C. Long, and H. B. Gooi, "Peer-to-peer energy trading in a prosumer-based community microgrid: A game-theoretic model," *IEEE Trans. Ind. Electron.*, vol. 66, no. 8, pp. 6087–6097, Aug. 2019.

[10] F. Moret and P. Pinson, "Energy collectives: A community and fairness based approach to future electricity markets," *IEEE Trans. Power Syst.*, vol. 34, no. 5, pp. 3994–4004, Sep. 2019.

[11] N. Wang, W. Xu, Z. Xu, and W. Shao, "Peer-to-peer energy trading among microgrids with multidimensional willingness," *Energies*, vol. 11, no. 12, p. 3312, Nov. 2018.

[12] A. Roy, A. Bruce, and I. MacGill, "The potential value of peer-to-peer energy trading in the Australian national electricity market," in *Proc. Asia–Pacific Solar Res. Conf.*, vol. 2016, 2016, pp. 1–13.

[13] L. A. Zadeh, "Fuzzy sets," *Inf. Control*, vol. 8, no. 3, pp. 338–353, 1965.

[14] M. T. Tang, G. H. Tzeng, and S. W. Wang, "A hierarchy fuzzy MCDM method for studying electronic marketing strategies in the information service industry," *J. Int. Inf. Manage.*, vol. 8, no. 1, p. 1, 1999.

[15] A. Mohaghar, M. R. Fathi, M. K. Zarchi, and A. Omidian, "A combined VIKOR—Fuzzy AHP approach to marketing strategy selection," *Bus. Manage. Strategy*, vol. 3, no. 1, p. 13, Dec. 1969.

[16] C.-T. Lin, C. Lee, and C.-S. Wu, "Fuzzy group decision making in pursuit of a competitive marketing strategy," *Int. J. Inf. Technol. Decis. Making*, vol. 9, no. 2, pp. 281–300, Mar. 2010.

[17] M. H. Gholami and M. S. Esfahani, "An integrated analytical framework based on fuzzy DEMATEL and fuzzy ANP for competitive market strategy selection," *Int. J. Ind. Syst. Eng.*, vol. 31, no. 2, p. 137, 2019.

[18] Y. L. Lee, D. F. Chang, and B. Wu, "Fuzzy evaluating management performance and marketing strategies in community colleges," *Int. J. Innov. Comput. Inf. Control*, vol. 8, no. 10, pp. 7405–7413, 2012.

[19] B. Oztaysi, T. Gurbuz, E. Albayrak, and C. Kahraman, "Target marketing strategy determination for shopping malls using fuzzy ANP," *J. Multiple-Valued Log. Soft Comput.*, vol. 27, p. 595, Nov. 2016.

[20] W.-R. Zhang, "Bipolar fuzzy sets and relations: A computational framework for cognitive modeling and multiagent decision analysis," in *Proc. NAFIPS/IFIS/NASA 1st Int. Joint Conf. North Amer. Fuzzy Inf. Process. Soc. Biannual Conf. Ind. Fuzzy Control Intell. Syst. Conf., NASA Joint Technol. Wo*, Dec. 1994, pp. 305–309.

[21] M. Riaz, D. Pamucar, A. Habib, and N. Jamil, "Innovative bipolar fuzzy sine trigonometric aggregation operators and SIR method for medical tourism supply chain," *Math. Problems Eng.*, vol. 2022, pp. 1–17, Jun. 2022.

[22] C. Jana, M. Pal, and J.-Q. Wang, "Bipolar fuzzy Dombi aggregation operators and its application in multiple-attribute decision-making process," *J. Ambient Intell. Humanized Comput.*, vol. 10, no. 9, pp. 3533–3549, Sep. 2019.

[23] C. Jana, M. Pal, and J.-Q. Wang, "Bipolar fuzzy dombi prioritized aggregation operators in multiple attribute decision making," *Soft Comput.*, vol. 24, no. 5, pp. 3631–3646, Mar. 2020.

[24] G. Wei, F. E. Alsaadi, T. Hayat, and A. Alsaedi, "Bipolar fuzzy Hamacher aggregation operators in multiple attribute decision making," *Int. J. Fuzzy Syst.*, vol. 20, no. 1, pp. 1–12, Jan. 2018.

[25] D. Karabasevic, P. Radanov, D. Stanujkic, G. Popovic, and B. Predic, "Going green: Strategic evaluation of green ICT adoption in the textile industry by using bipolar fuzzy MULTIMOORA method," *Industria Textila*, vol. 72, no. 1, pp. 3–10, Feb. 2021.

[26] B. Alsolame and N. O. Alshehri, "Extension of VIKOR method for MCDM under bipolar fuzzy set," *Int. J. Anal. Appl.*, vol. 18, no. 6, pp. 989–997, 2020.

[27] M. Akram and M. Arshad, "Bipolar fuzzy TOPSIS and bipolar fuzzy ELECTRE-I methods to diagnosis," *Comput. Appl. Math.*, vol. 39, no. 1, pp. 1–21, Mar. 2020.

[28] N. Deva and A. Felix, "Designing DEMATEL method under bipolar fuzzy environment," *J. Intell. Fuzzy Syst.*, vol. 41, no. 6, pp. 7257–7273, Dec. 2021.
[29] C. Jana and M. Pal, "Extended bipolar fuzzy EDAS approach for multi-criteria group decision-making process," *Comput. Appl. Math.*, vol. 40, no. 1, pp. 1–15, Feb. 2021.
[30] R. Liu, L.-X. Hou, H.-C. Liu, and W. Lin, "Occupational health and safety risk assessment using an integrated SWARA-MABAC model under bipolar fuzzy environment," *Comput. Appl. Math.*, vol. 39, no. 4, pp. 1–17, Dec. 2020.
[31] R. R. Yager, "Prioritized aggregation operators," *Int. J. Approx. Reasoning*, vol. 48, no. 1, pp. 263–274, Apr. 2008.
[32] R. R. Yager, "Prioritized OWA aggregation," *Fuzzy Optim. Decis. Making*, vol. 8, no. 3, pp. 245–262, Sep. 2009.
[33] X. Yu and Z. Xu, "Prioritized intuitionistic fuzzy aggregation operators," *Inf. Fusion*, vol. 14, no. 1, pp. 108–116, Jan. 2013.
[34] M. S. A. Khan, S. Abdullah, A. Ali, and F. Amin, "Pythagorean fuzzy prioritized aggregation operators and their application to multi-attribute group decision making," *Granular Comput.*, vol. 4, no. 2, pp. 249–263, Apr. 2019.
[35] M. Riaz, D. Pamucar, H. M. Athar Farid, and M. R. Hashmi, "Q-rung orthopair fuzzy prioritized aggregation operators and their application towards green supplier chain management," *Symmetry*, vol. 12, no. 6, p. 976, Jun. 2020.
[36] J. Aczél and C. Alsina, "Characterizations of some classes of quasilinear functions with applications to triangular norms and to synthesizing judgements," *Aequationes Mathematicae*, vol. 25, no. 1, pp. 313–315, Dec. 1982.
[37] M. Sarfraz, K. Ullah, M. Akram, D. Pamucar, and D. Božanić, "Prioritized aggregation operators for intuitionistic fuzzy information based on Aczel–Alsina T-norm and T-Conorm and their applications in group decision-making," *Symmetry*, vol. 14, no. 12, p. 2655, Dec. 2022.
[38] T. Senapati, G. Chen, and R. R. Yager, "Aczel-Alsina aggregation operators and their application to intuitionistic fuzzy multiple attribute decision making," *Int. J. Intell. Syst.*, vol. 37, no. 2, pp. 1529–1551, 2022.
[39] T. Senapati, "Approaches to multi-attribute decision-making based on picture fuzzy Aczel–Alsina average aggregation operators," *Comput. Appl. Math.*, vol. 41, no. 1, p. 40, Feb. 2022.
[40] T. Mahmood and U. Rehman, "A novel approach towards bipolar complex fuzzy sets and their applications in generalized similarity measures," *Int. J. Intell. Syst.*, vol. 37, no. 1, pp. 535–567, Jan. 2022.
[41] T. Mahmood, U. U. Rehman, A. Jaleel, J. Ahmmad, and R. Chinram, "Bipolar complex fuzzy soft sets and their applications in decision-making," *Mathematics*, vol. 10, no. 7, p. 1048, Mar. 2022.
[42] J. Gwak, H. Garg, N. Jan, and B. Akram, "A new approach to investigate the effects of artificial neural networks based on bipolar complex spherical fuzzy information," *Complex Intell. Syst.*, vol. 9, no. 4, pp. 4591–4614, Aug. 2023.
[43] T. Mahmood, U. Ur Rehman, Z. Ali, and T. Mahmood, "Hybrid vector similarity measures based on complex hesitant fuzzy sets and their applications to pattern recognition and medical diagnosis," *J. Intell. Fuzzy Syst.*, vol. 40, no. 1, pp. 625–646, Jan. 2021.
[44] Ö. Özer, "Hamacher prioritized aggregation operators based on complex picture fuzzy sets and their applications in decision-making problems," *J. Innov. Res. Math. Comput. Sci.*, vol. 1, no. 1, pp. 33–54, 2022.
[45] Z. Ali, "Decision-making techniques based on complex intuitionistic fuzzy power interaction aggregation operators and their applications," *J. Innov. Res. Math. Comput. Sci.*, vol. 1, no. 1, pp. 107–125, 2022.

UBAID UR REHMAN received the M.Sc., M.S., and Ph.D. degrees in mathematics from International Islamic University Islamabad, Pakistan, in 2018, 2020, and 2023, respectively. He has more than 62 published articles in reputed journals. His research interests include algebraic structures, aggregation operators, similarity measures, soft set, bipolar fuzzy set, complex fuzzy set, bipolar complex fuzzy set, fuzzy logic, fuzzy decision making, and their applications.

FATEN LABASSI received the Ph.D. degree in algebraic topology from the University of Tunis, in 2013. She is currently an Assistant Professor with the Department of Mathematics and Statistics, Imam Mohammed Ibn Saud Islamic University, Riyadh. She has published several articles in international journals and is recognized for her significant contributions to the field of mathematics. Her research has made her a respected figure among her colleagues and students. She actively participates in conferences and workshops, sharing her expertise and fostering collaboration within the mathematical community. She is a highly respected mathematician and educator, known for her unwavering dedication to research and teaching.

TURKI ALSURAIHEED received the Ph.D. degree from The University of Sheffield, U.K., in 2019. He is currently an Assistant Professor with the Department of Mathematics, King Saud University, Riyadh, Saudi Arabia. He has contributed to the academic community through numerous research articles published in various internationally renowned journals. His research interests include the applications of ring and module theory, multiplication modules, multiplication rings, and fuzzy algebra.

TAHIR MAHMOOD received the Ph.D. degree in mathematics from Quaid-i-Azam University Islamabad, Pakistan, in 2012, in the field of Fuzzy Algebra. He is an Associate Professor of Mathematics at the Department of Mathematics and Statistics, International Islamic University Islamabad, Pakistan. More than 330 research publications on his credit with more than 9050 citations, more than 800 impact factor, H-index 47, and i10-index 169. He has produced 60 M.S. students and nine Ph.D. students. His areas of interest are algebraic structures, fuzzy algebraic structures, decision making, and generalizations of fuzzy sets. Currently he is working as an editorial board member of three impact factor journals.

MERAJ ALI KHAN received the Ph.D. degree from the Department of Mathematics, Aligarh Muslim University, Aligarh, India. He is currently a Professor with the Department of Mathematics and Statistics, Imam Mohammad Ibn Saud Islamic University, Riyadh, Saudi Arabia. Prior to this, he was with the Department of Mathematics, University of Tabuk, Saudi Arabia, and the School of Mathematics, Thapar Institute of Engineering and Technology, Punjab, India. With over a hundred research articles published in various reputed international journals.