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RESEARCH ARTICLE

Finite/Fixed-Time Synchronization for Stochastic Multilayer Networks With Pinning Control

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ABSTRACT In recent years, there has been a strong focus on the stability and control of stochastic networks, particularly single-layer networks. However, with the emergence of multi-layer networks, the research in this area has expanded significantly. Multi-layer networks have gained much attention due to their relevance in network science. In this paper, we address the synchronization of stochastic multi-layer networks (SMLNs) using finite-time (FnT) and fixed-time (FxT) methods with pinning control. Firstly, we provide the conditions for achieving FxT synchronization in SMLNs. Secondly, we present the conditions for FnT synchronization in SMLNs. Additionally, two kinds of finite-time convergent values of the system are given, which can be any given time. Furthermore, we discuss the relationship between the coupling matrix and the network layers, particularly in the context of minimum convergence time. This analysis has practical implications for real-world applications. Finally, we present numerical simulations to demonstrate the effectiveness of our proposed method.

INDEX TERMS Stochastic multi-layer networks, synchronization, finite-time control, fixed-time control.

I. INTRODUCTION

As is well known, synchronization, as a very common natural phenomenon, has been widely applied in many fields during its evolution and development [1], [2]. Although many important achievements have been made in various studies and applications of complex networks, most of the research results are mainly based on achieving a single structure and function of a single network (also known as a single-layer network) [3], [4], [5], [6], [7], [8], [9]. Among them, the complex system in reality was abstracted as a single-layer network, ignoring its possible multiple structures and functions: the coupling or overlapping interaction of multiple networks such as transportation networks between cities, commodity trade networks between countries and regions, and so on. Various multilayer networks have become ubiquitous. Funk and Jansen pointed out that studying MLNs as single-layer networks and ignored their multilayer nature would lead to erroneous conclusions [10]. Therefore, in order to meet the research needs of real complex systems coupled by different structures and functions, significant changes had taken place in complex networks and related research fields, with the focus shifting from singular-layer networks to multi-layer networks (MLNs).

At present, the study of MLNs has become one of the most popular research directions in the field of complex networks. Scholars obtained many meaningful results on the structure and properties of MLNs, such as Domenico et al. discussion of the physical properties of diffusion processes in MLNs [11]. Gao et al. studied the robustness of a network [12]. Pang et al. showed that the edge dynamics of MLNs were structurally more controllable than single-layer networks of the same size [13]. Boccaletti et al. discussed the structure and dynamics of MLN [14]. Mucha et al. discussed

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community structures in time-varying, multi-scale, and multiple networks [15]. However, some excellent results had also emerged in the research on the stability and synchronisation of MLN [16]. For example, Suykens et al. derived the robust local stability conditions of multi-layer recurrent neural networks through the robustness analysis of linear system under matrix inequalities and nonlinear disturbances [17]. Rakshit et al. used the principal stability function method to explore the synchronization of neurons within and between layers of multi-layer hypernetworks, and obtained the stability conditions for synchronization and the robustness of synchronization states to initial conditions [18]. Yao et al. designed a nonlinear hydraulic system controller based on a MLNs to achieve asymptotic tracking of various disturbances. Theoretical analysis showed that the controller has semi global asymptotic stability [19]. Liu et al. conducted a study on an encryption scheme that relies on the synchronization of a two-layer complex dynamic network [20]. Similarly, He et al. explored the synchronization control of nonlinear multi-agent systems using the MLN approach [21]. Additionally, Wei et al. delved into the synchronization of double-layer networks that were affected by delayed nodes and noise disturbances [22]. Sevilla-Escoboza et al. studied the problem of synchronising MLNs across layers [23]. Wang et al. discussed how inter-node synchronisation and full synchronisation could be achieved in MLNs under a directed-switched communication topology with Lur'e-type dynamics [24]. Blaha et al. deliberated the clustering synchronization of multi-layer Colpitts oscillator networks in their experiments [25].

It is expensive and unnecessary that controllers are applied to all nodes in practical engineering. By selectively applying control to a small number of nodes in the network, the entire network exhibits the desired behavior. Compared to controlling all nodes, applying local feedback control to a small number of nodes in the network can be more attractive and desirable due to its lower cost. Therefore, we can use the method of pinning control for SMLNs, and achieve control of the entire SMLNs by controlling a few nodes. At present, there were few research results on MLNs containment control. For example, Lu et al. studied the network synchronization problem of networks using pinning control methods [26]. Zhou et al. further investigated the clustering synchronisation of multiple stochastic subnets based on a pinning control approach [27]. Ning et al. studied the generalized synchronization of two-layer networks based on pinning control methods [28]. Wang et al. studied the mean square synchronization of random MLNs by pinning control methods [29]. Of course, the majority of these research findings are predicated on the assumption that the control time tending towards infinity. However, in engineering, it is often necessary to realise control of the network in a limited period of time, which has led the researchers to pay more attention to the theory of FnT control in the study of MLNs. For example, Sun et al. conducted a study on the FxT event-triggered synchronous control of multi-layer Kuramoto oscillator networks [30]. Zhang et al. conducted a study on a specific type of multilayer nonlinear coupled complex networks that exhibited intermittent feedback FnT synchronization [31]. Tan et al. provided a detailed analysis of a synchronization control method for two-layer dynamic networks that incorporates FxT stochastic mean square synchronization [32]. Tang et al. discussed the concept of adaptive FnT mixed inter layer synchronization for time-varying coupled delay two-layer complex networks [33].

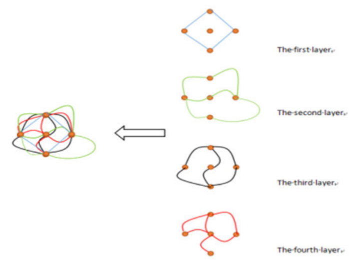

FIGURE 1. The schematic diagram of the four-layer SMLNs with five node.

In addition, in reality, due to the uncertainty of the environment, nodes are usually affected by random disturbances, such as infectious disease epidemics in turbulent biology in social networks, and Stochastic resonance systems in chemistry. The existence of stochastic term generally exacerbates the instability of the system, so in the synchronization control of the system, it is necessary to design appropriate controllers to ensure system stability. It is not difficult to see that studying the FnT time constraint control of SMLNs will be a more meaningful topic. For example, a four-layer network structure diagram is shown in Figure 1. Different colors indicate different layers. In this paper we assume that each layer has the same random disturbance.

The primary objective of this note is to establish criteria for FnT/FxT synchronization in SMLNs. The main work and contributions of this study are outlined as follows: 1) Based on FxT/FnT stable theorems of stochastic nonlinear system, the sufficient conditions for FxT/FnT synchronization of SMLNs are given. At the same time, the concrete expressions of the finite settling time of two kinds of convergence time are also given. 2) At present, there is little literature on the relationship among the finite-time convergence of multi-layer networks, the number of pinningcontrol nodes and system structure. Under the condition of achieving minimum convergence time, this paper provides the relationship among finite-time convergence, the number of pinningcontrol nodes and system structure, which will help to optimize the control of MLNs, for example, how to select the number of control nodes under the minimum control time according to the network structure.

The structure of this paper is as follows. In Section II, the model and several lemmas are provided. In Section III, some FnT pth moment quasi-synchronization conditions for SNNs are introduced. An illustrative example is showcased in Section IV. In the last part of the paper, the conclusions from the research are summarized.

II. MODEL STATEMENT AND PRELIMINARIES

In this paper, we shall consider $(\Omega, \Im, P)$ as a complete probability space, accompanied by a filtration denoted as $\{\Im_t\}_{t\geq 0}$, which meets the standard conditions. Let $\omega = \{\omega(t), t \geq 0\}$ be an m-dimensional $\Im_t$ -adapted Brownian motion. The space R^n refers to the real n-dimensional space, while R^+ represents all nonnegative real numbers. The space $R^{n\times m}$, on the other hand, denotes the space of $n \times m$ matrices. For a given vector or matrix $y \in R^n$, $|y|$ denotes the Euclidean norm $|y| = \left(\sum_{i=1}^n y_i^2\right)^{1/2}$. The transpose of a vector or a matrix A is denoted by A^T. ε denotes mathematical expectation.

Consider the following SMLNs,

$$\dot{\mu}_i(t) = f\left(\mu_i(t)\right) + c\sum_{k=1}^m \sum_{j=1}^N B_{ij}^{(k)} \Gamma_k \mu_j(t) + \sum_{k=1}^m \sigma_i\left(t, \mu_i(t)\right) \dot{\omega}_i(t), \tag{1}$$

where $\mu_i(t) = (\mu_{i1}(t), \mu_{i2}(t), \cdots, \mu_{in}(t))^T \in R^n$, $f(\cdot)$ are nonlinear functions; $B^{(k)} = \left(B_{ij}^{(k)}\right)_{N\times N}$ $(k = 1, 2, \cdots, m)$ is the coupling matrix describing topological structure of the kth layer and satisfies $B_{ii}^{(k)} = -\sum_{j=1, j\neq i}^N B_{ij}^{(k)}$, $i = 1, 2, \cdots, N$. Γ_k represents the inner coupling matrix of the kth layer. $\omega_i(t) = (\omega_{i1}(t), \cdots, \omega_{in}(t))^T \in R^n$ are n-dimensional Brownian motions, $\sigma(t) = \left(\sigma_{ij}(t, \mu_i(t))\right)_{n\times n} : R^+ \times R^n \times R^n \to R^{n\times n}$ is the noise intensity matrix, and it is linear, c is coupling strength.

For SMLNs (1), a response SMLNs can be constructed as follows:

$$\dot{\vartheta}_i(t) = f\left(\vartheta_i(t)\right) + c\sum_{k=1}^m \sum_{j=1}^N B_{ij}^{(k)} \Gamma_k \vartheta_j(t) + \sum_{k=1}^m \sigma_i\left(t, \vartheta_i(t)\right) \dot{\omega}_i(t) + u_i(t), \tag{2}$$

where $\vartheta_i = (\vartheta_{i1}, \vartheta_{i2}, \cdots, \vartheta_{in})^T \in R^n$, $u_i(t) = (u_{i1}(t), u_{i2}(t), \cdots, u_{in}(t))^T \in R^n$ is the controller.

Let $\varpi_i(t) = \vartheta_i(t) - \mu_i(t)$, the error system can be illustrated in the following manner:

$$\begin{aligned} d\varpi(t) &= \left[F\left(\vartheta(t)\right) - F\left(\mu(t)\right) + c\sum_{k=1}^m \left(B^{(k)} \otimes \Gamma_k\right) \varpi(t) + u(t)\right] dt \\ &\quad + \sum_{k=1}^m \varphi^{(k)}(t) d\omega(t), \end{aligned} \tag{3}$$

where $\varpi(t) = \left(\varpi_1^T(t), \varpi_2^T(t), \cdots, \varpi_N^T(t)\right)^T$, $F(\mu(t)) = \left((f(\mu_1(t)))^T, (f(\mu_2(t)))^T, \cdots, (f(\mu_N(t)))^T\right)^T$, $u(t) = \left(u_1^T(t), u_2^T(t), \cdots, u_N^T(t)\right)^T$, $\varpi_i(t) = (\varpi_{i1}(t), \cdots, \varpi_{in}(t))^T$.

To arrive at the primary outcome, the following preliminaries are given.

(A1): Suppose $f(\cdot)$ satisfies the following Lipschitz condition

$$\left|f\left(\vartheta_i(t)\right) - f\left(\mu_i(t)\right)\right| \leq \eta_1 \varpi_i(t), \ \eta_1 \in R^+.$$

(A2): The noise intensity matrix satisfies the following conditions

$$trace\left(\sigma^T\left(t, \varpi(t)\right) \sigma\left(t, \varpi(t)\right)\right) \leq \eta_2 \sum_{i=1}^n |\varpi_i(t)|^2, \ \eta_2 \in R^+.$$

Consider the following stochastic nonlinear system (SNS)

$$d\vartheta(t) = f_1\left(\vartheta(t)\right) dt + f_2\left(\vartheta(t)\right) d\omega(t), \tag{4}$$

where $\vartheta(t) \in R^n$ is the state vector, $f_1(\cdot) : R^n \to R^n$ and $f_2(\cdot) : R^n \to R^{n\times m}$ are continuous functions, $\omega(\cdot)$ is an m-dimensional Brown moment. $f_1(0) = 0, f_2(0) = 0$, it is granted that the SNS (4) has a unique global solution denoted by $\vartheta(t, \vartheta_0)$, $0 \leq t < \infty$, where ϑ_0 is the initial state.

For each $\upsilon(t) \in C^{2,1}\left(R^n \times R^+, R^+\right)$, the operator $\mathrm{L}\upsilon(t)$ relative to the SNS (4) is

$$\mathbf{L}\upsilon(t) = \frac{\partial\upsilon(t)}{\partial\vartheta(t)} \cdot f_1 + \frac{1}{2} trace\left(f_2^T \cdot \frac{\partial^2\upsilon(t)}{\partial\vartheta^2(t)} \cdot f_2\right),$$

where $(\partial\upsilon(t)/\partial\vartheta(t)) = ((\partial\upsilon(t)/\partial\vartheta_1(t)), \cdots, (\partial\upsilon(t)/\partial\vartheta_n(t)))$, $\left(\partial^2\upsilon/\partial\vartheta^2\right) = \left(\left(\partial^2\upsilon/\partial\vartheta_j\partial\vartheta_k\right)\right)_{n\times n} (j, k = 1, 2, \cdots, n)$.

Definition 1 [33]: The equilibrium $\vartheta(t) = 0$ of the SNS (4) is FnT stable if

$$\lim_{t\to T(\vartheta(0))} \varepsilon\left|\vartheta(t)\right| = 0, \ \ \forall\vartheta(0) \in R^n,$$

where $T(\vartheta(0)) > 0$ is associated with the initial condition of the system.

Definition 2 [34]: The equilibrium $\vartheta(t) = 0$ of the SNS (4) is FxT stable if

$$\lim_{t\to T} \varepsilon\left|\vartheta(t)\right| = 0, \ \ \forall\vartheta(0) \in R^n,$$

where $T > 0$ is a constant.

Lemma 1 [35]: If $\alpha \in R^n$, $\beta \in R^n$, then $\forall\phi > 0$,

$$\alpha'\beta + \beta'\alpha \leq \phi\alpha'\alpha + \phi^{-1}\beta'\beta.$$

Lemma 2 [36]: If $\chi_i \geq 0, i = 1, 2, \cdots, N, \ 0 < \delta \leq 1, \ \gamma > 1$, then

$$\sum_{\mathrm{i}=1}^{\mathrm{N}} \chi_{\mathrm{i}}^{\delta} \geq \left(\sum_{\mathrm{i}=1}^{\mathrm{N}} \chi_{\mathrm{i}}\right)^{\delta}, \sum_{\mathrm{i}=1}^{\mathrm{N}} \chi_{\mathrm{i}}^{\gamma} \geq \mathrm{N}^{1-\gamma}\left(\sum_{\mathrm{i}=1}^{\mathrm{N}} \chi_{\mathrm{i}}\right)^{\gamma}.$$

Lemma 3 [37]: The linear matrix inequality (LMI) $H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{pmatrix} < 0$ is synonymous with either of the following two conditions:

$$(1)\ H_{11} < 0, H_{22} - H_{12}^T H_{11}^{-1} H_{12} < 0,$$
$$(2)\ H_{22} < 0, H_{11} - H_{12} H_{22}^{-1} H_{12}^T < 0,$$

where $H_{11} = H_{11}^T$ and $H_{22} = H_{22}^T$.

Lemma 4 [38]: Consider the SNS (4), if there exists a function $\upsilon(\vartheta(t)) = \vartheta^2(t)$ that is regular, positive definite and radically unbounded, as well as real numbers $\kappa_1 > 0,\ 0 < p < 1$, such that

$$\mathbf{L}\upsilon\left(\vartheta(t)\right) \leq -\kappa_1 \upsilon^p\left(\vartheta(t)\right), \tag{5}$$

then the origin of the SNS (4) is FnT stochastic stable, and

$$\varepsilon\left[T\right] \leq \frac{\upsilon(0)}{\kappa_1}\frac{1}{1-p}.$$

Lemma 5 [39]: For the SNS (4), if there exists a continuous and positive-definite function $\upsilon\left(\vartheta(t)\right) = \vartheta^2(t)$ that fulfills the following condition satisfies**:**

$$\mathbf{L}\upsilon\left(\vartheta(t)\right) \leq -\varphi_1 \upsilon^p\left(\vartheta(t)\right) - \varphi_2 \upsilon^q\left(\vartheta(t)\right), \tag{6}$$

then the SNS (4) is FnT stochastic stable, and

$$\varepsilon\left[T\right] \leq \frac{1}{1-p}\frac{1}{\varphi_1} + \frac{1}{q-1}\frac{1}{\varphi_2},$$

where $\varphi_1 > 0,\ \varphi_2 > 0,\ q > 1,\ 0 < p < 1$.

Remark 1: At present, the stability concepts related to convergence speed mainly include exponential stability, finite time stability, and fixed time stability. For example, for the dynamic system $\dot{x}(t) = u(x(t))$, to ensure its exponential stability, proportional control is generally used $u(x(t)) = -kx$, to ensure its finite time stability, low power control is generally used $u(x(t)) = -|x|^a sign(x(t)),\ 0 \leq a < 1$ to ensure its fixed time stability, double power control is generally used $u(x(t)) = -|x|^a sign(x(t)) - |x(t)|^b sign(x(t)),\ 0 \leq a < 1,\ b > 1$. Our system is FxT and FnT stable.

III. FNT/FXT SYNCHRONIZATION OF SMLNS

Theorem 1: Under (A1)-(A2), if $\lambda_{max}(B^{(k)} \otimes \Gamma_k)_{N-l} < -\frac{1}{cm}\left(\eta_1 + \frac{1}{2}\eta_2\right), k^* > \lambda_{max}\left(H_{11} - H_{12}H_{N-l}^{-1}H_{12}^T\right)$, then the error system (3) is FxT stability by the controller

$$u_i(t) = \begin{cases} -k_i\varpi_i(t) - \hat{k}_i\varpi_i^q(t) - \overline{k_i}\varpi_i^p(t), \\ \qquad i = 1, 2, \cdots, l, \\ 0, \quad i = l+1, l+2, \cdots, N, \end{cases} \tag{7}$$

where $\lambda_{min}\left(\overline{K_N} \otimes I_N\right) \geq \frac{2}{(1-p)T^*}2^{\frac{1-p}{2}}$, $\lambda_{min}\left(\hat{K}_N \otimes I_N\right) \geq \frac{2}{(q-1)T^*}2^{\frac{1-q}{2}}(Nn)^{\frac{q-1}{2}}$, $\overline{K_N} = diag(\overbrace{\overline{k_1}, \cdots, \overline{k_l}}^{l}, \overbrace{0, \cdots, 0}^{N-l})$, $\hat{K}_N = diag(\overbrace{\hat{k}_1, \cdots, \hat{k}_l}^{l}, \overbrace{0, \cdots, 0}^{N-l})$, $K_N = diag(\overbrace{k_1, \cdots, k_l}^{l}, \overbrace{0, \cdots, 0}^{N-l})$, $k^* = min_{1\leq i\leq l}\left(k_i\right)$, k_i is a positive numbers which is to be determined, and

$$\varepsilon\left[T\right] \leq T^*,$$

where $T^* > 0$ is any given time.

Proof: Let

$$\upsilon(t) = \frac{1}{2}\sum_{i=1}^{N}\varpi_i^T(t)\varpi_i(t), \tag{8}$$

the time derivative of $\upsilon(t)$ along the trajectory of system (8) can be calculated as a random differential using the Itô's formula

$$d\upsilon(t) = \mathbf{L}\upsilon(t)dt + \varpi^T(t)\sigma\left(t, \varpi(t)\right)d\omega(t),$$

where the differential operator

$$\begin{aligned} &\mathbf{L}\upsilon(t) \\ &= \sum_{i=1}^{N}\varpi_i(t)\Big[f\left(\vartheta_i(t)\right) - f\left(\mu_i(t)\right) + c\sum_{k=1}^{m}\sum_{j=1}^{N} \\ &\quad \times B_{ij}^{(k)}\Gamma_k\varpi_j(t) + u_i(t)\Big] + \frac{1}{2}trace\left(\left(\varphi(t)\right)^T\varphi(t)\right). \end{aligned} \tag{9}$$

By using (A1)-(A2) , yields

$$\begin{aligned} &\mathbf{L}\upsilon(t) \\ &\leq \sum_{i=1}^{N}\Big[\eta_1\varpi_i^T(t)\varpi_i(t) + c\sum_{k=1}^{m}\sum_{j=1}^{N}B_{ij}^{(k)} \\ &\quad \times \Gamma_k\varpi_i^T(t)\varpi_j(t) - k_i\varpi_i^T(t) \\ &\quad \times \varpi_i(t) - \hat{k}_i\varpi_i^T(t)\varpi_i^q(t) - \overline{k_i}\varpi_i^T(t)\varpi_i^p(t)\Big] \\ &\quad + \frac{1}{2}\eta_2\sum_{i=1}^{N}\varpi_i^T(t)\varpi_i(t) \\ &\leq \sum_{i=1}^{N}\Big[\eta_1\varpi_i^T(t)\varpi_i(t) + c\sum_{k=1}^{m}\sum_{j=1}^{N}B_{ij}^{(k)} \\ &\quad \times \Gamma_k\varpi_i^T(t)\varpi_j(t)\Big] \\ &\quad - \sum_{i=1}^{l}\Big[k_i\varpi_i^T(t)\varpi_i(t)\Big] - \sum_{i=1}^{l}\Big[\hat{k}_i\varpi_i^T(t)\varpi_i^q(t)\Big] \\ &\quad - \sum_{i=1}^{l}\Big[\overline{k_i}\varpi_i^T(t)\varpi_i^p(t)\Big] + \frac{1}{2}\eta_2\sum_{i=1}^{N}\varpi_i^T(t)\varpi_i(t) \\ &\leq \varpi^T(t)\Big[\Big(\Big(\eta_1 + \frac{1}{2}\eta_2\Big)I_N + c\sum_{k=1}^{m}B^{(k)} \otimes \Gamma_k - K_N\Big) \\ &\quad \otimes I_N\Big] \\ &\quad \times \varpi(t) - \lambda_{min}\left(\hat{K}_N \otimes I_N\right)\varpi^T(t)\varpi^q(t) \\ &\quad - \lambda_{min}\left(\overline{K_N} \otimes I_N\right)\varpi^T(t)\varpi^p(t) \end{aligned} \tag{10}$$

Let $\Omega = \left(\eta_1 + \frac{1}{2}\eta_2\right)I_N + c\sum_{k=1}^{m}B^{(k)} \otimes \Gamma_k - K_N$, $K_N = diag(\overbrace{k_1, \cdots, k_l}^{l}, \overbrace{0, \cdots, 0}^{N-l})$, $\hat{K}_N = diag(\overbrace{\hat{k}_1, \cdots, \hat{k}_l}^{l}, \overbrace{0, \cdots, 0}^{N-l})$, $\overline{K_N} = diag(\overbrace{\overline{k_1}, \cdots, \overline{k_l}}^{l}, \overbrace{0, \cdots, 0}^{N-l})$, $H = \left(\eta_1 + \frac{1}{2}\eta_2\right)I_N + c\sum_{k=1}^{m}B^{(k)}$, then $\Omega = H - K_N$. From Lemma 3, $H - K_N = \begin{bmatrix} H_{11} - K_l & H_{12} \\ H_{12}^T & H_{N-l} \end{bmatrix} < 0$ where $K_l = diag\left(k_1, \cdots, k_l\right)$.

For

$$\begin{aligned} &\lambda(H)_{N-l} \\ &= \lambda\Big(\Big(\eta_1 + \frac{1}{2}\eta_2\Big)I_N + c\sum_{k=1}^{m}B^{(k)} \otimes \Gamma_k\Big)_{N-l} \\ &\leq \eta_1 + \frac{1}{2}\eta_2 + cm\lambda_{max}\left(B^{(k)} \otimes \Gamma_k\right)_{N-l}. \end{aligned}$$

If $\lambda_{max}\left(B^{(k)}\otimes\Gamma_k\right)_{N-l} < -\frac{1}{cm}\left(\eta_1+\frac{1}{2}\eta_2\right)$, then $H_{N-l} < 0$. From Lemma 3, if $S < 0$, then $(H_{11}-K_l)-H_{12}H_{N-l}^{-1}H_{12}^T < 0$, which means $k^* > \lambda_{max}\left(H_{11}-H_{12}H_{N-l}^{-1}H_{12}^T\right)$. Thus, by choosing a suitable k^* to make $\Omega < 0$, that is

$$\left(\left(\eta_1+\frac{1}{2}\eta_2\right)I_N+c\sum_{k=1}^{m}B^{(k)}\otimes\Gamma_k-K_N\right)\otimes I_N < 0.$$

So,

$$\begin{aligned}
&\mathbf{L}\upsilon(t)\\
&\le \varpi^T(t)\left[\left(\left(\eta_1+\frac{1}{2}\eta_2\right)I_N+c\sum_{k=1}^{m}B^{(k)}\otimes\Gamma_k-K_N\right)\right.\\
&\quad\left.\otimes I_N\right]\varpi(t)-\lambda_{min}\left(\hat{K}_N\otimes I_N\right)\varpi^T(t)\varpi^q(t)\\
&\quad-\lambda_{min}\left(\overline{K_N}\otimes I_N\right)\varpi^T(t)\varpi^p(t)\\
&\le -\lambda_{min}\left(\hat{K}_N\otimes I_N\right)\varpi^T(t)\varpi^q(t)\\
&\quad-\lambda_{min}\left(\overline{K_N}\otimes I_N\right)\varpi^T(t)\varpi^p(t).
\end{aligned}$$

When $0 < p < 1$, $\varpi^T(t)\varpi^p(t) \ge 2^{\frac{p+1}{2}}(\upsilon(t))^{\frac{p+1}{2}}$, and $q > 1$, $\varpi^T(t)\varpi^q(t) \ge 2^{\frac{q+1}{2}}(Nn)^{\frac{1-q}{2}}(\upsilon(t))^{\frac{q+1}{2}}$, so it has

$$\begin{aligned}
\mathbf{L}\upsilon(t) \le& -\lambda_{min}\left(\overline{K_N}\otimes I_N\right)2^{\frac{p+1}{2}}(\upsilon(t))^{\frac{p+1}{2}}\\
&-\lambda_{min}\left(\hat{K}_N\otimes I_N\right)2^{\frac{q+1}{2}}(Nn)^{\frac{1-q}{2}}(\upsilon(t))^{\frac{q+1}{2}}.
\end{aligned}$$

From Lemma 5, the origin of the SNS (3) is FxT stochastic stable, and

$$\begin{aligned}
\varepsilon[T] \le& \frac{1}{1-\frac{p+1}{2}}\frac{1}{2^{\frac{p+1}{2}}\lambda_{min}\left(\overline{K_N}\otimes I_N\right)}\\
&+\frac{1}{\frac{q+1}{2}-1}\frac{1}{\lambda_{min}\left(\hat{K}_N\otimes I_N\right)2^{\frac{q+1}{2}}(Nn)^{\frac{1-q}{2}}},
\end{aligned}$$

when $\lambda_{min}\left(\overline{K_N}\otimes I_N\right) \ge \frac{2}{(1-p)T^*}2^{\frac{1-p}{2}}$, $\lambda_{min}\left(\hat{K}_N\otimes I_N\right) \ge \frac{2}{(q-1)T^*}2^{\frac{1-q}{2}}(Nn)^{\frac{q-1}{2}}$,

$$\varepsilon[T] \le T^*.$$

Remark 2: In theorem 1, $\varepsilon[T] \le T^*$ is obviously independent of the initial value of the system, so the error system (3) converges in FxT.

Theorem 2: Under (A1)-(A2), if $\lambda_{max}\left(B^{(k)}\otimes\Gamma_k\right)_{N-l} < -\frac{1}{cm}\left(\eta_1+\frac{1}{2}\eta_2\right)$, $k^* > \lambda_{max}\left(H_{11}-H_{12}H_{N-l}^{-1}H_{12}^T\right)$, then the error system (3) is FnT stability by the controller

$$u_i(t)=\begin{cases}-k_i\varpi_i(t)-\overline{k_i}\varpi_i^p(t), & i=1,2,\cdots,l\\ 0, & i=l+1,l+2,\cdots,N\end{cases} \tag{11}$$

where $\lambda_{min}\left(\overline{K_N}\otimes I_N\right) \ge \frac{1}{(1-p)T^*}\sum_{i=1}^{N}\varpi_i^T(0)\varpi(0)$, $K_N = diag(\overbrace{k_1,\cdots,k_l}^{l},\overbrace{0,\cdots,0}^{N-l})$, $\overline{K_N}=diag(\overbrace{\overline{k_1},\cdots,\overline{k_l}}^{l},\overbrace{0,\cdots,0}^{N-l})$, $k^* = min_{1\le i\le l}(k_i)$, k_i is a positive numbers which are to be determined, and

$$\varepsilon[T] \le T^*,$$

where $T^* > 0$ is any given time.

Proof: Let

$$\begin{aligned}
&\upsilon(t)\\
&=\frac{1}{2}\sum_{i=1}^{N}\varpi_i^T(t)\varpi_i(t).\mathbf{L}\upsilon(t)\le\sum_{i=1}^{N}\left[\eta_1\varpi_i^T(t)\varpi_i(t)\right.\\
&\quad+c\sum_{k=1}^{m}\sum_{j=1}^{N}B_{ij}^{(k)}\Gamma_k\varpi_i^T(t)\varpi_j(t)-k_i\varpi_i^T(t)\varpi_i(t)\\
&\quad\left.-\overline{k_i}\varpi_i^T(t)\varpi_i^p(t)\right]+\frac{1}{2}\eta_2\sum_{i=1}^{N}\varpi_i^T(t)\varpi_i(t)\\
&\le\sum_{i=1}^{N}\left[\eta_1\varpi_i^T(t)\varpi_i(t)+c\sum_{k=1}^{m}\sum_{j=1}^{N}B_{ij}^{(k)}\right.\\
&\quad\left.\times\Gamma_k\varpi_i^T(t)\varpi_j(t)\right]-\sum_{i=1}^{l}\left[k_i\varpi_i^T(t)\varpi_i(t)\right]\\
&\quad-\sum_{i=1}^{l}\left[\overline{k_i}\varpi_i^T(t)\varpi_i^p(t)\right]+\frac{1}{2}\eta_2\sum_{i=1}^{N}\varpi_i^T(t)\varpi_i(t)\\
&\le\varpi^T(t)\left[\left(\left(\eta_1+\frac{1}{2}\eta_2\right)I_N+c\sum_{k=1}^{m}B^{(k)}\otimes\Gamma_k-K_N\right)\right.\\
&\quad\left.\otimes I_N\right]\varpi(t)-\lambda_{min}\left(\overline{K_N}\otimes I_N\right)\varpi^T(t)\varpi^p(t).
\end{aligned} \tag{12}$$

Similar to the process of proving Theorem 1, by choosing a suitable $k^* = min_{1\le i\le l}(k_i)$ to make

$$\left(\left(\eta_1+\frac{1}{2}\eta_2\right)I_N+c\sum_{k=1}^{m}B^{(k)}\otimes\Gamma_k-K_N\right)\otimes I_N < 0.$$

So,

$$\begin{aligned}
\mathbf{L}\upsilon(t)\le&\varpi^T(t)\left[\left(\left(\eta_1+\frac{1}{2}\eta_2\right)I_N+c\sum_{k=1}^{m}B^{(k)}\otimes\Gamma_k\right.\right.\\
&\left.\left.-K_N\right)\otimes I_N\right]\varpi(t)-\lambda_{min}\left(\overline{K_N}\otimes I_N\right)\varpi^T(t)\varpi^p(t)\\
&\le-\lambda_{min}\left(\overline{K_N}\otimes I_N\right)\varpi^T(t)\varpi^p(t)\\
&\le-\lambda_{min}\left(\overline{K_N}\otimes I_N\right)\upsilon(t)^{\frac{p+1}{2}}.
\end{aligned}$$

From Lemma 4, the origin of the SNS (3) is FnT stochastic stable, and

$$\varepsilon[T]\le\frac{\upsilon(0)}{\lambda_{min}\left(\overline{K_N}\otimes I_N\right)}\times\frac{1}{1-\frac{p+1}{2}},$$

when $\lambda_{min}\left(\overline{K_N}\otimes I_N\right) \ge \frac{1}{(1-p)T^*}\sum_{i=1}^{N}\varpi_i^T(0)\varpi_i(0)$,

$$\varepsilon[T]\le\frac{\upsilon(0)}{\frac{1}{(1-p)T^*}\sum_{i=1}^{N}\varpi_i^T(0)\varpi_i(0)}\times\frac{1}{1-\frac{p+1}{2}},$$

so

$$\varepsilon[T]\le T^*.$$

Remark 3: In Theorem 2,there is $\varepsilon[T] \le T^*$ only when $\lambda_{min}\left(\overline{K_N}\otimes I_N\right) \ge \frac{1}{(1-p)T^*}\sum_{i=1}^{N}\varpi_i^T(0)\varpi_i(0)$, so the convergence time of the system is still dependent on the initial value of the system.

Remark 4: FnTcontrol generally means that the settling time of the system convergence is related to the initial value

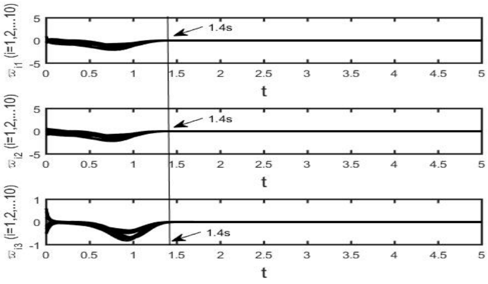

FIGURE 2. Synchronous evolution curve.

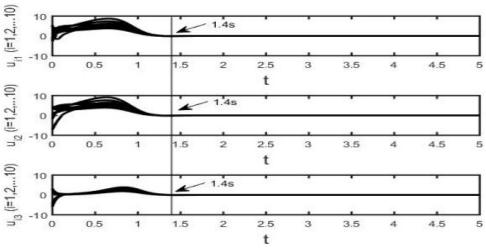

FIGURE 3. The controller evolution curve.

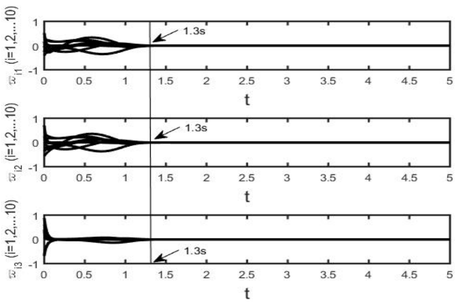

FIGURE 4. Synchronous evolution curve.

of the system, while the FxTcontrol means that the settling time of the system convergence is unrelated to the initial value of the system, but only related to the parameters of the system. Therefore, theorem 1 shows that the error system (3) is FxTstable fromRemark 2, while theorem 2 shows that the error system (3) is FnT stable from Remark 3.

Remark 5: It can be seen from theorems 1-2 that the upper bound of the settling time can be any value. However, in theorem 1, once the settling time T^* is fixed, it will correspondingly affect the control intensity $\overline{K_N}$ and $\hat{K}_N$. In particular, in theorem 2, the control intensity $\overline{K_N}$ is not only related to the settling time T^*, but also to the initial value of the system.

Remark 6: According to theorems 1-2, in numerical simulation, after the settling time T^* is given, the values of other parameters can be determined by the conditions of theorems 1-2.

Remark 7: From the conditions of theorems 1-2, it can be seen that under the condition of achieving minimum convergence time in multi-layer networks, Theorems 1-2 provides the connection among FnT, the number of constrained control nodes and the system structure, which will help further optimize the control of multi-layer networks.

Remark 8: In [40] and [41], the FnT control of multi-layer networks is discussed, but the pinning control method is

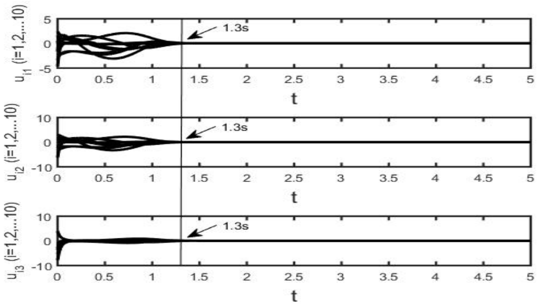

FIGURE 5. The controller evolution curve.

ignored. In [42] and [43] the pinning control of multi-layer networks is discussed, but the FnT control method is ignored. In [40], [44], [45], [46], [47], [48], and [49], the connection between the FnT, the number of pinning control nodes, and the structure of the network is also not discussed.

IV. ILLUSTRATIVE EXAMPLE

The network node is assumed to be the Rossler system, i.e,

$$\begin{cases} \dot{s}_{i1} = -\left(s_{i2} + s_{i3}\right) \\ \dot{s}_{i2} = s_{i1} + 0.2 s_{i2} \\ \dot{s}_{i3} = 0.2 - 5.7 s_{i3} + s_{i1} s_{i3}, \end{cases}$$

considering the boundedness of the above system and based on numerical simulations, one can obtain $\eta_1 = 38.1$. If $\sigma(t, \varpi) = 0.1 diag\{\varpi_{i1}, \varpi_{i2}, \varpi_{i3}\}$, $\eta_2 = 0.1$, $N = 10$, $n = 3$, $\Gamma_k = I$, $m = 2$ and as shown in the equation at the top of the next page. If all initial values are rand [0,1], let $k_i = 12$, $\hat{k}_i = 8$, $\overline{k_i} = 9$, $c = 4.5$, $\mathrm{T}^* = 1.5$, $\mathrm{p} = 0.6$, $\mathrm{q} = 1.5$. Base on the conditions of Theorem 1, by simple calculation, let the number of pinning controlling nodes be $l = 5$, one has $\lambda_{max}(B^{(k)} \otimes \Gamma_k)_{N-l} = -5.4821 < -\frac{1}{4.5 \times 2}\left(38.1 + \frac{1}{2} \times 0.1\right) = -4.2389$, $\lambda_{min}\left(\hat{K}_N \otimes I_N\right) \geq \frac{2}{(q-1)T^*} 2^{\frac{1-q}{2}} (Nn)^{\frac{q-1}{2}} = 5.2480$, $\lambda_{min}\left(\overline{K_N} \otimes I_N\right) \geq \frac{2}{(1-p)T^*} 2^{\frac{1-p}{2}} = 3.8290$, which are satisfying. Fig.2 shows synchronization error convergence time is approximately 1.4, less than 1.5. Clearly, the synchroniza-tion error (3) is stable at zero under the pinning controllers (7) with $l = 5$. Fig.3 shows that the controller evolution curve asymptotically approaches 0. The validity of Theorem 1 is verified by the implementation of numerical simulations.

For the simulation of theorem 2, if all initial values are rand [0,1], let $k_i = 15$, $\overline{k_i} = 12$, $c = 4.5$, $\mathrm{T}^* = 1.5$, $\mathrm{p} = 0.6$. Similar to the calculation of theorem 1, let the number of pinning controlling nodes be $l = 5$, which satisfies the conditions of Theorem 2. Fig.4 shows synchronization error convergence time is approximately 1.3, less than 1.5. So the synchronization error (3) is stable at zero by the pinning controllers (11) with $l = 5$. Fig.5 shows that the controller evolution curve asymptotically approaches 0. The validity of Theorem 2 is verified by the implementation of numerical simulations.

Remark 9: The numerical simulation in this paper only verifies the FnT/FxTstability of multi-layer networks. For the asymptotic or exponential stability of multi-layer networks,

$$B^{(1)} = B^{(2)} = \begin{pmatrix} -6 & 2 & 1 & -1 & 1 & 0 & 0 & 2 & 0 & 1 \\ 2 & -7 & 1 & 0 & 1 & 2 & 0 & 2 & 0 & -1 \\ 1 & 1 & -5 & 1 & 2 & 1 & 0 & 0 & -2 & 1 \\ -1 & 0 & 1 & -6 & 0 & 1 & 2 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 & -4 & -2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 1 & -2 & -4 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & -1 & -3 & 0 & 2 & 0 \\ 2 & 2 & 0 & 2 & 1 & 1 & 0 & -7 & 0 & -1 \\ 0 & 0 & -2 & 1 & 1 & 1 & 2 & 0 & -5 & 2 \\ 1 & -1 & 2 & 0 & 0 & 0 & 0 & -1 & 2 & -3 \end{pmatrix}.$$

we need to further study the asymptotic or exponential stability conditions of multi-layer networks, and then complete the numerical simulation according to the asymptotic or exponential stability conditions, and compare the stable types from the simulation diagram, which will be our next research work.

V. CONCLUSION

This paper specifically concentrated on the synchronization of FnT/FxT in SMLNs using pinning control. It successfully derived new criteria for achieving FnT/FxT synchronization of SMLNs with pinning control. In addition, the paper also explored the correlation between the intensity of control and the different layers of the network, taking into account the minimum convergence time. This analysis will be of great value for practical application. Finally, the theoretical findings presented in the paper were validated through numerical simulations. How to find an optimal pinning control strategy among the number of pinning control nodes, feedback again and FnT convergence rate, which will be our future work.

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