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RESEARCH ARTICLE

Bayesian and E-Bayesian Reliability Analysis of Improved Adaptive Type-II Progressive Censored Inverted Lindley Data

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ABSTRACT Lately, a novel improved adaptive progressive censored strategy of Type-II was developed that can ensure that the experiment duration does not overextend a specific span. When a sample is formed using such censoring, three estimation issues of the model parameter and certain reliability metrics of the inverted Lindley lifetime distribution are taken into consideration. In addition to the traditional likelihood methodology, Bayesian and E-Bayesian methodologies with squared error loss are considered. The asymptotic distribution of frequentist estimates over the empirical Fisher information is employed to get the estimated confidence intervals for each parameter. A technical procedure called Monte-Carlo Markov-Chain is operated to provide the required Bayes and E-Bayes estimations as well as to construct their credible intervals. We deliver extensive empirical comparisons to explain the applicability and usefulness of the various suggested strategies. Lastly, two actual data collections gathered from the engineering and medical disciplines are analyzed to verify the offered model's relevance and viability in the context of reality. The study findings suggest that, in order to get the required estimations, the E-Bayesian paradigm via the Metropolis-Hastings sampler is preferable in comparison to the other approaches.

INDEX TERMS New inverted Lindley, improved adaptive progressive Type-II censoring, likelihood, Bayesian and E-Bayesian estimations, MCMC technique.

I. INTRODUCTION

Enhancing some lifespan models as well as employing them effectively to solve modelling issues in a range of fields, including but not restricted to engineering, health, and finance, among several others, has been the focus of significant work in recent years. Lindley [1] developed the Lindley distribution (LD), which is a hybrid of the gamma and exponential models. Data with non-monotone forms may not be as relevant for the LD because it is only suitable for fitting data with an increasing hazard rate function (HRF). Therefore, Sharma et al. [2] presented the inverted-Lindley (IL) model with an inverse (upside-down) bathtub-shaped HRF as an inverted version of the LD.

Consider that the random variable $X > 0$ pursues the IL distribution, utilize IL(ϕ) as its symbol, and let ϕ be a scale parameter with $\bar{\phi} = 1/(1+\phi)$. Then, the probability density function (PDF) that corresponds to X is defined as

$$f(x;\phi) = \frac{\phi\bar{\phi}e^{-\frac{\phi}{x}}(1+x)}{x^3},\ x>0, \phi>0, \tag{1}$$

and its cumulative distribution function (CDF) is

$$F(x;\phi) = \left(1+\frac{\bar{\phi}}{x}\right)e^{-\frac{\phi}{x}}. \tag{2}$$

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The reliability function (RF) and HRF of the IL model at a given time t are, respectively, given by

$$R(t;\phi) = 1 - \left(1 + \frac{\bar{\phi}}{t}\right) e^{-\frac{\phi}{t}} \tag{3}$$

and

$$h(t;\phi) = \frac{\left(\frac{\phi}{t}\right)^2 (1+t)}{\left[t(1+\phi)\left(e^{\frac{\phi}{t}} - 1\right) - \phi\right]}. \tag{4}$$

Considering the IL distribution as the parent distribution, many authors considered some of its estimation concerns. Using a Type-I censored sample, Basu et al. [3] examined different of its estimation concerns. Basu et al. [4] studied the IL model via a progressive hybrid censored strategy with binomial removals. Based on hybrid censoring, Basu et al. [5] examined the conventional and Bayesian predictions for the IL distribution. Using ranked set sampling, Hassan et al. [6] considered the estimation of the IL reliability factor. Evaluating the PDF and CDF for the IL distribution was handled by Asgharzadeh et al. [7]. Given an adaptive Type-II progressively censored (AT-IIPC) sample, Alotaibi et al. [8] discussed the IL distribution.

In light of the limited time and financial commitment, full-life testing is not practical in engineering investigations or life tests. This has led to the development and widespread usage of censored experiments that terminate when a certain percentage of the units yield. A popular technique that has caught the interest of researchers and reliability specialists is the progressive Type-II censoring (PT-IIC) strategy. Under this system, n products are assigned to a life test; in advance, the researcher uses the progressive censoring scheme (PCS) $\underline{S} = (S_1, \ldots, S_m)$ to get the number m of failure units. Following the initial failure $X_{1:m:n}$, S_1 of the test's surviving products will be randomly drawn. Proceeding, S_2 of the remaining products is randomly extracted upon the occurrence of the subsequent failure $X_{2:m:n}$, and so forth. Eventually, all of the remaining units, indicated by S_m, are drawn from the test when the m^{th} failure appears with lifetime $X_{m:m:n}$. Numerous writers took this plan into consideration, for instance Sultan et al. [9], Mondal and Kundu [10], Wu and Gui [11] and Abo-Kasem et al. [12]. The AT-IIPC strategy is a more general censoring plan in which the PT-IIC plan can be acquired as a particular matter. Ng et al. [13] suggested the AT-IIPC approach to improve statistical inference efficiency. Similar to the PT-IIC plan, the PCS $\underline{S} = (S_1, \ldots, S_m)$ and the failure size m are preassigned before beginning the trial; however, testing time is authorized to go beyond a preset limit τ_1. The PCS is changed in this instance by pausing to remove any units until the final failure time $X_{m:m:n}$. At this stage, all remaining units have to be removed from the test. For more detail regarding the AT-IIPC strategy, one can refer to Nassar and Abo-Kasem [14], Kohansal and Shoaee [15], Ren and Gui [16] and Schmiedt and Cramer [17].

Ng et al. [13] highlighted that the AT-IIPC plan works well in the area of statistical inference when the test's overall duration is not an important factor. On the other hand, if the testing period is very long, as in the case of reliable products, the AT-IIPC scheme will not guarantee a suitable overall test length. To address this issue, Yan et al. [18] proposed an innovative censorship technique called an improved AT-IIPC (IAPT-IIC) method. This strategy can successfully guarantee that the trial ends within the time frame specified. Regarding the number of n units, prefixed m, and PCS $\underline{S} = (S_1, \ldots, S_m)$, this scheme's description is comparable to that of the PT-IIC and AT-IIPC. But two thresholds are offered: τ_1 and τ_2, where $\tau_1 < \tau_2$. Upon seeing the i^{th} failure time $X_{i:m:n}$, S_i units are arbitrarily removed.

In this approach, the experiment can end in three different ways, denoted by C-I, C-II and C-III, as shown below

- C-I: The PT-IIC plan can be seen in this case, where the test stops at $X_{m:m:n}$ if $X_{m:m:n} < \tau_1$.
- C-II: The AT-IIPC plan is demonstrated in this situation, where the test terminates at $X_{m:m:n}$ if $\tau_1 < X_{m:m:n} < \tau_2$. Here, the PCS is changed by setting $S_{r_1+1} = \cdots = S_{m-1} = 0$, after acquiring $X_{r_1:m:n}$, where r_1 is the observed failures over τ_1. Following that, every last unit is eliminated at the moment of m^{th} failure.
- C-III: The experiment is terminated at τ_2 if $\tau_1 < \tau_2 < X_{m:m:n}$. After getting $X_{r_1:m:n}$, there are no more withdrawals, i.e. $S_{r_1+1} = \cdots = S_{r_2-1} = 0$, where $r_2 < m$ is the observed failures over τ_2. All of the leftover units, namely $S^\bullet = n - r_2 - \sum_{i=1}^{r_1} S_i$, are eliminated at τ_2.

Regarding the IAPT-IIC plan, not much work has been completed. The notable works are those of Nassar and Elshahhat [19], Elshahhat and Nassar [20], Elbatal et al. [21] and Dutta and Kayal [22]. Let $x_i = x_{i:m:n}$, $i = 1, \ldots, m$ for simplicity of notation. Then, based on the observed IAPT-IIC sample $\underline{x} = (x_1 < \cdots < x_{r_1} < \tau_1 < \cdots < x_{r_2} < \tau_2)$, with PCS $\underline{S} = (S_1, \ldots, S_{r_1}, 0, \ldots, 0, S^*)$, the likelihood function (LF) becomes

$$L(\phi|\underline{x}) = \prod_{i=1}^{M_2} f(x_i) \prod_{i=1}^{M_1} [1 - F(x_i)]^{S_i} [1 - F(\tau)]^{S^*}, \tag{5}$$

where M_1, M_2, τ and S^* are specified as follow

$$M_1 = \begin{cases} m, & \text{for C-I} \\ r_1, & \text{for C-II} \\ r_1, & \text{for C-III,} \end{cases} \qquad M_2 = \begin{cases} m, & \text{for C-I} \\ m, & \text{for C-II} \\ r_2, & \text{for C-III.} \end{cases}$$

and

$$\tau = \begin{cases} 0, & \text{for C-I} \\ x_m, & \text{for C-II} \\ \tau_2, & \text{for C-III,} \end{cases} \qquad S^* = \begin{cases} n - m - \sum_{i=1}^{m} S_i, & \text{for C-I} \\ n - m - \sum_{i=1}^{r_1} S_i, & \text{for C-II} \\ n - r_2 - \sum_{i=1}^{r_1} S_i, & \text{for C-III.} \end{cases}$$

In this work, we aim to investigate the estimation issues of the IL distribution based on IAPT-IIC data, considering the flexibility of the IL distribution in modeling real data and the efficiency of the IAPT-IIC scheme in ending the experiment. One classical and two Bayesian estimation

approaches are considered for this purpose. We compare three point estimations for the unknown parameter as well as the two reliability indices. Additionally, we explore three interval estimation techniques for different parameters. It is worth mentioning that this is the first instance where calculations for the IL distribution, along with its reliability indices, have been compared. We anticipate that this will capture the attention of a wide range of readers. We can outline our study's objectives as follows:

1) From classical perspective, we obtain the maximum likelihood estimates (MLEs) as well as approximate confidence intervals (ACIs) of ϕ, RF, and HRF.
2) Apply the Bayesian method via the squared error loss function (SELF) to get the Bayes estimates (BEs) of ϕ, RF, and HRF. The Bayes credible intervals (BCIs) are also computed.
3) Obtaining the Expected-BEs (EBEs) and Expected-BCIs (EBCIs) of ϕ, RF, and HRF using the SELF by implementing the E-Bayesian method as an alternative to the Bayesian approach.
4) Comparing the performance of the different point and interval estimation techniques using simulation analysis. This comparison will help us identify the most efficient method for evaluating the IL distribution in the presence of IAPT-IIC data.
5) Highlighting the practical value of various techniques through the analysis of two real data sets in engineering and medicine.

The rest of this study is organized as follows: Section II reports the MLEs and ACIs of the IL distribution. In Section III, the Bayesian estimations of the IL distribution are considered. Section IV covers the EBEs and EBCIs. Section V presents the numerical results and the design of the simulation. Section VI offers two applications related to the engineering field. The paper concludes in Section VII.

II. LIKELIHOOD ESTIMATION

Acquiring an IAPT-IIC sample $\underline{\mathbf{x}}$ with PCS $\underline{S} = (S_1, \ldots, S_{r_1}, 0, \ldots, 0, S^*)$, one can express the joint LF in (5) using the PDF and CDF given by (1) and (2), respectively, as follow

$$L(\phi|\underline{x}) = (\phi\bar{\phi})^{M_2}[w(\tau;\phi)]^{S^*}e^{-\phi u(\underline{x})}\prod_{i=1}^{M_1}[w(x_i;\phi)]^{S_i}, \quad (6)$$

where $x_i = 1/x_i$, $w(x_i;\phi) = e^{\phi x_i} - (1+\bar{\phi}x_i)$ and $u(\underline{x}) = \sum_{i=1}^{M_2} x_i + \sum_{i=1}^{M_1} S_i x_i + S^*\tau$. The natural logarithm of (6) is

$$l(\phi|\underline{x}) = M_2\log(\phi\bar{\phi}) - \phi u(\underline{x}) + S^*\log[w(\tau;\phi)] + \sum_{i=1}^{M_1} S_i\log[w(x_i;\phi)]. \quad (7)$$

The MLE of ϕ, denoted by $\hat{\phi}$, is the solution of the subsequent formula

$$\frac{dl(\phi|\underline{x})}{d\phi} = \frac{M_2}{\phi} + \frac{M_2}{\phi(1+\phi)} - u(\underline{x}) + S^*w_1(\tau;\phi) + \sum_{i=1}^{M_1} S_i w_1(x_i;\phi) = 0, \quad (8)$$

where $w_1(x_i;\phi) = \frac{x_i[(1+\phi)^2e^{\phi x_i}-1]}{(1+\phi)^2w(x_i;\phi)}$. It is crucial to point out that Equation (8) is unable to be solved analytically. As such, it is hard to get the MLE ϕ in explicit structure. Numerous numerical schemes like the Newton-Raphson method can yield the needed estimate. From (3) and (4), the MLEs $\hat{R}(t)$ and $\hat{h}(t)$ employing the invariance feature can be acquired as

$$\hat{R}(t) = 1 - \left(1 + \frac{\hat{\phi}}{t}\right)e^{-\frac{\hat{\phi}}{t}}$$

and

$$\hat{h}(t) = \frac{\left(\frac{\hat{\phi}}{t}\right)^2(1+t)}{\left[t(1+\hat{\phi})\left(e^{\frac{\hat{\phi}}{t}}-1\right)-\hat{\phi}\right]}.$$

Obtaining the ACIs of ϕ, together with the RF and HRF, is the next task in this section. To accomplish this, we use the asymptotic characteristics linked to the MLEs.

We initially take the inverse of the observed Fisher's matrix $(\widehat{V}_1)$ in order to estimate the variance of $\hat{\phi}$ where

$$\widehat{V}_1 = \left(-\frac{d^2l(\phi|\underline{x})}{d\phi^2}\right)^{-1}_{\phi=\hat{\phi}},$$

where

$$\frac{d^2l(\phi|\underline{x})}{d\phi^2} = \frac{M_2}{(1+\phi)^2} - \frac{2M_2}{\phi^2} + S^*w_2(\tau;\phi) + \sum_{i=1}^{M_1} S_i w_2(x_i;\phi),$$

where $w_2(x_i;\phi) = \frac{x_i[x_i(1+\phi)^3e^{\phi x_i}+2]}{(1+\phi)^3w(x_i;\phi)} - w_1^2(x_i;\phi)$. It is simple to find the standard error for the estimate $\hat{\phi}$, represented by $\widehat{SE}_1$, by taking the square root of $\widehat{V}_1$. Then, the ACI of ϕ can be expressed as, given any significance level, say α,

$$\hat{\phi} \pm z_{\frac{\alpha}{2}}\,\widehat{SE}_1,$$

with $z_{\frac{\alpha}{2}}$ is the upper $(\frac{\alpha}{2})^{th}$ standard normal percentile point.

However, in order to acquire these ACIs for the RF and HRF, we have to first acquire the SEs linked to their relevant MLEs. This can be accomplished by using the delta method (DM). The DM is a very commonly used approach to approximate the variances of complex functions.

To apply it, we require to get the derivatives of $R(t)$ and $h(t)$ regarding the unknown parameter ϕ as given below

$$\frac{dR(t)}{d\phi} = \frac{\bar{\phi}e^{-\frac{\phi}{t}}[1+v(t;\phi)]}{tq(t;\phi)}$$

and

$$\frac{dh(t)}{d\phi} = \frac{\phi(1+t)\left\{e^{-\frac{\phi}{t}}[v(t;\phi)+\phi^2]-v(t;\phi)\right\}}{t^2\left\{\phi+q(t;\phi)[1-e^{-\frac{\phi}{t}}]\right\}^2}.$$

where $q(t;\phi) = t(1+\phi)$ and $v(t;\phi) = q(t;\phi) + 2t$. Let $\widehat{D}_1 = \frac{dR(t)}{d\phi}\big|_{\phi=\hat{\phi}}$ and $\widehat{D}_2 = \frac{dh(t)}{d\phi}\big|_{\phi=\hat{\phi}}$. Then, the respective estimated variances of the MLEs $\hat{R}(t)$ and $\hat{h}(t)$ can be obtained as

$$\widehat{V}_2 \approx \widehat{D}_1\widehat{V}_1\widehat{D}_1^\top \text{ and } \widehat{V}_3 \approx \widehat{D}_2\widehat{V}_1\widehat{D}_2^\top.$$

Now, we can calculate the ACIs for $R(t)$ and $h(t)$ as

$$\hat{R}(t) \pm z_{\frac{\alpha}{2}} \widehat{SE}_2, \text{ and } \hat{h}(t) \pm z_{\frac{\alpha}{2}} \widehat{SE}_3,$$

respectively, where $\widehat{SE}_j = \sqrt{\widehat{V}_j}, j = 2, 3$.

III. BAYESIAN ESTIMATION

The BEs and BCIs for ϕ, $R(t)$, and $h(t)$ are examined in this section. The ability of the Bayesian method to integrate historical data about the unknown parameter with the available observed data makes it a very valuable estimation technique. The technique improves statistical inference's effectiveness when compared with classical ones. From the perspective of the Bayesian paradigm, determining the prior knowledge which indicates our understanding about the studied parameter is the first step.

In this instance, we use the same assumption as Sharma et al. [2], who took the gamma PDF into account as a prior distribution for ϕ, in the following form

$$g(\phi) \propto \phi^{a-1} e^{-b\phi}, \ \phi > 0, a, b > 0. \tag{9}$$

The observed data provided by the LF, as given by (6), can be combined with previously available information, as given by the prior knowledge in (9), to derive the posterior PDF of ϕ as follows

$$G(\phi|\underline{x}) = \frac{\phi^{a+M_2-1}\bar{\phi}^{M_2}}{A}[w(\tau;\phi)]^{S^*} e^{-\phi[b+u(\underline{x})]} \times \prod_{i=1}^{M_1}[w(x_i;\phi)]^{S_i}, \tag{10}$$

where $A = \int_0^\infty L(\phi|\underline{x})g(\phi)d\phi$.

For the parameter ϕ or any function on it, say $\pi(\phi)$, the BE using the SELF can be acquired as

$$\tilde{\pi}_B(\phi) = A^{-1}\int_0^\infty \pi(\phi)\phi^{a+M_2-1}\bar{\phi}^{M_2}[w(\tau;\phi)]^{S^*} e^{-\phi[b+u(\underline{x})]} \times \prod_{i=1}^{M_1}[w(x_i;\phi)]^{S_i} d\phi. \tag{11}$$

As predicted, the ratio of integrals in (11) prevents finding a closed form for the BE $\tilde{\pi}_B(\phi)$. To create a solution around this problem and obtain both the necessary BE and the BCI, we recommend to use the MCMC process.

Before using the MCMC technique, it is required to ascertain whether the posterior distribution matches any well-known distribution. It is obvious that the posterior distribution in (10) cannot be described by any well-known distribution. The Metropolis-Hastings (M-H) procedure is suited in the present instance for collecting the necessary samples from (10) with normal proposal distribution (NPD). To collect MCMC samples and obtain the BEs and BCIs, follow the steps reported in Algorithm 1.

Algorithm 1 The MCMC Method in Bayesian Setup

Input: Size of iterations $\mathbb{H}$ and the size of burn-in $\mathbb{H}^\star$
Input: Set $j = 1$
Input: Set $\phi^{(0)} = \hat{\phi}$
Output: Get $\phi^{(j)}$ by M-H steps from $G(\phi|\underline{x})$
1: **for** $j \leftarrow 1 to \mathbb{H}$ **do**
2: Create ϕ^* from $N(\hat{\phi}, \widehat{V}_1)$
3: Obtain $\xi_\phi = \min\left\{1, \frac{G(\phi^*|\underline{x})}{G(\phi^{(j-1)}|\underline{x})}\right\}$
4: Generate a variate u from $U(0, 1)$
5: **if** $(u \leqslant \xi_\phi)$ **then**
6: Set $\phi^{(j)} = \phi^*$
7: **end if**
8: **if** $(u > \xi_\phi)$ **then**
9: Set $\phi^{(j)} = \phi^{(j-1)}$
10: **end if**
11: **end for**
12: **return** $j = j + 1$
13: **return** Repeat j to $\mathbb{H}$ and set $\bar{\mathbb{H}} = \mathbb{H} - \mathbb{H}^\star$
Output: Get $\left(\phi^{(j)}, R^{(j)}, h^{(j)}\right)$, $j = \mathbb{H}^\star + 1, \ldots, \mathbb{H}$
Output: Compute the BE of ϕ as $\tilde{\phi}_B = \frac{1}{\bar{\mathbb{H}}}\sum_{i=\mathbb{H}^\star+1}^{\mathbb{H}} \phi^{(j)}$
Output: Compute the BE of $R(t)$ as $\tilde{R}_B = \frac{1}{\bar{\mathbb{H}}}\sum_{i=\mathbb{H}^\star+1}^{\mathbb{H}} R^{(j)}$
Output: Compute the BE of $h(t)$ as $\tilde{h}_B = \frac{1}{\bar{\mathbb{H}}}\sum_{i=\mathbb{H}^\star+1}^{\mathbb{H}} h^{(j)}$
14: **for** $j \leftarrow \mathbb{H}^\star + 1 to \mathbb{H}$ **do**
15: Order $\phi^{(j)}$ as $\phi^{[\mathbb{H}^\star+1]} < \cdots < \phi^{[\mathbb{H}]}$
16: Order $R^{(j)}$ as $R^{[\mathbb{H}^\star+1]} < \cdots < R^{[\mathbb{H}]}$
17: Order $h^{(j)}$ as $h^{[\mathbb{H}^\star+1]} < \cdots < h^{[\mathbb{H}]}$
18: **end for**
Output: Get the BCI of ϕ as $\left\{\phi^{[(\bar{\mathbb{H}})\frac{\alpha}{2}]}, \phi^{[(\bar{\mathbb{H}})(1-\frac{\alpha}{2})]}\right\}$
Output: Get the BCI of $R(t)$ as $\left\{R^{[(\bar{\mathbb{H}})\frac{\alpha}{2}]}, R^{[(\bar{\mathbb{H}})(1-\frac{\alpha}{2})]}\right\}$
Output: Get the BCI of $h(t)$ as $\left\{h^{[(\bar{\mathbb{H}})\frac{\alpha}{2}]}, h^{[(\bar{\mathbb{H}})(1-\frac{\alpha}{2})]}\right\}$

IV. E-BAYESIAN ESTIMATION

The hyper-parameters in the standard Bayesian estimation are regarded as constants. The method of E-Bayesian estimation is another strategy that considers the hyper-parameters as random variables with known probability distributions. Han [23] is the first to examine this method in order to obtain the exponential distribution estimation. Numerous authors have up to this point, explored this method for examining specific E-Bayesian estimation problems related to various statistical models. Consider, for instance, Okasha and Wang Han [24], Yousefzadeh [25] and Athirakrishnan and Abdul-Sathar [26]. According to the E-Bayesian approach, the EBE is obtained by taking the expected value of the standard BE with respect to the joint hyper-parameter distribution. Let $p(a, b)$ be the joint hyper-parameters probability distribution. Then, for the IL distribution with the IAPT-IIC data, the EBE of $\pi(\phi)$, can be expressed as

$$\tilde{\pi}_{EB}(\phi) = \int\int_\Omega \tilde{\pi}_B(\phi)\, p(a, b)\, da\, db, \tag{12}$$

where Ω refer to the domain of the hyper-parameters a and b, while $\tilde{\pi}_B(\phi)$ refers to the BE of the parameter $\pi(\phi)$. It is important to build the joint PDF of the hyper-parameters (a, b) to ensure that the prior distribution of the ϕ is a decreasing function of ϕ, as noted by Han [23]. It is evident that the gamma distribution is capable of achieving this feature when $0 < a < 1$ and $b > 0$. Thus, in our case, the beta distribution is chosen to be the probabilistic density of the hyper-parameter a.

Alternatively, the uniform probability distribution on the interval $(0, c)$ is used as the distribution of the hyper-parameter b. Following this, the combined PDF of a and b (say $p(a, b)$) can be formulated as

$$p(a, b) = \frac{a^{\kappa-1}(1-a)^{\nu-1}}{c\,B(\kappa, \nu)}, 0 < a < 1, 0 < b < c, \kappa, \nu > 0. \tag{13}$$

From (11) and (13), one can get the EBE of $\pi(\phi)$ as given below

$$\tilde{\pi}_{EB}(\phi) = \int_0^c \int_0^1 \tilde{\pi}_B(\phi) \frac{a^{\kappa-1}(1-a)^{\nu-1}}{c\,B(\kappa, \nu)}\, da\, db. \tag{14}$$

Analogously to the Case of obtaining the BEs and BCIs in the preceding section, the integrals in (14) render the work of obtaining the EBEs extremely challenging. We consider applying the MCMC approach to obtain the necessary estimations in order to get around this problem as proposed by Nassar et al. [27]. This MCMC procedure work by generating the parameters a and b from the beta and uniform densities instead of setting their values as constants. Rather than setting the weights of the hyper-parameters a and b as constants, the MCMC algorithm generates them from the beta and uniform distributions. The required samples can be generated, and the EBEs and EBCIs of the relevant parameters can be obtained by following the steps described in Algorithm 2.

Algorithm 2 The MCMC Method in E-Bayesian Setup

Input: Size of iterations $\mathbb{D}$ and the size of burn-in $\mathbb{D}^\star$
Input: Set $j = 1$
Input: Set $\phi^{(0)} = \hat{\phi}$
Input: Get $a^{(j)}$ and $b^{(j)}$ from beta and uniform distributions, respectively
Output: Get $\phi^{\bullet(j)}$ by M-H steps from $G(\phi|\underline{x})$
Output: Get $R^{\bullet(j)}$ and $h^{\bullet(j)}$ by setting $\phi^{\bullet(j)}$ in place of ϕ.
1: **return** $j = j + 1$
2: **return** Repeat j to $\mathbb{D}$ and set $\bar{\mathbb{D}} = \mathbb{D} - \mathbb{D}^\star$
Output: Get $\left(\phi^{\bullet(j)}, R^{\bullet(j)}, h^{\bullet(j)}\right)$, $j = \mathbb{D}^\star + 1, \ldots, \mathbb{D}$
Output: Get the EBE of ϕ as $\tilde{\phi}_{EB} = \frac{1}{\bar{\mathbb{D}}} \sum_{i=\mathbb{D}^\star+1}^{\mathbb{D}} \phi^{\bullet(j)}$
Output: Get the EBE of $R(t)$ as $\tilde{R}_{EB} = \frac{1}{\bar{\mathbb{D}}} \sum_{i=\mathbb{D}^\star+1}^{\mathbb{D}} R^{\bullet(j)}$
Output: Get the EBE of $h(t)$ as $\tilde{h}_{EB} = \frac{1}{\bar{\mathbb{D}}} \sum_{i=\mathbb{D}^\star+1}^{\mathbb{D}} h^{\bullet(j)}$
3: **for** $j \leftarrow \mathbb{D}^\star + 1 to \mathbb{D}$ **do**
4: Order $\phi^{\bullet(j)}$ as $\phi^{\bullet[\mathbb{D}^\star+1]} < \cdots < \phi^{\bullet[\mathbb{D}]}$
5: Order $R^{\bullet(j)}$ as $R^{\bullet[\mathbb{D}^\star+1]} < \cdots < R^{\bullet[\mathbb{D}]}$
6: Order $h^{\bullet(j)}$ as $h^{\bullet[\mathbb{D}^\star+1]} < \cdots < h^{\bullet[\mathbb{D}]}$
7: **end for**
Output: Get the EBCI of ϕ as $\left\{\phi^{\bullet[(\bar{\mathbb{D}})\frac{\alpha}{2}]}, \phi^{\bullet[(\bar{\mathbb{D}})(1-\frac{\alpha}{2})]}\right\}$
Output: Get the EBCI of $R(t)$ as $\left\{R^{\bullet[(\bar{\mathbb{D}})\frac{\alpha}{2}]}, R^{\bullet[(\bar{\mathbb{D}})(1-\frac{\alpha}{2})]}\right\}$
Output: Get the EBCI of $h(t)$ as $\left\{h^{\bullet[(\bar{\mathbb{D}})\frac{\alpha}{2}]}, h^{\bullet[(\bar{\mathbb{D}})(1-\frac{\alpha}{2})]}\right\}$

V. NUMERICAL COMPARISONS

We utilize simulations in this part to assess the accuracy and practicality of the estimates for a IL's parameters and reliability measurements. We use the IAPT-IIC mechanism 1,000 times to evaluate the findings of ϕ, $R(t)$, or $h(t)$ from IL(0.75). When $t = 0.1$, the estimates of $R(0.1) = 0.99708$ and $h(0.1) = 019613$ are considered.

We also choose values for n which can be 40 or 80, $\tau_1(= 0.5, 1.5)$, $\tau_2(= 0.8, 2.8)$, and m is the percentage of failures such as 50% or 75% of each n. Also, for every combination of numbers (n, m), we also look at different PT-IIC designs $\underline{S}$ as

$$\text{S1} : \left(n-m, 0^{m-1}\right), \ \text{S2} : \left(0^{\frac{m}{2}-1}, n-m, 0^{\frac{m}{2}}\right),$$

and

$$\text{S3} : \left(0^{m-1}, n-m\right),$$

where, for example, 1^m means 1 is repeated m times. Once we collect 1,000 IAPT-IIC samples, we will use the 'maxLik' package (by Henningsen and Toomet [28]) to find the acquired classical point and interval estimations for all unknown parameters.

Next, the BEs and EBEs as well as their BCIs and EBCIs of ϕ, $R(t)$, or $h(t)$ are computed by the 'coda' (by Plummer et al. [29]) package. To show the implementation of the proposed gamma information, by simulating 5,000 past-complete samples (with $n = 50$) from IL(0.75), we took the values of (a, b) as (89.28626,117.5833). In E-Bayes' calculations, we set (κ, ν, c) as (10,5,1). Following Section III, we set $\mathbb{H} = 12,000$ and $\mathbb{B} = 2,000$. The average point estimates (APEs), root mean squared errors (RMSEs), mean absolute biases (MABs), average confidence lengths (AILs), and coverage percentages (CPs) of ϕ (for example) are obtained by

$$\text{APE}(\dot{\phi}) = \frac{1}{1000} \sum_{i=1}^{1000} \dot{\phi}^j,$$

$$\text{RMSE}(\dot{\phi}) = \sqrt{\frac{1}{1000} \sum_{i=1}^{1000} \left(\dot{\phi}^j - \phi\right)^2}$$

$$\text{MAB}(\dot{\phi}) = \frac{1}{1000} \sum_{i=1}^{1000} \left|\dot{\phi}^j - \phi\right|,$$

$$\text{AIL}_{:(1-\alpha)}(\phi) = \frac{1}{1000}\sum_{i=1}^{1000}\left(\mathcal{U}_{\dot{\phi}^i} - \mathcal{L}_{\dot{\phi}^i}\right),$$

$$\text{CP}_{(1-\alpha)}(\phi) = \frac{1}{1000}\sum_{i=1}^{1000}\mathbf{1}^{**}_{\left(\mathcal{L}_{\dot{\phi}^i}:\mathcal{U}_{\dot{\phi}^i}\right)}(\phi),$$

respectively, where $\dot{\phi}^j$ is the desired estimate of the parameter ϕ at i^{th} sample, $\mathbf{1}^{**}(\cdot)$ is an indicator.

In Tables 1-3, the APE, RMSE, and MAB are listed in the 1st, 2nd, and 3rd columns, respectively. In Tables 4-6, the AIL and CP are listed in the 1st and 2nd columns, respectively. From Tables 1-6, we can report the following comments in terms of the lowest level of RMSEs, MABs, and AILs as well as the highest level of CPs:

- All received estimates for ϕ, $R(t)$, or $h(t)$ created by all suggested methodologies behaved well.
- As n raises, all estimates of ϕ, $R(t)$, and $h(t)$ act well. The same note is also reached when m increases or $\sum_{i=1}^{m} R_i$ deceases.
- As τ_1 and τ_2 increase, all simulated findings of ϕ, $R(t)$, or $h(t)$ become even better.
- Based on the prior information via gamma distribution, the point and credible interval estimations formed by the Bayes' setup for all studied parameters perform satisfactorily when compared to the likelihood estimates.
- Based on the uniform and gamma distributions as probabilistic distributions for the hyper-parameters, the point and credible findings acquired using the E-Bayes' setup for all studied parameters operate effectively when versus those created from the Bayesian or likelihood approaches.
- Comparing the suggested removal plans Si for $i = 1, 2, 3$, the offered results of ϕ (founded by S1) and of $R(t)$ or $h(t)$ (using S3) serve more acceptable than others.
- As a guideline, in the case of data generated by the suggested mechanism, the studied IL model parameters should be estimated using the E-Bayes approach.

TABLE 1. Point estimations of ϕ.

(n, m)	S	MLE			BE			EBE		
		APE	RMSE	MAB	APE	RMSE	MAB	APE	RMSE	MAB
					$(\tau_1, \tau_2) = (0.5, 0.8)$					
(40,20)	S1	0.9148	0.1379	0.0919	0.8032	0.0980	0.0732	0.8315	0.0766	0.0659
	S2	0.8072	0.1422	0.1117	0.7604	0.1120	0.0866	0.7841	0.0858	0.0714
	S3	0.8161	0.1581	0.1239	0.7608	0.1261	0.1020	0.7857	0.1054	0.0866
(40,30)	S1	0.7585	0.1179	0.0881	0.7739	0.0871	0.0659	0.7683	0.0640	0.0546
	S2	0.7499	0.1234	0.0892	0.7704	0.0889	0.0676	0.7635	0.0725	0.0566
	S3	0.7653	0.1291	0.0913	0.7646	0.0925	0.0709	0.7713	0.0751	0.0585
(80,40)	S1	0.8032	0.1025	0.0832	0.8751	0.0764	0.0606	0.7963	0.0543	0.0435
	S2	0.7445	0.1081	0.0853	0.7577	0.0805	0.0636	0.7610	0.0579	0.0477
	S3	0.7583	0.1130	0.0868	0.7598	0.0832	0.0649	0.7679	0.0619	0.0515
(80,60)	S1	0.8238	0.0828	0.0690	0.7737	0.0653	0.0515	0.7841	0.0470	0.0374
	S2	0.8061	0.0960	0.0762	0.7651	0.0657	0.0523	0.7740	0.0505	0.0399
	S3	0.8417	0.0990	0.0808	0.7609	0.0715	0.0564	0.7876	0.0521	0.0429
					$(\tau_1, \tau_2) = (1.5, 2.8)$					
(40,20)	S1	0.8558	0.1349	0.0921	0.7967	0.0968	0.0721	0.8082	0.0712	0.0627
	S2	0.8154	0.1415	0.0988	0.7549	0.1033	0.0747	0.7862	0.0762	0.0681
	S3	0.8130	0.1557	0.1166	0.7555	0.1174	0.0828	0.7830	0.0926	0.0775
(40,30)	S1	0.7691	0.1085	0.0837	0.7827	0.0855	0.0639	0.7730	0.0625	0.0518
	S2	0.7539	0.1134	0.0881	0.7584	0.0856	0.0663	0.7653	0.0652	0.0555
	S3	0.7574	0.1200	0.0901	0.7593	0.0896	0.0678	0.7676	0.0689	0.0575
(80,40)	S1	0.8269	0.0951	0.0787	0.8074	0.0734	0.0554	0.8121	0.0516	0.0405
	S2	0.7476	0.0987	0.0824	0.7548	0.0747	0.0588	0.7628	0.0541	0.0430
	S3	0.7553	0.1027	0.0826	0.7559	0.0797	0.0617	0.7669	0.0596	0.0495
(80,60)	S1	0.8882	0.0786	0.0639	0.8040	0.0615	0.0507	0.8246	0.0466	0.0341
	S2	0.8148	0.0862	0.0715	0.7566	0.0651	0.0520	0.7796	0.0498	0.0353
	S3	0.8281	0.0886	0.0777	0.7575	0.0697	0.0545	0.7796	0.0509	0.0377

TABLE 2. Point estimations of $R(t)$.

(n, m)	S	MLE			BE			EBE		
		APE	RMSE	MAB	APE	RMSE	MAB	APE	RMSE	MAB
					$(\tau_1, \tau_2) = (0.5, 0.8)$					
(40,20)	S1	0.9978	0.0475	0.0306	0.9963	0.0339	0.0224	0.9975	0.0245	0.0187
	S2	0.9977	0.0439	0.0285	0.9965	0.0328	0.0214	0.9975	0.0221	0.0168
	S3	0.9991	0.0403	0.0266	0.9974	0.0303	0.0195	0.9984	0.0193	0.0162
(40,30)	S1	0.9959	0.0357	0.0242	0.9967	0.0283	0.0194	0.9973	0.0171	0.0154
	S2	0.9956	0.0325	0.0224	0.9968	0.0248	0.0181	0.9972	0.0164	0.0146
	S3	0.9959	0.0269	0.0210	0.9971	0.0226	0.0167	0.9973	0.0163	0.0141
(80,40)	S1	0.9961	0.0246	0.0195	0.9967	0.0221	0.0164	0.9973	0.0152	0.0131
	S2	0.9961	0.0232	0.0172	0.9968	0.0210	0.0150	0.9971	0.0136	0.0122
	S3	0.9977	0.0219	0.0164	0.9986	0.0197	0.0149	0.9979	0.0131	0.0112
(80,60)	S1	0.9984	0.0190	0.0156	0.9968	0.0181	0.0145	0.9976	0.0123	0.0101
	S2	0.9979	0.0183	0.0154	0.9969	0.0163	0.0134	0.9974	0.0113	0.0094
	S3	0.9982	0.0175	0.0135	0.9973	0.0154	0.0121	0.9976	0.0103	0.0081
					$(\tau_1, \tau_2) = (1.5, 2.8)$					
(40,20)	S1	0.9978	0.0447	0.0289	0.9961	0.0311	0.0210	0.9974	0.0216	0.0172
	S2	0.9979	0.0418	0.0275	0.9963	0.0266	0.0196	0.9976	0.0185	0.0159
	S3	0.9986	0.0361	0.0245	0.9975	0.0252	0.0186	0.9980	0.0180	0.0153
(40,30)	S1	0.9955	0.0335	0.0240	0.9964	0.0244	0.0179	0.9972	0.0158	0.0150
	S2	0.9958	0.0313	0.0210	0.9965	0.0230	0.0173	0.9972	0.0149	0.0143
	S3	0.9963	0.0250	0.0195	0.9973	0.0219	0.0160	0.9974	0.0137	0.0137
(80,40)	S1	0.9960	0.0225	0.0183	0.9965	0.0211	0.0154	0.9972	0.0135	0.0128
	S2	0.9962	0.0220	0.0166	0.9967	0.0207	0.0148	0.9972	0.0124	0.0121
	S3	0.9982	0.0202	0.0157	0.9979	0.0195	0.0147	0.9982	0.0113	0.0110
(80,60)	S1	0.9982	0.0175	0.0153	0.9967	0.0174	0.0143	0.9975	0.0103	0.0100
	S2	0.9981	0.0167	0.0148	0.9967	0.0161	0.0131	0.9975	0.0089	0.0091
	S3	0.9990	0.0156	0.0132	0.9979	0.0145	0.0112	0.9984	0.0083	0.0079

VI. DATA APPLICATIONS

This part of the study examines two groups of actual data from engineering and medical areas to notice how the estimating strategies operate in genuine life.

A. ENGINEERING DATA

We shall analyze here an engineering data set describing the failure durations for fifty parts (in 1000 h) presented by Murthy et al. [30]. Henceforth, we'll refer to this type of data as Murthy's data set. Table 7 presents the new transformed time points after each item is multiplied by ten times.

First, we need to see if the IL model works well with the provided data or not. Before continuing, we calculate the MLE(standard error (Std.Er)) of ϕ and Kolmogorov-Smirnov (K-S) with its p-value. Therefore, the MLE(Std.Er) of ϕ is 2.1788(0.2491) and the K-S (p-value) is 0.1586(0.162). Therefore, it can be concluded that the IL distribution is an effective model for analyzing Murthy's data. To indicate that the MLE of ϕ exists and is unique, Figure 1(a) depicts the log-likelihood function of ϕ. It suggests that the acquired MLE $\hat{\phi} \cong 2.1788$ from the complete data exists and is also unique. We offer this estimate as a trustworthy starting point for future calculations. Figures 1(b)-1(c) display graphs of estimated reliability and density functions. These graphs reveal the same outcomes as Figure 1(a).

To evaluate the acquired theoretical findings of ϕ, $R(t)$, and $h(t)$ from Murthy's data group, by fixing $m = 25$ and different options of $\underline{S}$ and τ_i for $i = 1, 2$, three IAPT-IIC data sets are created; see Table 8. Subsequently, from Table 8, the point estimates (along with their Std.Ers) as well as the interval estimates (with their widths) developed by the maximum likelihood, Bayes, and E-Bayes approaches of ϕ, $R(t)$, and $h(t)$ at $t = 1$ are computed; see Table 9. In Table 9, for each parameter, the first row shows the obtained measure using the

TABLE 3. Point estimations of $h(t)$.

(n, m)	S	MLE			BE			EBE		
		APE	RMSE	MAB	APE	RMSE	MAB	APE	RMSE	MAB
		$(\tau_1, \tau_2) = (0.5, 0.8)$								
(40,20)	S1	0.1494	0.2261	0.1582	0.1462	0.1680	0.1195	0.1681	0.1002	0.0890
	S2	0.2187	0.2114	0.1494	0.1513	0.1443	0.1135	0.1649	0.0917	0.0826
	S3	0.2165	0.1962	0.1398	0.1703	0.1422	0.1056	0.1383	0.0891	0.0758
(40,30)	S1	0.2601	0.1774	0.1325	0.2109	0.1316	0.0990	0.1839	0.0861	0.0746
	S2	0.2619	0.1687	0.1282	0.2020	0.1288	0.0966	0.1864	0.0839	0.0708
	S3	0.2446	0.1537	0.1206	0.1902	0.1180	0.0913	0.1753	0.0796	0.0679
(80,40)	S1	0.2373	0.1396	0.1130	0.2115	0.1105	0.0904	0.1837	0.0777	0.0645
	S2	0.2432	0.1371	0.1117	0.2074	0.1076	0.0848	0.1912	0.0757	0.0612
	S3	0.0979	0.1302	0.1016	0.1560	0.1069	0.0825	0.1446	0.0755	0.0592
(80,60)	S1	0.2056	0.1097	0.0958	0.1132	0.1042	0.0806	0.1604	0.0745	0.0589
	S2	0.1991	0.1028	0.0930	0.1447	0.0959	0.0799	0.1768	0.0729	0.0585
	S3	0.1261	0.0985	0.0864	0.1805	0.0890	0.0763	0.1632	0.0706	0.0558
		$(\tau_1, \tau_2) = (1.5, 2.8)$								
(40,20)	S1	0.2258	0.2140	0.1496	0.2364	0.1315	0.1028	0.1719	0.0932	0.0832
	S2	0.1420	0.2025	0.1435	0.2278	0.1244	0.0987	0.1674	0.0907	0.0743
	S3	0.1014	0.1798	0.1316	0.1654	0.1215	0.0970	0.1165	0.0857	0.0703
(40,30)	S1	0.2450	0.1752	0.1284	0.2219	0.1205	0.0967	0.1786	0.0804	0.0693
	S2	0.2526	0.1491	0.1218	0.2185	0.1180	0.0961	0.1889	0.0792	0.0667
	S3	0.2265	0.1284	0.1092	0.1784	0.1115	0.0895	0.1820	0.0782	0.0652
(80,40)	S1	0.2416	0.1173	0.1057	0.2197	0.1093	0.0880	0.1822	0.0766	0.0645
	S2	0.2376	0.1163	0.1017	0.2130	0.1070	0.0842	0.1884	0.0761	0.0613
	S3	0.1276	0.1104	0.0974	0.1422	0.1062	0.0820	0.1270	0.0754	0.0595
(80,60)	S1	0.1248	0.1059	0.0943	0.2131	0.1013	0.0801	0.1704	0.0733	0.0591
	S2	0.1349	0.1021	0.0900	0.2098	0.0931	0.0781	0.1689	0.0702	0.0583
	S3	0.0748	0.0945	0.0823	0.1434	0.0863	0.0690	0.1181	0.0670	0.0526

TABLE 4. Interval estimations of ϕ.

(n, m)	S	ACI		BCI		EBCI	
		AIL	CP	AIL	CP	AIL	CP
		$(\tau_1, \tau_2) = (0.5, 0.8)$					
(40,20)	S1	0.397	0.950	0.342	0.956	0.239	0.961
	S2	0.420	0.947	0.364	0.953	0.247	0.960
	S3	0.433	0.946	0.380	0.951	0.255	0.958
(40,30)	S1	0.332	0.955	0.312	0.960	0.212	0.964
	S2	0.346	0.954	0.322	0.959	0.219	0.963
	S3	0.377	0.952	0.339	0.957	0.226	0.962
(80,40)	S1	0.303	0.960	0.278	0.965	0.197	0.970
	S2	0.308	0.959	0.289	0.964	0.198	0.968
	S3	0.318	0.957	0.307	0.961	0.206	0.965
(80,60)	S1	0.262	0.966	0.244	0.970	0.181	0.973
	S2	0.276	0.963	0.254	0.968	0.182	0.973
	S3	0.299	0.961	0.261	0.967	0.191	0.972
		$(\tau_1, \tau_2) = (1.5, 2.8)$					
(40,20)	S1	0.392	0.950	0.336	0.957	0.232	0.962
	S2	0.417	0.949	0.358	0.955	0.243	0.961
	S3	0.429	0.948	0.375	0.953	0.250	0.960
(40,30)	S1	0.322	0.956	0.311	0.961	0.210	0.965
	S2	0.336	0.955	0.318	0.960	0.215	0.964
	S3	0.375	0.952	0.328	0.958	0.222	0.963
(80,40)	S1	0.293	0.961	0.266	0.966	0.186	0.971
	S2	0.300	0.960	0.275	0.965	0.198	0.970
	S3	0.310	0.958	0.303	0.962	0.201	0.966
(80,60)	S1	0.256	0.967	0.240	0.971	0.172	0.974
	S2	0.265	0.964	0.245	0.970	0.179	0.973
	S3	0.286	0.962	0.254	0.967	0.184	0.972

TABLE 5. Interval estimations of $R(t)$.

(n, m)	S	ACI		BCI		EBCI	
		AIL	CP	AIL	CP	AIL	CP
		$(\tau_1, \tau_2) = (0.5, 0.8)$					
(40,20)	S1	0.160	0.952	0.122	0.957	0.086	0.962
	S2	0.152	0.953	0.116	0.959	0.080	0.963
	S3	0.141	0.955	0.103	0.961	0.072	0.965
(40,30)	S1	0.129	0.957	0.095	0.962	0.065	0.967
	S2	0.113	0.960	0.089	0.963	0.060	0.968
	S3	0.085	0.962	0.072	0.965	0.054	0.970
(80,40)	S1	0.079	0.963	0.068	0.967	0.052	0.971
	S2	0.076	0.964	0.059	0.969	0.050	0.972
	S3	0.074	0.964	0.049	0.970	0.048	0.973
(80,60)	S1	0.071	0.965	0.045	0.971	0.043	0.974
	S2	0.062	0.967	0.042	0.972	0.038	0.975
	S3	0.057	0.969	0.034	0.975	0.027	0.977
		$(\tau_1, \tau_2) = (1.5, 2.8)$					
(40,20)	S1	0.154	0.953	0.118	0.959	0.076	0.964
	S2	0.144	0.955	0.109	0.960	0.072	0.965
	S3	0.135	0.957	0.099	0.962	0.069	0.966
(40,30)	S1	0.124	0.959	0.093	0.963	0.065	0.968
	S2	0.103	0.961	0.085	0.964	0.058	0.969
	S3	0.081	0.963	0.072	0.965	0.052	0.971
(80,40)	S1	0.076	0.965	0.062	0.969	0.050	0.972
	S2	0.074	0.965	0.054	0.970	0.049	0.973
	S3	0.072	0.966	0.048	0.971	0.045	0.974
(80,60)	S1	0.069	0.967	0.040	0.973	0.037	0.975
	S2	0.057	0.969	0.037	0.975	0.032	0.976
	S3	0.047	0.971	0.031	0.9760	0.023	0.978

TABLE 6. Interval estimations of $h(t)$.

(n, m)	S	ACI		BCI		EBCI	
		AIL	CP	AIL	CP	AIL	CP
		$(\tau_1, \tau_2) = (0.5, 0.8)$					
(40,20)	S1	0.777	0.852	0.634	0.859	0.326	0.886
	S2	0.747	0.857	0.558	0.864	0.316	0.888
	S3	0.699	0.862	0.525	0.869	0.304	0.891
(40,30)	S1	0.649	0.867	0.507	0.873	0.299	0.892
	S2	0.541	0.872	0.478	0.876	0.293	0.893
	S3	0.461	0.877	0.421	0.880	0.288	0.895
(80,40)	S1	0.428	0.883	0.388	0.886	0.284	0.896
	S2	0.419	0.886	0.375	0.889	0.281	0.897
	S3	0.408	0.888	0.313	0.893	0.277	0.900
(80,60)	S1	0.380	0.892	0.264	0.899	0.246	0.906
	S2	0.350	0.896	0.244	0.904	0.224	0.909
	S3	0.322	0.901	0.221	0.908	0.179	0.912
		$(\tau_1, \tau_2) = (1.5, 2.8)$					
(40,20)	S1	0.751	0.855	0.617	0.862	0.315	0.888
	S2	0.711	0.860	0.542	0.865	0.309	0.891
	S3	0.646	0.865	0.513	0.871	0.300	0.893
(40,30)	S1	0.621	0.869	0.482	0.875	0.298	0.894
	S2	0.532	0.874	0.449	0.878	0.291	0.895
	S3	0.456	0.880	0.399	0.884	0.285	0.896
(80,40)	S1	0.423	0.884	0.352	0.890	0.280	0.899
	S2	0.414	0.887	0.297	0.898	0.278	0.900
	S3	0.401	0.889	0.271	0.900	0.259	0.904
(80,60)	S1	0.369	0.896	0.254	0.903	0.236	0.909
	S2	0.328	0.899	0.225	0.908	0.191	0.911
	S3	0.310	0.903	0.194	0.911	0.155	0.915

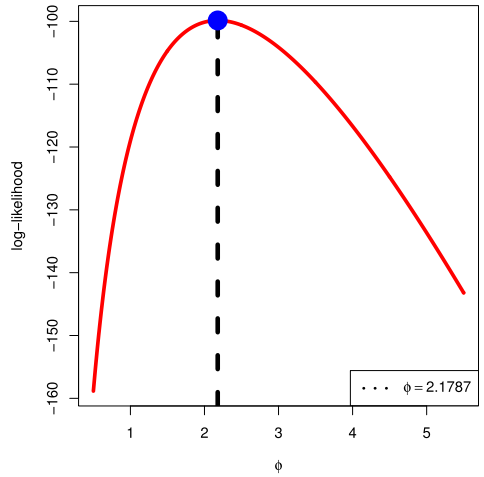
MLE. The second row is based on the BE, while the third row is based on the EBE.

All Bayes and E-Bayes evaluations are done by setting $\mathbb{H} = 40,000$ and $\mathbb{B} = 10,000$. To develop the E-Bayes inferences, without loss of generality, we assumed that $\kappa = \nu = c = 1$. The results in Table 9 illustrate that the EBEs and EBCIs outperform all other estimates in terms of lowest Std.Ers and Widths.

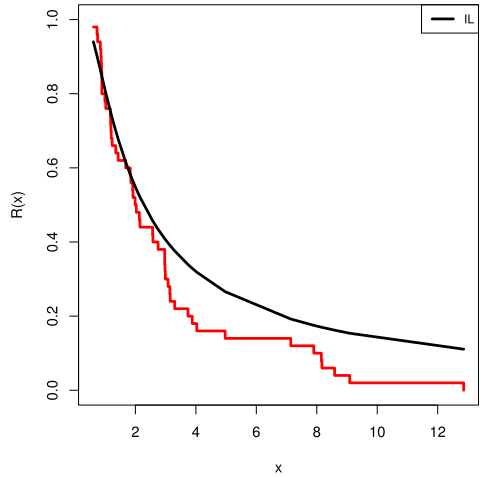
Figure 2 illustrates the log-likelihood curve for all IAPT-IIC samples from Murthy's data for varied choices of ϕ, demonstrating that the newly obtained $\hat{\phi}$ exists and is unique. It supports the same conclusions provided in Table 9 and advises using the value of $\hat{\phi}$ in every sample as a starting guess for the needed Bayes' and E-Bayes' assessments. Based on sample A (as an example), Figures 3 and 4 show density and trace plots of ϕ, RF, and HRF to show their convergence state. The solid and dashed lines represent the sample mean and 95% credible boundaries, respectively. Figures 3 and 4 demonstrate that the collected Markovian variates of all parameters converge adequately. It likewise indicates that the Bayes iterations of ϕ are substantially symmetrical, while those for RF and HRF are negatively and positively skewed, respectively. The similar conclusion is also drawn in the case of E-Bayes iterations. The plots established in instances B and C are offered as supplemental materials.

TABLE 7. Time points of 50 components.

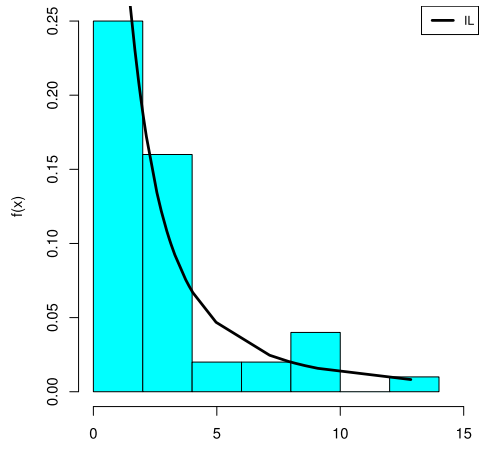
0.61	0.73	0.75	0.84	0.86	0.87	0.88	0.89	0.89	0.89
0.99	1.02	1.17	1.18	1.19	1.20	1.23	1.35	1.43	1.68
1.83	1.85	1.91	1.92	1.99	2.03	2.13	2.15	2.57	2.58
2.75	2.97	2.97	2.98	2.99	3.08	3.14	3.15	3.30	3.74
3.88	4.03	4.97	7.14	7.90	8.15	8.17	8.59	9.09	12.86



(a) log-likelihood



(b) reliability



(c) density

FIGURE 1. Diagrams for fit of the IL model from Murthy's data.

B. MEDICAL DATA

In this part, from the medical domain, we analyze a realistic data set consisting of 45 annually survival spans of a set of patients who acquired only chemotherapy therapy; see Bekker et al. [31]. In Table 10, each number in the patients' data is scaled up by hundred to facilitate handling. Before delving into the developed estimations, one question is whether the IL lifetime model coincides with the provided data or not. From Table 10, the MLE(Std.Er) of ϕ is 42.005(6.1277), while the K-S(p-value) is 0.1586(0.186). This means that the IL model suits the patient's data group satisfactorily. Figure 5(a) exhibits that the acquired MLE $\hat{\phi} \cong 42.005$ from the full patient data exists and is also unique. To perform the next calculations, we use this estimate as a good starting value. Figures 5(b)-5(c), which display estimated reliability and density lines, support the fitting result.

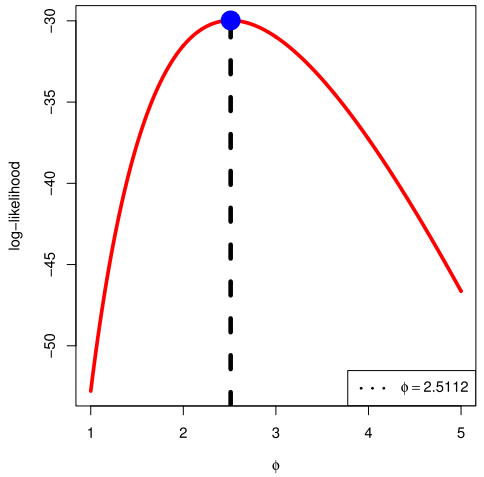
Just like the same calculation scenarios presented in Subsection VI-A, from Table 10, different IAPT-IIC samples with specified $m = 20$ and different alternatives of τ_i, $i = 1, 2$, and S_i, $i = 1, \ldots, m$ are created; see Table 11. In Table 12, the results of ϕ, $R(t)$, and $h(t)$ (at $t = 20$)

TABLE 8. Three IAPT-IIC samples from Murthy's data.

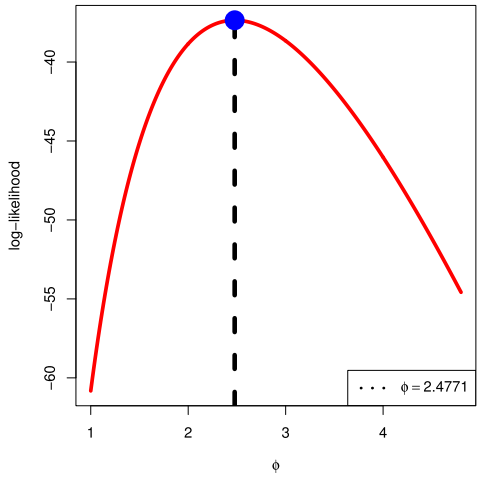
Sample	$\underline{\mathbf{S}}$	τ_1	τ_2	τ	(M_1, M_2)	S^*
D1[A]	$(5^5, 0^{20})$	0.8	1.7	1.7	(2,15)	25
Observed data		0.61, 0.75, 0.84, 0.87, 0.88, 0.89, 0.89, 0.99, 1.17, 1.18, 1.20, 1.23, 1.35, 1.43, 1.68				
D1[B]	$(0^{10}, 5^5, 0^{10})$	1.2	2.1	2.1	(13,18)	17
Observed data		0.61, 0.73, 0.75, 0.84, 0.86, 0.87, 0.88, 0.89, 0.89, 0.89, 0.99, 1.17, 1.91, 1.23, 1.43, 1.68, 1.85, 1.91				
D1[C]	$(0^{20}, 5^5)$	2.6	3.1	3.1	(24,25)	5
Observed data		0.61, 0.73, 0.75, 0.84, 0.86, 0.87, 0.88, 0.89, 0.89, 0.89, 0.99, 1.02, 1.17, 1.18, 1.19, 1.20, 1.23, 1.35, 1.43, 1.68, 1.83, 1.92, 2.15, 2.58, 2.97				

TABLE 9. Estimates of ϕ, $R(t)$, and $h(t)$ from Murthy's data.

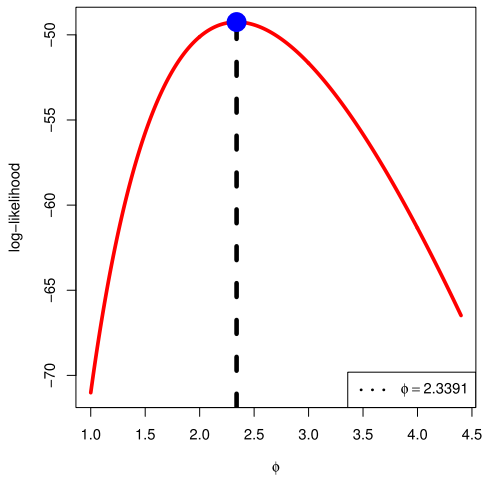
Sample	Par.	MLE BE EBE		ACI BCI EBCI		
		Est.	Std.Er	Lower	Upper	Width
A	ϕ	2.5113	0.3185	1.8871	3.1355	1.2484
		2.5236	0.2193	2.1017	2.9616	0.8599
		2.5227	0.2182	2.1050	2.9588	0.8538
	$R(1)$	0.8608	0.0422	0.7780	0.9436	0.1656
		0.8595	0.0293	0.7949	0.9096	0.1147
		0.8594	0.0292	0.7955	0.9093	0.1138
	$h(1)$	0.3387	0.0693	0.2029	0.4745	0.2717
		0.3388	0.0475	0.2518	0.4380	0.1861
		0.3390	0.0473	0.2523	0.4371	0.1848
B	ϕ	2.4770	0.2997	1.8897	3.0644	1.1748
		2.4878	0.2146	2.0786	2.9200	0.8414
		2.4891	0.2130	2.0745	2.9163	0.8418
	$R(1)$	0.8562	0.0410	0.7758	0.9366	0.1608
		0.8548	0.0296	0.7904	0.9059	0.1155
		0.8550	0.0294	0.7896	0.9056	0.1159
	$h(1)$	0.3462	0.0664	0.2161	0.4764	0.2603
		0.3465	0.0473	0.2590	0.4443	0.1853
		0.3462	0.0471	0.2596	0.4454	0.1857
C	ϕ	2.3390	0.2732	1.8035	2.8746	1.0711
		2.3494	0.2041	1.9587	2.7586	0.7999
		2.3514	0.2034	1.9578	2.7605	0.8027
	$R(1)$	0.8360	0.0424	0.7529	0.9192	0.1664
		0.8347	0.0319	0.7656	0.8901	0.1245
		0.8350	0.0318	0.7654	0.8903	0.1249
	$h(1)$	0.3779	0.0651	0.2504	0.5055	0.2551
		0.3780	0.0484	0.2883	0.4777	0.1894
		0.3775	0.0482	0.2879	0.4780	0.1901



(a) Sample A



(b) Sample B



(c) Sample C

FIGURE 2. The log-likelihoods of ϕ from Murthy's data.

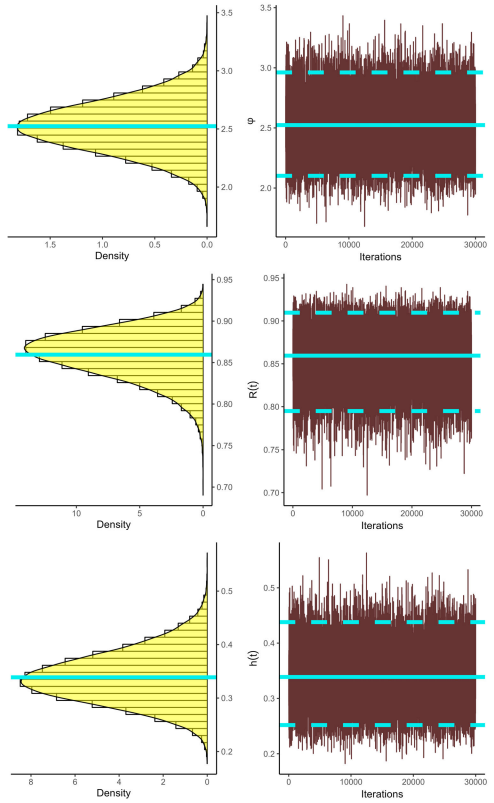


FIGURE 3. The density (left) and trace (right) plots of ϕ, $R(t)$, and $h(t)$ using BE method from Murthy's data.

TABLE 10. Survival times for 45 patients.

4.7	11.5	12.1	13.2	16.4	19.7	20.3	26.0
28.2	29.6	33.4	39.5	45.8	46.6	50.1	50.7
52.9	53.4	54.0	64.1	64.4	69.6	84.1	86.3
109.9	121.9	127.1	132.6	144.7	148.5	155.3	158.1
158.9	217.8	234.3	241.6	244.4	282.5	283.0	357.8
365.8	374.3	397.8	400.3	403.30			

TABLE 11. Three IAPT-IIC data sets from patient data.

Sample	$\underline{\mathbf{S}}$	τ_1	τ_2	τ	(M_1, M_2)	S^*
D2[A]	$(5^5, 0^{15})$	5	50	50	(1,10)	30
Observed data		4.700, 11.50,13.20, 16.40, 19.70, 26.00, 29.60, 33.40, 45.80, 46.60				
D2[B]	$(0^7, 5^5, 0^8)$	30	55	55	(9,15)	20
Observed data		4.700, 11.50, 12.10, 13.20, 16.40, 19.70, 20.30, 26.00, 29.60, 33.40, 45.80, 46.60, 50.10, 52.90, 53.40				
D2[C]	$(0^{15}, 5^5)$	70	90	90	(18,20)	10
Observed data		4.700, 11.50, 12.10, 13.20, 16.40, 19.70, 20.30, 26.00, 28.20, 29.60, 33.40, 39.50, 45.80, 46.60, 50.10, 50.70, 54.00, 64.10, 84.10, 86.30				

are presented. The results in Table 12 demonstrate that the EBEs and EBCIs for the various parameters function better than all others. To demonstrate the existence and uniqueness properties of $\hat{\phi}$, Figure 6 displays the log-likelihood curves of ϕ. It indicates that the acquired estimates of $\hat{\mu}$ based on all samples exist and are unique.

To see the convergence status of the acquired Bayes and E-Bayes estimates, employing sample A (for instance),

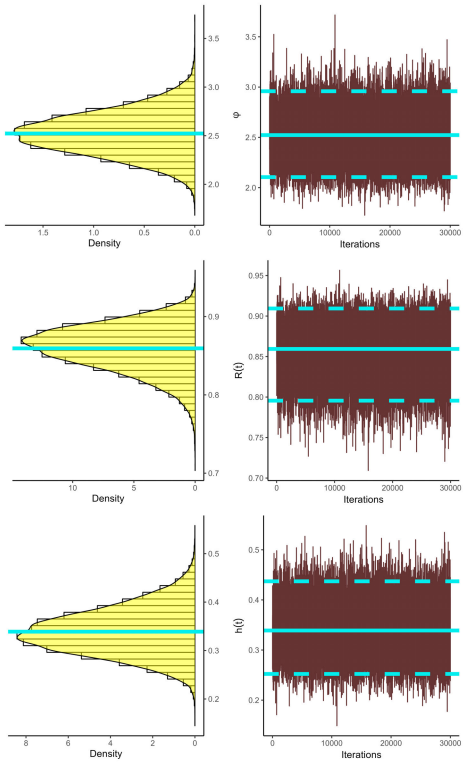


FIGURE 4. The density (left) and trace (right) plots of ϕ, $R(t)$, and $h(t)$ using EBE method from Murthy's data.

TABLE 12. Estimates of ϕ, $R(t)$, and $h(t)$ from patient data.

Sample	Par.	MLE BE EBE		ACI BCI EBCI		
		Est.	Std.Er	Lower	Upper	Width
A	ϕ	45.989	5.9316	34.363	57.614	23.252
		45.987	0.3017	45.396	46.580	1.1842
		45.989	0.2997	45.401	46.575	1.1744
	$R(20)$	0.8948	0.0312	0.8336	0.9559	0.1223
		0.8948	0.0016	0.8916	0.8978	0.0062
		0.8948	0.0016	0.8916	0.8978	0.0062
	$h(20)$	0.0132	0.0026	0.0081	0.0184	0.0104
		0.0132	0.0001	0.0130	0.0135	0.0005
		0.0132	0.0001	0.0130	0.0135	0.0005
B	ϕ	41.119	8.3886	24.678	57.561	32.883
		41.118	0.3016	40.527	41.711	1.1837
		41.119	0.3494	40.437	41.806	1.3696
	$R(20)$	0.8658	0.0563	0.7555	0.9761	0.2205
		0.8658	0.0020	0.8618	0.8697	0.0079
		0.8658	0.0023	0.8611	0.8703	0.0092
	$h(20)$	0.0156	0.0043	0.0072	0.0240	0.0168
		0.0156	0.0002	0.0153	0.0159	0.0006
		0.0156	0.0002	0.0152	0.0159	0.0007
C	ϕ	42.164	5.9316	30.539	53.790	23.252
		42.163	0.3016	41.572	42.756	1.1838
		42.164	0.3491	41.480	42.848	1.3677
	$R(20)$	0.8726	0.0378	0.7986	0.9466	0.1480
		0.8726	0.0019	0.8688	0.8763	0.0075
		0.8726	0.0022	0.8682	0.8769	0.0087
	$h(20)$	0.0150	0.0029	0.0093	0.0208	0.0116
		0.0150	0.0001	0.0148	0.0153	0.0006
		0.0150	0.0002	0.0147	0.0154	0.0007

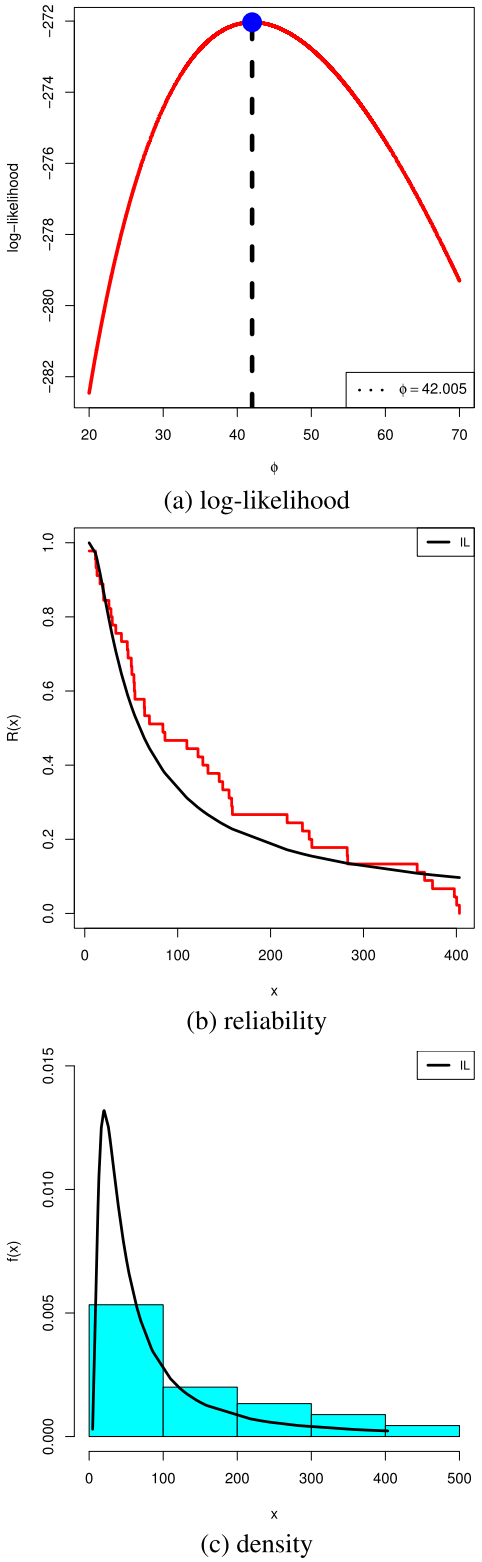
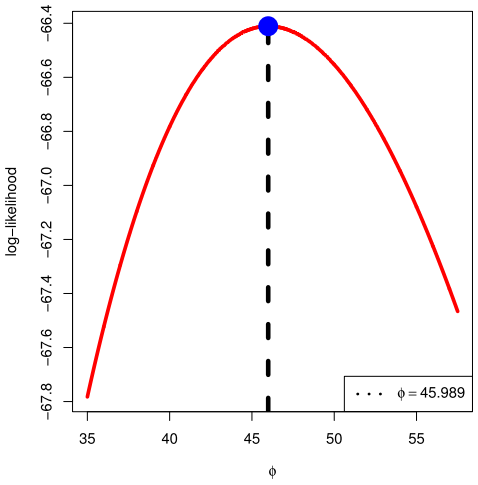
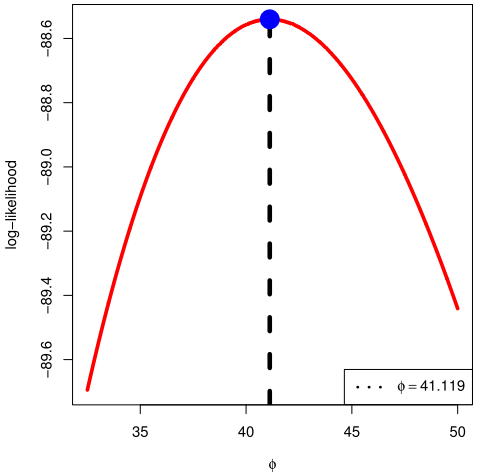


FIGURE 5. Diagrams for fit of the IL model from patient data.

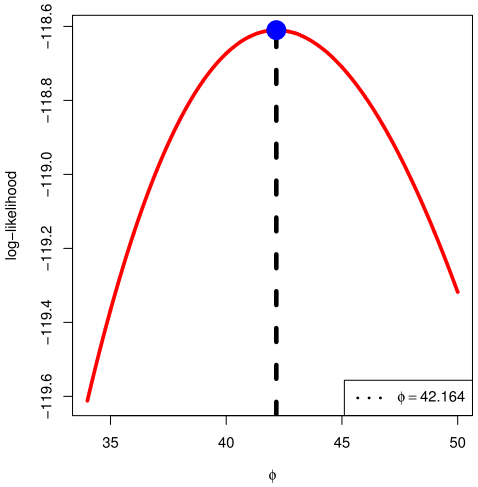
Figures 7 and 8 display both the density and trace plots of ϕ, RF and HRF. It shows that the simulated Bayes and E-Bayes variates converge well. It also shows that all Bayes and E-Bayes findings of ϕ, $R(t)$, and $h(t)$ are symmetric, as well as supporting the same characteristics listed in Table 12. Additionally, in the supplemental file, the Bayes and E-Bayes



(a) Sample A



(b) Sample B



(c) Sample C

FIGURE 6. The log-likelihoods of ϕ from patient data.

plots of ϕ, RF and HRF based utilizing samples B and C are given.

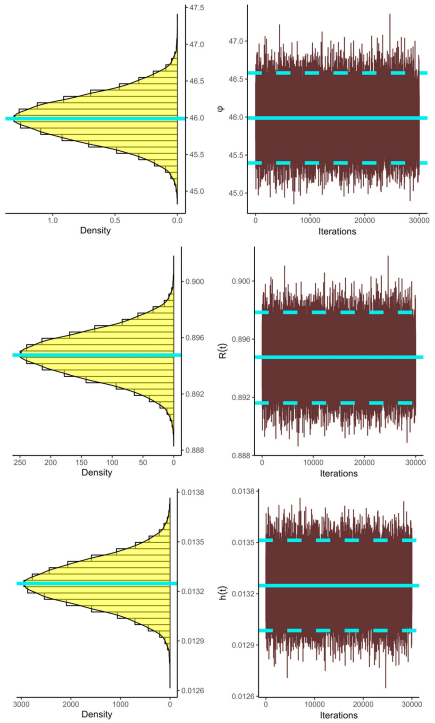


FIGURE 7. The density (left) and trace (right) plots of ϕ, $R(t)$, and $h(t)$ using BE method from patient data.

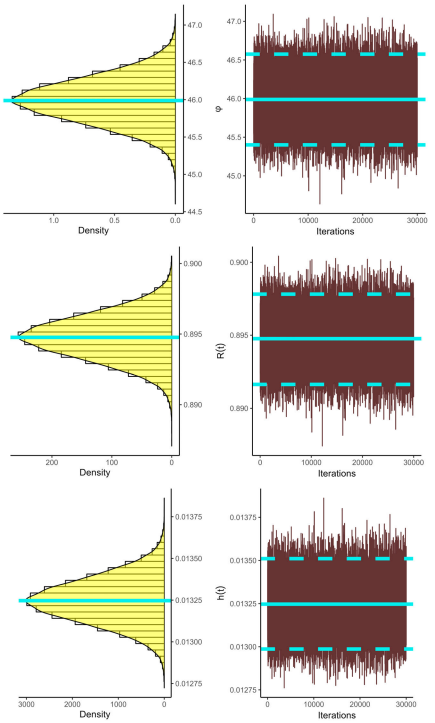


FIGURE 8. The density (left) and trace (right) plots of ϕ, $R(t)$, and $h(t)$ using EBE method from patient data.

VII. CONCLUDING REMARKS

Performing a novel censoring plan, namely an improved adaptive Type-II progressively censoring, this paper compared various estimations using the maximum likelihood,

Bayesian, and E-Bayesian estimation approaches for the inverted Lindley model. The asymptotic confidence and credible ranges using both Bayesian and E-Bayesian approaches for each unknown parameter are also obtained. Utilizing the squared error loss under the presumption of gamma prior, all point and interval estimations developed using both Bayesian setups are computed via the MCMC approximation procedure. Several Monte Carlo simulations are conducted based on four accuracy metrics to notice the efficiency of the various point and interval approaches. Two real-life applicable examples using data groups from the engineering and medical domains are discussed to confirm the relevancy of the proposed setups to real-life statuses. The numerical investigation yields indicated that the E-Bayes paradigm supplies acceptable point estimates and adequate credible intervals compared to its competitive paradigms. As a result, the analysis results using the proposed censoring scheme from the engineering and medical data sets provide a good justification for the inverted Lindley model, support the simulation conclusions, and demonstrate the applicability of the presented strategies in various practical situations.

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CONFLICT OF INTEREST

The authors declare no conflict of interest.

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