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RESEARCH ARTICLE

A Novel T-G IFE Method for Two Dimensional Semi-Linear Elliptic Interface Problems Based on Coarse Grid Correction

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ABSTRACT In this paper, a novel Two-Grid (T-G) algorithm is proposed and analyzed for semi-linear interface problems in two dimension. To linearize the Immersed Finite Element Method (IFEM) equations, a T-G method based on some Newton iteration approach and correction method is investigated. It is shown that the algorithm can achieve asymptotically optimal approximation as long as the mesh sizes satisfy $H = \mathcal{O}(h^{1/3})$ in L^p norm (for H^1 norm, it even suffices to take $H = \mathcal{O}(h^{1/5})$). As a result, solving such a large class of nonlinear equation will not be much more difficult than solving one linearized equation.

INDEX TERMS Interface problem, coarse gird correction, immersed finite element method, nonlinear problem, Newton iteration.

I. INTRODUCTION

Let Ω be a convex polygonal domain in $\mathcal{R}^2$ and $\Omega_1 \subset \Omega$ be an open domain with C^2 boundary $\Gamma = \partial\Omega_1 \subset \Omega$. Let $\Omega_2 = \Omega \backslash \Omega_1$. We consider the following semi-linear elliptic interface problem:

$$-\nabla \cdot (\beta \nabla u) = f(x, u), \quad x \in \Omega. \tag{1}$$

The system is subjected to the boundary condition:

$$u = 0, \quad \text{on } \partial\Omega \tag{2}$$

and the homogeneous jump conditions on the interface,

$$[u] = 0, \quad \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] = 0, \quad \text{across } \Gamma, \tag{3}$$

where $[v]$ is defined as the jump of v across the interface Γ by $[v](x) = v_1(x) - v_2(x)$, $x \in \Gamma$, with $v_i = v|_{\Omega_i}$ the restrictions of v to Ω_i, $i = 1, 2$, and $\mathbf{n}$ the unit outward normal to the boundary $\partial\Omega_1$. For ease of exposition, we assume that the coefficient function β is positive and piecewise constant, i.e.,

$$\beta(x) = \beta_1 \ \text{ for } \ x \in \Omega_1; \quad \beta(x) = \beta_2 \ \text{ for } \ x \in \Omega_2. \tag{4}$$

The semi-linear interface problem (1)-(3) occurs frequently as the involved partial differential equations used to simulate many basic physical phenomenons, which have applications in many physical and engineering problems, such as fluid dynamics [1], [2], [3], seismo-acoustics [4], and electromagnetics [5].

In this paper, we focus on speeding up the iterations by using two-grid approaches [6], [7], [8]. There are lots of literatures concerning about the interface problems by different treatments, but, as far as we know, there are few results about two-grid methods for semi-linear interface problems by Finite Element Methods (FEMs) [9], [10] (or Immersed Finite Element Methods (IFEMs) [11], [12]). In [12], we present two efficient two-grid (T-G) algorithms. It is of great theoretical interest that a further coarse grid correction after the fine grid correction can actually improve the accuracy. That is, we first solve a nonlinear problem by

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applying the Newton-like iteration on a (cheap) coarse grid and then solve a linear elliptic system on a (expensive) fine grid and solving one more linear equation on the coarse space. It is shown that the algorithm can achieve asymptotically optimal approximation as long as the mesh sizes satisfy $H = \mathcal{O}(h^{1/3})$ (for H^1 norm, it even suffices to take $H = \mathcal{O}(h^{1/5})$).

The remainder of the article is organized as follows: Section II, we introduce the weak form and the IFE approximation. Section III presents our novel two-grid algorithm and gives its error estimates. Numerical tests are presented in Section IV.

II. WEAK FORM AND IFEM

Let $L^2(\Omega)$ be the set of square-integrable functions on Ω with usual norm $\|\cdot\|$. Furthermore, let $(\cdot,\cdot)$ denote the L^2 inner product, scalar and vector, and $\langle\cdot,\cdot\rangle_{\partial\Omega}$ present the $L^2(\partial\Omega)$ inner product with norm $\|\cdot\|_{\partial\Omega}$.

We shall also use the standard Sobolev space $W^{m,p}(\Omega)$ with a norm $\|\cdot\|_{m,p}$ given by $\|\phi\|_{m,p}^p = \sum_{|\alpha|\le m}\|D^\alpha\phi\|_{L^p(\Omega)}^p (1 \le p < \infty)$ and $\|\phi\|_{m,\infty} = \max_{|\alpha|\le m} ess\,\sup_{x\in\Omega}|\partial^\alpha u|$, where the multi index $\alpha = (\alpha_1, \alpha_2)$ and $|\alpha| = \alpha_1 + \alpha_2$. For $p = 2$, we define $H^m(\Omega) = W^{m,2}(\Omega)$, $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$ and $\|\cdot\|_{0,\infty} = \|\cdot\|_{L^\infty}$. We will also write $f(x,\xi) := f(\xi)$ and $\partial f(x,\xi)/\partial\xi := f'(\xi)$ for simplicity.

For the analysis, we introduce the following space, for $r \ge 1$, $1 \le p < \infty$,

$$\widetilde{W}^{r,p}(\Omega) := \{u \in W^{1,p}(\Omega) : u \in W^{r,p}(\Omega_s),\ s = 1, 2\}$$

equipped with the norm

$$\|u\|_{\widetilde{W}^{r,p}(\Omega)}^p := \|u\|_{W^{r,p}(\Omega_1)}^p + \|u\|_{W^{r,p}(\Omega_2)}^p,\ \forall u \in \widetilde{W}^{r,p}(\Omega).$$

For $p = 2$, we denote $\widetilde{H}^r(\Omega) := \widetilde{W}^{r,2}(\Omega)$, equipped with the norm

$$\|u\|_{\widetilde{H}^r(\Omega)}^2 := \|u\|_{H^r(\Omega_1)}^2 + \|u\|_{H^r(\Omega_2)}^2,\ \forall u \in \widetilde{H}^r(\Omega).$$

The weak form for the semi-linear interface problem (1) - (3) reads: find $u \in H_h$ satisfies,

$$a_h(u, v) = (f(u), v), \quad \forall v \in V_h, \tag{5}$$

where

$$a_h(u, v) = \sum_{T\in\mathcal{T}_h}\left(\int_T \beta\nabla u\cdot\nabla v dx - \int_{\partial T}\beta\nabla u\cdot\mathbf{n} v ds\right), \quad \forall u, v \in H_h(\Omega), \tag{6}$$

where $H_h(\Omega) := \{v|\ v|_T \in H^1(T),\ \forall T \in \mathcal{T}_h\}$ and $H_h(\Omega)$ is equiped with the broken H^1 semi-norm, which is $\|v\|_{1,h} := \left(\sum_{T\in\mathcal{T}_h}|\sqrt{\beta}\nabla v|_{L^2(T)}^2\right)^{1/2}$.

Next, we construct an immersed finite element space $V_h(\Omega) \subset H_h(\Omega)$. First, on each element $K \in \mathcal{T}_h$, we let

$$V_h(K) = span\{\phi_j(x),\quad 1 \le j \le 3,\quad x \in K\},$$

where ϕ_j, $1 \le j \le 3$ are standard linear nodal basis functions if K is a non-interface element; otherwise, if K is an interface element, ϕ_j, $1 \le j \le 3$ are piecewise linear basis functions discussed in [13] and [14]. Then, we can define the immersed finite element space over the whole solution domain Ω as follows:

$$V_h(\Omega) = \{v|v \text{ satisfies conditions (I) - (III) given below}\}$$

(I) $v|_K \in V_h(K)$, $\forall K \in \mathcal{T}_h$;
(II) v is continuous at every mesh point $X \in \mathcal{N}_h$;
(III) $v|_{\partial\Omega} = 0$.

Now, we propose the IFE approximation: find $u_h \in V_h(\Omega)$ satisfies the following equation,

$$a_h(u_h, \phi) = (f(u_h), \phi), \forall\phi \in V_h. \tag{7}$$

Throughout this paper, we assume the nonlinear term $f(x, u)$ has second order derivative with respect to its second argument u. Here, we don't need the nonlinear term such the condition

$$|f'(u)| \le C|u| \quad \text{and} \quad |f''(u)| \le C, \forall u \in \mathcal{R}. \tag{8}$$

We only need the following weaker assumptions on the nonlinear term $f(x, u)$.

Assumption 2.1: Assume $f(x, u) : \Omega \times \mathcal{R} \to \mathcal{R}$ is a Carathéodory function, which satisfies the barrier-sign conditions in its second argument: there exist constants $\alpha, \beta \in \mathcal{R}$ with $\alpha \le \beta$, such that

$$\begin{cases} f(x,u) \ge 0, & \text{if } u \ge \beta,\\ f(x,u) \le 0, & \text{if } u \le \alpha, \end{cases} \quad \text{a.e. in } \Omega.$$

This assumption guarantees the L^∞ boundedness of the weak solution u and the numerical solution u_h solved by (5) and (7), respectively. That is to say there exists some $u_1, u_2 \in \mathcal{R}$, $u_1 < u_2$, we have $u_1 \le u, u_h \le u_2$. In order to give nonlinearity some local Lipschitz property, we introduce the following additional hypothetical conditions.

Assumption 2.2: Assume $f(u)$ is locally monotone, namely, $f'(u) \ge 0$, for any $u \in [u_1, u_2]$.

Assumption 2.1 and Assumption 2.2 ensure the boundness of $f'(u)$ and $f''(u)$, for any u between u_1 and u_2. Then, the error estimates of IFE solution derive by (7) can be established [12].

Lemma 2.1: Let $u \in H_0^1(\Omega) \cap \widetilde{W}^{3,p}(\Omega)$ $(2 \le p < \infty)$ and $u_h \in V_h$ be the solution to problem (1)-(3) and the IFE equations (7), respectively. Then, for some positive constant C,

$$\|u - u_h\|_{1,h} \le Ch\|u\|_{\widetilde{H}^3(\Omega)}, \tag{9}$$

and

$$\|u - u_h\|_{L^p(\Omega)} \le Ch^2\|u\|_{\widetilde{W}^{3,p}(\Omega)}. \tag{10}$$

III. T-G ALGORITHM AND ERROR ESTIMATES

In this section, we will present the novel Coarse grid Correction Two-Grid (CC T-G) method and error analysis. The fundamental ingredient in this scheme is another immersed FE space $V_H(H \gg h)$ defined on a coarser quasi-uniform triangulation of Ω. Setting $A_H(v,\phi) = a_h(v,\phi) - (f'(u_H)v,\phi)$. Then, we present the CC T-G method which has three steps as follows.

Algorithm 3.1: Step 1: On the coarse grid $\mathcal{T}_H$, compute $u_H \in V_H$ satisfing the following nonlinear system,

$$a_H(u_H, v_H) = (f(u_H), v_H), \quad \forall v_H \in V_H. \tag{11}$$

Step 2: On the fine grid $\mathcal{T}_h$, compute $U_h \in V_h$ to satisfy the following linear system:

$$A_H(U_h, v_h) = (f(u_H) - f'(u_H)u_H, v_h), \quad \forall v_h \in V_h. \tag{12}$$

Step 3: On the coarse grid $\mathcal{T}_H$, solve the following linear system for $e_H \in V_H$:

$$A_H(e_H, v_H) = \frac{1}{2}\left(f''(u_H)(U_h - u_H)^2, v_H\right), \quad \forall v_H \in V_H. \tag{13}$$

Set $u^h = U_h + e_H$.

The new feature of the above Algorithm 3.1 mainly lies in step 3 where a further coarse grid correction is performed. Corresponding to the form $A_H(\cdot,\cdot)$, we define a projection: $H_h(\Omega) \mapsto V_H$ by

$$A_H(\phi, Q_H v) = A_H(\phi, v), \quad \forall \phi \in V_H, \ v \in H_h(\Omega).$$

By the interpolation properties of IFE functions [13], [14] and the coercivity of $a_h(\cdot,\cdot)$ [12], it can be easily shown that there exists $H_0 > 0$, if $H \le H_0$, Q_H is well defined and satisfies

$$\|w - Q_H w\| + H\|w - Q_H w\|_{1,h} \le C(H_0)H\|w\|_{1,h}, \ \forall w \in H_h. \tag{14}$$

Lemma 3.1: For any $\chi \in V_h$,

$$A_H(u^h, \chi) = \left(f(u_H) - f'(u_H)u_H + \frac{1}{2}f''(u_H)(U_h - u_H)^2, \chi\right) + \frac{1}{2}\left(f''(u_H)(U_h - u_H)^2, Q_H\chi - \chi\right). \tag{15}$$

Proof: By the definition of Q_H and e_H,

$$A_H(e_H, \chi) = A_H(e_H, Q_H\chi) = \frac{1}{2}\left(f''(u_H)(U_h - u_H)^2, Q_H\chi\right).$$

Summarize (12) and (13), (15) can be easily derived.

Then, we introduce the theoretical result for two-step T-G solution, which has been obtained (see reference [12], Lemma 5.1).

Lemma 3.2: Let $u_h \in V_h$ be the solution to (7) on $\mathcal{T}_h$ and $U_h \in V_h$ be the approximated solution obtained by (12). Then we have the following estimate

$$\|u_h - U_h\|_{1,h} \le CH^4\|u\|_{\widetilde{W}^{3,4}(\Omega)}, \tag{16}$$

for some positive constant C.

The estimate in Lemma 3.2 is already quite remarkable because of the high power on the coarse mesh size H. But more remarkable estimates will be seen in the next lemma.

Lemma 3.3: Let $u_h \in V_h$ be the solutions of (7) on $\mathcal{T}_h$, and $u^h \in V_h$ the approximated solution obtained by Algorithm 3.1. Then we have the following estimate

$$\|u_h - u^h\|_{1,h} \le CH^5\|u\|_{\widetilde{W}^{3,4}(\Omega)}, \tag{17}$$

for some positive constant C.

Proof: By the definition of u_h and the Taylor expansion, we have

$$a_h(u_h, \zeta) - (f'(u_H)u_h, \zeta) = \left(f(u_H) - f'(u_H)u_H + \frac{1}{2}f''(u_H)(u_h - u_H)^2, \zeta\right) + \left(O(u_h - u_H)^3, \zeta\right),$$

which, together with (15) gives that for any $\zeta \in V_h$

$$A_H(u_h - u^h, \zeta) = \frac{1}{2}\left(f''(u_H)((u_h - u_H)^2 - (U_h - u_H)^2), \zeta\right) + \frac{1}{2}\left(f''(u_H)(U_h - u_H)^2, \zeta - Q_H\zeta\right) + \left(O(u_h - u_H)^3, \zeta\right). \tag{18}$$

By the Hölder inequality and the well known Sobolev inequality,

$$\begin{aligned}&\left((u_h - u_H)^2 - (U_h - u_H)^2, \zeta\right)\\ &\le \|(u_h - U_h)(u_h - u_H + U_h - u_H)\|_{0,\frac{6}{5}}\|\zeta\|_{0,6}\\ &\le \|u_h - U_h\|_{0,3}\|u_h - u_H + U_h - u_H\|_{0,2}\|\zeta\|_{0,6}\\ &\le \|u_h - U_h\|_{1,h}(\|u_h - u_H\| + \|U_h - u_H\|)\|\zeta\|_{0,6}.\end{aligned} \tag{19}$$

It follows from the convergence result (10) and Lemma 3.2,

$$\left((u_h - u_H)^2 - (U_h - u_H)^2, \zeta\right) \le CH^6\|u\|_{\widetilde{W}^{3,4}(\Omega)}\|u\|_{\widetilde{W}^{3,2}(\Omega)}\|\zeta\|_{1,h}. \tag{20}$$

By the Schwarz inequality, (14) and Lemma 3.2,

$$\begin{aligned}&\frac{1}{2}\left(f''(u_H)(U_h - u_H)^2, \zeta - Q_H\zeta\right)\\ &\le C\|U_h - u_H\|_{0,4}^2\|\zeta - Q_H\zeta\|\\ &\le CH^5\|u\|_{\widetilde{W}^{3,4}(\Omega)}^2\|\zeta\|_{1,h},\end{aligned} \tag{21}$$

where we have used (14) in the last step.

By Hölder inequality

$$\begin{aligned}\left(O(u_h - u_H)^3, \zeta\right) &\le \|(u_h - u_H)^3\|_{0,\frac{4}{3}}\|\zeta\|_{0,4}\\ &\le \|u_h - u_H\|_{0,4}^3\|\zeta\|_{1,h}\\ &\le CH^6\|u\|_{\widetilde{W}^{3,4}(\Omega)}^3\|\zeta\|_{1,h}.\end{aligned} \tag{22}$$

Then, Lemma 3.3 follows immediately from (20), (21) and (22). This completes the proof.

From (9), Lemma 3.3 and the triangle inequality, we can easily get the following theorem.

Theorem 3.1: Let $u \in H_0^1(\Omega) \cap \widetilde{W}^{3,4}(\Omega)$ be the solution of (5), and $u^h \in V_h$ be the solution of Algorithm 3.1. We have the following estimate

$$\|u - u^h\|_{1,h} \le C(h + H^5)\|u\|_{\widetilde{W}^{3,4}(\Omega)}, \tag{23}$$

for some positive constant C.

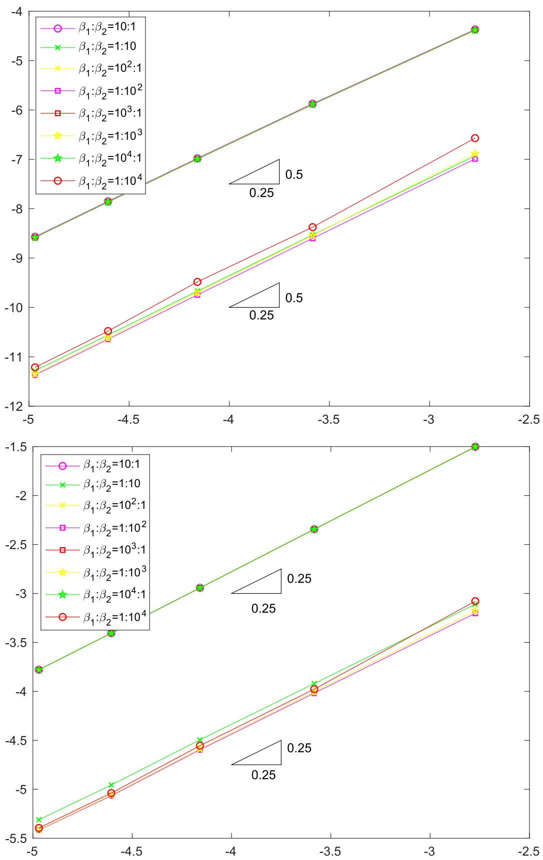

FIGURE 1. Log-log errors of IFE solutions in L^2 norm (above) and H^1 norm (below) for different diffusion coefficient ratios.

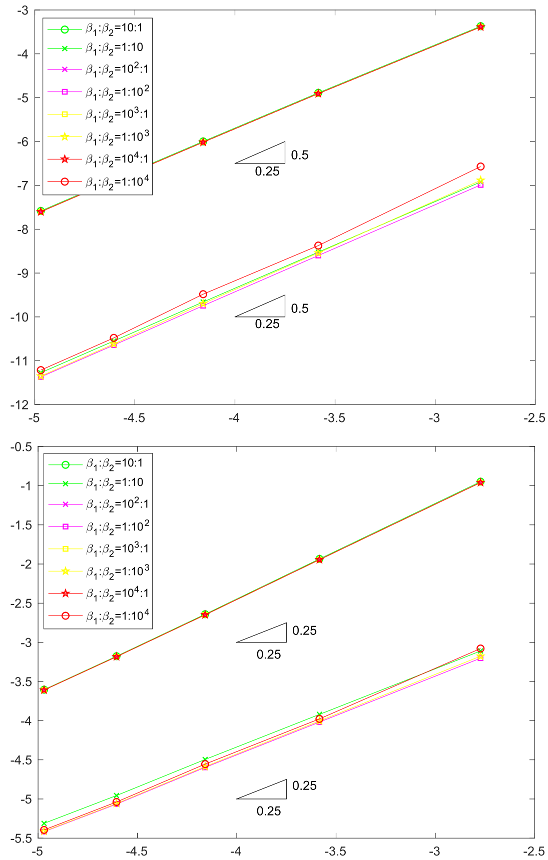

FIGURE 2. Log-log errors of T-G IFE solutions in L^2 norm (above) and H^1 norm (below) for different diffusion coefficient ratios, $h = H^2$.

Next, we derive the $L^p(2 \leq p < \infty)$ norm error estimate of $u_h - u^h$.

Lemma 3.4: Let $u_h \in V_h$ be the solution to (7) on $\mathcal{T}_h$ and $u^h \in V_h$ be the approximated solution obtained by Algorithm 3.1. Then we have the following estimate

$$\|u_h - u^h\|_{L^p(\Omega)} \leq CH^6 \|u\|_{\widetilde{W}^{3,4}(\Omega)}, \tag{24}$$

for $2 \leq p < \infty$ and some positive constant C.

Proof: To derive the estimate in $L^p(\Omega)$ norm, we use a duality argument by considering the auxiliary problem: find $\mu \in \widetilde{H}^2(\Omega)$ such that

$$-\nabla \cdot (\beta \nabla \mu) - f'(u_H)\mu = u_h - u^h, \quad in\ \Omega, \tag{25}$$

with the same conditions as (2)-(3). Given $2 \leq p < \infty$, set $q = p/(p-1) \in (1, 2]$. By the regularity of the weak solution u, there exists $H_0 > 0$, and $C(H_0)$, if $H \leq H_0$, we know that [8]

$$\|\mu\|_{\widetilde{H}^2(\Omega)} \leq C(H_0) \left\| u_h - u^h \right\|_{L^q(\Omega)}, \tag{26}$$

where we have used the regularity assumption(formula (27) in [12]) of the auxiliary problem.

Along with (25), let us also introduce the IFE approximation: find $\mu_h \in V_h$ satisfying

$$a_h(\mu_h, v_h) - \left(f'(u_H)\mu_h, v_h\right) = \left(u_h - u^h, v_h\right), \quad \forall v_h \in V_h. \tag{27}$$

Then, multiplying $v_h = u_h - u^h$ to both sides of (25), we have

$$\begin{aligned}
&\left(u_h - u^h, u_h - u^h\right) \\
&\quad = a_h(\mu, u_h - u^h) - \left(f'(u_H)\mu, u_h - u^h\right) \\
&\quad = a_h(\mu - \mu_h, u_h - u^h) - \left(f'(u_H)(\mu - \mu_h), u_h - u^h\right) \\
&\qquad + a_h(\mu_h, u_h - u^h) - \left(f'(u_H)\mu_h, u_h - u^h\right) \\
&\quad =: K_1 + K_2 + K_3 + K_4.
\end{aligned} \tag{28}$$

By continuity of $a_h(\cdot, \cdot)$, we have

$$\begin{aligned}
|K_1 + K_2| &\lesssim M(1 + \|f\|_{1,\infty})\|\mu - \mu_h\|_{1,h}\|u_h - u^h\|_{1,h} \\
&\leq ChH^5 \|\mu\|_{\widetilde{H}^2(\Omega)} \|u\|_{\widetilde{W}^{3,4}} \\
&\leq CH^6 \|u\|_{\widetilde{W}^{3,4}} \|u_h - u^h\|_{L^q(\Omega)},
\end{aligned} \tag{29}$$

where we have used Lemma 3.3 and (26) in the last step.

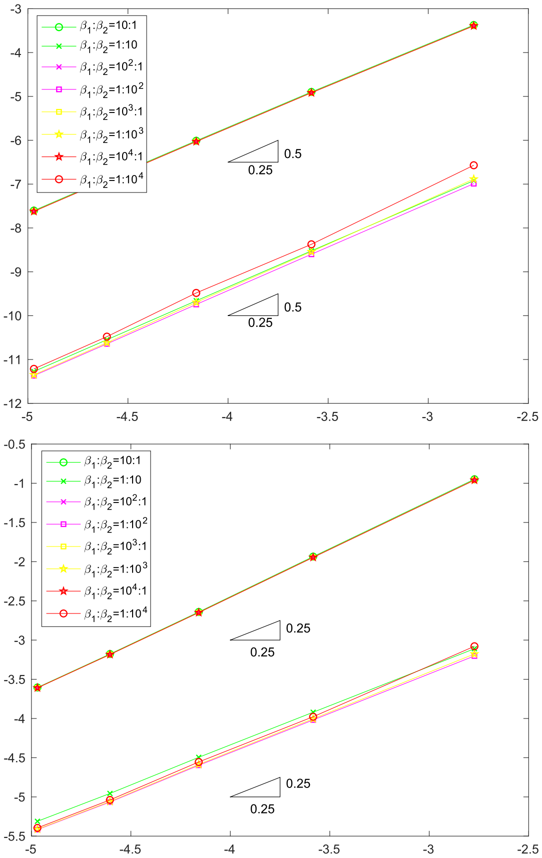

FIGURE 3. Log-log errors of CC T-G IFE solutions in L^2 norm (above) and H^1 norm (below) for different diffusion coefficient ratios, $h = H^2$.

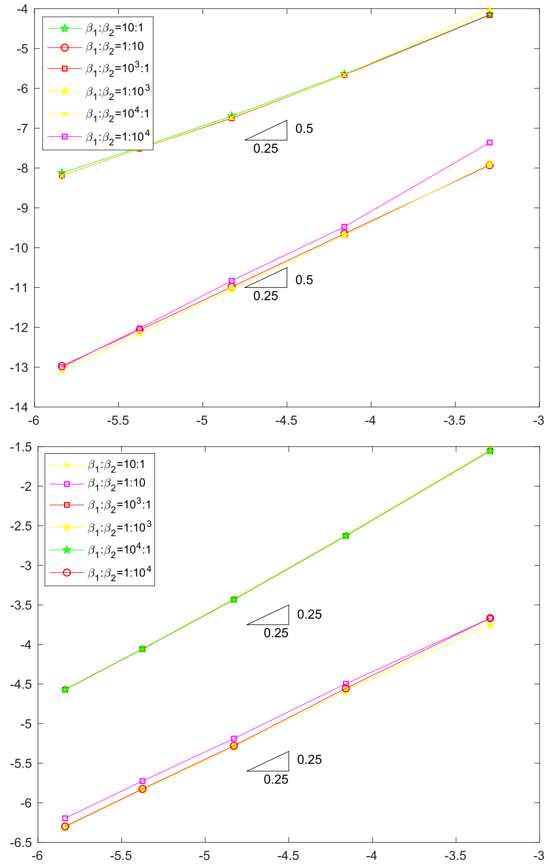

FIGURE 4. Log-log errors of T-G IFE solutions in L^2 norm (above) and H^1 norm (below) for different diffusion coefficient ratios, $h = H^3$.

For $|K_3+K_4|$, combining (18), (20), (21) and (22), we have

$$A_H(u_h - u^h, \zeta) \lesssim H^6\|u\|_{\widetilde{W}^{2,4}(\Omega)}\|\zeta\|_{1,h} + H^4\|u\|_{\widetilde{W}^{3,4}(\Omega)}\|(I-Q_H)\zeta\|.$$

It follows

$$\begin{aligned}|K_3+K_4| &\le CH^6\|u\|_{\widetilde{W}^{3,4}(\Omega)}\|\mu_h\|_{1,h} + H^4\|u\|_{\widetilde{W}^{3,4}(\Omega)}\|(I-Q_H)\mu_h\| \\ &\le CH^6\|u\|_{\widetilde{W}^{3,4}(\Omega)}\|\mu_h\|_{1,h} + H^4\|u\|_{\widetilde{W}^{3,4}(\Omega)}(\|(I-Q_H)\mu\| + H\|\mu-\mu_h\|_{1,h}) \\ &\le CH^6\|u\|_{\widetilde{W}^{3,4}(\Omega)}\left\|u_h - u^h\right\|_{L^q(\Omega)}. \end{aligned} \tag{30}$$

Then, Lemma 3.4 follows immediately from the error estimates for K_i, with $i = 1, 2, 3, 4$. This finishes the proof of Lemma 3.4.

Finally, we can easily obtain the following L^p $(2 \le p < \infty)$ norm error estimate of Algorithm 3.1.

Theorem 3.2: Let $u \in H_0^1(\Omega) \cap \widetilde{W}^{3,4}(\Omega)$ be the solution of (5), and $u^h \in V_h$ be the solution of Algorithm 3.1. Then, We have

$$\left\|u-u^h\right\|_{L^p(\Omega)} \le C(h^2+H^6)\|u\|_{\widetilde{W}^{3,4}(\Omega)}, \tag{31}$$

for some positive constant C.

Proof: The theorem can be easily proved by the error estimates of IFE solutions in L^p norm (10), (24) in Lemma 3.4, and the triangle inequality.

According to Theorem 3.1 and Theorem 3.2, it suffices to take $H = \mathcal{O}(h^{1/3})$, while guaranteeing the optimal (or nearly optimal) approximation for the discretization u^h in both H^1 and L^p norm, and for H^1 norm, it even suffices to take $H = \mathcal{O}(h^{1/5})$.

IV. NUMERICAL EXPERIMENTS

In this section, we present numerical results to verify the effectiveness and robustness of our proposed scheme. All the experiments are computed with double precision and are performed on a desktop computer with a Intel Core i7-9700 CPU, 3.00 GHz, and 8 GB memory. We consider solving the following test semi-linear equations:

$$-\nabla\cdot(\beta_i\nabla u_i) + u_i^3 = f_i, \ \text{ in } \Omega_i, \ \ i = 1, 2, \tag{32}$$

where the boundary condition and interface jump are subjected to (2)-(3).

Example 4.1: In this example, take the domain $\Omega = (-1, 1)\times(-1, 1)$, the interface Γ being the circle centered at $(0, 0)$ with radius r_0, so that $\Omega_1 = \{(x, y) \in \mathcal{R}^2 | x^2+y^2 < r_0^2\}$,

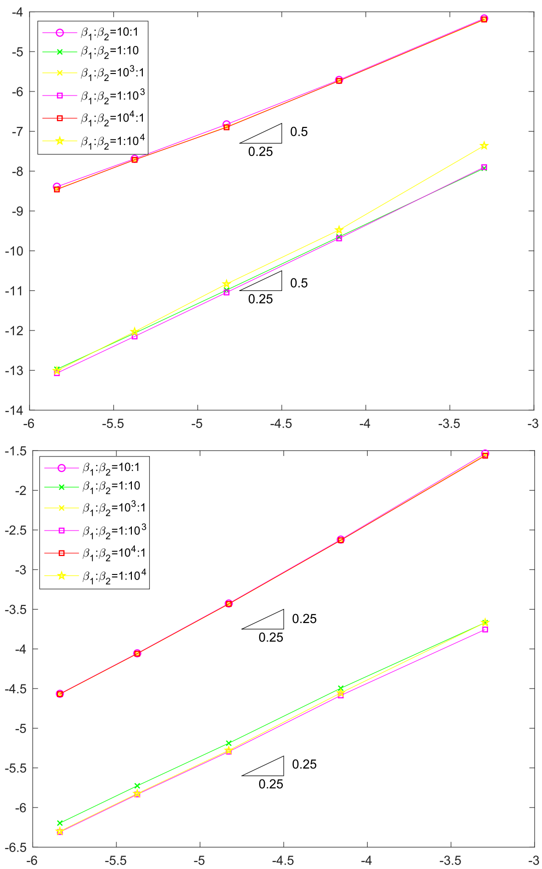

FIGURE 5. Log-log errors of CC T-G IFE solutions in L^2 norm (above) and H^1 norm (below) for different diffusion coefficient ratios, $h = H^3$.

$\Omega_2 = \Omega \backslash \Omega_1$. For the exact solution, we choose

$$u = \begin{cases} r^\alpha/\beta_1, & (x, y) \in \Omega_1, \\ r^\alpha/\beta_2 + (1/\beta_1 - 1/\beta_2) r_0^\alpha, & \text{otherwise}, \end{cases} \tag{33}$$

where $r = \sqrt{x^2 + y^2}$, $\alpha = 3$, $r_0 = 0.5$, and f_i is suitably chosen such the exact solution.

First, we verify that the IFE solutions have optimal convergence orders in both L^2 norm and semi-H^1 norm. For convergence orders in L^3 and L^4 norms, we have observed the similar behavior. In Figure 1, we display log-log errors of IFE solutions with various diffusion coefficients ratios, which illustrate that the IFE solutions of semi-linear interface problems (1)-(3) have second order convergence in L^2 norm and first order convergence in semi-H^1 norm.

For the relationship $h = H^2$, log-log errors of T-G IFE solutions and CC T-G IFE solutions with different diffusion coefficient ratios are given in Figure 2 and Figure 3, seperately. We know that both the T-G IFEM and CC T-G IFEM have optimal convergence rate in L^2 and semi-H^1 norms with the coarse grid mesh size $H = h^{1/2}$.

Log-log Errors of T-G IFEM and CC T-G IFEM with $H = h^{1/3}$ are shown in Figure 4 and Figure 5. We know that both T-G IFEM and CC T-G IFEM have optimal convergence orders, which are consistent with our theoretical results.

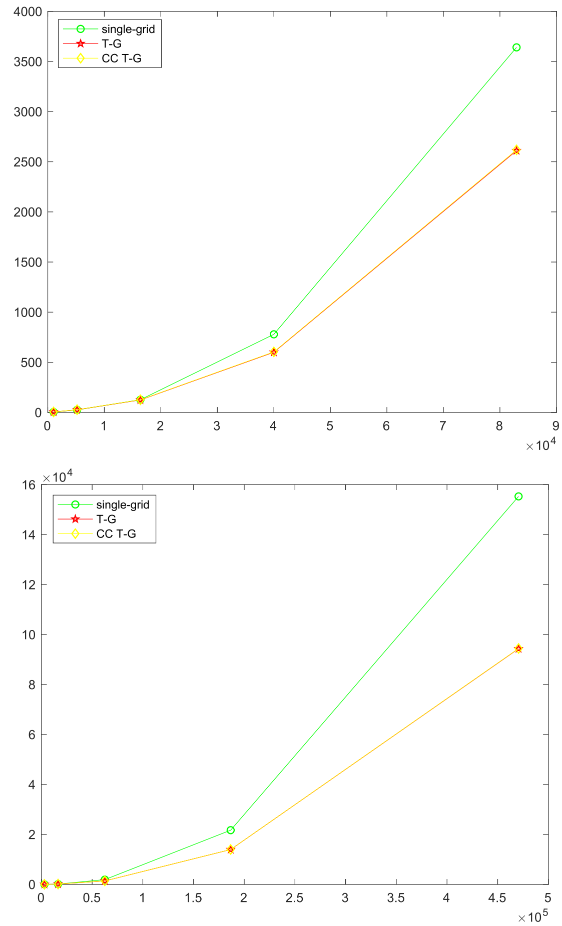

FIGURE 6. A comparison of computing time for IFEM on single grid, T-G IFEM, and CC T-G IFEM with $h = H^2$ (above) and $h = H^3$ (below).

However, errors of T-G IFEM in L^2 norm are bigger than the errors of CC T-G IFE solutions.

For $h = H^4$, we compute IFEM solution by Newton method on mesh size $h = 1/256$. The errors in L^2 norm and semi-H^1 norm are $4.017e-06$ and $2.510e-03$, respectively. By contrast with Table 1, we find that CC T-G IFEM is more robust than T-G IFEM. In semi-H^1 norm, the errors of T-G IFEM and CC T-G IFEM are consistent with that of IFEM on single grid. For L^2 norm, the errors of T-G IFEM and CC T-G IFEM is bigger than the errors on single grid. The three step coarse grid correction T-G IFEM has smaller errors than two-step T-G IFEM. The similar behavior has been derived for different diffusion coefficients. It should be pointed out

TABLE 1. Error of T-G IFEM and CC T-G IFEM, with fixed fine grid mesh size $h = 1/256$, $\beta_1 : \beta_2 = 1 : 10^4$.

H	h	$\|u - U_h\|_{L^2}$	$\|u - U_h\|_{1,h}$	$\|u - u^h\|_{L^2}$	$\|u - u^h\|_{1,h}$
1/4	1/256	4.534e-06	2.510e-03	4.497e-06	2.510e-03
1/8	1/256	4.077e-06	2.510e-03	4.063e-06	2.510e-03
1/16	1/256	4.023e-06	2.510e-03	4.021e-06	2.510e-03
1/32	1/256	4.017e-06	2.510e-03	4.017e-06	2.510e-03
1/64	1/256	4.017e-06	2.510e-03	4.017e-06	2.510e-03

that the smaller of the mesh size, the advantage of CC T-G IFEM is more obvious.

To verify the convergence order of T-G IFEM with $h = H^5$, we compute the IFE solutions on mesh size $h = 1/243$. The errors on L^2 norm and semi-H^1 norm with $\beta_1 : \beta_2 = 1 : 10^4$ are $4.432e-06$ and $2.510e-03$, respectively. From Table 2, we know that $|u - u^h|_{1,h}$ has the convergence accuracy as IFE solutions. But for T-G IFEM with $h = H^5$, $|u - U_h|_{1,h}$ has been increased at the sixth place after the decimal point. Thus, we verify the CC T-G IFEM have optimal convergence accuracy in H^1 norm with $H = h^{1/5}$. Moreover, we show the computing time of IFEM on single grid, T-G IFEM, and CC T-G IFEM in Figure 6. T-G IFEMs have greatly saved the calculation time and the third correction step expends little computational cost. In a word, the CC T-G IFEM can greatly improve computation accuracy by adding a small calculation cost.

TABLE 2. Error of T-G IFEM and CC T-G IFEM, with fixed fine grid mesh size $h = 1/243$, $\beta_1 : \beta_2 = 1 : 10^4$.

H	h	$\|u-U_h\|_{L^2}$	$\|u-U_h\|_{1,h}$	$\|u-u^h\|_{L^2}$	$\|u-u^h\|_{1,h}$
1/3	1/243	6.577e-06	2.880e-03	6.444e-06	2.880e-03
1/9	1/243	4.509e-06	2.880e-03	4.507e-06	2.880e-03
1/27	1/243	4.470e-06	2.880e-03	4.470e-06	2.880e-03
1/81	1/243	4.470e-06	2.880e-03	4.470e-06	2.880e-03

V. CONCLUSION

In this paper, we present a novel two-grid algorithm for semi-linear elliptic interface problems solved by IFEM. Optimal error estimates of the coarse grid correction two-grid solution in both H^1 and L^p norm are derived. It shows that the same accuracy of the IFE solutions are obtained by using the relationship $h = \mathcal{O}(H^3)$ (it even suffices to take $h = \mathcal{O}(H^5)$ for H^1 norm) between the fine grid and the coarse grid. The key ingredient of the two-grid method is that we use the correction technique on the coarse grid. Furthermore, we know that a very coarse grid space is sufficient for nonlinear problem that are dominated by linear part. In our future work, we will consider two-grid algorithms for more complex system.

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

USE OF AI TOOLS DECLARATION

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

REFERENCES

[1] W. J. Layton, F. Schieweck, and I. Yotov, "Coupling fluid flow with porous media flow," *SIAM J. Numer. Anal.*, vol. 40, no. 6, pp. 2195–2218, 2002.
[2] Z. Li and K. Ito, *The Immersed Interface Method. Numerical Solutions of PDEs Involving Interfaces and Irregular Domains*. Philadelphia, PA, USA: Society for Industrial and Applied Mathematics, 2006.
[3] H. Wang, D. Liang, R. E. Ewing, S. L. Lyons, and G. Qin, "An approximation to miscible fluid flows in porous media with point sources and sinks by an Eulerian–Lagrangian localized adjoint method and mixed finite element methods," *SIAM J. Sci. Comput.*, vol. 22, no. 2, pp. 561–581, 2000.
[4] J. M. Collis, W. L. Siegmann, F. B. Jensen, M. Zampolli, E. T. Küsel, and M. D. Collins, "Parabolic equation solution of seismo-acoustics problems involving variations in bathymetry and sediment thickness," *J. Acoust. Soc. Amer.*, vol. 123, no. 1, pp. 51–55, 2008.
[5] E. C. Whipple, "Potentials of surfaces in space," *Rep. Prog. Phys.*, vol. 44, no. 11, pp. 1197–1250, 2000.
[6] J. Xu, "Two-grid discretization techniques for linear and nonlinear PDEs," *SIAM J. Numer. Anal.*, vol. 33, no. 5, pp. 1759–1777, 1996.
[7] Y. Huang and Y. Chen, "A multi-level iterative method for solving finite element equations of nonlinear singular two-point boundary value problems," *Natural Sci. J. Xiangtan Univ.*, vol. 16, pp. 23–26, Jun. 1994.
[8] J. Xu, "A novel two-grid method for semilinear elliptic equations," *SIAM J. Sci. Comput.*, vol. 15, no. 1, pp. 231–237, 1994.
[9] Y. Chen, Y. Wang, Y. Huang, and L. Fu, "Two-grid methods of expanded mixed finite-element solutions for nonlinear parabolic problems," *Appl. Numer. Mathe.*, vol. 144, pp. 204–222, Oct. 2019.
[10] M. J. Holst, S. Ryan, and Y. Zhu, "Two-grid methods for semilinear interface problems," *Numer. Methods Partial Differ. Equ.*, vol. 29, no. 5, pp. 1729–1748, 2012.
[11] Y. Wang, Y. Chen, and Y. Huang, "A two-grid method for semi-linear elliptic interface problems by partially penalized immersed finite element methods," *Math. Comput. Simul.*, vol. 169, pp. 1–15, Mar. 2020.
[12] Y. Wang, Y. Chen, Y. Huang, and Y. Liu, "Two-grid methods for semi-linear elliptic interface problems by immersed finite element methods," *Appl. Math. Mech. English Ed.*, vol. 40, no. 11, pp. 1657–1676, 2019.
[13] Z. Li, T. Lin, Y. Lin, and R. C. Rogers, "An immersed finite element space and its approximation capability," *Numer. Methods Partial Differ. Equ.*, vol. 20, no. 3, pp. 338–367, 2004.
[14] Z. Li, T. Lin, and X. Wu, "New Cartesian grid methods for interface problems using the finite element formulation," *Numer. Math.*, vol. 96, no. 1, pp. 61–98, 2003.

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