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On Concept Lattices for Numberings

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Abstract: The theory of numberings studies uniform computations for families of mathematical objects. In this area, computability-theoretic properties of at most countable families of sets $\mathcal{S}$ are typically classified via the corresponding Rogers upper semilattices. In most cases, a Rogers semilattice cannot be a lattice. Working within the framework of Formal Concept Analysis, we develop two new approaches to the classification of families $\mathcal{S}$. Similarly to the classical theory of numberings, each of the approaches assigns to a family $\mathcal{S}$ its own concept lattice. The first approach captures the cardinality of a family $\mathcal{S}$: if $\mathcal{S}$ contains more than 2 elements, then the corresponding concept lattice $\mathrm{FC}_1(\mathcal{S})$ is a modular lattice of height 3, such that the number of its atoms to the cardinality of $\mathcal{S}$. Our second approach gives a much richer environment. We prove that for any countable poset P, there exists a family $\mathcal{S}$ such that the induced concept lattice $\mathrm{FC}_2(\mathcal{S})$ is isomorphic to the Dedekind-MacNeille completion of P. We also establish connections with the class of enumerative lattices introduced by Hoyrup and Rojas in their studies of algorithmic randomness. We show that every lattice $\mathrm{FC}_2(\mathcal{S})$ is anti-isomorphic to an enumerative lattice. In addition, every enumerative lattice is anti-isomorphic to a sublattice of the lattice $\mathrm{FC}_2(\mathcal{S})$ for some family $\mathcal{S}$.

Key words: theory of numberings; concept lattice; index set; complete lattice; enumerative lattice; Formal Concept Analysis

1 Introduction

The theory of numberings investigates uniform computational procedures for families of mathematical objects. Let $\mathcal{S}$ be an at most countable family. A numbering ν of the family $\mathcal{S}$ is a surjective map from the set of natural numbers ω onto $\mathcal{S}$.

Numberings have emerged as an important methodological tool with the rise of the modern formal notion of algorithmic computation. Gödel[1] employed an effective numbering of first-order formulae in the proof of his seminal incompleteness theorems. Kleene[2] (see also Theorem XXII in Ref. [3]) constructed the celebrated numbering of the family of all partial recursive functions—this is a list $\{\varphi_e(x)\}_{e\in\omega}$ enumerating all unary partial recursive functions. The key property of the numbering is that the binary function $\psi(e,x) := \varphi_e(x)$ is also partial recursive. In the 1950's, the foundations of the modern theory of numberings were developed by Kolmogorov and Uspenskii[4], Uspenskii[5], and independently by Rogers[6].

A key classification tool in the theory of numberings is provided by the notion of a Rogers semilattice. In order to put things into perspective, here we briefly discuss Rogers semilattices for computable families of sets. We refer the reader to Ref. [7] for the background

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on computability theory.

Let $\mathcal{S}$ be a family of computably enumerable (or c.e., for short) sets, i.e., each set A from $\mathcal{S}$ is a subset of ω which can be enumerated by a Turing machine. A numbering ν of the family $\mathcal{S}$ is computable if the set $\{(k,x) : k \in \omega,\ x \in \nu(k)\}$ is computably enumerable. One can say that a computable numbering provides a uniform algorithmic enumeration for the family $\mathcal{S}$. A family $\mathcal{S}$ is called computable if it admits a computable numbering.

A natural preorder on numberings is provided by the notion of reducibility. A numbering ν is reducible to a numbering μ, denoted by $\nu \leqslant \mu$, if there is a total computable function $f(x)$ such that $\nu(k) = \mu(f(k))$, for all $k \in \omega$. Informally speaking, a reduction $\nu \leqslant \mu$ is realized by an algorithmic procedure, which given a ν-index of an object $A \in \mathcal{S}$, outputs a μ-index of A. Two numberings ν and μ are equivalent, denoted by $\nu \equiv \mu$, if $\nu \leqslant \mu$ and $\mu \leqslant \nu$. By $[\nu]_\equiv$ we denote the $\equiv$-equivalence class of the numbering ν.

For a computable family $\mathcal{S}$, its Rogers semilattice $\mathcal{R}(\mathcal{S})$ is the following poset: the domain of $\mathcal{R}(\mathcal{S})$ is the set

$$\left\{[\nu]_\equiv : \nu \text{ is a computable numbering of } \mathcal{S}\right\},$$

and the ordering of $\mathcal{R}(\mathcal{S})$ is induced by the reducibility $\leqslant$. It is known that $\mathcal{R}(\mathcal{S})$ is an upper semilattice. On the other hand[8], if the poset $\mathcal{R}(\mathcal{S})$ contains at least two elements, then $\mathcal{R}(\mathcal{S})$ cannot be a lattice (i.e., there exists a pair $\{a, b\}$ from $\mathcal{R}(\mathcal{S})$, such that the pair does not have an infimum).

Rogers semilattices allow one to compare algorithmic properties of different computable families. For example, the following fact is well-known: If a finite family $\mathcal{S} = \{A_0, A_1, \ldots, A_n\}$ of c.e. sets has the property

$$\forall\, i \text{ and } \forall\, j,\ (i \neq j \rightarrow A_i \setminus A_j \neq \phi) \tag{1}$$

then the semilattice $\mathcal{R}(\mathcal{S})$ contains only one element. Roughly speaking, one can say that all families $\mathcal{S}$ satisfying Formula (1) exhibit the same behavior, if we talk about their algorithmic enumerations.

We should emphasize that in general, studying isomorphism types of Rogers semilattices is notoriously hard. For example, to our best knowledge, there is still no complete answer to the following question. Let $\mathcal{S}_0$ and $\mathcal{S}_1$ be finite families of c.e. sets. When are the Rogers semilattices $\mathcal{R}(\mathcal{S}_0)$ and $\mathcal{R}(\mathcal{S}_1)$ isomorphic?

We refer the reader to Refs. [9, 10] for the latest results on the question above. Further background on Rogers semilattices of computable families can be found, e.g., in Refs. [11−13].

In this paper, we develop two new approaches to classification of families of sets $\mathcal{S}$ and their numberings. These approaches are based on Formal Concept Analysis[14]. Similarly to the notion of Rogers semilattice, for an at most countable family $\mathcal{S}$, we introduce the posets $\mathrm{FC}_i(\mathcal{S})$, where $i \in \{1, 2\}$. In contrast to the classical Rogers semilattices, the introduced posets are complete lattices.

The paper is arranged as follows. Section 2 contains the necessary preliminaries.

Section 3 gives a brief overview of our first approach: the lattices $\mathrm{FC}_1(\mathcal{S})$. For a family $\mathcal{S}$, a concept Δ from $\mathrm{FC}_1(\mathcal{S})$ has the following property: the extent of Δ is a subset of ω, and the intent of Δ is a subfamily of $\mathcal{S}$. We give a complete characterization of all possible isomorphism types of $\mathrm{FC}_1(\mathcal{S})$ (Theorem 1). If $\mathcal{S}$ has only one element, then $\mathrm{FC}_1(\mathcal{S})$ is a one-element lattice; otherwise, $\mathrm{FC}_1(\mathcal{S})$ is a modular lattice of height 3 with precisely κ atoms, where κ is the cardinality of the family $\mathcal{S}$.

Section 4 discusses our second approach (the lattices $\mathrm{FC}_2(\mathcal{S})$). All results of Section 4 (except Proposition 1) are new.

We show that the lattices $\mathrm{FC}_2(\mathcal{S})$ are closely connected to the enumerative lattices introduced by Hoyrup and Rojas[15] in their investigations of algorithmic randomness on computable metric spaces. We show that every lattice $\mathrm{FC}_2(\mathcal{S})$ is anti-isomorphic to an enumerative lattice (Theorem 2). On the other hand, we prove that every enumerative lattice is anti-isomorphic to a sublattice of some lattice $\mathrm{FC}_2(\mathcal{S})$ (Theorem 3).

In addition, we prove the following: if a lattice $\mathcal{L}$ is the Dedekind-MacNeille completion of some countable poset, then $\mathcal{L}$ is isomorphic to the lattice $\mathrm{FC}_2(\mathcal{S})$ for some family $\mathcal{S}$ (Proposition 2). This fact gives us a large list of examples of the lattices $\mathrm{FC}_2(\mathcal{S})$: in particular, this shows that our second approach provides a much richer environment than the approach of $\mathrm{FC}_1(\mathcal{S})$.

We also give an example of an uncountable complete lattice which is not isomorphic to any $\mathrm{FC}_2(\mathcal{S})$ (Proposition 3). This gives an answer to an open question from Ref. [16].

In Section 5, we consider the complexity of the following isomorphism problem: For two families $\mathcal{S}$ and $\mathcal{T}$, when are the corresponding lattices $\mathrm{FC}_2(\mathcal{S})$ and $\mathrm{FC}_2(\mathcal{T})$ isomorphic?

Theorem 4 proves that the index set associated with the problem is Σ^1_1-hard (i.e., any set from the class Σ^1_1 of the analytical hierarchy is many-one reducible to the set). We give a new proof of Theorem 4: this proof employs the known facts on computable linear orders.

Section 6 concludes the paper.

2 Preliminary

Following the usual conventions of computability theory, by ω we denote the set of natural numbers. For a set X, the power set of X is denoted by $P(X)$. By $\mathrm{card}(X)$ we denote the cardinality of X.

We assume that the reader is familiar with the basic notions of computability theory and computable structure theory. We refer to the Refs. [7, 17] for the background.

2.1 Formal concept analysis

The preliminaries on Formal Concept Analysis follow Ref. [18]. For the background in lattice theory, we refer to Refs. [19, 20].

Recall that a formal context $K=(G, M, I)$ consists of the set of objects G, the set of attributes M, and the incidence relation $I \subseteq G \times M$. If K is a formal context and $A \subseteq G$, then

$$\alpha_K(A) := \{m \in M : (\forall g \in A)\, [(g, m) \in I]\}.$$

For $B \subseteq M$, we have

$$\beta_K(B) := \{g \in G : (\forall m \in B)\, [(g, m) \in I]\}.$$

If the triple K is clear from the discussion, then we omit the subscript K, e.g., we write $\alpha(A)$ in place of $\alpha_K(A)$.

A formal concept of the context K is a pair (A, B), such that $A \subseteq G$, $B \subseteq M$, $B = \alpha_K(A)$, and $A = \beta_K(B)$. For a formal concept $\varDelta = (A, B)$, A is called the extent of $\varDelta$, and B is the intent of $\varDelta$.

The ordering of the concepts of K is defined as follows:

$$(A_0, B_0) \leqslant (A_1, B_1) \ \Leftrightarrow\ A_0 \subseteq A_1 \ \Leftrightarrow\ B_0 \supseteq B_1.$$

The basic theorem on concept lattices (see Ref. [18]) establishes the following:

(1) The ordering on the set of all concepts of K induces a complete lattice. This lattice is called the concept lattice of K, and we denote it by $L(K)$.

(2) Let $\mathcal{L}$ be a complete lattice. Consider the formal context $K_{\mathcal{L}} = (\mathcal{L}, \mathcal{L}, \leqslant_{\mathcal{L}})$. Then the lattice $L(K_{\mathcal{L}})$ is isomorphic to $\mathcal{L}$. In addition, every concept of $K_{\mathcal{L}}$ is of the form

$$(\hat{a}, \check{a}) = (\{b : b \leqslant_{\mathcal{L}} a\}, \{c : a \leqslant_{\mathcal{L}} c\})$$

for some element $a \in \mathcal{L}$.

Let $(P, \leqslant)$ be a poset. A function $f: P \to P$ is a closure operator (on $(P, \leqslant)$) if it satisfies the following properties:

(1) $x \leqslant f(x)$;

(2) if $x \leqslant y$, then $f(x) \leqslant f(y)$;

(3) $f(f(x)) = f(x)$.

An element $x \in P$ is called closed (with respect to f) if $f(x) = x$.

Let $K = (G, M, I)$ be a formal context. Then the function $\beta \circ \alpha$ is a closure operator on the poset $(P(G), \subseteq)$ (see, e.g., Proposition 8 in Ref. [18]). In addition, the set of extents of formal concepts of K contains precisely the $\beta \circ \alpha$-closed elements of $(P(G), \subseteq)$.

2.2 Related work in the theory of numberings

There is a large body of literature on Rogers semilattices of computable families. Here we mention only a few classical results in this area. Let $\mathcal{S}$ be a computable family of c.e. sets. Since there are only countably many c.e. sets, the Rogers semilattice $\mathcal{R}(\mathcal{S})$ is at most countable. In addition, it is not hard to observe that there are at most countably many isomorphism types of Rogers semilattices (for computable families).

Khutoretskii[21] proved the following: if the Rogers semilattice $\mathcal{R}(\mathcal{S})$ contains more than one element, then $\mathcal{R}(\mathcal{S})$ is infinite. The aforementioned result of Selivanov[8] shows that an infinite semilattice $\mathcal{R}(\mathcal{S})$ cannot be a lattice. Ershov and Lavrov[22] (see also Refs. [9, 11]) proved that there exist finite families $\mathcal{S}_i$, $i \in \omega$, of c.e. sets, such that the Rogers semilattices $\mathcal{R}(\mathcal{S}_i)$ are pairwise non-isomorphic. V'yugin[23] proved that there are infinitely many pairwise non-elementarily equivalent Rogers semilattices of computable families.

Goncharov and Sorbi[24] started developing the theory of generalized computable numberings. This area has become a fruitful line of research which focuses on numberings in various computability-theoretic hierarchies, and the corresponding Rogers semilattices. Nowadays, a plethora of results are known for Rogers semilattices in the following hierarchies:

- Arithmetical hierarchy—see Refs. [25−27];

- Hyperarithmetical hierarchy[28, 29];
- Ershov hierarchy[30–32];
- Analytical hierarchy[33–35].

3 Overview of the First Approach

Our first approach is based on the following definition:

Definition 1 Let $\mathcal{S}$ be an at most countable family, and let ν be a numbering of the family $\mathcal{S}$. Consider the relation

$$I_\nu := \{(n, \nu(n)) : n \in \omega\}.$$

By $\mathrm{FC}_1(\mathcal{S})$ we denote the concept lattice of the formal context $K = (\omega, \mathcal{S}, I_\nu)$.

It turns out that, informally speaking, the isomorphism type of the lattice $\mathrm{FC}_1(\mathcal{S})$ encodes only the cardinality of the family $\mathcal{S}$. This is witnessed by the following result.

For a natural number $n \geqslant 2$, let M_n be a modular lattice of height 3 with n atoms. By M_ω we denote a modular lattice of height 3 with countably many atoms.

Theorem 1 Let $\mathcal{S}$ be an at most countable, non-empty family. If $\mathcal{S}$ contains only one element, then $\mathrm{FC}_1(\mathcal{S})$ is a one-element lattice. Otherwise, $\mathrm{FC}_1(\mathcal{S})$ is isomorphic to $M_{\mathrm{card}(\mathcal{S})}$.

The proof of Theorem 1 is based on the following general lattice-theoretic lemma. Suppose that f is a surjective map from a set X onto a set Y, where $\mathrm{card}(Y) \geqslant 2$. Then one can show that the concept lattice $L(X, Y, \Gamma_f)$, where Γ_f is the graph of the map f, is isomorphic to the lattice $M_{\mathrm{card}(Y)}$.

The content of the current paper is focused on new results, thus, we omit the formal proof of Theorem 1. The full proof of Theorem 1 is published in Section 3 of Ref. [16].

Note that Theorem 1 justifies our choice of notations in Definition 1—indeed, the isomorphism type of the poset $\mathrm{FC}_1(\mathcal{S})$ does not depend on the choice of a numbering ν. Informally speaking, the lattice $\mathrm{FC}_1(\mathcal{S})$ is an invariant, which provides some kind of characterization for all possible numberings of $\mathcal{S}$.

4 The Second Approach

In this section, we only work with at most countable, non-empty families $\mathcal{S} \subset P(\omega)$. Our second approach is based on the following definition:

Definition 2 Let ν be a numbering of a family $\mathcal{S}$. Consider a binary relation

$$Q_\nu = \{(x, n) : n \in \omega,\ x \in \nu(n)\} \subseteq \omega \times \omega.$$

By $\mathrm{FC}_2(\mathcal{S})$ we denote the concept lattice $L(\omega, \omega, Q_\nu)$.

Intuitively speaking, our second approach is more expressive than the approach of Section 3: here the relation Q_ν is not necessarily the graph of a surjective function and hence, the restrictions provided by Theorem 1 do not apply to $\mathrm{FC}_2(\mathcal{S})$.

First, we establish the following useful result:

Lemma 1 Let $\mathcal{S}$ be a family of subsets of ω and let ν be a numbering of the family $\mathcal{S}$. For a set $X \subseteq \omega$, its $\beta \circ \alpha$-closure in the formal context (ω, ω, Q_ν) satisfies the following:

$$\beta \circ \alpha(X) = \bigcap \{Z \in \mathcal{S} : X \subseteq Z\} \tag{2}$$

Proof For a set $X \subseteq \omega$, we have

$$\begin{aligned}\alpha(X) &= \{n : (\forall x \in X)\,[(x, n) \in Q_\nu]\} = \\ &\{n : X \subseteq \nu(n)\}, \\ \beta \circ \alpha(X) &= \{x : \forall n\,[X \subseteq \nu(n) \to x \in \nu(n)]\} = \\ &\bigcap \{\nu(n) : X \subseteq \nu(n)\} = \\ &\bigcap \{Z \in \mathcal{S} : X \subseteq Z\}.\end{aligned}$$

Lemma 1 is proved. ■

The following result is the consequence of Lemma 1.

Proposition 1 Let $\mathcal{S}$ be a family of subsets of ω. The structure $\mathrm{FC}_2(\mathcal{S})$ is well-defined, i.e., the isomorphism type of the lattice $\mathrm{FC}_2(\mathcal{S})$ does not depend on the choice of a numbering ν.

Proof By Lemma 1, the $\beta \circ \alpha$-closure of a given set X does not depend on the choice of a numbering ν. Hence, it is easy to show that the isomorphism type of $\mathrm{FC}_2(\mathcal{S})$ also does not depend on ν. ■

4.1 Connections with enumerative lattices

Here we show that the lattices $\mathrm{FC}_2(\mathcal{S})$ are closely connected to the enumerative lattices introduced by Hoyrup and Rojas[15].

A numbered set is a pair (A, γ), where A is a non-empty, at most countable set, and γ is a numbering of the set A.

Definition 3 (Definition 3.1.1 of Ref. [15]) An enumerative lattice is a triple $(L, \leqslant, \boldsymbol{A})$, such that:

- $(L, \leqslant)$ is a complete lattice;
- $\boldsymbol{A} = (A, \gamma)$ is a numbered set satisfying $A \subseteq L$;
- Every element $b \in L$ is the supremum of some subset of A.

For a detailed discussion of enumerative lattices, we refer the reader to Refs. [15, 36]. In this section, by the isomorphism type of an enumerative lattice $(L, \leqslant, \boldsymbol{A})$

we mean the isomorphism type of the lattice $(L, \leqslant)$.

First, we obtain the following result:

Theorem 2 Let $\mathcal{S} \subset P(\omega)$ be a non-empty, at most countable family. Then the lattice $\mathrm{FC}_2(\mathcal{S})$ is anti-isomorphic to an enumerative lattice.

Proof For a family $\mathcal{S}$, we consider the lattice $\mathrm{FC}_2(\mathcal{S})$. As in Definition 2, we choose some numbering ν of the family $\mathcal{S}$.

We define an enumerative lattice $(L, \leqslant_0, \boldsymbol{A})$ as follows:

- Elements of L are precisely the $\beta \circ \alpha$-closed subsets of ω;
- $a \leqslant_0 b$ if and only if $a \supseteq b$;
- The numbered set $\boldsymbol{A}$ is equal to $(\mathcal{S}, \nu)$.

Since $\beta \circ \alpha$-closed subsets of ω are precisely the extents of formal concepts (in the context (ω, ω, Q_ν)), it is clear that $(L, \leqslant_0)$ is a complete lattice which is anti-isomorphic to $\mathrm{FC}_2(\mathcal{S})$.

Suppose that $Y \in \mathcal{S}$. Then by Eq. (2), we have

$$Y \subseteq \beta \circ \alpha(Y) = \bigcap \{Z \in \mathcal{S} : Y \subseteq Z\} \subseteq Y.$$

Hence, the set Y is $\beta \circ \alpha$-closed. Consequently, we have $\mathcal{S} \subseteq L$.

In addition, Eq. (2) implies that every element $b \in L$ is the supremum (with respect to $\leqslant_0$) of the set $\{z \in \mathcal{S} : b \subseteq z\}$. We conclude that the triple $(L, \leqslant_0, \boldsymbol{A})$ is an enumerative lattice which is anti-isomorphic to our lattice $\mathrm{FC}_2(\mathcal{S})$. Theorem 2 is proved. ■

Second, we establish another interesting connection with enumerative lattices:

Theorem 3 Let $(L, \leqslant, \boldsymbol{A})$ be an enumerative lattice. Then there exists a family $\mathcal{S} \subset P(\omega)$ such that the lattice $(L, \leqslant)$ is anti-isomorphic to a sublattice of $\mathrm{FC}_2(\mathcal{S})$.

Proof We consider an enumerative lattice $(L, \leqslant, \boldsymbol{A})$, where $\boldsymbol{A} = (A, \gamma)$ is a numbered set. We define a numbered set $(\mathcal{S}, \nu)$ via the numbering ν: for $k \in \omega$, we put

$$\nu(k) = \{m \in \omega : \gamma(m) \nleqslant \gamma(k)\}.$$

Note that $\nu(k) \subseteq \nu(\ell)$ if and only if $\gamma(k) \geqslant \gamma(\ell)$. In addition, every set $\nu(k)$ is $\beta \circ \alpha$-closed.

For an element $a \in L$, by $\downarrow_\gamma a$ we denote the set

$$\downarrow_\gamma a = \{m \in \omega : \gamma(m) \leqslant a\}.$$

By $(\downarrow_\gamma a)^c$ we denote its complement, i.e., $\omega \setminus (\downarrow_\gamma a)$. Notice that $(\downarrow_\gamma \gamma(k))^c = \nu(k)$, for all $k \in \omega$.

For an arbitrary element a from L, we define the set

$$\psi(a) := \bigcap \{\nu(k) : (\downarrow_\gamma a)^c \subseteq \nu(k)\}.$$

By Eq. (2), every $\psi(a)$ is a $\beta \circ \alpha$-closed set, and hence, $\psi(a)$ can be viewed as an element of the lattice $\mathrm{FC}_2(\mathcal{S})$. In addition, we note that $\psi(\gamma(\ell)) = \nu(\ell)$, for each $\ell \in \omega$.

We show that the map ψ is an anti-isomorphism acting from $(L, \leqslant)$ onto some sublattice of $\mathrm{FC}_2(\mathcal{S})$.

If $a \leqslant b$, then $\downarrow_\gamma a \subseteq \downarrow_\gamma b$, $(\downarrow_\gamma a)^c \supseteq (\downarrow_\gamma b)^c$, and $\psi(a) \supseteq \psi(b)$.

Now suppose that $a \nleqslant b$. Since $(L, \leqslant, \boldsymbol{A})$ is an enumerative lattice, we have

$$a = \sup_L \{\gamma(k) : \gamma(k) \leqslant a\},$$

and there exists k_0, such that $\gamma(k_0) \leqslant a$ and $\gamma(k_0) \nleqslant b$. Hence, $(\downarrow_\gamma a)^c \subseteq \nu(k_0)$ and $k_0 \in (\downarrow_\gamma b)^c$. This implies that $\psi(a) \subseteq \nu(k_0)$ and $k_0 \in \psi(b)$. On the other hand, since $k_0 \notin \nu(k_0)$, we deduce that $\psi(a) \not\supseteq \psi(b)$.

The discussed argument shows that the map ψ anti-isomorphically embeds $(L, \leqslant)$ into $\mathrm{FC}_2(\mathcal{S})$. Theorem 3 is proved. ■

4.2 Dedekind-MacNeille completions

Let $(P, \leqslant)$ be a poset. For a set $A \subseteq P$, one defines:

$$A^u = \{x \in P : (\forall a \in A)(a \leqslant x)\},$$
$$A^\ell = \{x \in P : (\forall a \in A)(x \leqslant a)\}.$$

By $\mathrm{DM}(P)$ we denote the set

$$\mathrm{DM}(P) = \{A \subseteq P : (A^u)^\ell = A\}.$$

The poset $(\mathrm{DM}(P), \subseteq)$ is called the Dedekind-MacNeille completion of the poset $(P, \leqslant)$.

The following results are known (see Theorems 7.40 and 7.41 in Ref. [19]):

(1) The map $\varphi: x \mapsto \downarrow x = \{y \in P : y \leqslant x\}$ is an order embedding from $(P, \leqslant)$ into $(\mathrm{DM}(P), \subseteq)$. In addition, φ preserves all infima and suprema which exist in P.

(2) The poset $(\mathrm{DM}(P), \subseteq)$ is a complete lattice.

(3) If P itself is a complete lattice, then P is isomorphic to its Dedekind-MacNeille completion.

In addition, the Dedekind-MacNeille completion of a Boolean algebra is also a Boolean algebra (see Theorem XII.2.13 in Ref. [20]).

We obtain the following result which provides us with a rich class of examples of lattices $\mathrm{FC}_2(\mathcal{S})$:

Proposition 2 Let $(P, \leqslant)$ be at most countable poset. Then there exists a family $\mathcal{S} \subset P(\omega)$ such that the lattice $\mathrm{FC}_2(\mathcal{S})$ is isomorphic to the Dedekind-MacNeille completion $(\mathrm{DM}(P), \subseteq)$.

Proof Without loss of generality, one may assume that P is a subset of ω. The desired family $\mathcal{S}$ contains all sets of the form

$$\downarrow b = \{x \in P : x \leqslant b\}, \text{ for } b \in P.$$

By Eq. (2), a set $X \subseteq \omega$ is $\beta \circ \alpha$-closed (in the context (ω, ω, Q_ν)) if and only if

$$X = \bigcap \{\downarrow b : X \subseteq \downarrow b\} = \bigcap \{\downarrow b : b \in X^u\} = (X^u)^\ell.$$

Therefore, the Dedekind-MacNeille completion $\mathrm{DM}(P)$ contains precisely the $\beta \circ \alpha$-closed subsets of ω. This fact implies that the lattice $(\mathrm{DM}(P), \subseteq)$ is isomorphic to $\mathrm{FC}_2(\mathcal{S})$. Proposition 2 is proved. ■

4.3 Examples of lattices $\mathrm{FC}_2(\mathcal{S})$

First, we list some examples which are provided by Proposition 2. Each of the following complete lattices can be realized as the lattice $\mathrm{FC}_2(\mathcal{S})$ for some family $\mathcal{S}$.

(1) For every at most countable complete lattice, note that another proof of this fact is given in Theorem 4.1.(b) of Ref. [16].

(2) The real unit interval [0, 1]. This is the Dedekind-MacNeille completion of the countable poset $([0,1] \cap Q, \leqslant)$—see Example 7.44.(1) in Ref. [19].

(3) The Boolean algebra $\mathcal{P}(\omega)$ of all subsets of ω. This is the completion of the countable poset

$$(\{\{k\} : k \in \omega\} \cup \{\omega \setminus \{k\} : k \in \omega\}; \subseteq),$$

see Example 7.44.(3) of Ref. [19].

(4) The quotient Boolean algebra of Borel sets (of reals) modulo meager sets. This is the completion of the countable atomless Boolean algebra—see Ref. [37].

On the other hand, we give an example of an uncountable complete lattice $\mathcal{L}$, such that Zermelo-Fraenkel set theory with Choice (ZFC) proves that $\mathcal{L}$ is not isomorphic to any $\mathrm{FC}_2(\mathcal{S})$. This provides an answer to Problem 5.2 of Ref. [16].

For a linear order $(L, \leqslant)$, by L^* we denote the corresponding reverse ordering, i.e., $L^* = (L, \geqslant)$. As usual, by ω_1 we denote the least uncountable ordinal.

Proposition 3 Let α be an uncountable successor ordinal. Then the linear order α^* is a complete lattice which cannot be isomorphic to any lattice $\mathrm{FC}_2(\mathcal{S})$.

Proof First, we observe that α^* is indeed a complete lattice. Now, towards a contradiction, assume that α^* is isomorphic to the lattice $\mathrm{FC}_2(\mathcal{S})$ for some family $\mathcal{S} \subset P(\omega)$. Then there exists an anti-isomorphism ψ acting from the ordinal α onto $\mathrm{FC}_2(\mathcal{S})$.

Choose a numbering ν of the family $\mathcal{S}$. For $k \in \omega$, define $\beta_k := \psi^{-1}(\nu(k))$. Recall that by Eq. (2), we have

$$\psi(\omega_1) = \bigcap \{\nu(\ell) : \psi(\omega_1) \subseteq \nu(\ell)\}.$$

Since ψ is an anti-isomorphism, we deduce that

$$\omega_1 = \sup\{\beta_\ell : \omega_1 \geqslant \beta_\ell\}.$$

Therefore, there exists a countable increasing sequence of ordinals $(\gamma_i)_{i\in\omega}$, such that $\omega_1 = \lim_i \gamma_i$, $\lim_i \gamma_i$ is equal to $\sup\{\gamma_i : i \in w\}$. This means that the cofinality of ω_1 is countable. We note that this contradicts the following known fact: ZFC proves that ω_1 is a regular cardinal (see Corollary 5.3 in Ref. [38]).

5 Isomorphism Problem for Lattices $\mathrm{FC}_2(\mathcal{S})$

Let K be a class of algebraic structures. Within the framework of computable structure theory, the algorithmic complexity of the isomorphism problem on the class K is typically measured via the following approach. One fixes an appropriate numbering ν of the set of all computable members of K, and then one aims to obtain complexity bounds for the following index set:

$$\mathrm{Iso}(K) = \{(i, j) \in \omega \times \omega : \nu(i) \cong \nu(j)\}.$$

Typically, one wants to prove that the set $\mathrm{Iso}(K)$ is complete (with respect to many-one reducibility) in one of the levels of a familiar computability-theoretic hierarchy (e.g., the arithmetical hierarchy).

As an example of a recent application of index sets, we mention the following: Ref. [39] established that there is no reasonable syntactic characterization of the algebraic structures which have a polynomial-time-computable isomorphic copy. For a detailed discussion of index sets, we refer to the surveys of Refs. [40, 41].

In this section, we apply the discussed approach to measure the complexity of the isomorphism problem for the lattices $\mathrm{FC}_2(\mathcal{S})$.

Here we work with index sets in the setting of the theory of numberings. Within this framework, the systematic investigations of index sets were initiated by McLaughlin[42]. For the known results in this area, we refer to Ref. [43].

Recall that $\{\varphi_e\}_{e\in\omega}$ is Kleene's numbering of the family of all unary partial computable functions. As usual, for $e \in \omega$, W_e denotes the c.e. set $\mathrm{dom}(\varphi_e)$.

We consider the following effective listing. For $i, k \in \omega$, let

$$\theta_i(k) = \begin{cases} W_{\varphi_i(k)}, & \text{if } \varphi_i(k) \text{ is defined;} \\ \phi, & \text{otherwise.} \end{cases}$$

It is well-known that the list $\{\theta_i\}_{i\in\omega}$ enumerates all computable numberings of all computable families. In addition, there exists a total computable function

$h_0(x, y)$, such that $\theta_i(k) = W_{h_0(i,k)}$ for all i and k.

For $i \in \omega$, by $\mathcal{T}_i$ we denote the family of c.e. sets, which is indexed by the numbering θ_i, i.e.,

$$\mathcal{T}_i = \{\theta_i(k) : k \in \omega\}.$$

As usual, by ω_1^{CK} we denote the least non-computable ordinal. The standard linear ordering of the rationals is denoted by η.

A lower bound for the complexity of the isomorphism problem for the lattices $\mathrm{FC}_2(\mathcal{S})$ is provided by the following result.

Theorem 4 The index set

$$\mathrm{IFC} = \{(i, j) : \mathrm{FC}_2(\mathcal{T}_i) \cong \mathrm{FC}_2(\mathcal{T}_j)\}$$

is Σ_1^1-hard (i.e., every set from the level Σ_1^1 of the analytical hierarchy is many-one reducible to the set IFC).

Proof Recall that Harrison[44] proved that the linear order $\omega_1^{\mathrm{CK}} \cdot (1+\eta)$ has a computable isomorphic copy. Notice that the order $\omega_1^{\mathrm{CK}} \cdot (1+\eta)$ contains infinite descending chains.

In addition, the following fact is known (see Proposition 3.2 in Ref. [45]). For any Σ_1^1 set $X \subseteq \omega$, there exists a computable sequence $(\mathcal{L}_n)_{n\in\omega}$ of computable linear orders, such that

- if $n \in X$, then $\mathcal{L}_n \cong \omega_1^{\mathrm{CK}} \cdot (1+\eta)$;
- if $n \notin X$, then $\mathcal{L}_n$ is isomorphic to a computable ordinal.

In what follows, we assume that X is non-empty.

Given the sequence $(\mathcal{L}_n)_{n\in\omega}$, we define $\mathcal{M}_n := \mathcal{L}_n + 1$, for each $n \in \omega$. Without loss of generality, one may assume that for each n, the domain of the order $\mathcal{M}_n$ is equal to ω.

Similarly to Proposition 2, for a given $n \in \omega$, we define a computable family $\mathcal{S}_n$ of sets as follows. We put

$$\mathcal{S}_n = \{\nu_n(k) : k \in \omega\},$$

where $\nu_n(k) = \{x \in \omega : x \leqslant_{\mathcal{M}_n} k\}$. Then one can show that the lattice $\mathrm{FC}_2(\mathcal{S}_n)$ is isomorphic to the Dedekind-MacNeille completion of the order $\mathcal{M}_n$.

Consider a function $g(x)$, such that for every $n \in \omega$, the family $\mathcal{S}_n$ is equal to $\mathcal{T}_{g(n)}$. All the described transformations (i.e., mapping n to $\mathcal{L}_n$, transforming $\mathcal{L}_n$ into $\mathcal{M}_n$, and constructing the family $\mathcal{S}_n$ from the order $\mathcal{M}_n$) are effective. Thus, one can show that the function $g(x)$ is computable.

Fix a number $e_0 \in X$. Consider the following computable function:

$$h(x) := (g(e_0), g(x)).$$

We notice the following properties of the function h.

If $n \in X$, then we have $\mathcal{M}_n \cong \mathcal{M}_{e_0} \cong \omega_1^{\mathrm{CK}} \cdot (1+\eta) + 1$. Thus, the lattices $\mathrm{FC}_2(\mathcal{S}_n)$ and $\mathrm{FC}_2(\mathcal{S}_{e_0})$ are isomorphic, and $h(n) \in \mathrm{IFC}$.

If $n \notin X$, then $\mathcal{M}_n$ is isomorphic to a computable successor ordinal. Hence, $\mathcal{M}_n$ is a complete lattice, and its completion is isomorphic to $\mathcal{M}_n$. Therefore, $\mathrm{FC}_2(\mathcal{S}_n)$ is not isomorphic to $\mathrm{FC}_2(\mathcal{S}_{e_0})$, and $h(n) \notin \mathrm{IFC}$.

We conclude that the function h provides a many-one reduction from the set X to the index set IFC. Therefore, the set IFC is Σ_1^1-hard. Theorem 4 is proved.

6 Conclusion and Future Work

In this paper, we develop two new approaches to classification of at most countable families $\mathcal{S} \subset P(\omega)$. The approaches are based on the methods of Formal Concept Analysis and the theory of numberings.

The first approach assigns to a family $\mathcal{S}$ the concept lattice $\mathrm{FC}_1(\mathcal{S})$. We obtained a complete characterization of the isomorphism types of the lattices $\mathrm{FC}_1(\mathcal{S})$.

Within the second approach, the induced concept lattices $\mathrm{FC}_2(\mathcal{S})$ can realize a plethora of isomorphism types: in particular, for any countable poset P, its Dedekind-MacNeille completion is isomorphic to the lattice $\mathrm{FC}_2(\mathcal{S})$, for an appropriately chosen family $\mathcal{S}$. The lattices $\mathrm{FC}_2(\mathcal{S})$ are also closely connected to the enumerative lattices introduced by Hoyrup and Rojas.

The work leaves open several questions about the isomorphism types of the lattices $\mathrm{FC}_2(\mathcal{S})$. Here we state three of them:

(1) Is every enumerative lattice $\mathcal{E}$ anti-isomorphic to the lattice $\mathrm{FC}_2(\mathcal{S})$ for some family $\mathcal{S}$? Note that Theorem 3 only says that $\mathcal{E}$ is anti-isomorphic to a sublattice of $\mathrm{FC}_2(\mathcal{S})$.

(2) Find the computability-theoretic complexity of the index set IFC from Theorem 4. More formally, what is the many-one degree of IFC?

(3) Is every Σ_1^1 equivalence relation computably reducible to the isomorphism relation on the lattices $\mathrm{FC}_2(\mathcal{S})$?

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References

[1] K. Gödel, On formally undecidable propositions of Principia Mathematica and related systems I, *Monatsh. Math. Phys.*, vol. 38, no. 1, pp. 173–198, 1931.

[2] S. C. Kleene, On notation for ordinal numbers, *J. Symb. Log.*, vol. 3, no. 4, pp. 150–155, 1938.

[3] S. C. Kleene, *Introduction to Metamathematics*. New York, NY, USA: Van Nostrand, 1952.

[4] A. N. Kolmogorov and V. A. Uspenskii, On the definition of an algorithm, (in Russian), *Uspekhi Matematicheskikh Nauk*, vol. 13, no. 4, pp. 3–28, 1958.

[5] V. A. Uspenskii, Systems of denumerable sets and their enumeration, (in Russian), *Doklady Akademii Nauk SSSR*, vol. 105, pp. 1155–1158, 1958.

[6] H. Rogers, Gödel numberings of partial recursive functions, *J. Symb. Log.*, vol. 23, no. 3, pp. 331–341, 1958.

[7] R. I. Soare, *Turing Computability*: *Theory and Applications*. Berlin, Genmany: Springer, 2016.

[8] V. L. Selivanov, Two theorems on computable numberings, *Algebra Logic*, vol. 15, no. 4, pp. 297–306, 1976.

[9] Y. L. Ershov, Necessary isomorphism conditions for Rogers semilattices of finite partially ordered sets, *Algebra Logic*, vol. 42, no. 4, pp. 232–236, 2003.

[10] Y. L. Ershov, Rogers semilattices of finite partially ordered sets, *Algebra Logic*, vol. 45, no. 1, pp. 26–48, 2006.

[11] Y. L. Ershov, *Theory of Numberings*, (in Russian). Moscow, USSR: Nauka, 1977.

[12] Y. L. Ershov, Theory of numberings, in *Handbook of Computability Theory*, E. R. Griffor, ed. Amsterdam, the Netherlands: North-Holland, 1999, pp. 473–503.

[13] S. A. Badaev and S. S. Goncharov, The theory of numberings: Open problems, in *Computability Theory and Its Applications*, P. Cholak, S. Lempp, M. Lerman, and R. Shore, eds. Providence, RL, USA: American Mathematical Society, 2000, pp. 23–38.

[14] R. Wille, Restructuring lattice theory: an approach based on hierarchies of concepts, in *Ordered Sets*, I. Rival, ed. Dordrecht, Netherlands: D. Reidel Publishing Company, 1982, pp. 445–470.

[15] M. Hoyrup and C. Rojas, Computability of probability measures and Martin-Löf randomness over metric spaces, *Inf. Comput.*, vol. 207, no. 7, pp. 830–847, 2009.

[16] N. Bazhenov, M. Mustafa, and A. Nurakunov, On two types of concept lattices in the theory of numberings, in *Proc. Theory and Applications of Models of Computation, 17th Annual Conference*, Tianjin, China, 2022, pp. 79–92.

[17] C. J. Ash and J. F. Knight, *Computable Structures and the Hyperarithmetical Hierarchy*. Amsterdam, the Netherlands: Elsevier Science B.V., 2000.

[18] B. Ganter and R. Wille, *Formal Concept Analysis*: *Mathematical Foundations*. Berlin, Germany: Springer, 1999.

[19] B. A. Davey and H. A. Pristley, *Introduction to Lattices and Order, Second Edition*. Cambridge, UK: Cambridge University Press, 2002.

[20] R. Balbes and P. Dwinger, *Distributive lattices*. Columbia, MO, USA: University of Missouri Press, 1974.

[21] A. B. Khutoretskii, On the cardinality of the upper semilattice of computable enumerations, *Algebra Logic*, vol. 10, no. 5, pp. 348–352, 1971.

[22] Y. L. Ershov and I. A. Lavrov, The upper semilattice $L(\gamma)$, *Algebra Logic*, vol. 12, no. 2, pp. 93–106, 1973.

[23] V. V. V'yugin, On some examples of upper semilattices of computable enumerations, *Algebra Logic*, vol. 12, no. 5, pp. 287–296, 1973.

[24] S. S. Goncharov and A. Sorbi, Generalized computable numerations and nontrivial Rogers semilattices, *Algebra Logic*, vol. 36, no. 6, pp. 359–369, 1997.

[25] S. A. Badaev, S. S. Goncharov, and A. Sorbi, Elementary theories for Rogers semilattices, *Algebra Logic*, vol. 44, no. 3, pp. 143–147, 2005.

[26] S. A. Badaev, S. S. Goncharov, and A. Sorbi, Isomorphism types of Rogers semilattices for families from different levels of the arithmetical hierarchy, *Algebra Logic*, vol. 45, no. 6, pp. 361–370, 2006.

[27] S. Yu, Podzorov, Arithmetical *D*-degrees, *Sib. Math. J.*, vol. 49, no. 6, pp. 1109–1123, 2008.

[28] S. Badaev and S. Goncharov, Computability and numberings, in *New Computational Paradigms*, S. B. Cooper, B. Löwe, and A. Sorbi, eds. New York, NY, USA: Springer, 2008, pp. 19–34.

[29] N. Bazhenov, M. Mustafa, and M. Yamaleev, Elementary theories and hereditary undecidability for semilattices of numberings, *Arch. Math. Logic*, vol. 58, nos. 3&4, pp. 485–500, 2019.

[30] S. S. Goncharov, S. Lempp, and D. R. Solomon, Friedberg numberings of families of *n*-computably enumerable sets, *Algebra Logic*, vol. 41, no. 2, pp. 81–86, 2002.

[31] S. A. Badaev and S. Lempp, A decomposition of the Rogers semilattice of a family of d.c.e. sets, *J. Symb. Log.*, vol. 74, no. 2, pp. 618–640, 2009.

[32] I. Herbert, S. Jain, S. Lempp, M. Mustafa, and F. Stephan, Reductions between types of numberings, *Ann. Pure Appl. Logic*, vol. 170, no. 12, p. 102716, 2019.

[33] N. Bazhenov, S. Ospichev, and M. Yamaleev, Isomorphism types of Rogers semilattices in the analytical hierarchy, arXiv preprint arXiv: 1912.05226, 2019.

[34] N. A. Bazhenov, M. Mustafa, S. S. Ospichev, and M. M. Yamaleev, Numberings in the analytical hierarchy, *Algebra Logic*, vol. 59, no. 5, pp. 404–407, 2020.

[35] N. A. Bazhenov, M. Mustafa, and Z. Tleuliyeva, Theories of Rogers semilattices of analytical numberings, *Lobachevskii J. Math.*, vol. 42, no. 4, pp. 701–708, 2021.

[36] M. Hoyrup, Computability, randomness and ergodic theory on metric spaces, PhD dissertation, Université Paris

Diderot, Paris, France, 2008.
[37] B. Balcar and T. Jech, Weak distributivity, a problem of von Neumann and the mystery of measurability, *Bull. Symb. Log.*, vol. 12, no. 2, pp. 241–266, 2006.
[38] T. Jech, *Set Theory, The Third Millennium Edition*. Berlin, Germany: Springer, 2003.
[39] N. Bazhenov, M. Harrison-Trainor, I. Kalimullin, A. Melnikov, and K. M. Ng, Automatic and polynomial-time algebraic structures, *J. Symb. Log.*, vol. 84, no. 4, pp. 1630–1669, 2019.
[40] W. Calvert and J. F. Knight, Classification from a computable viewpoint, *Bull. Symb. Log.*, vol. 12, no. 2, pp. 191–218, 2006.
[41] E. B. Fokina, V. Harizanov, and A. Melnikov, Computable model theory, in *Turing's Legacy*: *Developments from Turing's Ideas in Logic*, R. Downey, ed. Cambridge, UK: Cambridge University Press, 2014, pp. 124−194.
[42] T. G. McLaughlin, The family of all recursively enumerable classes of finite sets, *Trans. Am. Math. Soc.*, vol. 155, no. 1, pp. 127–136, 1971.
[43] S. Wehner, Computable enumeration and the problem of repetition, PhD dissertation, Simon Fraser University, Burnaby, Canada, 1995.
[44] J. Harrison, Recursive pseudo well-orderings, *Trans. Am. Math. Soc.*, vol. 131, no. 2, pp. 526–543, 1968.
[45] W. M. White, On the complexity of categoricity in computable structures, *Math. Log. Q.*, vol. 49, no. 6, pp. 603–614, 2003.

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