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An Optimal Pricing and Ordering Policy with Trapezoidal-Type Demand Under Partial Backlogged Shortages

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Abstract: Based on the retail inventory operation of Heilan Home, this study incorporates the price factor into inventory environment involving trapezoidal time-varying products. A joint pricing and ordering issue with deteriorating items under partial backlogged shortages is firstly explored in a fixed selling cycle. The corresponding optimization model aiming at maximizing profit performance of inventory system is developed, the theoretical analysis of solving the model is further provided, and the modelling frame generalizes some inventory models in the existing studies. Then, a solving algorithm for the model is designed to determine the optimal price, initial ordering quantity, shortage time point, and the maximum inventory level. Finally, numerical examples are presented to illustrate the model, and the results show the robustness of the proposed model.

Key words: inventory; trapezoidal-type demand; pricing; deteriorating items; partial backlogged shortages

1 Introduction

In modern commercial activities, inventory is usually used to smooth the downstream demand and fulfill the upstream orders, which plays an important role in modern retail operation. A high product inventory usually provides customers a high service level, but not absolutely. For example, possessing too much inventory for ages may bring out the threat of obsolescence in product style and the substantial increase in holding costs, simultaneously. On the other hand, keeping too little inventory may result in the risk of shortages and profit loss from lost sales. As a result, it is crucial for retailers to manage inventory efficiently in today's business practice.

Pricing is one of the significant characteristics in the inventory-based research[1]. Especially when customer demand is fluctuating with time, the importance of the price characteristic is increased[2]. Trapezoidal demand, as a typical time-dependent pattern, is described as "with the advance of time, the demand for the items initially increases, then becomes stable, and finally decreases to a constant or zero"[3, 4]. Nowadays, due to continuous renewal of technology and fierce business competition, most short-cycle products basically follow this time trajectory in the terminal retail market[5]. In academia, many scholars have extensively explored inventory models considering trapezoidal-type demand[6–9]. In existing studies[10–12], the customer demand is often characterized as a function of time, stock level or selling price, separately. However, in the actual retail operation, the time and the selling price ought to be investigated jointly. The reason behind this observation is that the demand is closely related to the market stage of the product, storage status, and the selling price, namely, the customer demand may vary with the time, and meanwhile it may also vary when the selling price decreases or increases. Hence, integrating the selling price and trapezoidal-type

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demand, in the inventory system is vital to the real-life operatio[13, 14].

Our motivation stems from the retail inventory operation of Heilan Home (HLA), who, as a famous China's clothing retailer, mainly runs menswear products[15]. Most products displayed in HLA's store are seasonal and remain comparatively fixed sales period. The basic customer demand fits well into the trapezoidal time-varying distribution. Specifically, the demand will increase with time at the beginning when potential buyers are attracted by the products style, then keep steady once the products style is accepted, and finally decrease with time till the end of the sales season. In price, HLA sets a unified selling price for the same menswear style. However, if the product price increases, customers often turn to other brands, resulting in the decrease in demand. HLA's upstream manufacturers adopt the made-to-order mode. After coming in contact with HLA's staffs, we know that the dilemma of HLA inventory operation. On one hand, once the retailer's inventory system is out of stock, backorders will be produced by regional factories and delivered to the store at the end of the replenishment cycle, which often leads some customers switch to similar products sold by other retailers. On the other hand, if the initial order quantity is too large, then the total holding costs increases, and unsalable phenomenon also often occurs. Therefore, it is very important for HLA how to make trade-offs between shortages and the order fulfillment. However, existing related inventory models involving joint price- and trapezoidal-dependent demand[13, 14] ignore the impacts of this shortage factor and lost orders on the target performance of the inventory system. To the best of our knowledge, there is no previous study concerning inventory system optimization that focus on the effects of pricing, trapezoidal-type demand, deteriorating factor, and partial backlogged shortages on the ordering strategies, simultaneously.

To cover this research gap, we model a pricing and replenishment strategy issue based on the retail operation case of HLA. Specifically, we explore an inventory system aiming at maximizing profit performance with deteriorating items in a limited sales cycle, in which the trapezoidal time effect as a basic demand rate is adopted, and customer demand is quantified as a function of trapezoidal time and price. Inventory system permits shortages, the unmet customer demand in this interim period is partially backlogged, and the demand backlogged is satisfied at the end of the inventory replenishment period. Also, partial backlogging can generate losses of profits.

The rest of this paper is organized as follows. Section 2 mainly reviews the inventory literature on trapezoidal-type demand, inventory pricing issue, and inventory shortages. Section 3 gives the problem description and introduces notations as well as some related assumptions. Section 4 models an optimization problem and discusses the optimality of the model. Section 5 lists special cases. Section 6 designs a solving algorithm for the optimization model. Section 7 presents three numerical examples to show all the possible optimal values in the feasible region. Section 8 provides sensitivity analysis for gaining the robustness of the model and managerial implications. Section 9 summarizes the whole study and gives future research direction.

2 Related Work

The critical features of this article are the demand effect of the trapezoidal time, the pricing of products, and the application of partial backlogging. To highlight our contribution, three related research directions in the inventory-based literature are reviewed: trapezoidal-type demand, inventory pricing issue, and inventory shortages.

2.1 Trapezoidal-type demand

In the classical economic ordering quantity (EOQ) issue, the customer demand is usually characterized by a constant in the model assumption. However, for some fad or seasonal products, the demand reflected in each phase of their lifetime often varies over time. For example, the demand for these products in the introduction and growth phase gradually increases, while the demand observed in the mature phase is relatively stable, and thereafter, the demand in the decline phase decreases with time and is gradually withdrawn from the market. The demand that fits the above-mentioned time-varying feature in the product lifetime is often referred to trapezoidal-type demand, which is more general than other demand types including constant demand, increasing demand over time, decreasing demand over time, and ramp-type time-varing demand. In the last decade, the inventory models related to trapezoidal-type demand have attracted much more attention. Cheng and Wang[3] firstly explored an inventory model for deteriorating items with trapezoidal-type demand which considered a linearly piecewise function of the time. Based on a generalized trapezoidal-type demand, Cheng et al.[4] further developed an inventory repelenishment policy considering an exponential time-varying partial backlogging rate. Under trapezoidal-type demand Singh et al.[5] analyzed an EOQ issue with trade credit, where the product lifetime is asummed to be a random variable and follow a generalized Pareto distribution. Later, Uthayakumar and Rameswari[6] extended an ordering problem concerning trapezoidal-type demand to the economic production ordering quantity environment.

Considering a generalized trapezoidal-type demand function, a time-varying deterioration rate, and an extanded partial backlogging rate, Lin[7] further proposed an integrated inventory model and gave sufficient conditions for the global optimal solution of the model. In a time-dependent deterioration and backlogging setting, Wu et al.[16] also studied two ordering issues covering shortages and no shortages for trapezoidal-type products. More recently, some ralated studies with trapezoidal demand have also discussed inventory replenishment policies by investigating the maximum lifetime[8, 13], two-warehouse ordering and warehouse mode selection[9], permissible delay in payment[17], crisp and fuzzy environment[18], and one-time order inventory[19]. Howerer, the impact of the prcing factor on the inventory system has not been considered in the above-memtioned studies.

2.2 Inventory pricing

In today's business practice, customers in the purchasing state are more sensitive to the product price than ever before[11]. Usually, low-priced commodities generate higher customer demand and high-priced commodities result in lower demand, which will affect the decision-making of inventory managers in ratail operation. In the existing literature on price-dependent demand, linear price-dependent demand is most frequently assumed[1, 20–22]. Subsequent demand inclues logarithmic-concave-type demand[10, 23], exponential price-dependent demand[2, 11, 24], power price-dependent demand[25–27], and so on.

In addition, a lot of recent scholars have also devoted more attention to the replenishment system considering price-dependent demand. For example, focusing on ameliorating items, Mondal et al.[28] investigated a deteriorating issue concerning the price-dependent demand and no shortages. Then, Mishra et al.[29] explored a deterministic ordering model for deteriorating items under selling price-dependent demand, where holding cost is assumed to be time dependent. Considering a limited shelf/display space, Teng and Chang[30] further formulated an economic production quantity issues under deteriorating enviroment, in which the customer demand is characterized by the price factor and on-display stock, jointly. Under the production inventory environment, Sridevi et al.[31] built a random model under price-dependent demand and considered that the production rate follows a Weibull distribution. In an inflation setting, Rao and Rao[32] also analyzed an ordering issue. In their model, the lifetime is set to be a generalized Pareto distribution and the delay in payments is allowed. More recently, taking into account the credit financing and non-instantaneous deterioration, Jaggi et al.[33] further generalized a single inventory case to two storage facilities. For expired products, Khan et al.[1] proposed two inventory systems with no shortages and partial backlogged shortages under price-dependent demand. However, previous researches on price-dependent demand seldom considered the effects of trapezoidal-type demand products on inventory ordering decisions.

2.3 Inventory shortages

Inventory shortages are very common in retail industies. Within the inventory order cycle, holding cost usually accounts for a large proportion of the total inventory cost, and therefore, an inventory system allowing appropriate shortages is often less expensive to control than that of no any shortages[34]. Yang et al.[35] developed a lot-size ordering model considering inflation and shortages. Later, Chu et al.[36] explored a replenishment policy with a mixture of back orders and lost sales, and discussed the optimality of the model. Dye et al.[37] further extended the inventory model of Jaggi et al.[33] to any log-concave demand case. The common characteristic of the above-mentioned literature is that the system starts with shortages but ends with zero inventory. By contrast, there is also an inventory situation that starts with the maximum inventory but ends with shortages. For example, Dye[38] investigated a joint pricing and ordering inventory issue with time-varying deteriorating rate and partial backlogging. Abad[39] incorporated shortages cost and lost sales cost into a lot-size inventory model with the pricing. Moreover, related researches also dedicated to the inventory system with shortages facing the economic ordering quantity problem[40–42].

Generally, it is often assumed that unsatisfied demand is either completely lost or completely backlogged in existing literature on shortages. However, the partial backologging phenomenon often occurs in some real-life situations, i.e., some loyal purchasers still tend to wait although the waiting time is longer, while other impatient buyers will choose to go elsewhere. In recent years, a lot of researches have focused on exploring inventory ordering policies with shortages and partial backlogging. For example, considering the price and order size, Abad[43] proposed a model concerning partial backlogging. Then, focusing on stock-dependent demand rate, Dye and Ouyang[44] developed an EOQ model under time-varying partial backlogging. Based on a production inventory system, Giri et al.[45] formulated an European foundation for quality (EPQ) inventory model with partial backlogging and increasing demand. Assuming that deteriorating rate follows the weibull distribution and the demand rate is exponential declining, Pareek and Sharma[46] further investigated an inventory ordering issuse with partial backlogging. More recently, considering exponentially partial backlogging rate, Arif[47] explored an inventory replenishment policy with price-dependent demand. In a two-warehosue inventory setting, Gupta et al.[48] studied a retailer's ordering policy with partial backlogging rate and time-dependent deteriorating under permissible delay in payment.

The inventory features of existing studies are summarized in Table 1. Evidently, few scholars investigate joint pricing and ordering strategies with trapezoidal-type demand. Shah et al.[13] developed a model considering price-sensitive trapezoidal-type demand. However, their model ignores the impact of the shortages on inventory system. The work by Cheng and Wang[3] is closely related to this paper and particularly worth mentioning. Facing trapezoidal-type demand product, Cheng and Wang[3] explored an optimal inventory issue and analyzed how customer demand influence the total cost of the inventory system. However, it is assumed that the customer demand rate is only described as a piecewise function of time, and the pricing issue is not discussed in their model. In contrast, it is assumed that the demand is depended on time and price simultaneously in this paper, and we mainly focus on the impacts of the price and the time on the average total profit of the inventory

Table 1 Review of previous studies.

Reference	Shortage	Partial backlogging rate	Time dependent demand	Price dependent demand
Khan et al.[1]	Yes	Time-varying	No	Linear
San-José et al.[2]	Yes	No	Power	Logit
Cheng and Wang[3]	Yes	No	Trapezoidal	No
Cheng et al.[4]	Yes	Time-varying	Trapezoidal	No
Singh et al.[5]	Yes	No	Trapezoidal	No
Uthayakumar and Rameswari[6]	No	No	Trapezoidal	No
Lin[7]	Yes	Time-varying	Trapezoidal	No
Wu et al.[8]	Yes	Time-varying	Trapezoidal	No
Xu et al.[9]	Yes	Time-varying	Trapezoidal	No
Panda et al.[12]	Yes	Time-varying	Stock depend demand	Linear
Shah et al.[13]	No	No	Trapezoidal	Yes
Wu et al.[16]	Yes	Time-varying	Trapezoidal	No
Bhunia and Maiti[49]	Yes	No	Linear	No
Yang[50]	Yes	No	Constant	No
Skouri et al.[51]	Yes	Time-varying	Ramp	No
Agrawal and Banerjee[52]	Yes	Constant	Ramp	No
Agrawal et al.[53]	Yes	Constant	Ramp	No
Sarkar et al.[54]	Yes	Time-varying	Quadratic	No
Panda et al.[55]	No	No	Ramp	No
Jaggi et al.[56]	Yes	Time-varying	Linear	No
Present	Yes	Time-varying	Trapezoidal	Yes

system. Moreover, although shortages are permitted and unsatisfied customer demand is considered to be completely backlogged in their model, lost sales situation which ties up with inventory performance and will increase the retailer's operation cost is not considered. In this study, the partial backlogging rate is considered to be dependent on the customer waiting time, and the loss caused by lost sales is incorporated into the profit performance of the model.

The main contributions of this study are as follows. First, the price factor is incorporated into inventory ordering issues considering the trapezoidal time effect as a basic demand rate, and an inventory replenishment policy including ordering quantity, the maximum inventory level, selling price, and shortage time point is characterized. Second, the conditions for obtaining the optimal solution of the model are provided, and the corresponding algorithm for solving the model is presented. Third, the robustness and applicability of the model are illustrated. Fourth, the utilization of a multiplicative demand form enlarges the application scope of our inventory model. For example, by setting the relevant model parameters, our model can be applicable to some specific retail settings such as price-dependent demand, linear time-varying demand, ramp-type time-varying demand, full backordering, no shortages, among others.

3 Model Description

In this section, we describe a replenishment issue involving deteriorating product based on a joint price- and trapezoidal-dependent demand, and introduce necessary model notations and assumptions.

3.1 Problem description

Consider a continuous review retail system, where the retailer sells products kept in the system to the end customer in a finite inventory planning horizon. The retail replenishment for products is instantaneous and the lead time is zero. When products from the upstream manufacturer enter the retail system, they are exhausted because of the deterioration and customer demand until the inventory level is zero. Retail system allows shortages. The retailer is able to forecast customer demand and determine the initial ordering quantity according to the demand and shortages. Once shortages happen, backlogged demand is satsified at the end of the inventory cycle. Considering the impact of the waiting time on customer demand, the partial backlogging rate is adopted as $Z(x) = \mathrm{e}^{-\delta x}$, where x is the waiting time up to the next replenishment and $0 < \delta < 1$. This implies that the more cumulative unsatisfied customers in the waiting line, the more the amount of lost sales due to shortages. The exponential backlogging rate is widely used in the literature[4, 9, 49].

3.2 Model notations

The following notations including model paremetres, domain parameters, decision variables, and other variables are listed.

(1) Model parameters

A_0: denotes the fixed cost per cycle;

c: denotes the unit buying cost;

c_1: denotes the unit handing cost of the deteriorated item;

c_2: denotes the per unit inventory holding cost per unit time;

c_3: denotes the per unit shortage cost per unit time;

c_4: denotes the unit lost sales cost;

T: denotes the length of inventory cycle;

p^L: denotes the minimum lower bound of allowable price;

p^U: denotes the maximum upper bound of allowable price;

$I(t)$: denotes the inventory level;

(2) Domain parameters

D_1 denotes a region defined as $D_1 = \left\{(t_1, p)\middle|0 \leqslant t_1 \leqslant \mu_1;\ p^L \leqslant p \leqslant p^U\right\}$;

D_2 denotes a region defined as $D_2 = \left\{(t_1, p)\middle|\mu_1 \leqslant t_1 \leqslant \mu_2;\ p^L \leqslant p \leqslant p^U\right\}$;

D_3 denotes a region defined as $D_3 = \left\{(t_1, p)\middle|\mu_2 \leqslant t_1 \leqslant T;\ p^L \leqslant p \leqslant p^U\right\}$;

D: denotes a region defined as $D = \left\{(t_1, p)\middle|0 \leqslant t_1 \leqslant T;\ p^L \leqslant p \leqslant p^U\right\}$, that is $D = \bigcup_{i=1}^{3} D_i$.

(3) Decision variables

t_1: denotes the time point when the system starts to be out of stock;

p: denotes the selling price each unit, and $p \in [p^L, p^U]$;

Q_i: denotes the ordering amount per cycle for case with D_i;

S_i: denotes the maximum inventory level for case with D_i;

S: denotes the maximum inventory level per cycle in the entire system;

Q: denotes the total ordering amount per cycle in the entire system.

(4) Other variables

$AP_i(\cdot)$: denotes the average total profit for case with D_i;

$AP(\cdot)$: denotes the average total profit in the entire system on D.

3.3 Model assumptions

The items kept in the system are perishable, and the repair or replacement of deteriorated items is not considered during the storage period. The deterioration rate $\theta(t)$ is assumed to be time-dependent, and $0<\theta(t)<1$.

The demand rate is a general decreasing function of the selling price and varies trapezoidally with time. Here, we adopt $D(p,t)=A(t)d(p)$, where $d(p)$ satisfies $d'(p)<0$, $d''(p)\geqslant 0$, and $2d'(p)+d''(p)<0$; Trapezoidal demand $A(t)$ is shown in Fig. 1, that is,

$$A(t)=\begin{cases} a_1+b_1t, & 0\leqslant t\leqslant \mu_1; \\ d_0, & \mu_1\leqslant t\leqslant \mu_2; \\ a_2-b_2t, & \mu_2\leqslant t\leqslant T\leqslant \dfrac{a_2}{b_2}, \end{cases}$$

where μ_1 is the time point changing from the linearly increasing demand to the constant demand d_0, and μ_2 is the time point changing from d_0 to the linearly decreasing demand. This demand function of the price and the time can be employed to describe a real market phenomenon: the customer demand often decreases as the selling price increases, or it may change with time. The consideration of the multiplicative demand function is very useful especially for deteriorating items such as seasonal goods, clothes, and fad products[23]. In addition, the form of the multiplicative trapezoidal-type time effect is assumed to be a basic demand rate, which can essentially repesent the demand trajectory of short-cycle products[3, 6, 7].

Before the inventory system starts, the deterioration of transporting goods and the logistics cost from the supplier to the retail inventory system are ignored.

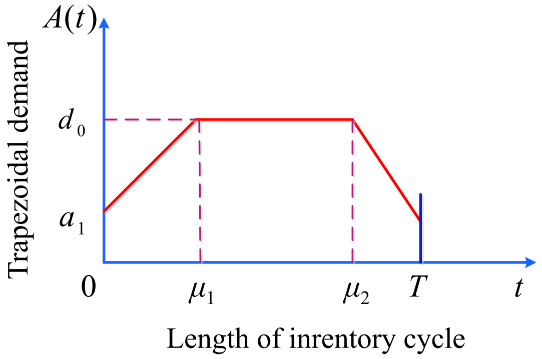

Fig. 1 A trapezoidal demand function of *t*.

4 Proposed Model and Model Analysis

The inventory model based on the inventory behavior of the system is firstly developed in this section. Then the properties of the model are analyzed and the optimal solutions to the model are given.

According to the above assumptions and problem description, Q units enter the system at the beginning of the inventory cycle (i.e., $t=0$). Beacause of the comprehensive effects of the customer demand and deterioration, $I(t)$ is depleted gradually in $(0,t_1)$, and it drops to zero at time point $t=t_1$. Shortages happen during $[t_1,T]$, and the cumulative shortages are partial backlogged at the rate of $Z(T-t)=\mathrm{e}^{-\delta(T-t)}$ until the end of the inventory cycle. Thus, the changes of the inventory level $I(t)$ in the closed interval $[0,T]$ can be described as

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t}=-A(t)d(p)-\theta(t)I(t),\ 0\leqslant t\leqslant t_1 \tag{1}$$

and

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t}=-Z(T-t)A(t)d(p),\ t_1\leqslant t\leqslant T \tag{2}$$

Considering possible values of μ_1, μ_2, t_1, p, and T, three different inventroy cases (see Fig. 2) are explored as follows.

4.1 Case with D_1

In this case, inventory depletion occurs in $[0,t_1]$ due to both the demand $(a_1+b_1t)d(p)$ and the deterioration $\theta(t)$, and therefore, $I(t)$ during $[0,t_1]$ can be described by

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t}=-(a_1+b_1t)d(p)-\theta(t)I(t),\ 0\leqslant t\leqslant t_1 \tag{3}$$

Solving Eq. (3) with the boundary condition $I(t_1)=0$, we have

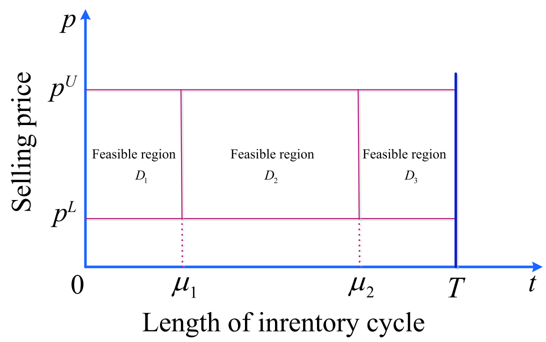

Fig. 2 Feasible region distribution of inventory system.

$$I(t) = -d(p)\mathrm{e}^{-\int_0^t \theta(x)\mathrm{d}x}, \int_{t_1}^{t} (a_1 + b_1 x)\mathrm{e}^{\int_0^x \theta(y)\mathrm{d}y}\mathrm{d}x, 0 \leqslant t \leqslant t_1 \quad (4)$$

In $[t_1, T]$, there is no deteriorating phenomenon in the system, the inventory depletion happens due to both the demand and the partial backlogging, and hence, the changes of $I(t)$ during $[t_1, T]$ satisfy Eqs. (5)–(7), respectively.

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -Z(T-t)(a_1 + b_1 t)d(p),\ t_1 \leqslant t \leqslant \mu_1 \quad (5)$$

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -Z(T-t)d_0 d(p),\ \mu_1 \leqslant t \leqslant \mu_2 \quad (6)$$

and

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -Z(T-t)(a_2 - b_2 t)d(p),\ \mu_2 \leqslant t \leqslant T \quad (7)$$

Solving Eqs. (5)–(7) with $I(t_1) = 0$, $I(\mu_1^-) = I(\mu_1^+)$, and $I(\mu_2^-) = I(\mu_2^+)$, we have

$$I(t) = -\frac{d(p)}{\delta^2}\mathrm{e}^{-\delta T}\left[\mathrm{e}^{\delta t}(a_1\delta + b_1\delta t - b_1) - \mathrm{e}^{\delta t_1}(a_1\delta + b_1\delta t_1 - b_1)\right], t_1 \leqslant t \leqslant \mu_1 \quad (8)$$

$$I(t) = -\frac{d(p)d_0}{\delta}\mathrm{e}^{-\delta T}\left(e^{\delta t} - \mathrm{e}^{\delta\mu_1}\right) + I\left(\mu_1^-\right), \mu_1 \leqslant t \leqslant \mu_2 \quad (9)$$

and

$$I(t) = -\frac{d(p)}{\delta^2}\mathrm{e}^{-\delta T}\left[\mathrm{e}^{\delta t}(a_2\delta - b_2\delta t + b_2) - \mathrm{e}^{\delta\mu_2}(a_2\delta - b_2\delta\mu_2 + b_2)\right] + I\left(\mu_2^-\right), \mu_2 \leqslant t \leqslant T \quad (10)$$

From Eq. (4), the maximum inventory level can be easily calculated as

$$S_1 = I(0) = d(p)\int_0^{t_1}(a_1 + b_1 x)\mathrm{e}^{\int_0^x \theta(y)\mathrm{d}y}\mathrm{d}x \quad (11)$$

Similarly, from Eqs. (10) and (11), the total order quantity per inventory cycle is expressed as

$$Q_1 = S_1 - I(T) \quad (12)$$

Moreover, the total quantity of lost sales in $[t_1, T]$ is calculated as

$$L_1 = d(p)\left\{\int_{t_1}^{\mu_1}[1 - Z(T-t)](a_1 + b_1 t)\mathrm{d}t + \int_{\mu_1}^{\mu_2}[1 - Z(T-t)]d_0\mathrm{d}t + \int_{\mu_2}^{T}[1 - Z(T-t)](a_2 - b_2 t)\mathrm{d}t\right\} \quad (13)$$

To sum up, from Eqs. (4) and (8)–(13), the related cost and total revenue in $[0, T]$ can be calculated as

- The setup cost: A_0;
- The ordering cost: $E_T = cQ_1$;
- The cost of deteriorated items: $D_T = c_1\left[S_1 - \int_0^{t_1} d(p)(a_1 + b_1 t)\mathrm{d}t\right]$;
- The inventory holding cost: $H_T = c_2\int_0^{t_1} I(t)\mathrm{d}t$;
- The inventory shortages cost: $B_T = -c_3\int_{t_1}^{T} I(t)\mathrm{d}t$;
- The opportunity cost caused by the lost sales: $O_T = c_4 L_1$;
- The total sales revenue:

$$F = p\left[\int_0^{t_1} d(p)(a_1 + b_1 t)\mathrm{d}t - I(T)\right].$$

Thus, the average total profit for case with the closed region D_1 is expressed by

$$AP_1(p, t_1) = \frac{F - H_T - B_T - O_T - D_T - E_T - A_0}{T} = \frac{1}{T}\left\{p\left[\int_0^{t_1} d(p)(a_1 + b_1 t)\mathrm{d}t - I(T)\right] - c_1\left[S_1 - \int_0^{t_1} d(p)(a_1 + b_1 t)\mathrm{d}t\right] - c[S_1 - I(T)] - c_2\int_0^{t_1} I(t)\mathrm{d}t + c_3\int_{t_1}^{T} I(t)\mathrm{d}t - c_4 d(p)\left[\int_{t_1}^{\mu_1}(a_1 + b_1 t)[1 - Z(T-t)]\mathrm{d}t + \int_{\mu_1}^{\mu_2} d_0[1 - Z(T-t)]\mathrm{d}t + \int_{\mu_2}^{T}(a_2 - b_2 t)[1 - Z(T-t)]\mathrm{d}t\right] - A_0\right\} \quad (14)$$

4.2 Case with D_2

In this case, the changes of $I(t)$ in $[0, t_1]$ are described, respectively, by

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -(a_1 + b_1 t)d(p) - \theta(t)I(t), 0 \leqslant t \leqslant \mu_1 \quad (15)$$

and

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -d_0 d(p) - \theta(t)I(t),\ \mu_1 \leqslant t \leqslant t_1 \quad (16)$$

Solving Eqs. (15) and (16) with $I(\mu_1^-) = I(\mu_1^+)$ and $I(t_1) = 0$, we have

$$I(t) = -d(p)\mathrm{e}^{-\int_0^t \theta(x)\mathrm{d}x}\left\{\int_{t_1}^{t}(a_1 + b_1 x)\mathrm{e}^{\int_0^x \theta(y)\mathrm{d}y}\mathrm{d}x + \int_{t_1}^{\mu_1}[d_0 - (a_1 + b_1 x)]\mathrm{e}^{\int_0^x \theta(y)\mathrm{d}y}\mathrm{d}x\right\}, 0 \leqslant t \leqslant \mu_1 \quad (17)$$

and

$$I(t) = -d(p)d_0\mathrm{e}^{-\int_0^t \theta(x)\mathrm{d}x}\int_{t_1}^{t}\mathrm{e}^{\int_0^x \theta(y)\mathrm{d}y}\mathrm{d}x, \mu_1 \leqslant t \leqslant t_1 \quad (18)$$

In $[t_1, T]$, the changes of $I(t)$ during this interval satisfy Eqs. (19) and (20).

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -Z(T-t)d_0d(p),\ t_1 \leqslant t \leqslant \mu_2 \tag{19}$$

and

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -Z(T-t)(a_2-b_2t)d(p),\ \mu_2 \leqslant t \leqslant T \tag{20}$$

Solving Eqs. (19) and (20) with $I(t_1)=0$ and $I(\mu_2^-)=I(\mu_2^+)$, we have

$$I(t) = -\frac{d(p)d_0}{\delta}[Z(T-t)-Z(T-t_1)],\ t_1 \leqslant t \leqslant \mu_2 \tag{21}$$

and

$$I(t) = -\frac{d(p)}{\delta^2}\mathrm{e}^{-\delta T}\left\{\mathrm{e}^{\delta t}[a_2\delta-b_2(t\delta-1)]- \mathrm{e}^{\delta\mu_2}[a_2\delta-b_2(\mu_2\delta-1)]\right\}+I(\mu_2^-),\ \mu_2 \leqslant t \leqslant T \tag{22}$$

From Eq. (17), the maximum inventory level can be obtained by

$$S_2 = d(p)\left\{\int_0^{t_1}(a_1+b_1x)\mathrm{e}^{\int_0^x\theta(y)\mathrm{d}y}\mathrm{d}x+ \int_{\mu_1}^{t_1}[d_0-(a_1+b_1x)]\mathrm{e}^{\int_0^x\theta(y)\mathrm{d}y}\mathrm{d}x\right\} \tag{23}$$

Similarly, from Eqs. (22) and (23), the total order quantity per inventory cycle is computed by

$$Q_2 = S_2 - I(T) \tag{24}$$

Moreover, the total quantity of lost sales during $[t_1,T]$ is calculated by

$$L_2 = d(p)\left\{\int_{t_1}^{\mu_2}d_0[1-Z(T-t)]\mathrm{d}t+ \int_{\mu_2}^{T}(a_2-b_2t)[1-Z(T-t)]\mathrm{d}t\right\} \tag{25}$$

Therefore, the average total profit with the closed region D_2 can be formulated as

$$\begin{aligned} &AP_2(t_1,p)= \\ &\frac{1}{T}\Big\{p\Big[\int_0^{\mu_1}d(p)(a_1+b_1t)\mathrm{d}t+\int_{\mu_1}^{t_1}d(p)d_0\mathrm{d}t-I(T)\Big]- \\ &c_2\int_0^{t_1}I(t)\mathrm{d}t+c_3\int_{t_1}^{T}I(t)\mathrm{d}t- \\ &c_4d(p)\Big\{\int_{t_1}^{\mu_2}[1-Z(T-t)]d_0\mathrm{d}t+ \\ &\int_{\mu_2}^{T}[1-Z(T-t)](a_2-b_2t)\mathrm{d}t\Big\}- \\ &c_1\Big[S_2-\int_0^{\mu_1}d(p)(a_1+b_1t)\mathrm{d}t-\int_{\mu_1}^{t_1}d(p)d_0\mathrm{d}t\Big]- \\ &c[S_2-I(T)]-A_0\Big\} \end{aligned} \tag{26}$$

4.3 Case with D_3

In this inventory case, the behaviors of $I(t)$ in $[0,t_1]$ satisfy Eqs. (27)–(29).

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -d(p)(a_1+b_1t)-\theta(t)I(t),\ 0 \leqslant t \leqslant \mu_1 \tag{27}$$

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -d_0d(p)-\theta(t)I(t),\ \mu_1 \leqslant t \leqslant \mu_2 \tag{28}$$

and

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -(a_2-b_2t)d(p)-\theta(t)I(t),\ \mu_2 \leqslant t \leqslant t_1 \tag{29}$$

Solving Eqs. (27)–(29) with $I(\mu_1^-)=I(\mu_1^+)$, $I(\mu_2^-)=I(\mu_2^+)$, and $I(t_1)=0$, we have

$$\begin{aligned} I(t) = &-d(p)\mathrm{e}^{-\int_0^t\theta(x)\mathrm{d}x}\Big\{\int_{t_1}^{t}(a_1+b_1x)\mathrm{e}^{\int_0^x\theta(y)\mathrm{d}y}\mathrm{d}x+ \\ &\int_{t_1}^{\mu_1}[d_0-(a_1+b_1x)]\mathrm{e}^{\int_0^x\theta(y)\mathrm{d}y}\mathrm{d}x+ \\ &\int_{t_1}^{\mu_2}[(a_2-b_2x)-d_0]\mathrm{e}^{\int_0^x\theta(y)\mathrm{d}y}\mathrm{d}x\Big\}, \\ &0 \leqslant t \leqslant \mu_1 \end{aligned} \tag{30}$$

$$\begin{aligned} I(t) = &-d(p)\mathrm{e}^{-\int_0^t\theta(x)\mathrm{d}x}\Big\{\int_{t_1}^{t}d_0\mathrm{e}^{\int_0^x\theta(y)\mathrm{d}y}\mathrm{d}x+ \\ &\int_{t_1}^{\mu_2}[(a_2-b_2x)-d_0]\mathrm{e}^{\int_0^x\theta(y)\mathrm{d}y}\mathrm{d}x\Big\}, \\ &\mu_1 \leqslant t \leqslant \mu_2 \end{aligned} \tag{31}$$

and

$$\begin{aligned} I(t) = &-d(p)\mathrm{e}^{-\int_0^t\theta(x)\mathrm{d}x}\int_{t_1}^{t}(a_2-b_2x)\mathrm{e}^{\int_0^x\theta(y)\mathrm{d}y}\mathrm{d}x, \\ &\mu_2 \leqslant t \leqslant t_1 \end{aligned} \tag{32}$$

In $[t_1,T]$, the inventory behavior of $I(t)$ is described by

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -Z(T-t)(a_2-b_2t)d(p),\ t_1 \leqslant t \leqslant T \tag{33}$$

Solving Eq. (33) with $I(t_1)=0$, we have

$$I(t) = -\frac{d(p)}{\delta^2}\mathrm{e}^{-\delta T}\left\{\mathrm{e}^{\delta t}[a_2\delta-b_2(t\delta-1)]- \mathrm{e}^{\delta t_1}[a_2\delta-b_2(t_1\delta-1)]\right\},\ t_1 \leqslant t \leqslant T \tag{34}$$

From Eq. (30), the maximum inventory level can be gained by

$$\begin{aligned} S_3 = &d(p)\Big\{\int_0^{t_1}(a_1+b_1x)\mathrm{e}^{\int_0^x\theta(y)\mathrm{d}y}\mathrm{d}x+ \\ &\int_{\mu_1}^{t_1}[d_0-(a_1+b_1x)]\mathrm{e}^{\int_0^x\theta(y)\mathrm{d}y}\mathrm{d}x+ \\ &\int_{\mu_2}^{t_1}[(a_2-b_2x)-d_0]\mathrm{e}^{\int_0^x\theta(y)\mathrm{d}y}\mathrm{d}x\Big\} \end{aligned} \tag{35}$$

Similarly, from Eqs. (34) and (35), the ordering quantity per inventory cycle can be computed by

$$Q_3 = S_3 - I(T) \tag{36}$$

In addition, the total quantity of lost sales in $[t_1,T]$ is calculated by

$$L_3 = d(p)\left\{\int_{t_1}^{T}(a_2 - b_2 t)[1 - Z(T-t)]\,\mathrm{d}t\right\} \tag{37}$$

Therefore, the average total profit for case with the closed region D_3 is expressed as

$$\begin{aligned}
&AP_3(t_1,p) = \\
&\frac{1}{T}\Big\{p\Big[\int_0^{\mu_1} d(p)(a_1+b_1 t)\,\mathrm{d}t + \int_{\mu_1}^{\mu_2} d(p)d_0\,\mathrm{d}t\Big] + \\
&p\Big[\int_{\mu_2}^{t_1} d(p)(a_2-b_2 t)\,\mathrm{d}t - I(T)\Big] - \\
&c_2\int_0^{t_1} I(t)\,\mathrm{d}t + c_3\int_{t_1}^{T} I(t)\,\mathrm{d}t - \\
&c_4 d(p)\Big\{\int_{t_1}^{T}(a_2-b_2 t)[1-Z(T-t)]\,\mathrm{d}t\Big\} - \\
&c_1\Big[S_3 - \int_0^{\mu_1} d(p)(a_1+b_1 t)\,\mathrm{d}t - \\
&\int_{\mu_1}^{\mu_2} d(p)d_0\,\mathrm{d}t - \int_{\mu_2}^{t_1} d(p)(a_2-b_2 t)\,\mathrm{d}t\Big] - \\
&c[S_3 - I(T)] - A_0\Big\}
\end{aligned} \tag{38}$$

Combining with the above discussion of the inventory profit performance in each case, the average total profit for this system in the region D can be summarized as

$$AP(t_1,p) = \begin{cases} AP_1(t_1,p), & (t_1,p)\in D_1; \\ AP_2(t_1,p), & (t_1,p)\in D_2; \\ AP_3(t_1,p), & (t_1,p)\in D_3 \end{cases} \tag{39}$$

where $AP_1(t_1,p)$, $AP_2(t_1,p)$, and $AP_3(t_1,p)$ are obtained by Eqs. (14), (26), and (38), respectively. Then, the nonlinear programming Model M for this system can be formulated as below:

$$\begin{aligned} &\text{Model M}: \quad \max \quad AP(t_1,p), \\ &\text{s.t.,} \quad (t_1,p)\in D \text{ and } D = \bigcup_{i=1}^{3} D_i \end{aligned} \tag{40}$$

To obtain the optimal solution to the Model M, we have the following theorems.

Theorem 4.1 In the Model M, the first-order necessary criteria for maximizing the objective function $AP(t_1,p)$ is equivalent to the criteria that $f(t_1,p)=0$ and $g(t_1,p)=0$, where $f(t_1,p)$ and $g(t_1,p)$ can be provided by Eqs. (47) and (58), respectively.

Proof Refer to proof in Appendix 1.

Theorem 4.1 suggests that the optimality of this model depends not only on the unit purchasing cost and the costs incurred by storage, shortages and lost sales, but also on the trapezoidal time and the price. These findings are essentially different from previous researches[3, 7], in which only the model with the trapezoidal time-varying demand is investigated from the perspective of inventory operation cost and the impact of selling price on the ratail inventory system is not considered.

Next, we will formally characterize the solutions to Eqs. (47) and (58). By using Eq. (47), define $F(t_1) = f(t_1,p)$ for any given selling price p. The following proposition can be obtained.

Proposition 4.1 For any given selling price p, $F(t_1)$ is a monotonically decreasing function in $t_1 \in [0,T]$.

Proof Refer to proof in Appendix 2. ■

From Proposition 4.1, for any given selling price p, it is easy to verify that $F(0) = p(1-\mathrm{e}^{-\delta T}) + c_3 T\mathrm{e}^{-\delta T} + c_4(1-\mathrm{e}^{-\delta T}) - c(1-\mathrm{e}^{-\delta T}) > 0$ and $F(T) = -c_1(\mathrm{e}^{\int_0^T \theta(y)dy} - 1) - c_2\int_0^T[\mathrm{e}^{\int_0^T \theta(y)dy + \int_0^t \theta(x)dx}]\mathrm{d}t - c[\mathrm{e}^{\int_0^T \theta(y)dy} - 1] < 0$. By the intermediate value theorem, there exists a unique time point t_1 satisfying $F(t_1)=0$. Thus, a judgment can be made based on the relationship between t_1 and p: for any given selling price p, there exists a unique time point t_1 satisfying $f(t_1,p)=0$, which implies that t_1 can be uniquely determined as a function of p, and thus, the function relationship between them can be described as $t_1 = t_1(p)$. In addition, taking the implicit derivative of two sides of the equation $f(t_1,p)=0$ with respect to p, we have $\frac{\mathrm{d}t_1}{\mathrm{d}p} = -\frac{\partial f(t_1,p)/\partial p}{\partial f(t_1,p)/\partial t_1} = -\frac{1-\mathrm{e}^{-\delta(T-t_1)}}{F'(t_1)} > 0$. Substituting $t_1 = t_1(p)$ into Eq. (58) and denoting $G(p) = g(t_1(p),p)$, we obtain the following result.

Proposition 4.2 If $d'(p)[\alpha(t_1) + p\gamma(t_1)] + d(p)\gamma(t_1) < 0$ holds, then $G(p)$ is a monotonically decreasing function in $p \in (p^L, p^U)$, where $\alpha(t_1)$ and $\gamma(t_1)$ can be provided by Eqs. (56) and (57), respectively.

Proof Refer to proof in Appendix 3. ■

In general, once the system is out of stock, shortages not only affect initial orders, but also reduce customers' loyalty to the brand. Thus, the cost of shortages in actual operation is significantly larger than other inventory costs[35, 39]. In this case, $\alpha(t_1) + p\gamma(t_1)$ is usually greater than 0. While in reality, the value of δ is usually small, thus $\gamma(t_1)$ is very close to 0. As a result, without losing generality, we provide an sufficient assumption $d'(p)[\alpha(t_1) + p\gamma(t_1)] + d(p)\gamma(t_1) < 0$, which implies that $G(p)$ is usually a monotonically decreasing function of p in this inventory system considering shortages.

Theorem 4.2 For $G(p)$, $p \in \left[p^L, p^U\right]$, the optimal results are characterized as follows:

(1) If $G\left(p^L\right) \leqslant 0$, then there exists a unique solution pair (t_1^*, p^*) that maximizes $AP(t_1, p)$, where $p^* = p^L$, and $t_1^* = t_1^{\#1}$ is the solution of $f\left(t_1, p^L\right) = 0$.

(2) If $G\left(p^U\right) \geqslant 0$, then there exists a unique solution pair (t_1^*, p^*) that maximizes $AP(t_1, p)$, where $p^* = p^U$, and $t_1^* = t_1^{\#2}$ is the solution of $f\left(t_1, p^U\right) = 0$.

(3) If $G\left(p^L\right) > 0$ and $G\left(p^U\right) < 0$, then there exists a unique solution pair (t_1^*, p^*) that maximizes $AP(t_1, p)$, where (t_1^*, p^*) is the solution of equations $f(t_1, p) = 0$ and $g(t_1, p) = 0$.

Proof Refer to proof in Appendix 4. ■

Theorem 4.2 gives the proof the existence and uniqueness of the optimal solution to Model M and indicates that for high-priced products, the input of more shortages is not good for the profit performance of this system. Similarly, for low-priced products, the behavior of less shortages is not beneficial to the average total profit. In addition, if the pricing is not restricted by the external retail market, the retailer may gain more profit by adopting moderate shortages.

Theorem 4.3 In the Model M, let (t_1^*, p^*) be the optimal solution that maximizes $AP(t_1, p)$ in the region D, we have

(1) If $(t_1^*, p^*) \in D_1$, then $AP(t_1^*, p^*) = AP_1(t_1^*, p^*)$;

(2) If $(t_1^*, p^*) \in D_2$, then $AP(t_1^*, p^*) = AP_2(t_1^*, p^*)$;

(3) If $(t_1^*, p^*) \in D_3$, then $AP(t_1^*, p^*) = AP_3(t_1^*, p^*)$.

5 Special Case

The following models in the existing studies are taken as special cases of Model M.

(1) When $p = 0$, $c = 0$, $\theta(t) = \theta$, $d(p) = 1$, and $Z(x) = 1$, then Model M is simplified as that in Cheng and Wang[3].

(2) When $p = 0$, $c = 0$, $\theta(t) = \theta$, $d(p) = 1$, $Z(x) = 1$, $a_1 = 0$, $b_1 = D_0$, $a_2 = D_0\mu$, $b_2 = 0$, and $\mu_1 = \mu < \mu_2$, then model M is the same as that in Ref. [57].

(3) When $p = 0$, $c = 0$, $\theta(t) = abt^{b-1}$, $d(p) = 1$, $Z(x) = 1$, $a_1 = 0$, $b_1 = D_0$, $a_2 = D_0\mu$, $b_2 = 0$, and $\mu_1 = \mu < \mu_2$, then Model M is reduced to that in Ref. [58].

(4) When $p = 0$, $c = 0$, $\theta(t) = abt^{b-1}$, $d(p) = 1$, $Z(x) = \dfrac{1}{1+\delta x}$, $a_1 = 0$, $b_1 = D_0$, $a_2 = D_0\mu$, $b_2 = 0$, and $\mu_1 = \mu < \mu_2$, then M is simplified as that in Ref. [59].

Integrate the findings of Theorem 4.1 to Theorem 4.3 mentioned above, a solving algorithm is formulated. The optimal solution pair (t_1^*, p^*) and $AP(t_1^*, p^*)$ in Model M are gained by using the following algorithm.

6 Algorithm 1

Algorithm 1 Solving algorithm for the optimization model

Input: The values of exogenous variables and initial variables.

Output: The optimal pair (t_1^*, p^*) and $AP(t_1^*, p^*)$.

1: Put $p = p^L$ into Eq. (47) to get $t_1 = t_1^{\#1}$, where $t_1^{\#1}$ is the solution of $f\left(t_1, p^L\right) = 0$. Jump to Step 4.

2: Put $p = p^U$ into Eq. (47) to get $t_1 = t_1^{\#2}$, where $t_1^{\#2}$ is the solution of $f\left(t_1, p^U\right) = 0$. Jump to Step 4.

3: Judge the signs of $G(p^L) = g(t_1^{\#1}, p^L)$ and $G(p^U) = g(t_1^{\#2}, p^U)$. One of three possible cases is executed as follows.

(1) If $G(p^L) \leqslant 0$, then $t_1^* = t_1^{\#1}$, $p^* = p^L$.

(2) If $G(p^U) \geqslant 0$, then $t_1^* = t_1^{\#2}$, $p^* = p^U$.

(3) If $G(p^L) > 0$ and $G(p^U) < 0$, then the equations $f(t_1, p) = 0$ and $g(t_1, p) = 0$ are solved by the Newton-Raphson method, get (t_1^*, p^*).

4: Determine the optimal pair (t_1^*, p^*) and $AP(t_1^*, p^*)$.

(1) If $\left(t_1^*, p^*\right) \in D_1$, then put $\left(t_1^*, p^*\right)$ into Eqs. (11), (12), and (14), get the optimal $S^* = S_1(t_1^*, p^*)$, $Q^* = Q_1(t_1^*, p^*)$, and $AP(t_1^*, p^*) = AP_1(t_1^*, p^*)$;

(2) If $\left(t_1^*, p^*\right) \in D_2$, then put $\left(t_1^*, p^*\right)$ into Eqs. (23), (24), and (26), get the optimal $S^* = S_2(t_1^*, p^*)$, $Q^* = Q_2(t_1^*, p^*)$, and $AP(t_1^*, p^*) = AP_2(t_1^*, p^*)$;

(3) If $\left(t_1^*, p^*\right) \in D_3$, then put $\left(t_1^*, p^*\right)$ into Eqs. (35), (36), and (38), get the optimal $S^* = S_3(t_1^*, p^*)$, $Q^* = Q_3(t_1^*, p^*)$, and $AP(t_1^*, p^*) = AP_3(t_1^*, p^*)$.

7 Numerical Example

In order to further illustrate the model from numerical study, three examples are implemented. Specifically, the first numerical example illustrates the case when the constraints on the inventory shortage time point t_1 and the selling price p are loose, i.e., the optimal solution pair exists within the feasible region D. The other two examples consider that the optimal solution solution pair happens at the boundary point of the feasible region D.

Example 7.1 In this example, the values of the exogenous parameters are set as follows: $A_0 = \$200$/order, $c = \$20$/unit, $c_1 = \$3$/unit, $c_2 = \$10$/unit/week, $c_3 = \$30$/unit/week, $c_4 = \$25$/unit, $d_0 = 130$ units, $\mu_1 = 6$ weeks, $\mu_2 = 10$ weeks, $T = 12$ weeks, $a_1 = 100$, $a_2 = 220$, $b_1 = 5$, $b_2 = 9$, $p^L = \$100$/unit, $p^U = \$120$/unit,

$Z(T-t)=\mathrm{e}^{-0.1(T-t)}$, $d(p)=a-bp=200-1.5p$, and $\theta(t)=mt=0.065t$.

According to the algorithm provided in Section 5, initialize the values of $p^L=100$ and $p^U=120$. By solving $f\left(t_1,p^L\right)=0$ and $f\left(t_1,p^U\right)=0$, we have $t_1^{\#1}=5.5391$ and $t_1^{\#2}=5.6483$. Compute $G(p^L)$ and $G(p^U)$. Since $G(p^L)=g(5.5391,100)=52\,218.9046>0$ and $G(p^U)=g(5.6483,120)=-21\,678.8431<0$, using the Newton-Raphson method to solve equations $f(t_1,p)=0$ and $g(t_1,p)=0$, we have $p^*=114.1498$ and $t_1^*=5.6172$. Noting that $(t_1^*,p^*)=(5.6172,114.1498)\in D_1$, substitue $(5.6172,114.1498)$ into Eqs. (12) and (14) and compute the function values of $S_1(5.6172, 114.1498)=27\,860.94$ units, $Q_1(5.6172,114.1498)=45\,019.27$ units and $AP_1(5.6172,114.1498)=\$56\,881.34$, and thus we have the maximum inventory level $S^*=27\,860.94$ units, the optimal order quantity $Q^*=45\,019.27$ units, and the optimal average total profit $AP^*=\$56\,881.34$ (see Fig. 3). In this example, Fig. 4 shows that the optimal solution pair (t_1^*,p^*) occurs at an interior point of the feasible region D.

Example 7.2 Consider the following parameter values: $c=\$24$/unit, $c_1=\$3.6$/unit, $c_2=\$12$/unit/week, $c_3=\$36$/unit/week and $c_4=\$30$/unit, the values of the remaining exogenous parameters are the same as those in Example 7.1.

Initialize the parameter values of the model. By solving $f\left(t_1,100\right)=0$ and $f\left(t_1,120\right)=0$, we have $t_1^{\#1}=5.4413$ and $t_1^{\#2}=5.5391$. Since $G(p^L)=g(5.4413,100)=86\,919.3325>0$ and $G(p^U)=g(5.5391,120)=13\,376.4301>0$, $p^*=p^U=120$ and $t_1^*=t_1^{\#2}=5.5391$. Substituing $\left(t_1^*,p^*\right)$ into Eqs. (23), (24), and (26), we have the maximum inventory level $S^*=18\,815.49$ units, the optimal $Q^*=30\,846.30$ units, and the optimal $AP^*=\$12\,501.89$ (see Fig. 5). In this example, Fig. 6 shows that the optimal solution pair (t_1^*,p^*) occurs at an upper boundary set of the feasible region D.

Example 7.3 The following parameter values are

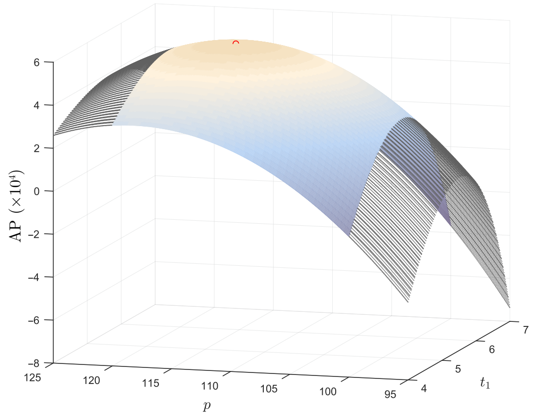

Fig. 3 Average total profit as a function of t_1 and p for Example 7.1.

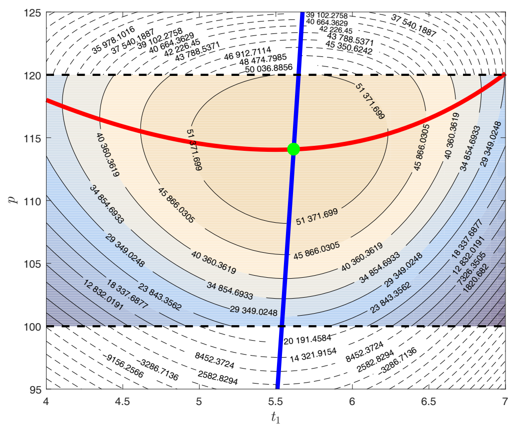

Fig. 4 Contour graph of optimal profit with t_1 and p for Example 7.1.

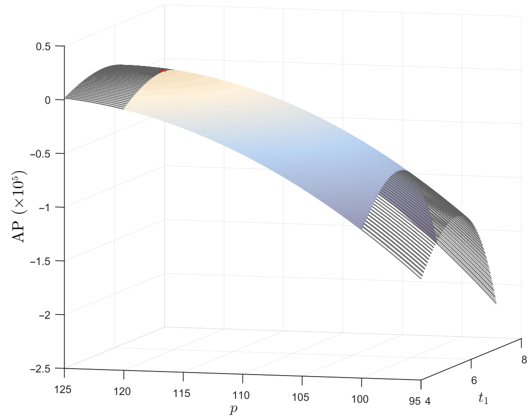

Fig. 5 Average total profit as a function of t_1 and p for Example 7.2.

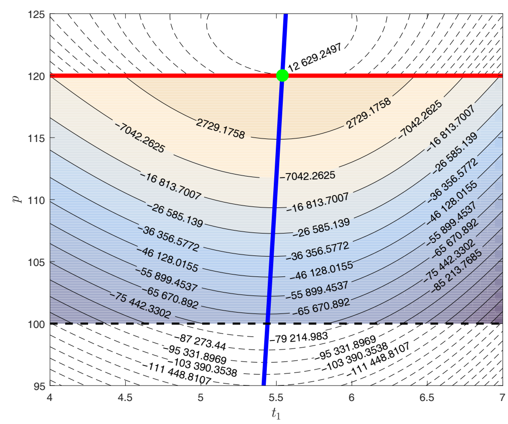

Fig. 6 Contour graph of optimal profit with t_1 and p for Example 7.2.

used: $c = \$14/\text{unit}$, $c_1 = \$2.1/\text{unit}$, $c_2 = \$7/\text{unit/week}$, $c_3 = \$21/\text{unit/week}$, $c_4 = \$17.5/\text{unit}$, the values of the remaining exogenous parameters are the same as those in Example 7.1.

Similarly, by solving $f(t_1, 100) = 0$ and $f(t_1, 120) = 0$, we have $t_1^{\#1} = 5.7635$ and $t_1^{\#2} = 5.8954$. Since $G(p^L) = g(5.7635, 100) = -48.4140 < 0$ and $G(p^U) = g(5.8954, 120) = -74\,752.0922 < 0$, $p^* = p^L = 100$ and $t_1^* = t_1^{\#1} = 5.7635$. Hence, the maximum inventory level is $S^* = 51103.10$ units, the optimal $Q^* = 80\,417.51$ units, and the optimal $AP^* = \$173\,129.01$ (see Fig. 7). In this example, Fig. 8 shows that the optimal solution pair (t_1^*, p^*) occurs at a lower boundary set of the feasible region D.

8 Sensitivity Analysis

The model parameters in the above numerical examples in Section 7 are all regarded as static value. It is crucial for inventory managers to know the impacts

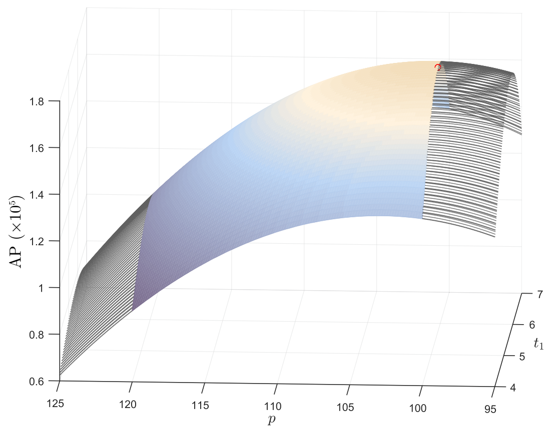

Fig. 7 The average total profit as a function of t_1 and p for Example 7.3.

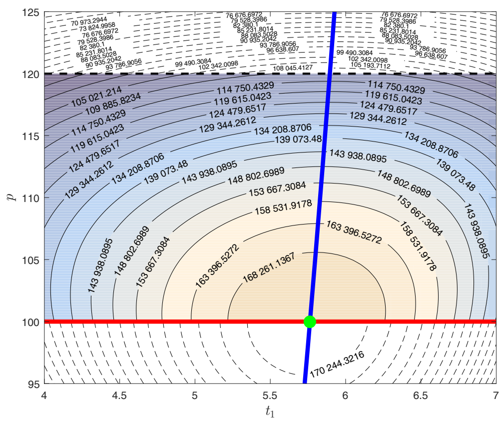

Fig. 8 Contour graph of optimal profit with t_1 and p for Example 7.3.

of the dynamic model parameters on the optimal solution. In view of this, sensitivity analysis is performed by varying one of θ, d_0, a, b, T, a_1, a_2, b_1, b_2, μ_1, and μ_2 while keeping other exogenous values fixed. The values of the parameters used in this section are the same as those in Example 7.1.

From Table 2, the robustness of the optimal solution can be easily observed as follows.

(1) The optimal p^* is insensitive to the exogenous parameters b_1, b_2, μ_1, and μ_2, but it is more sensitive to a, b, and T than d_0, a_1, and a_2.

(2) The optimal t_1^* is slightly sensitive to the exogenous parameter T, but it is nearly insensitive to others exogenous parameters.

(3) The optimal S^* is insensitive to the exogenous parameters μ_1 and μ_2, but it is slightly sensitive to d_0, b_1, and b_2, and particularly sensitive to a, b, a_1, a_2, and T.

(4) The optimal Q^* is insensitive to the exogenous parameter μ_1, but it is slightly sensitive to d_0, b_1, and b_2, and very sensitive to θ, a, b, T, a_1, a_2, and μ_2.

(5) The optimal AP^* is insensitive to the exogenous parameter μ_1, but it is slightly sensitive to d_0, b_1, and b_2, and highly sensitive to a, b, T, a_1, a_2, and μ_2.

On the whole, it can be inferred that the model M is nearly robust. However, it can also be found that the basic demand scale parameter a, the replenishment cycle T, and the price coefficient b have significant impacts on the total average profit compared to other exogenous parameters. In practice, for the potential market demand, inventory managers may use some effective methods such as moving average and exponential smoothing to improve the accuracy of demand forecasting according to the empirical data,. For the replenishment cycle T, inventory managers could choose the corresponding cost-volume-profit analysis to evaluate whether to fine-tune it.

9 Conclusion

In today's retail market, the customer demand is closely related to the market stage of the product, and thus the coordination of the pricing, time-varying factor, shortages, and the order fulfillment in the deteriorating inventory is not only essential but also utile. Our motivation stems from the investigation of HLA's inventory operation. Taking both pricing and trapezoidal-type demand into consideration, our study focuses on an inventory issue for deteriorating items in

Table 2 Sensitivity analysis for Example 7.1.

Parameter	Value	−30%	−20%	−10%	0%	10%	20%	30%
θ	p^*	111.3741	112.3791	113.3004	114.1498	114.9369	115.6694	116.3539
	t_1^*	6.2253	5.9995	5.7982	5.6172	5.4534	5.3042	5.1676
	S^*	33 638.4234	31 520.4681	29 600.5431	27 860.9420	26 278.2206	24 832.4289	23 506.6451
	Q^*	51 867.8679	49 427.5850	47 146.0262	45 019.2717	43 034.9318	41 180.6204	39 444.6212
	AP^*	76 697.8511	69 119.3201	62 576.4232	56 881.3417	51 887.1228	47 478.3934	43 563.4415
d_0	p^*	111.6722	112.5482	113.3726	114.1498	114.8838	115.5781	116.2359
	t_1^*	5.6038	5.6086	5.6130	5.6172	5.6211	5.6248	5.6283
	S^*	31 304.6197	30 091.1478	28 945.0795	27 860.9420	26 833.8404	25 859.3831	24 933.6187
	Q^*	47 288.0139	46 512.1712	45 756.4244	45 019.2717	44 299.3551	43 595.4437	42 906.4195
	AP^*	66 306.1953	62 960.6396	59 825.0179	56 881.3417	54 113.6275	51 507.6244	49 050.5846
a	p^*	-	100.7940	107.4691	114.1498	120.0000	120.0000	120.0000
	t_1^*	-	5.5436	5.5809	5.6172	5.6483	5.6483	5.6483
	S^*	-	8301.0427	17 957.0550	27 860.9420	39 175.6098	58 763.4147	78 351.2196
	Q^*	-	13 597.3861	29 211.1984	45 019.2717	62 942.8369	94 414.2554	125 885.6739
	AP^*	-	5 296.4208	24 217.9159	56 881.3417	103 214.6704	154 830.3390	206 446.0075
b	p^*	120.0000	120.0000	120.0000	114.1498	108.0762	103.0183	100.0000
	t_1^*	5.6483	5.6483	5.6483	5.6172	5.5842	5.5561	5.5391
	S^*	72 474.8782	54 845.8537	37 216.8293	27 860.9420	20 731.9435	13 790.3292	4703.8726
	Q^*	116 444.2483	88 119.9717	59 795.6951	45 019.2717	33 704.3270	22 536.3449	7711.5746
	AP^*	190 961.3070	144 507.2053	98 053.1036	56 881.3417	29 282.4810	12 098.0981	2591.3652
T	p^*	100.4370	104.5290	109.1893	114.1498	119.2649	120.0000	-
	t_1^*	4.8087	5.1377	5.4034	5.6172	5.7891	5.9009	-
	S^*	35 544.8016	35 091.7991	32 414.4931	27 860.9420	21 774.2955	21 513.4733	-
	Q^*	54 171.8097	55 225.9428	51 929.0991	45 019.2717	35 255.4897	34 886.8527	-
	AP^*	177 021.9856	135 550.6437	93 264.5765	56 881.3417	29 257.5777	8297.5554	-
a_1	p^*	116.8378	115.8160	114.9283	114.1498	113.4616	112.8489	112.2997
	t_1^*	5.6315	5.6261	5.6214	5.6172	5.6135	5.6102	5.6072
	S^*	17 888.5572	21 146.5469	24 475.0905	27 860.9420	31 293.9763	34 766.3212	38 271.7635
	Q^*	31 125.6548	35 740.2632	40 372.6119	45 019.2717	49 677.6427	54 345.7177	59 021.9228
	AP^*	34 284.2061	41 586.2758	49 133.7747	56 881.3417	64 794.1495	72 845.0140	81 012.4055
a_2	p^*	-	118.5584	116.0087	114.1498	112.7343	111.6203	110.7208
	t_1^*	-	5.6407	5.6271	5.6172	5.6096	5.6035	5.5987
	S^*	-	13 480.7082	20 488.8131	27 860.9420	35 464.9366	43 225.7105	51 097.4411
	Q^*	-	24 704.1087	34 815.7945	45 019.2717	55 280.3524	65 579.9337	75 906.5106
	AP^*	-	24 592.1429	40 104.5482	56 881.3417	74 469.7768	92 610.2692	111 143.2895
b_1	p^*	114.3786	114.3005	114.2243	114.1498	114.0771	114.0060	113.9365
	t_1^*	5.6184	5.6180	5.6176	5.6172	5.6168	5.6164	5.6160
	S^*	26 366.9648	26 864.5023	27 362.5004	27 860.9420	28 359.8109	28 859.0916	29 358.7695
	Q^*	42 755.6418	43 509.7920	44 264.3402	45 019.2717	45 774.5727	46 530.2299	47 286.2306
	AP^*	53 583.7982	54 681.2075	55 780.4096	56 881.3417	57 983.9438	59 088.1588	60 193.9320
b_2	p^*	113.5506	113.7372	113.9365	114.1498	114.3786	114.6245	114.8897
	t_1^*	5.6140	5.6150	5.6160	5.6172	5.6184	5.6197	5.6212
	S^*	32 364.4343	30 860.0487	29 358.7695	27 860.9420	26 366.9648	24 877.2999	23 392.4865
	Q^*	51 828.8147	49 556.1771	47 286.2306	45 019.2717	42 755.6418	40 495.7364	38 240.0162
	AP^*	66 858.5300	63 520.0916	60 193.9320	56 881.3417	53 583.7982	50 303.0021	47 040.9204
μ_1	p^*	114.2072	114.2102	114.1736	114.1498	114.1727	114.2377	114.3365

(to be continued)

Table 2 Sensitivity analysis for Example 7.1. (continued)

Parameter	Value	−30%	−20%	−10%	0%	10%	20%	30%
	t_1^*	5.6175	5.6175	5.6173	5.6172	5.6173	5.6177	5.6182
	S^*	28 249.0113	27 996.3534	27 868.7458	27 860.9420	27 828.9121	27 738.0663	27 600.0244
	Q^*	45 474.0383	45 154.7482	45 023.2138	45 019.2717	44 992.7828	44 923.0317	44 830.6454
	AP^*	57 084.9703	56 759.5240	56 793.3938	56 881.3417	56 787.7487	56 531.2290	56 163.9295
μ_2	p^*	114.7002	114.6958	114.3365	114.1498	114.2072	113.6684	112.0067
	t_1^*	5.6201	5.6201	5.6182	5.6172	5.6175	5.6146	5.6056
	S^*	27 091.1101	27 097.2975	27 600.0244	27 860.9420	28 249.0113	30 028.1787	33 861.5935
	Q^*	46 127.7908	44 756.9377	44 830.6454	45 019.2717	45 474.0383	48 125.3635	54 267.1845
	AP^*	57 360.8399	55 242.1872	56 163.9295	56 881.3417	57 084.9703	62 137.9513	76 706.9519

a fixed selling cycle, where the system allows shortages and partial backlogging rate is quantitatively described as a decreasing function concerning the waiting time. The existence and uniqueness of the optimal solution to the model is discussed, and a solving algorithm for the model is designed to determine the optimal price, initial ordering quantity, shortage time point, and the maximum inventory level. Numerical examples are presented to show all the possible optimal values in the feasible region. Sensitivity analysis is tested to illustrate the model robustness and its application.

There still exist some limitations in this paper. For future research, an inventory model with the variable inventory cycle will be considered. Moreover, some epochal inventory features such as stochastic demand, product promotion, trade credit, investment in shopping experience, and environmental regulation, will also be incorporated in this research.

Appendix

Appendix 1: Proof of Theorem 4.1.

To determine the optimal t_1 and p in the Model M, for each branch function of $AP(t_1,p)$, we take the first-order derivative of $AP_i(t_1,p)$ $(i=1,\ 2,\ 3)$ with respect to t_1 and p respectively as follows:

$$\frac{\partial AP_1(t_1,p)}{\partial t_1}=\frac{d(p)(a_1+b_1t_1)f(t_1,p)}{T} \tag{A1}$$

$$\frac{\partial AP_2(t_1,p)}{\partial t_1}=\frac{d(p)d_0 f(t_1,p)}{T} \tag{A2}$$

$$\frac{\partial AP_3(t_1,p)}{\partial t_1}=\frac{d(p)(a_2-b_2t_1)f(t_1,p)}{T} \tag{A3}$$

$$\frac{\partial AP_1(t_1,p)}{\partial p}=\frac{d'(p)K_1(t_1)+[d(p)+pd'(p)]M_1(t_1)}{T} \tag{A4}$$

$$\frac{\partial AP_2(t_1,p)}{\partial p}=\frac{d'(p)K_2(t_1)+[d(p)+pd'(p)]M_2(t_1)}{T} \tag{A5}$$

$$\frac{\partial AP_3(t_1,p)}{\partial p}=\frac{d'(p)K_3(t_1)+[d(p)+pd'(p)]M_3(t_1)}{T} \tag{A6}$$

where

$$\begin{aligned} f(t_1,p)=&p[1-Z(T-t_1)]-c_1\left[\mathrm{e}^{\int_0^{t_1}\theta(y)\mathrm{d}y}-1\right]-\\ &c_2\left[\int_0^{t_1}\mathrm{e}^{\int_0^{t_1}\theta(y)\mathrm{d}y-\int_0^{t}\theta(x)\mathrm{d}x}\mathrm{d}t\right]+\\ &c_3\left[\int_{t_1}^{T}Z(T-t)\,\mathrm{d}t\right]+c_4[1-Z(T-t_1)]-\\ &c\left[\mathrm{e}^{\int_0^{t_1}\theta(y)\mathrm{d}y}-Z(T-t_1)\right] \end{aligned} \tag{A7}$$

$$\begin{aligned} &K_1(t_1)=-\\ &c_1\left[\frac{S_1}{d(p)}-\int_0^{t_1}(a_1+b_1t)\,\mathrm{d}t\right]-\frac{c}{d(p)}[S_1-I(T)]-\\ &\frac{c_2}{d(p)}\int_0^{t_1}I(t)\,\mathrm{d}t+\frac{c_3}{d(p)}\int_{t_1}^{T}I(t)\,\mathrm{d}t-\\ &c_4\left[\int_{t_1}^{\mu_1}(a_1+b_1t)[1-Z(T-t)]\,\mathrm{d}t+\right.\\ &\int_{\mu_1}^{\mu_2}d_0[1-Z(T-t)]\,\mathrm{d}t+\\ &\left.\int_{\mu_2}^{T}(a_2-b_2t)[1-Z(T-t)]\,\mathrm{d}t\right] \end{aligned} \tag{A8}$$

$$\begin{aligned} &K_2(t_1)=-\frac{c_2}{d(p)}\int_0^{t_1}I(t)\,\mathrm{d}t+\frac{c_3}{d(p)}\int_{t_1}^{T}I(t)\,\mathrm{d}t-\\ &c_4\left\{\int_{t_1}^{\mu_2}[1-Z(T-t)]d_0\mathrm{d}t+\right.\\ &\left.\int_{\mu_2}^{T}[1-Z(T-t)](a_2-b_2t)\,\mathrm{d}t\right\}-\\ &c_1\left[\frac{S_2}{d(p)}-\int_0^{\mu_1}(a_1+b_1t)\,\mathrm{d}t-\int_{\mu_1}^{t_1}d_0\mathrm{d}t\right]-\\ &\frac{c}{d(p)}[S_2-I(T)] \end{aligned} \tag{A9}$$

$$K_3(t_1) = -\frac{c_2}{d(p)}\int_0^{t_1} I(t)\mathrm{d}t + \frac{c_3}{d(p)}\int_{t_1}^{T} I(t)\mathrm{d}t - c_4\left\{\int_{t_1}^{T}(a_2-b_2t)[1-Z(T-t)]\mathrm{d}t\right\}$$

$$K_3(t_1) = -\frac{c_2}{d(p)}\int_0^{t_1} I(t)\mathrm{d}t + \frac{c_3}{d(p)}\int_{t_1}^{T} I(t)\mathrm{d}t - c_4\left\{\int_{t_1}^{T}(a_2-b_2t)[1-Z(T-t)]\mathrm{d}t\right\} - c_1\left[\frac{S_3}{d(p)} - \int_0^{\mu_1}(a_1+b_1t)\mathrm{d}t - \int_{\mu_1}^{\mu_2} d_0\mathrm{d}t - \int_{\mu_2}^{t_1}(a_2-b_2t)\mathrm{d}t\right] - \frac{c}{d(p)}[S_3 - I(T)] \tag{A10}$$

$$M_1(t_1) = \int_0^{t_1}(a_1+b_1t)\mathrm{d}t + \frac{\mathrm{e}^{-\delta T}}{\delta^2}\left[\mathrm{e}^{\delta T}(a_2\delta - b_2\delta T + b_2) - b_2\mathrm{e}^{\delta\mu_2} - b_1\mathrm{e}^{\delta\mu_1} - \mathrm{e}^{\delta t_1}(a_1\delta + b_1\delta t_1 - b_1)\right] \tag{A11}$$

$$M_2(t_1) = \int_0^{\mu_1}(a_1+b_1t)\mathrm{d}t + d_0(t_1-\mu_1) + \frac{\mathrm{e}^{-\delta T}}{\delta^2}\left[\mathrm{e}^{\delta T}(a_2\delta - b_2\delta T + b_2) - b_2\mathrm{e}^{\delta\mu_2} - d_0\delta\mathrm{e}^{\delta t_1}\right] \tag{A12}$$

and

$$M_3(t_1) = \int_0^{\mu_1}(a_1+b_1t)\mathrm{d}t + \int_{\mu_1}^{\mu_2} d_0\mathrm{d}t + \int_{\mu_2}^{t_1}(a_2-b_2t)\mathrm{d}t + \frac{\mathrm{e}^{-\delta T}}{\delta^2}\left\{\mathrm{e}^{\delta T}[a_2\delta - b_2(\delta T - 1)] - \mathrm{e}^{\delta t_1}[a_2\delta - b_2(\delta t_1 - 1)]\right\} \tag{A13}$$

Next, we explore the properties of $\frac{\partial AP_i(t_1,p)}{\partial t_1}$ $(i = 1, 2, 3)$. It is easy to see from Eqs. (41)–(43) that $\frac{\partial AP_i(t_1,p)}{\partial t_1}$ has similar function structure. From the assumptions mentioned in Section 3.3 that $a_1 + b_1\mu_1 = d_0 = a_2 - b_2\mu_2$, we easily derive that the function $\frac{\partial AP(t_1,p)}{\partial t_1}$ is continuous in domain D and can be integratedly written as $\frac{\partial AP(t_1,p)}{\partial t_1} = \frac{1}{T}D(p,t_1)f(t_1,p)$. Since $D(p,t_1) > 0$, then $\frac{\partial AP(t_1,p)}{\partial t_1} = 0$ is just equivalent to $f(t_1,p) = 0$. Furthermore, we investigate the properties of $\frac{\partial AP_i(t_1,p)}{\partial p}$ $(i = 1, 2, 3)$. It's not hard to find from Eqs. (44)–(46) that $\frac{\partial AP_i(t_1,p)}{\partial p}$ also has similar function behavior. For the purpose of integrating these equations, we firstly analyze the functional characteristics of $K_i(t_1)$ and $M_i(t_1)$ $(i = 1, 2, 3)$. It is easy to check from Eqs. (48)–(53) that $\lim_{t_1\to\mu_1^-} K_1(t_1) = \lim_{t_1\to\mu_1^+} K_2(t_1)$, $\lim_{t_1\to\mu_2^-} K_2(t_1) = \lim_{t_1\to\mu_2^+} K_3(t_1)$, $\lim_{t_1\to\mu_1^-} M_1(t_1) = \lim_{t_1\to\mu_1^+} M_2(t_1)$, and $\lim_{t_1\to\mu_2^-} M_2(t_1) = \lim_{t_1\to\mu_2^+} M_3(t_1)$. Letting

$$K(t_1) = \begin{cases} K_1(t_1), & 0 \leqslant t_1 \leqslant \mu_1; \\ K_2(t_1), & \mu_1 \leqslant t_1 \leqslant \mu_2; \\ K_3(t_1), & \mu_2 \leqslant t_1 \leqslant T \end{cases} \tag{A14}$$

and

$$M(t_1) = \begin{cases} M_1(t_1), & 0 \leqslant t_1 \leqslant \mu_1; \\ M_2(t_1), & \mu_1 \leqslant t_1 \leqslant \mu_2; \\ M_3(t_1), & \mu_2 \leqslant t_1 \leqslant T \end{cases} \tag{A15}$$

we can gain that $K(t_1)$ and $M(t_1)$ are continuous on the interval $[0,T]$. Also, it is easy to see from Eqs. (48)–(53) that $K_i(t_1) < 0$ and $M_i(t_1) > 0$, $i = 1, 2, 3$. Hence, for $t_1 \in [0,T]$, we have $K(t_1) < 0$ and $M(t_1) > 0$. Addtionally, from Eqs. (48)–(50), we can derive $K_1'(t_1) = (a_1 + b_1t_1)\alpha(t_1)$, $K_2'(t_1) = d_0\alpha(t_1)$, and $K_3'(t_1) = (a_2 - b_2t_1)\alpha(t_1)$, where

$$\alpha(t_1) = -\mathrm{e}^{\int_0^{t_1}\theta(y)\mathrm{d}y}\left[c + c_2\int_0^{t_1}\mathrm{e}^{-\int_0^{t}\theta(x)\mathrm{d}x}\mathrm{d}t + c_1\right] + \mathrm{e}^{-\delta(T-t_1)}[c + c_3(T-t_1) - c_4] + c_1 + c_4 \tag{A16}$$

Noting $\lim_{t_1\to\mu_1^-} K_1'(t_1) = \lim_{t_1\to\mu_1^+} K_2'(t_1)$ and $\lim_{t_1\to\mu_2^-} K_2'(t_1) = \lim_{t_1\to\mu_2^+} K_3'(t_1)$, we thus gain $K(t_1)$ is differentiable on the interval $(0,T)$ and can be integrately written as $K'(t_1) = A(t_1)\alpha(t_1)$. Similarly, we also derive that $M(t_1)$ is differentiable on the interval $(0,T)$ and can be written as $M'(t_1) = A(t_1)\gamma(t_1)$, where

$$\gamma(t_1) = 1 - \mathrm{e}^{-\delta(T-t_1)} \tag{A17}$$

As a result, from the properties of $K(t_1)$ and $M(t_1)$ defined above, $\frac{\partial AP(t_1,p)}{\partial p}$ can also be written as $\frac{\partial AP(t_1,p)}{\partial p} = \frac{1}{T}g(t_1,p)$, where

$$g(t_1,p) = d'(p)K(t_1) + [d(p) + pd'(p)]M(t_1) \tag{A18}$$

which implies that $\frac{\partial AP(t_1,p)}{\partial p} = 0$ is equivalent to $g(t_1,p) = 0$.

This ends with the proof of Theorem 4.1.

Appendix 2: Proof of Proposition 4.1.

The first-order derivative of $F(t_1)$ concerning t_1 is $F'(t_1) = -c[\theta(t_1)\mathrm{e}^{\int_0^{t_1}\theta(y)\mathrm{d}y} - \delta\mathrm{e}^{-\delta(T-t_1)}] - c_1\theta(t_1)\mathrm{e}^{\int_0^{t_1}\theta(y)\mathrm{d}y} - c_2[\theta(t_1)\mathrm{e}^{\int_0^{t_1}\theta(y)\mathrm{d}y}\int_0^{t_1}\mathrm{e}^{\int_0^{t}-\theta(x)\mathrm{d}x}dt + 1] - c_3\mathrm{e}^{-\delta(T-t_1)}[1 - \delta(T - t_1)] - \delta(c_4 + p)\mathrm{e}^{-\delta(T-t_1)}$. From the assumption mentioned before, we easily derive $\dfrac{\mathrm{d}F(t_1)}{\mathrm{d}t_1} < 0$. Hence, $F(t_1)$ is a strictly decreasing function in $t_1 \in (0,T)$.

This ends with the proof of Proposition 4.1.

Appendix 3: Proof of Proposition 4.2.

The first-order derivative of $G(p)$ concerning selling price p is $\dfrac{\mathrm{d}G(p)}{\mathrm{d}p} = \dfrac{\partial g(t_1,p)}{\partial t_1}\dfrac{\mathrm{d}t_1}{\mathrm{d}p} + \dfrac{\partial g(t_1,p)}{\partial p}$. After simplication, we have $\dfrac{\mathrm{d}G(p)}{\mathrm{d}p} = A(t_1)\{d'(p)[\alpha(t_1)+ p\gamma(t_1)]+d(p)\gamma(t_1)\}\dfrac{\mathrm{d}t_1}{\mathrm{d}p}+d''(p)K(t_1)+[2d'(p)+pd''(p)]M(t_1)$. When $d'(p)[\alpha(t_1)+p\gamma(t_1)]+d(p)\gamma(t_1)<0$, since $A(t_1)>0$, $\dfrac{\mathrm{d}t_1}{\mathrm{d}p}>0, K(t_1)<0$, and $M(t_1)>0$, using the basical assumptions $d''(p)\geqslant 0$ and $2d'(p)+pd''(p)<0$, it can be derived that $\dfrac{\mathrm{d}G(p)}{\mathrm{d}p}<0$. Hence, $G(p)$ is a monotonically decreasing function of p.

This ends the proof Proposition 4.2.

Appendix 4: Proof of Theorem 4.2.

(1) From Eq. (26), combining with Proposition 4.3, we have $\dfrac{\partial AP(t_1,p)}{\partial p} = \dfrac{g(t_1,p)}{T} = \dfrac{G(p)}{T} < \dfrac{G(p^L)}{T} \leqslant 0$, which implies that for any given t_1, $AP(t_1,p)$ is strictly decreasing in $p\in\left(p^L,p^U\right)$. Hence, the optimal selling price $p^*=p^L$. Substituting $p^*=p^L$ into Eq. (26), it is clear from $\dfrac{\mathrm{d}AP(t_1,p^L)}{\mathrm{d}t_1}=0$ that there exists a unique solution $t_1^*=t_1^{\#1}$ satisfying $f\left(t_1,p^L\right)=0$. Furthermore, $\dfrac{\mathrm{d}^2AP(t_1,p^L)}{\mathrm{d}^2t_1}|_{t_1=t_1^*} = \dfrac{f'\left(t_1^*,p^L\right)}{T}<0$, and therefore, (t_1^*,p^*) is the unique optimal solution of Model M.

(2) Similarly, $\dfrac{\partial AP(t_1,p)}{\partial p} = \dfrac{g(t_1,p)}{T} = 0\dfrac{G(p)}{T} > \dfrac{G(p^U)}{T} \geqslant 0$, which shows that for any given t_1, $AP(t_1,p)$ is strictly increasing in $p\in\left(p^L,p^U\right)$. Thus, $p^*=p^U$. Substituting $p^*=p^U$ into Eq. (26), it is clear from $\dfrac{\mathrm{d}AP(t_1,p^U)}{\mathrm{d}t_1}=0$ that there exists a unique solution $t_1^*=t_1^{\#2}$ satisfying $f\left(t_1,p^U\right)=0$. Moreover, $\dfrac{\mathrm{d}^2AP(t_1,p^U)}{\mathrm{d}^2t_1}|_{t_1=t_1^*} = \dfrac{f'\left(t_1^*,p^U\right)}{T}<0$, and therefore, second-order sufficient conditions indicate that (t_1^*,p^*) is the unique optimal solution of Model M.

(3) When $G\left(p^L\right)>0$, then $G\left(p^U\right)<0$, intermediate value theorem indicates that there exists a unique solution $p=p^*$ such that $G(p)=0$. Then, substituting $p=p^*$ into Eq. (26) and combining with Proposition 4.1, it's also not hard to find from $f(t_1,p^*)=0$ that there exists a unique solution $t_1=t_1^*$ such that $F(t_1)=0$ for a given $p=p^*$.

Furthermore, from the discussion mentioned before, we easily obtain $\left.\dfrac{\partial^2AP(t_1,p)}{\partial t_1^2}\right|_{(t_1^*,p^*)} = \dfrac{D(t_1,p)}{T}\left.\dfrac{\partial f(t_1,p)}{\partial t_1}\right|_{(t_1^*,p^*)}<0$, $\left.\dfrac{\partial^2AP(t_1,p)}{\partial p^2}\right|_{(t_1^*,p^*)} = \dfrac{1}{T}\left.\dfrac{\partial g(t_1,p)}{\partial p}\right|_{(t_1^*,p^*)}<0$, and $\left.\dfrac{\partial^2AP(t_1,p)}{\partial t_1\,\partial p}\right|_{(t_1^*,p^*)} = \dfrac{D(t_1^*,p^*)[1-\tau(T-t^*)]}{T}$. Using the assumption $0<\delta<1$ and combining with the sion before, we have $\left|\left.\dfrac{\partial^2AP(p,t_1)}{\partial t_1^2}\right|_{(t_1^*,p^*)}\right| > \left|\left.\dfrac{\partial^2AP(p,t_1)}{\partial t_1\,\partial p}\right|_{(t_1^*,p^*)}\right|$ and $\left|\left.\dfrac{\partial^2AP(p,t_1)}{\partial p^2}\right|_{(t_1^*,p^*)}\right| > \left|\left.\dfrac{\partial^2AP(p,t_1)}{\partial t_1\,\partial p}\right|_{(t_1^*,p^*)}\right|$. Thus, the determinant of the Hessian matrix at the stationary point (t_1^*,p^*) is

$$\det(\mathbf{H}) = \left(\left.\dfrac{\partial^2AP(p,t_1)}{\partial t_1^2}\right|_{(t_1^*,p^*)}\right)\left(\left.\dfrac{\partial^2AP(p,t_1)}{\partial p^2}\right|_{(t_1^*,p^*)}\right) - \left[\left.\dfrac{\partial^2AP(p,t_1)}{\partial t_1\,\partial p}\right|_{(t_1^*,p^*)}\right]^2 > 0.$$

As a result, we gain that (t_1^*,p^*) is the unique optimal solution of Model M.

This ends the proof Theorem 4.2.

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