Networked Evolutionary Game Analysis on Strategy Consensus under Pinning Control Based on Degree

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***Abstract*—This paper studies the strategy consensus problem of networked evolutionary agmes (NEGs) under degree-based pinning control. Firstly, two necessary and sufficient conditions are given for the strategy consensus of NEGs based on different premises. Secondly, an algorithm is obtained to get the minimal-agent control based on degree for making the NEGs achieve the strategy consensus. Thirdly, as application, we present two sufficient conditions for the strategy consensus of NEGs with two kinds of special networked graph: wheel graph and $f_{n\times 3}$ flower graph. Finally, an illustrative example is employed to show the effective of the obtained results.**

***Index Terms*—networked evolutionary games, pinning control, strategy consensus, semi-tensor product of matrices**

I. Introduction

Evolutionary games theory was first proposed for describing the evolutionary of living things in nature [1]. Since then, evolutionary games theory has been widely developed, representative scholars are Maynard Smith and Price [2]. With the rapid development of evolutionary games theory, many related problems in large and small fields have been effectively solved, such as biology, economics, mathematics and so on [3], [4].

Although evolutionary games theory has been widely developed [5], most of them are assumed to be played against each player. Considering the fact that in most practical scenarios, each player only plays with her or his neighbors. Based on above background in evolutionary games, NEGs have appeared and been extensively studied [6].

There is a very important problem in NEGs, that is, the problem of strategy consensus. In short, the main purpose of the consensus problem is to design an effective control so that all agents reach a consistent strategy. This problem is not only applied in networked evolutionary games, but also in multi-agent system [7], economical systems [8], and so forth [9].

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Recently, professor Cheng proposed an effective mathematical tool of the semi-tensor product of matrices (STP). It breaks the limitation of the original matrix product dimension. Cheng et al used the STP method to transform the evolutionary dynamical of NEGs into the algebraic form of the logical dynamical system [6]. Based on this theoretical method, a large number of studies on Boolean networks and NEGs emerged [10], [11].

Research has shown that for biological systems or complex networks, when a part of the nodes are imposed control, it may be easier to show the good performance of the system or network's characteristic [12]. For example, imposing control over the states Mdm2 and Wipl can promote apoptosis [13]. This control strategy is called pinning control. However, studies of NEGs using pinning controls have not yielded relevant results [14], [15].

To address this challenge, we study the strategy consensus problem under fixed pinning control. In comparison with the previous studies, our contribution is three folds.

(i) For the first time, the pinning control based on fixed strategy is transformed into its algebraic form, and the necessary and sufficient conditions of its strategy consensus are given.
(ii) Second, an algorithm is given to find the minimum number of agents under the pinning control with degree.
(iii) Thirdly, as an application, the sufficient conditions of the NEG with wheel graph and $f_{n\times 3}$ flower graph for strategy consensus problem are given.

The rest of this paper is organized as follows. Basic notations and knowledge are introduced in Section 2. The main results are presented in Section 3. An effective example is addressed in Section 4. Finally, conclusion is presented in Section 5.

II. Preliminaries

We first provide some notations that are needed in the sequel: $Row_i(A)$ represents the i-th column of matrix A. $Row(A)$ denotes the set of all columns of matrix A. $*$ denotes the Khatri-Rao product of matrices. $Blk_i(A)$ denotes the i-th $n \times p$ block of an $n \times mp$ matrix A. $\ltimes$ denotes the STP.

An NEG, denoted by $((N, E), G, \Xi)$, consists of

(i) (N, E) is a graph, where $N = \{1, 2, \cdots, n\}$ is the set of nodes and $E \subseteq N \times N$ is the set of edges;

(ii) $G = (N, S, \{C_i, i \in N\})$ is the fundamental network game (FNG), where $N = \{1, 2, \cdots, n\}$ is the set of players, $S := \prod_{i=1}^{n} S_i$ is the set of profiles, S_i is the strategy set of player i. Assume that the payoff matrix of the FNG about the player i has the following form:

$$C^i = \begin{pmatrix} c^i_{1,1} & \cdots & c^i_{1,k} \\ \vdots & & \vdots \\ c^i_{k,1} & \cdots & c^i_{k,k} \end{pmatrix}; \quad (1)$$

(iii) Ξ is an strategy update rule (SUR), which is determined by a set of mappings:

$$x_i(t+1) = g_i(x_j(t), \tau_j(t); j \in U(i) \cup \{i\}), t \in \mathbb{N}, \quad (2)$$

where $U(i)$ is the set of neighbors of player $i \in N$, and the overall payoff $\tau_i(t)$ of player i at time t is computed by:

$$\tau_i(t) = \frac{\Sigma_{j \in U(i)} C^i_{x_i(t), x_j(t)}}{|U(i)|}, i \in N. \quad (3)$$

Throughout this paper, we just consider the following SUR, namely unconditional imitation with fixed priority: the strategy of player i at time $t+1$, denoted by $x_i(t+1)$, is selected from the best strategy among the strategies of its neighbors $j \in U(i)$ and itself at time t, that is,

$$x_i(t+1) = \{x_{j'}(t) : j' = \arg \max_{j \in U(i) \cup \{i\}} c_j(x(t))\}. \quad (4)$$

Consider an NEG $((N, E), G, \Xi)$ and assume $S_i = \mathcal{D}_k$, $i \in N$. Using the STP method and letting $x_i(t) = l \sim \delta_k^l$, $l \in \mathcal{D}_k$, Cheng et al. [6] established an algebraic form of (4):

$$x_i(t+1) = M_i x(t), i \in N, \quad (5)$$

where $M_i \in \mathcal{L}_{k \times k^n}$. Multiplying all the n equations together, we derive the following algebraic state space representation of the considered NEG:

$$x(t+1) = Lx(t), \quad (6)$$

where $L = M_1 * M_2 * \cdots * M_n \in \mathcal{L}_{k^n \times k^n}$.

The definition of strategy consensus is presented below.

Definition 1: The NEG $((N, E), G, \Xi)$ is said to achieve strategy consensus at $\gamma \in \mathcal{D}_k$, if there exists an integer $T \in \mathbb{N}$ such that $x_i(t; x(0)) = \gamma, \forall\, i \in N$ holds for any integer $t \geq T$ and any initial strategy profile $x(0) \in \mathcal{D}_k^n$.

III. Main Result

A. Strategy consensus of NEGs via pinning control

The set of pinning players in the NEG is denoted by Θ, that is, $x_\theta(t) = \gamma$, $\forall\, \theta \in \Theta$, $\forall\, t \in \mathbb{N}$. Furthermore, for the convenience of studying, we label the players in N according to the following rule: $p \leq q$ if and only if $|U(p)| \geq |U(q)|$.

In the following, using the vector form of logical variables and setting $x(t) = \ltimes_{i=1}^{n} x_i(t)$, where $x_i(t) = \varepsilon \sim \delta_k^\varepsilon$, one can convert system (6) into an equivalent algebraic form with only one high-degree node, that is, player 1:

$$\begin{aligned} x(t+1) &= L \ltimes_{i=1}^{n} x_i(t) \\ &= Lx_1(t) \ltimes_{i=2}^{n} x_i(t) \\ &= Blk_\gamma(L) \ltimes_{i=2}^{n} x_i(t) \\ &:= \widetilde{L}\widetilde{x}(t), \end{aligned}$$

where $\widetilde{L} = Blk_\gamma(L)$ and $\widetilde{x}(t) = \ltimes_{i=2}^{n} x_i(t)$.

Noticing that $x_1(t+1) = \delta_k^\gamma$, we have

$$\begin{aligned} \widetilde{L}\widetilde{x}(t) &= x_1(t+1)\widetilde{x}(t+1) \\ &= \delta_k^\gamma \widetilde{x}(t+1). \end{aligned}$$

Furthermore,

$$(\delta_k^\gamma)^\top \widetilde{L}\widetilde{x}(t) = (\delta_k^\gamma)^\top \delta_k^\gamma \widetilde{x}(t+1), \quad (7)$$

which together with $(\delta_k^\gamma)^\top \delta_k^\gamma = 1$ shows that

$$\widetilde{x}(t+1) = (\delta_k^\gamma)^\top \widetilde{L}\widetilde{x}(t). \quad (8)$$

Based on Definition 1 and (8), we derive the following result.

Theorem 1: The NEG $((N, E), G, \Xi)$ with pinning control player 1 achieves strategy consensus at $\gamma \in \mathcal{D}_k$, if and only if

$$Row_\sigma\{[(\delta_k^\gamma)^\top \widetilde{L}]^{k^{n-1}}\} = \mathbf{1}^\top_{k^{n-1}}, \quad (9)$$

where $\sigma = 1 + (\gamma - 1)\frac{k^{n-1}-1}{k-1}$.

proof: (Necessity) Suppose that the NEG $((N, E), G, \Xi)$ achieves strategy consensus. Then, by Definition 1, there exists an integer $T \in \mathbb{N}$ such that $x_i(t; x(0)) = \delta_k^\gamma, \forall\, i \in N$ hold for any integer $t \geq T$ and any initial strategy profile $x(0) \in \mathcal{D}_k^n$. Since player 1 is a pinning control player, one obtains:

$$x_1(t) = \delta_k^\gamma, \ \forall\, t \in \mathbb{N}.$$

Therefore, one can conclude from (8) that

$$\begin{aligned} \widetilde{x}(t) &= (\delta_k^\gamma)^\top \widetilde{L}\widetilde{x}(t-1) \\ &= [(\delta_k^\gamma)^\top \widetilde{L}]^t \widetilde{x}(0), \ \forall\, t \geq T. \end{aligned}$$

Thus, we have

$$Row_\sigma\{[(\delta_k^\gamma)^\top \widetilde{L}]^t\} = \mathbf{1}^\top_{k^{n-1}}.$$

In the following, we prove that $t \leq k^{n-1}$. In fact, if $t > k^{n-1}$, there exists an initial profile $\widetilde{x}(0)$, an integer $T' > k^{n-1}$. We have $\widetilde{x}(t) \neq \widetilde{x}(e), \ \forall\ 0 \leq t \leq T' - 1$ and $x(T') = x_e$. Due to system (8) has k^{n-1} different profiles, there exists two integers $0 \leq t_1 \leq t_2 \leq T' - 1$, we obtain $\widetilde{x}(t_1) = \widetilde{x}(t_2) \neq x_e$.

Thus, the trajectory of system (8) starting from $\widetilde{x}(0) = \widetilde{x}(t_1)$ forms a loop, that is

$$\widetilde{x}(t_1) \to \widetilde{x}(t_1+1) \to \cdots \to \widetilde{x}(t_2).$$

Therefore, there is a contradiction to the Definition 1, which together with the proof of necessity shows that (9) holds.

(Sufficiency) Suppose that (9) holds. We need to prove that the system achieves strategy consensus at $\gamma \in \mathcal{D}_k$. In fact, from (9) it is easy to see that

$$[(\delta_k^\gamma)^\top \widetilde{L}]_{i,j}^{k^{n-1}} = 0, \forall\ i \neq \sigma, j = 1, \cdots, k^{n-1},$$

and

$$[(\delta_k^\gamma)^\top \widetilde{L}]_{i,j}^{k^{n-1}} = 0, i = \sigma, j = 1, \cdots, k^{n-1}.$$

Thus, we know that the profile of the system starting from any initial profile $\widetilde{x}(0) \in \mathcal{D}_k^{n-1}$ will reach the profile $x_e = (\delta_k^\gamma)^n$ and then stays at x_e forever. □

Assume that there are $\eta, 1 < \eta \leq n$, pinning control players and fix their strategies to δ_k^γ, NEGs can reach the strategy consensus. By generalizing formula (7), we can get

$$((\delta_k^\gamma)^\eta)^\top \widetilde{L}\widetilde{x}(t) = ((\delta_k^\gamma)^\eta)^\top (\delta_k^\gamma)^\eta \widetilde{x}(t+1). \quad (10)$$

Corollary 1: The NEG$((N,E),G,\Xi)$ under pinning control achieves strategy consensus, if and only if

$$Row_\sigma\{[((\delta_k^\gamma)^\eta)^\top \widetilde{L}]^{k^{n-\eta}}\} = \mathbf{1}_{k^{n-\eta}}^\top, \quad (11)$$

where $\sigma = 1 + (\gamma - 1)\frac{k^{n-\eta}-1}{k-1}$, η is the number of pinning control players.

Based on the above Theorem 1 and Corollary 1, below, we consider a more general case. We establish Algorithm 1 to get the minimal-agent pinning control under degree.

Algorithm 1 Construction algorithm of NEG's strategy consensus to obtain minimal-agent pinning control under degree

Require: Given $N = [1, \cdots, n]$, $C = [\ ]$
Ensure: $r = |C_i|$
Step 1: Verify whether the NEG$((N,E),G,\Xi)$ can reach the strategy consensus.
Step 2: If it is not satisfied, let $C_1 = [1]$, $N_1 = [2, \cdots, n]$, and verify whether the NEG satisfies (9) according to the Theorem 1.
Step 3: If it is still not satisfied, enter the follow loop.
for $i = 2, \cdots, n$ **do**
$N_i = [i+1, \cdots, n]$ and $C_i = [1, \cdots, i]$
$r = |C_i|$
if The NEG satisfies formula (11) **then**
return
end if
end for
Step 4: Until there exists i such that the NEG satisfies the condition of Corollary 1 under the premise of $N_i = [i+1, \cdots, n]$, $C_i = [1, \cdots, i]$ and $r = |C_i|$, the NEG reaches the strategy consensus.

Remark 1: As it can be seen from formulas (9) and (11), STP method is used to study the strategy consensus of NEGs, but the major drawback is that when players have k strategies, the dimension of the matrix is very large, resulting in MATLAB can not be solved, which encourages us to carry out the following research.

B. Special Networked Graph

As an application, we study two kinds of special networked structures. In this section, we give the sufficient conditions for these special networked graphs to only impose pinning control on the node with the largest degree to make the NEG reach strategy consensus.

Chemical compounds and drags are often modeled as graph where each vertex represents an atom of molecule, and covalent bounds between the corresponding vertices. An indicator defined over this molecular graph, the Wiener index, has been shown to be strongly correlated to various chemical properties of the compounds. Paper [16] determined the Wiener index of gear fan graph, wheel graph and so on.

First, we study a special network structure in graph theory: wheel graph.

Definition 2: Place a vertex in a circle of order $n-1$, connect this vertex with all vertices on the circle, and the resulting simple graph of order n is called a wheel graph of order n [17].

Without loss of generality, taking W_7 as an example, its networked graph is as follows:

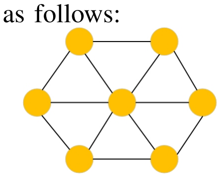

Fig 1: The networked graph of W_7.

In the following, based on the need of the research, we give the definition of the payoff vector and payoff matrix.

Definition 3: Given $t \in \mathbb{N}$, the profiles $x(t) \in \mathcal{D}_k^n$, the profile payoff vector for player $i \in N$ in the networked evolutionary game is defined as

$$\Pi(i, x(t)) = \big[[\Pi(i,x(t))]_1, \cdots, [\Pi(i,x(t))]_n\big], \quad (12)$$

where

$$[\Pi(i,x(t))]_j = \begin{cases} c_{i,j}(x_i(t), x_j(t)),\ if (i,j) \in E, \\ 0,\ if (i,j) \notin E. \end{cases} \quad (13)$$

Further, the profile payoff matrix is defined as:

$$\Pi(t) = \begin{pmatrix} \Pi(1, x(t)) \\ \vdots \\ \Pi(n, x(t)) \end{pmatrix}. \quad (14)$$

Based on the characteristics of the wheel graph, player 1 is designated as the pinning control player, and the sufficient condition for strategy consensus of the NEG is provided.

Theorem 2: For the wheel graph, if there exists the positive integer $\mu \in N$ such that for any profile $x'(\mu) \in \{x(\mu) | x_1(\mu) = \gamma, \forall\ x_i(\mu) \in \mathcal{D}_k, i \in \{2, \cdots, n\}\}$, we have

$$\frac{1}{n-1}\sum_{i=1}^{n} \Pi_{i,1}(\mu) \geq \frac{1}{3}\sum_{i=1}^{n} \Pi_{i,j}(\mu), \forall\ j \in \{2, \cdots, n\}, \quad (15)$$

then the NEG achieves strategy consensus.

proof: For W_n, player 1 has $n-1$ neighbors. In conjunction with the definition of the payoff matrix Π for the first round of the game, $Row_1(\Pi)$ represents the payoff between player 1 and player $i, i \in \{2, \cdots, n\}$ in the first round of the game.

Likewise, player $i, i \in \{2, \cdots, n\}$ has three neighbours. Therefore, in the first round of the game, when the equation (15) holds, we have:

$$x_i(t) = x_1(t), \forall\, i \in \{2, \cdots, n\}, \forall\, t \geq \mu.$$

Thus, the NEG achieves strategy consensus. □

The following, we will study another special type of network graph in graph theory: flower networked graph. First of all, we give the definition of flower graph.

Definition 4: A graph G is called an $f_{n\times m}$ flower graph if it has n vertices which form an n-cycle and n sets of $m-2$ vertices each which form m-cycles around the n-cycle so that each m-cycle uniquely intersects the n-cycle on a single edge. This graph will be denoted by $f_{n\times m}$ [18].

Here, we delve into the study of a relatively simple floral pattern networked graph: $f_{n\times 3}$ flower graph. The n nodes forming the central cycle have a degree of 4, while the remaining nodes have a degree of 2. Now, we will begin studying the simple flower networked graph. Similarly, taking $f_{6\times 3}$ as an example, its networked graph is:

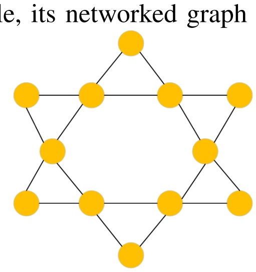

Fig 2: The networked graph of $f_{6\times 3}$.

Remark 2: Based on the characteristics of the $f_{n\times 3}$ flower graph, player $i, i \in \{1, \cdots, n\}$ has the same degree, which makes the selection of control players more complex compared to the wheel graph.

Specially, in the study of $f_{n\times 3}$ flower graph, the generalization of the situation payoff vector, defining its form as:

$$\widehat{\Pi}(i, z(t)) = \left[[\widehat{\Pi}(i, z(t))]_1, \cdots, [\widehat{\Pi}(i, z(t))]_{2n}\right], \tag{16}$$

where, $z(t)$ represents the situation of networked evolutionary game at time t with networked graph $f_{n\times 3}$.

Theorem 3: For the $f_{n\times 3}$ flower graph, if there exists the moment ν, for any profiles $z'(\nu) \in \left\{z'(\nu) | z_\kappa(\nu) \in \mathcal{D}_k, \forall\, \kappa \in \{1, \cdots, n\}, \forall\, r \in \{n+1, \cdots, 2n\}\right\}$, the following holds:

$$\frac{1}{4}\sum_{i=1}^{2n}[\widehat{\Pi}(\kappa, z'(\nu))]_i \geq \frac{1}{2}\sum_{i=1}^{2n}[\widehat{\Pi}(r, z'(\nu))]_j, \tag{17}$$

thus, the NEG achieves strategy consensus.

proof: According to the properties of the $f_{n\times 3}$ flower graph, player $i, i \in \{1, \cdots, n\}$ has 4 neighboring players. Player $j \in \{n+1, \cdots, 2n\}$ has 3 neighboring players. According to equation (3):

$$\tau_\kappa(\nu) = \frac{1}{4}\sum_{i=1}^{2n}[\widehat{\Pi}(\kappa, z'(\nu))]_i, \kappa \in \{1, \cdots, n\}, \tag{18}$$

$$\tau_r(\nu) = \frac{1}{3}\sum_{i=1}^{2n}[\widehat{\Pi}(r, z'(\nu))]_j, \forall\, r \in \{n+1, \cdots, 2n\}. \tag{19}$$

According to the equation (18) and (19), for $\forall\, \kappa \in \{1, \cdots, n\}$, $\forall\, r \in \{n+1, \cdots, 2n\}$, $\tau_\kappa(\nu+1) \geq \tau_r(\nu+1)$, we have:

$$x_r(\nu+1) = x_\kappa(\nu+1) = \gamma, \forall\, t \geq \nu,$$

where, $\forall\, \kappa \in \{1, \cdots, n\}$, $\forall\, r \in \{n+1, \cdots, 2n\}$.

Below, we prove the above equation for $\forall\, t \geq \nu+1$ using mathematical induction. Assuming that $x_\alpha(t) = \gamma, \forall\, \alpha \in \{1, \cdots, 2n\}$ holds when $t = k$, then according to equation (18) and (19), $x_i(k+1) = x_{j_i}(k) = \gamma$ holds, where $j_i = j^*$. By the induction hypothesis, we can conclude $x_r(t) = \gamma$, $\forall\, r \in \{1, \cdots, 2n\}$, $\forall\, t \geq \nu+1, t \in \mathbb{N}$.

In summary, the NEG with $f_{n\times 3}$ as the networked graph achieves strategy consensus under the degree-based pinning control. □

Remark 3: The method of STP is compared to the method of payoff matrix for the strategy consensus on NEGs based on pinning control. The algorithm complexity decreases from 2^n to n, significantly addressing the shortcomings mentioned in Remark 1, thereby improving algorithm efficiency and effectiveness. Future research will focus on extending this method to NEGs with general networked graph.

IV. Illustrative Example

Example 1: Consider an NEG$((N, E), G, \Xi)$, which has the following items.

- The player set $N = \{1, 2, 3, 4\}$, and the strategy set $S_i = \{1, 2\}, i = 1, 2, 3, 4$.
- The networked topological structure for the players is Figure 1.
- The basic game is the boxed pigs game whose payoff double matrix is shown in TABLE 1.
- The evolutionary rule is (4).

TABLE I
PAYOFF MATRIX OF BOXED PIGS GAME

	M	F
M	$(3, 1)$	$(2, 2)$
F	$(4, -1)$	$(0, 0)$

The following table establishes the algebraic form of NEG, where ”M” and ”F” are denoted by ”1” and ”2”, respectively.

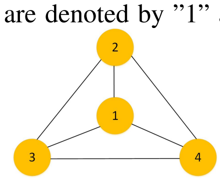

Fig 3: The networked graph of Example 1.

TABLE II
PAYOFFS→DYNAMICS OF THE NEG

$x(t)$	1111	1112	1121	1122
$c_1(t)$	3	$\frac{8}{3}$	$\frac{8}{3}$	$\frac{7}{3}$
$c_2(t)$	3	$\frac{8}{3}$	$\frac{7}{3}$	$\frac{7}{3}$
$c_3(t)$	3	$\frac{8}{3}$	4	$\frac{8}{3}$
$c_4(t)$	3	4	$\frac{7}{3}$	$\frac{8}{3}$
$x_1(t+1)$	1	1	1	1
$x_2(t+1)$	1	2	2	2
$x_3(t+1)$	1	2	2	2
$x_4(t+1)$	1	2	2	2
$x(t)$	1211	1212	1221	1222
$c_1(t)$	$\frac{8}{3}$	$\frac{7}{3}$	$\frac{7}{3}$	2
$c_2(t)$	4	$\frac{8}{3}$	$\frac{8}{3}$	$\frac{4}{3}$
$c_3(t)$	$\frac{8}{3}$	$\frac{7}{3}$	$\frac{8}{3}$	$\frac{4}{3}$
$c_4(t)$	$\frac{8}{3}$	$\frac{8}{3}$	$\frac{7}{3}$	$\frac{4}{3}$
$x_1(t+1)$	1	1	1	1
$x_2(t+1)$	2	2	2	1
$x_3(t+1)$	2	2	2	1
$x_4(t+1)$	2	2	2	1

According to the following table, we can get

$$M_1 = \delta_2[1\ 1\ 1\ 1\ 1\ 1\ 1\ 1], M_2 = \delta_2[1\ 2\ 2\ 2\ 2\ 2\ 2\ 2],$$
$$M_3 = \delta_2[1\ 2\ 2\ 2\ 2\ 2\ 2\ 2], M_4 = \delta_2[1\ 2\ 2\ 2\ 2\ 2\ 2\ 2].$$

Thus, we have

$$\widetilde{L} = \delta_{16}[1\ 8\ 8\ 8\ 8\ 8\ 8\ 1].$$

According to the Theorem 1, when player 1's strategy is δ_2^1 and $r = 1$, we can get

$$(\delta_2^1)^T \widetilde{L} = \delta_8[1\ 8\ 8\ 8\ 8\ 8\ 8\ 1].$$

After calculation, it can be obtained that

$$Row_1[(\delta_2^1)^T \widetilde{L}]^8 = \mathbf{1}_8^T.$$

Therefore, according to the Theorem 1, the NEG can reach strategy consensus under the degree-based pinning control.

Remark 4: Figure 4 illustrates a comparison of the state transition diagram for NEG under degree-based pinning control (Case 1) and non-degree-based pinning control (Case 2). As can be seen from Figure 4, without the use of the degree based pinning control study, the strategy consensus to the δ_{16}^1 cannot be achieved, which proves the effectiveness of this method.

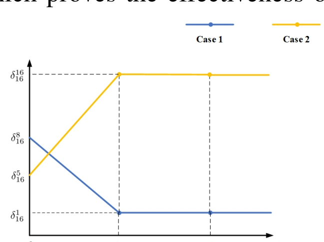

Fig 4: State transition diagram of Example 1.

V. CONCLUSION

In this paper, we have studied the strategy consensus of NEGs under the pinning control. By transforming the algebraic form, the necessary and sufficient conditions for the strategy consensus of NEGs under the degree-based pinning control are given. In addition, the design algorithm of constructing minimal-agent pinning control under degree is given. Finally, by defining the payoff matrix, the strategy consensus problem of NEGs with two kinds of special network topologies is studied. Further results will look at how this can be done on general network graphs.

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