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Direct Approaches for Representations of Various Algebraic Domains via Closure Spaces

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ABSTRACT
In this paper, F-augmented closure spaces are generalized to F-closure spaces, and the concept of F-closed sets are introduced. Properties of their ordered structures are investigated. Representations of various algebraic domains such as algebraic lattices, algebraic L-domains, BF-domains via F-closure spaces are considered. As applications of these methods, more direct approaches to representing various algebraic domains via classical closure space are given, respectively. F-relations between F-closure spaces are defined and properties of them are examined. It is also proved that the category of algebraic domains with Scott continuous maps is equivalent to that of F-closure spaces with F-relations.

KEYWORDS
closure space; algebraic domain; algebraic lattice; algebraic L-domain; BF-domain; categorical equivalence

1 Introduction

Domain theory, developed from continuous lattices introduced by Scott[1] in the 1970s as a denotational model for functional languages, is one of the important research fields of theoretical computer science[2]. Mutual transformations and infiltrations of the mathematical structures of orders and topologies are the basic features of domain theory. A closure system on a set is a family of subsets which is closed under arbitrary intersections. An underlying set equipped with closure system is called a closure space. Thus, closure spaces are generalized topological spaces. Many studies have shown that closure spaces are closely related to domain theory and play a key role in various completions of ordered structures (see Refs. [3–6]).

Representing ordered structures by families of sets is an interesting research topic in domain theory[7]. To represent/characterize ordered structures, or domains, one can use a suitable and familiar family of their structures ordered by the set-theoretic inclusion with some general way. Closure spaces were successfully used in representing various lattices. For example, Birkhoff's famous representation theorem for finite distributive lattices[8] and Stone's duality theorem for Boolean algebras[9]. Another fact is that the closed sets of an algebraic closure space (called a topped algebraic intersection structure in Ref. [10]) ordered by set-

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theoretic inclusion is an algebraic lattice, and conversely, every algebraic lattice is isomorphic to the family of all closed sets of an appropriate algebraic closure space ordered by set-theoretic inclusion[10]. These facts are well-known results in representation theory of ordered structures.

Recently, Li et al.[11] generalized algebraic closure spaces to continuous closure spaces and gave representations for continuous lattices. In Ref. [12], Guo and Li defined F-augmented closure spaces and provided alternative representations of algebraic domains. In Ref. [13], Wu et al. provided a new approach to represent algebraic domains and algebraic L-domains by algebraic closure spaces. In Ref. [14], Su and Li generalized the representations of algebraic lattices by means of algebraic closure spaces to the fuzzy setting. In Ref. [15], Yao and Li proposed the notion of BF-closure spaces and gave representations of BF-domains by means of families of morphisms. In Ref. [16], Zhang et al. provided a representation for arithmetic semilattices by closure spaces. Convex spaces are special algebraic closure spaces with empty set as a closed set[17]. Shen et al.[18] gave representations for algebraic lattices by sober convex spaces. In Ref. [19], Yao and Zhou gave representations for join-semilattices by sober convex spaces. The above work shows that closure spaces played an important role in representing ordered structures and various domains. For more discussion of representations for various domains via closure spaces, please refer to Refs. [20–22].

As mentioned, there are more than one method for representing algebraic domains in terms of closure spaces. However, each of the above representations is based on closure spaces with additional conditions or structures, such as algebraic closure spaces and F-augmented closure spaces. These representations mean that algebraic domains were not directly generated from classical closure spaces. It is thus natural to ask if there is a direct representation of algebraic domains by classical closure spaces. In order to solve this problem, we introduce the concept of F-closure spaces, which is more general than F-augmented closure spaces, and give representations for algebraic domains. It will be seen that classical closure spaces are special F-closure spaces. Based on this fact, we naturally obtain direct methods for representing algebraic domains via classical closure space. Moreover, being different from the method in Ref. [15], a set-theoretic method without using morphisms to represent BF-domains is given with the notion of bifinite F-closure spaces, which is the second innovation of this paper. Finally, we introduce the notion of F-relations between two F-closure spaces which is different from the F-morphisms defined in Ref. [12]. Since the compositions of F-relations are precisely the composition of binary relations, the proof of categorical results will be simplified to some extent.

This paper is organized as follows: In Section 2, we recall some basic notions in domain theory and closure spaces. In Section 3, we introduce the concepts of F-closure spaces and F-closed sets, discuss their properties and then give the representation theorem for algebraic domains, algebraic lattices and algebraic semilattices. In Section 4, we give set-theoretic representations for algebraic L-domains and BF-domains, which are objects of two maximal cartesian closed full subcategories of algebraic domains respectively (see Ref. [23]). In section 5, we give direct approaches to representing various algebraic domains via classical closure space. In section 6, we give the concept of F-relations and discuss the relationship between F-relations and Scott continuous maps. In Section 7, we establish an equivalence between the category of algebraic domains with Scott continuous maps and that of F-closure spaces with F-relations.

2 Preliminary

In this section, we recall some basic notions and results in lattice theory, domain theory and closure spaces. For notions not explicitly defined herein, please refer to Refs. [2, 3, 10, 24].

For a set U and $X \subseteq U$, we use $\mathcal{P}(U)$ to denote the power set of U, and $\mathcal{P}_{fin}(U)$ to denote the family of all nonempty finite subsets of U. The symbol $F \subseteq_{fin} X$ means that F is a finite subset of X.

Next, let us recall the notions of closure spaces which are taken from Refs. [3, 10].

A closure system is a family $\mathscr{C}$ of subsets of a set X that is closed under arbitrary intersections (including empty intersection). The pair $(X, \mathscr{C})$ is called a closure space and $C \in \mathscr{C}$ is called a closed set of $(X, \mathscr{C})$.

A closure operator on a set X is a map $c : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ satisfying:

(1) for all $A, B \in \mathcal{P}(X)$, $A \subseteq B \Longrightarrow c(A) \subseteq c(B)$;

(2) for all $A \in \mathcal{P}(X)$, $A \subseteq c(A)$;

(3) for all $A \in \mathcal{P}(X)$, $c(c(A)) = c(A)$.

Given a closure operator c on X, the related closure system $\mathscr{C}_c$ on X is obtained by defining

$$\mathscr{C}_c = \{A \in \mathcal{P}(X) \mid A = c(A)\}.$$

Conversely, given a closure system $\mathscr{C}$ on X, the related closure operator $c_{\mathscr{C}}$ is obtained by defining

$$\forall B \in \mathcal{P}(X),\ c_{\mathscr{C}}(B) = \cap\{C \in \mathscr{C} \mid B \subseteq C\}.$$

It is well known that the closure operator induced by the $\mathscr{C}_c$ is c, and the closure system induced by $c_{\mathscr{C}}$ is $\mathscr{C}$. That is,

$$c_{\mathscr{C}_c} = c;\ \mathscr{C}_{c_{\mathscr{C}}} = \mathscr{C}.$$

So there is a one-to-one correspondence between closure systems on X and closure operators on X. For a closure space $(X, \mathscr{C})$, $B \subseteq X$, we use $\overline{B}$ to denote $c_{\mathscr{C}}(B) = \cap\{C \in \mathscr{C} \mid B \subseteq C\}$. Clearly,

$$\mathscr{C} = \{B \in \mathcal{P}(X) \mid B = \overline{B}\} = \{\overline{B} \mid B \subseteq X\}.$$

Noticing that $c_{\mathscr{C}}$ defined above is a closure operator, we have Lemma 2.1.

Lemma 2.1 Let $(X, \mathscr{C})$ be a closure space. Then for all $A, B \in \mathcal{P}(X)$, we have

(1) $A \subseteq B \Longrightarrow \overline{A} \subseteq \overline{B}$;

(2) $A \subseteq \overline{A}$;

(3) $\overline{\overline{A}} = \overline{A}$.

Next, we recall some terminology and results of order and domain theory.

Let $(L, \leqslant)$ be a poset. A principal ideal of L is a set of the form $\downarrow x = \{y \in L \mid y \leqslant x\}$. For $A \subseteq L$, we write $\downarrow A = \{y \in L \mid \exists x \in A, y \leqslant x\}$ and $\uparrow A = \{y \in L \mid \exists x \in A, x \leqslant y\}$. A subset A is a lower set (resp., an upper set) if $A = \downarrow A$ (resp., $A = \uparrow A$). We say that z is a lower bound (resp., an upper bound) of A if $A \subseteq \uparrow z$ (resp., $A \subseteq \downarrow z$). A subset B is said to be up-bounded if B has an upper bound. The supremum of A is the least upper bound of A, denoted by $\bigvee A$ or $\sup A$. The infimum of A is the greatest lower bound of A, denoted by $\bigwedge A$ or $\inf A$. If $A \subseteq \downarrow x \subseteq L$, then we use $\bigvee_x A$ to denote the supremum of A in sub-poset $(\downarrow x, \leqslant)$, where the order inherits from L. A nonempty subset D of L is directed if every finite subset of D

has an upper bound in D. A poset L is a directed complete partially ordered set (dcpo, for short) if every directed subset of L has a supremum. A poset L is said to be pointed if L has a least element. A subset E of the directed set D is cofinal, if for every $d \in D$, there exists $e \in E$ such that $d \leqslant e$.

Lemma 2.2 Let D be a directed subset of a poset P and D has a supremum $\sup D$. If $\bigcup_{i=1}^{n} A_i = D$, then there is some A_j $(1 \leqslant j \leqslant n)$ which is a cofinal subset of D, and $\sup A_j = \sup D$.

Proof Assume that for all $i \in \{1,2,\ldots,n\}$, A_i is not a cofinal subset of D. Then for every $i \in \{1,2,\ldots,n\}$, select $d_i \in D$ such that there is no element in A_i greater than d_i. Since D is directed and $\{d_i \mid 1 \leqslant i \leqslant n\}$ is finite, there is $d \in D = \bigcup_{i=1}^{n} A_i$ such that $\{d_i \mid 1 \leqslant i \leqslant n\} \subseteq \downarrow d$. Hence, there is $j \in \{1,2,\ldots,n\}$ such that $d \in A_j$ and $d_j \leqslant d$, contradicting to the choice of d_j. Thus, there is some A_j $(1 \leqslant j \leqslant n)$ which is a cofinal subset of D. Clearly, $\sup A_j = \sup D$. □

Recall that in a poset P, we say that x way-below y, written $x \ll y$, if for any directed set D having a supremum with $\sup D \geqslant y$, there is some $d \in D$ such that $x \leqslant d$. If $x \ll x$, then x is called a compact element of P. The set $\{x \in P \mid x \ll x\}$ is denoted by $K(P)$. The set $\{y \in P \mid x \ll y\}$ will be denoted by $\twoheaduparrow x$ and $\{y \in P \mid y \ll x\}$ denoted by $\twoheaddownarrow x$. A poset P is said to be continuous (resp., algebraic) if for all $x \in P$, $\twoheaddownarrow x$ is directed (resp., $\downarrow x \cap K(P)$ is directed) and $x = \bigvee \twoheaddownarrow x$ (resp., $x = \bigvee(\downarrow x \cap K(P))$). If a dcpo P is continuous (resp., algebraic), then P is called a continuous domain (resp., an algebraic domain). A semilattice (resp., sup-semilattice) is a poset in which every pair of elements has an infimum (resp., a supremum). A complete lattice is a poset in which every subset has a supremum (equivalently, has an infimum). A subset B of of a poset P is called a basis of P if for all $x \in P$, there is $B_x \subseteq B \cap \twoheaddownarrow x$ such that B_x is directed and $\sup B_x = x$. It is well known that a poset P is continuous if and only if (iff) it has a basis, and that P is algebraic iff $K(P)$ is a basis.

Let L and P be two dcpos, and let $f: L \longrightarrow P$ be a map. If for every directed subset $D \subseteq L$, $f(\bigvee D) = \bigvee f(D)$, then f is called a Scott continuous map. It is customary to denote the set of all Scott continuous maps from L to P by $[L \longrightarrow P]$.

Lemma 2.3 Let P be a poset, $A \subseteq P$ and s, t be two upper bounds of A. If the supremums $\bigvee_s A$ and $\bigvee_t A$ exist and $\bigvee_s A \leqslant t$, then $\bigvee_s A = \bigvee_t A$.

Proof It follows from $\bigvee_s A \leqslant t$ that $\bigvee_s A$ is a upper bound of A in $\downarrow t$. Thus $\bigvee_t A \leqslant \bigvee_s A$. By $\bigvee_t A \leqslant \bigvee_s A \leqslant s$, we have $\bigvee_t A$ is a upper bound of A in $\downarrow s$. So $\bigvee_s A \leqslant \bigvee_t A$. Thus $\bigvee_s A = \bigvee_t A$. □

Lemma 2.4[2] Let P be a poset. Then for all $x, y, u, z \in P$,

(1) $x \ll y \Rightarrow x \leqslant y$;

(2) $u \leqslant x \ll y \leqslant z \Rightarrow u \ll z$;

(3) if $x \ll y$, $y \ll z$ and $x \vee y$ exists, then $x \vee y \ll z$;

(4) if P has a least element $\perp$, then $\perp \ll x$.

Definition 2.5[2,25] (1) A poset P is called a cusl, if any finite up-bounded subset A of P has a supremum.

(2) A poset P is called a bc-poset, if any up-bounded subset B of P has a supremum.

(3) A poset P is called an sL-cusl, if for all $p \in P$, $\downarrow p$ is a sup-semilattice.

(4) A poset P is called an L-cusl, if for all $p \in P$, $\downarrow p$ is a sup-semilattice with a bottom element.

(5) A continuous domain L is called acontinuous sL-domain (sL-domain, for short), if for all $x \in L$, the

set $\downarrow x$ is a sup-semilattice.

(6) A continuous domain L is called a continuous L-domain (L-domain, for short), if for all $x \in L$, the set $\downarrow x$ is a complete lattice.

(7) If a bc-poset L is also a continuous domain, then L is called a bc-domain. An algebraic bc-domain is called a Scott domain.

It is well-known that a bc-domain is a pointed L-domain and that a pointed sL-domain is a pointed L-domain. However, a pointed L-domain may not be a bc-domain. Clearly, a continuous domain L is a pointed L-domain iff L is a pointed sL-domain.

Lemma 2.6 Let L be an L-domain and for all $x \in L$ and let $\perp_x$ be the least element of $\downarrow x$. Then $\perp_x$ is a compact element.

Proof It is obvious that $\perp_x$ is a minimal element. If $D \subseteq L$ is directed and $\perp_x \leqslant \bigvee D = t$, then by the minimality of $\perp_x$ and $\perp_x \leqslant t$, we have $\perp_x = \perp_t$. Let $d \in D$. It follows from $d \leqslant t$ that $\perp_x = \perp_t \leqslant d$, showing that $\perp_x \in K(L)$. □

Definition 2.7[2] Let L be a dcpo.

(1) An approximate identity for L is defined to be a directed set $\mathcal{D} \subseteq [L \longrightarrow L]$ satisfying $\sup \mathcal{D} = id_L$, where id_L is the identity on L, and the symbol $[L \longrightarrow L]$ stands for the set of all Scott continuous maps from L to L.

(2) A Scott continuous map $\delta : L \longrightarrow L$ is said to be finitely separating if there exists a finite set M_δ such that for each $x \in L$, there exists $m \in M_\delta$ satisfying $\delta(x) \leqslant m \leqslant x$.

(3) If there is an approximate identity for L consisting of finitely separating maps, then L is called an FS-domain.

(4) If L is an algebraic FS-domain, then L is called a BF-domain.

Lemma 2.8[2] Let L be a dcpo.

(1) If $\mathcal{D} \subseteq [L \longrightarrow L]$ is an approximate identity for L, then $\mathcal{D}' = \{\delta^2 = \delta \circ \delta \mid \delta \in \mathcal{D}\}$ is also an approximate identity.

(2) If $\delta \in [L \longrightarrow L]$ is finitely separating, then $\delta(x) \ll x$ for all $x \in L$. Thus an FS-domain is a continuous domain.

Definition 2.9[2] Let P be a poset, $k : P \longrightarrow P$ a monotone map. If k satisfies

(1) $k(x) \leqslant x$ $(\forall x \in P)$;

(2) $k(k(x)) = k(x)$ $(\forall x \in P)$,

then k is called a kernel operator.

Lemma 2.10[2] For a dcpo L, the following statements are equivalent:

(1) L is a BF-domain;

(2) L is an algebraic domain and has an approximate identity consisting of maps with a finite range;

(3) L has an approximate identity consisting of kernel operators with a finite range.

3 F-closure Space and Algebraic Domain

In this section, we introduce the notion of F-closure spaces and F-closed sets, and then give representation theorems for algebraic domains and algebraic lattices in new approaches.

Definition 3.1[12] Let $(X,\mathscr{C})$ be a closure space, $\emptyset \neq \mathcal{F} \subseteq \mathcal{P}_{fin}(X) \cup \{\emptyset\}$. The triple $(X,\mathscr{C},\mathcal{F})$ is called a finite-subset-selection augmented closure space (for short, F-augmented closure space) if, for any $F \in \mathcal{F}$ and $B \subseteq_{fin} \overline{F}$, there exists $F' \in \mathcal{F}$ such that $B \subseteq F' \subseteq \overline{F}$.

Definition 3.2 Let $(X,\mathscr{C})$ be a closure space, $\emptyset \neq \mathcal{F} \subseteq \mathcal{P}_{fin}(X) \cup \{\emptyset\}$. Then the triple $(X,\mathscr{C},\mathcal{F})$ is called an F-closure space.

An F-augmented closure space must be an F-closure space. But the following example shows that an F-closure space may not be an F-augmented closure space.

Example 3.3 Let $X = \mathbb{N} = \{0,1,2,\ldots\}$ equipped with the order of numbers, $\mathscr{C} = \{A \subseteq \mathbb{N} \mid \downarrow A = A\}$ and $\mathcal{F} = \{\{5\}\}$. Then $(X,\mathscr{C},\mathcal{F})$ is an F-closure space. For $K = \{0\} \subseteq_{fin} \overline{\{5\}} = \{0,1,\ldots,5\}$, but there is no $F \in \mathcal{F}$ such that $K \subseteq F$. This shows that an F-closure space may not be an F-augmented closure space.

Definition 3.4 Let $(X,\mathscr{C},\mathcal{F})$ be an F-closure space, $E \subseteq X$. If for any $K \subseteq_{fin} E$, there always exists $F \in \mathcal{F}$ such that $K \subseteq \overline{F} \subseteq E$, then E is called an F-closed set of $(X,\mathscr{C},\mathcal{F})$. The collection of all F-closed sets of $(X,\mathscr{C},\mathcal{F})$ is denoted by $\mathfrak{C}(X,\mathscr{C},\mathcal{F})$.

Proposition 3.5 Let $(X,\mathscr{C},\mathcal{F})$ be an F-closure space. The following statements hold:

(1) For all $F \in \mathcal{F}$, we have $\overline{F} \in \mathfrak{C}(X,\mathscr{C},\mathcal{F})$;

(2) If $E \in \mathfrak{C}(X,\mathscr{C},\mathcal{F})$, $B \subseteq_{fin} E$, then $\overline{B} \subseteq E$;

(3) If $\{E_i\}_{i\in I} \subseteq (\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)$ is directed, then $\bigcup_{i\in I} E_i \in \mathfrak{C}(X,\mathscr{C},\mathcal{F})$;

(4) The poset $(\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)$ is a dcpo.

Proof It is routine to check by Definition 3.4. □

Proposition 3.6 Let $(X,\mathscr{C},\mathcal{F})$ be an F-closure space, $E \subseteq X$. The the following statements are equivalent:

(1) $E \in \mathfrak{C}(X,\mathscr{C},\mathcal{F})$;

(2) The family $\mathcal{A} = \{\overline{F} \mid F \in \mathcal{F} \text{ and } F \subseteq E\}$ is directed, and $E = \bigcup \mathcal{A}$;

(3) There exists family $\{F_i\}_{i\in I} \subseteq \mathcal{F}$ such that $\{\overline{F_i}\}_{i\in I}$ is directed and $E = \bigcup_{i\in I} \overline{F_i}$.

Proof $(1) \Rightarrow (2)$: If $E = \emptyset \in \mathfrak{C}(X,\mathscr{C},\mathcal{F})$, then $\emptyset \in \mathcal{F}$ and $\overline{\emptyset} = \emptyset$. So $\mathcal{A} = \{\emptyset\}$ is directed and $E = \bigcup \mathcal{A}$. If $E \neq \emptyset$, then $\mathcal{A} \neq \emptyset$. Let $F_1, F_2 \in \mathcal{F}$ with $F_1 \subseteq E$ and $F_2 \subseteq E$. Then $F_1 \cup F_2 \subseteq_{fin} E$. It follows from $E \in \mathfrak{C}(X,\mathscr{C},\mathcal{F})$ that there exists $F_3 \in \mathcal{F}$ such that $F_1 \cup F_2 \subseteq \overline{F_3} \subseteq E$. By Lemma 2.1, we have $\overline{F_1}, \overline{F_2} \subseteq \overline{F_3}$ and $\overline{F_3} \in \mathcal{A}$. Thus $\mathcal{A}$ is directed. To prove $E = \bigcup \mathcal{A}$, let $x \in E$. It follows from $E \in \mathfrak{C}(X,\mathscr{C},\mathcal{F})$ that there exists $F \in \mathcal{F}$ such that $\{x\} \subseteq \overline{F} \subseteq E$. Noticing that $F \subseteq \overline{F} \subseteq E$, we have that $\overline{F} \in \mathcal{A}$ and $x \in \bigcup \mathcal{A}$, showing that $E \subseteq \bigcup \mathcal{A}$. By Proposition 3.5 (2), we have $\bigcup \mathcal{A} \subseteq E$. Thus $E = \bigcup \mathcal{A}$.

$(2) \Rightarrow (3)$: Trivial.

$(3) \Rightarrow (1)$: If $B \subseteq_{fin} E = \bigcup_{i\in I} \overline{F_i}$. Noticing that $\{\overline{F_i}\}_{i\in I}$ is directed and B is finite, we have that there exists $i \in I$ such that $B \subseteq \overline{F_i} \subseteq E$, showing that $E \in \mathfrak{C}(X,\mathscr{C},\mathcal{F})$. □

The following result characterizes the way-below relation in $(\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)$

Proposition 3.7 Let $(X,\mathscr{C},\mathcal{F})$ be an F-closure space, $E_1, E_2 \in (\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)$. Then $E_1 \ll E_2$ if and only if there exists $F \in \mathcal{F}$, such that $E_1 \subseteq \overline{F} \subseteq E_2$.

Proof $\Rightarrow$: It follows from $E_2 \in \mathfrak{C}(X,\mathscr{C},\mathcal{F})$ and Proposition 3.5(3) that there exists directed family $\{\overline{F_i}\}_{i\in I}$ such that $E_2 = \bigcup_{i\in I}\overline{F_i}$, where $\{F_i\}_{i\in I} \subseteq \mathcal{F}$. By $E_1 \ll E_2$, there exists $i_0 \in I$ such that $E_1 \subseteq \overline{F_{i_0}} \subseteq E_2$.

$\Leftarrow$: For any directed family $\{E_i\}_{i\in I} \subseteq \mathfrak{C}(X,\mathscr{C},\mathcal{F})$, if $E_2 \subseteq \bigvee_{i\in I} E_i = \bigcup_{i\in I} E_i$, then by the assumption, there exists $F \in \mathcal{F}$ such that $E_1 \subseteq \overline{F} \subseteq E_2$. Therefore $F \subseteq \overline{F} \subseteq \bigcup_{i\in I} E_i$. Noticing that $\{E_i\}_{i\in I}$ is directed and F is a finite set, we know that there exists $i_0 \in I$ such that $F \subseteq E_{i_0}$. By Proposition 3.5(2), we have $\overline{F} \subseteq E_{i_0}$. Since $E_1 \subseteq \overline{F}$, we have $E_1 \subseteq E_{i_0}$, showing that $E_1 \ll E_2$. □

Corollary 3.8 Let $(X,\mathscr{C},\mathcal{F})$ be an F-closure space. Then $E \in (\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)$ is a compact element iff there exists $F \in \mathcal{F}$ such that $E = \overline{F}$. That is $K(\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)) = \{\overline{F} \mid F \in \mathcal{F}\}$.

Proof The proof follows directly from Proposition 3.7. □

Corollary 3.9 For an F-closure space $(X,\mathscr{C},\mathcal{F})$, $\{\overline{F} \mid F \in \mathcal{F}\}$ is a basis of $((\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)$.

Proof The proof follows from Proposition 3.6. □

Theorem 3.10 Let $(X,\mathscr{C},\mathcal{F})$ be an F-closure space. Then $(\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)$ is an algebraic domain.

Proof The proof follows from Proposition 3.5(4) and Corollaries 3.8 and 3.9. □

Next, we consider the reverse case. Given an algebraic domain $(L,\leqslant)$, let $\mathcal{F}_L = \{F \subseteq_{fin} K(L) \mid F \text{ has a greatest element } c_F\}$ and $\mathscr{C}_L$ be the family of all lower sets of $(K(L),\leqslant)$, that is $\mathscr{C}_L = \{A \subseteq K(L) \mid A = \downarrow A \cap K(L)\}$. The family $\mathscr{C}_L$ forms an Alexandrov topology on $K(L)$.

Theorem 3.11 Let $(L,\leqslant)$ be an algebraic domain. Then $(K(L),\mathscr{C}_L,\mathcal{F}_L)$ is an F-closure space, and $\mathfrak{C}(K(L),\mathscr{C}_L,\mathcal{F}_L) = \{\downarrow x \cap K(L) \mid x \in L\}$.

Proof Clearly, $(K(L),\mathscr{C}_L)$ is a closure space and $(K(L),\mathscr{C}_L,\mathcal{F}_L)$ is an F-closure space. It is easy to see that for all $F \in \mathcal{F}_L$, $\overline{F} = \downarrow c_F \cap K(L)$. Let $E \in \mathscr{C}_L$, it follows from Proposition 3.6 that there exists a directed set $D \subseteq K(L)$ such that $E = \bigcup\{\downarrow d \cap K(L) \mid d \in D\}$. Since L is an algebraic domain, we have $E = \downarrow \bigvee D \cap K(L)$, showing that $\mathfrak{C}(K(L),\mathscr{C}_L,\mathcal{F}_L) \subseteq \{\downarrow x \cap K(L) \mid x \in L\}$. Conversely, let $x \in L$ and $K \subseteq_{fin} \downarrow x \cap K(L)$. Since L is an algebraic domain, we have that $\downarrow x \cap K(L)$ is directed. Thus there exists $y \in \downarrow x \cap K(L)$ such that $K \subseteq \downarrow y \cap K(L)$. Set $F = \{y\} \in \mathcal{F}_L$. We have $K \subseteq \downarrow y \cap K(L) = \overline{F} \subseteq \downarrow x \cap K(L)$, showing that $\downarrow x \cap K(L) \in \mathfrak{C}(K(L),\mathscr{C}_L,\mathcal{F}_L)$. Thus $\{\downarrow x \cap K(L) \mid x \in L\} \subseteq \mathfrak{C}(K(L),\mathscr{C}_L,\mathcal{F}_L)$. To sum up, $\mathfrak{C}(K(L),\mathscr{C}_L,\mathcal{F}_L) = \{\downarrow x \cap K(L) \mid x \in L\}$. □

Given an algebraic domain L, we call $(K(L),\mathscr{C}_L,\mathcal{F}_L)$ the induced F-closure space corresponding to L.

Theorem 3.12 (Representation Theorem I: for algebraic domains) A dcpo $(L,\leqslant)$ is an algebraic domain iff there exists some F-closure space $(X,\mathscr{C},\mathcal{F})$ such that $(L,\leqslant) \cong (\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)$.

Proof $\Leftarrow$: Follows directly from Theorem 3.10.

$\Rightarrow$: Let L be an algebraic domain, $(K(L),\mathscr{C}_L,\mathcal{F}_L)$ the induced F-closure space. Define a map $f\colon (L,\leqslant) \to (\mathfrak{C}(K(L),\mathscr{C}_L,\mathcal{F}_L),\subseteq)$ such that $\forall x \in L$, we have $f(x) = \downarrow x \cap K(L)$. Since L is an algebraic domain and $\mathfrak{C}(K(L),\mathscr{C}_L,\mathcal{F}_L) = \{\downarrow x \cap K(L) \mid x \in L\}$, we have that $f\colon (L,\leqslant) \to (\mathfrak{C}(K(L),\mathscr{C}_L,\mathcal{F}_L),\subseteq)$ is an order isomorphism, showing that $(L,\leqslant) \cong (\mathfrak{C}(K(L),\mathscr{C}_L,\mathcal{F}_L),\subseteq)$. □

Next, in order to give representations for pointed algebraic domains, algebraic lattices and algebraic semilattices, we add some appropriate conditions to F-closure spaces respectively.

Theorem 3.13 (Representation Theorem II: for pointed algebraic domains) Let $(L,\leqslant)$ be a dcpo. Then L is a pointed algebraic domain iff there exists some F-closure space $(X,\mathscr{C},\mathcal{F})$ with a least element $O\in\{\overline{F}\mid F\in\mathcal{F}\}$ such that $(L,\leqslant)\cong(\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)$.

Proof $\Leftarrow$: It follows directly from Proposition 3.5(4) and Theorem 3.10 that $(\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)$ is an algebraic domain with O being the least element in $(\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)$. It follows from $(L,\leqslant)\cong(\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)$ that L is a pointed algebraic domain.

$\Rightarrow$: Given a pointed algebraic domain $(L,\leqslant)$ with the least element $\perp$. Then the induced F-closure space $(K(L),\mathscr{C}_L,\mathcal{F}_L)$ satisfies that there is the least element $\{\perp\}\in\{\overline{F}\mid F\in\mathcal{F}_L\}$, as desired. □

Lemma 3.14 Let $(L,\leqslant)$ be a continuous domain and let $B\subseteq L$ be a basis. If $(B,\leqslant)$ is a sup-semilattice, then $(L,\leqslant)$ is a sup-semilattice.

Proof Let $(B,\leqslant)$ be a sup-semilattice. For any $x,y\in L$, set $D=\{a\vee_B b\mid a\in\twoheaddownarrow x\cap B, b\in\twoheaddownarrow y\cap B\}$, where $a\vee_B b$ denotes the supremum of a and b in $(B,\leqslant)$. Clearly, D is directed and $\bigvee D$ exists. It is obvious that $x,y\leqslant\bigvee D$. Let $x,y\leqslant z$. Then for all $a\in\twoheaddownarrow x\cap B$ and $b\in\twoheaddownarrow y\cap B$, we have that $a\ll z$, $b\ll z$. Since B is a basis, we have $\twoheaddownarrow z\cap B$ is directed. Thus there exists $t\in\twoheaddownarrow z\cap B$ such that $a,b\leqslant t$. Therefore $a\vee_B b\leqslant t$. Noticing that $a\vee_B b\in D$, we have $\bigvee D\leqslant\bigvee(\twoheaddownarrow z\cap B)=z$. This shows that $\bigvee D$ is the least upper bound of x and y, namely, $x\vee y=\bigvee D$. Thus L is a sup-semilattice. □

Theorem 3.15 (Representation Theorem III: for algebraic lattices) Let $(L,\leqslant)$ be a dcpo. Then L is an algebraic lattice iff there exists some F-closure space $(X,\mathscr{C},\mathcal{F})$ satisfying that $(\{\overline{F}\mid F\in\mathcal{F}\},\subseteq)$ is a sup-semilattice with a least element and that $(L,\leqslant)\cong(\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)$.

Proof $\Leftarrow$: By Theorem 3.13, we have that $(\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)$ is a pointed algebraic domain. It follows from Lemma 3.14 and Corollary 3.9 that dcpo $(\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)$ is a sup-semilattice. Thus $(\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)$ is a complete lattice, hence an algebraic lattice.

$\Rightarrow$: Given an algebraic lattice $(L,\leqslant)$. By Lemma 2.4, we have that $K(L)$ is a sup-semilattice with a least element. It follows from

$$(\{\overline{F}\mid F\in\mathcal{F}_L\},\subseteq)=(\{\downarrow a\cap K(L)\mid a\in K(L)\},\subseteq)\cong(K(L),\leqslant),$$

that $(\{\overline{F}\mid F\in\mathcal{F}_L\},\subseteq)$ is a sup-semilattice with a least element. □

Theorem 3.16 (Representation Theorem IV: for algebraic semilattices) Let $(X,\mathscr{C},\mathcal{F})$ be an F-closure space. If $(X,\mathscr{C},\mathcal{F})$ satisfies

$$(*)\qquad \overline{F_1}\cap\overline{F_2}\in\mathfrak{C}(X,\mathscr{C},\mathcal{F})(\forall F_1,F_2\in\mathcal{F}),$$

then $(\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)$ is an algebraic semilattice.

Conversely, if L is an algebraic semilattice, then the induced F-closure space $(L,\mathscr{C}_L,\mathcal{F}_L)$ satisfies condition $(*)$.

Proof Firstly, we prove the first half of the theorem. It suffices to prove that $(\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)$ is a semilattice. Let $E_1,E_2\in\mathfrak{C}(X,\mathscr{C},\mathcal{F})$. Then there exist directed families $\{\overline{F_{1i}}\}_{i\in I}$ and $\{\overline{F_{2j}}\}_{j\in J}$ such that $\bigcup_{i\in I}\overline{F_{1i}}=E_1$ and $\bigcup_{j\in J}\overline{F_{2j}}=E_2$, where $\{F_{1i}\}_{i\in I}\subseteq\mathcal{F}$ and $\{F_{2j}\}_{j\in J}\subseteq\mathcal{F}$. By completely distributive law, we have

$$E_1 \cap E_2 = (\cup_{i\in I}\overline{F_{1i}}) \cap (\cup_{j\in J}\overline{F_{2j}}) = \cup_{\phi\in\Phi}(\overline{F_{1\phi(1)}} \cap \overline{F_{2\phi(2)}}),$$

where $\Phi = \{\phi : \{1,2\} \to I \cup J \mid \phi(1) \in I, \phi(2) \in J\}$. By the directedness of $\{\overline{F_{1i}}\}_{i\in I}$ and $\{\overline{F_{2j}}\}_{j\in J}$, we have that $\{\overline{F_{1\phi(1)}} \cap \overline{F_{2\phi(2)}}\}_{\phi\in\Phi}$ is directed. It follows from condition $(*)$ and Proposition 3.5(3) that $E_1 \cap E_2 \in \mathfrak{C}(X,\mathscr{C},\mathscr{F})$. Thus, $(\mathfrak{C}(X,\mathscr{C},\mathscr{F}),\subseteq)$ is a semilattice.

Conversely, for an algebraic semilattice L, let $F_1, F_2 \in \mathscr{F}_L$. we have

$$\overline{F_1} \cap \overline{F_2} = \downarrow c_{F_1} \cap \downarrow c_{F_2} \cap K(L) = (\downarrow c_{F_1} \wedge c_{F_2}) \cap K(L).$$

By Theorem 3.11, $\overline{F_1} \cap \overline{F_2} \in \mathfrak{C}(K(L),\mathscr{C}_L,\mathscr{F}_L)$, showing that $(K(L),\mathscr{C}_L,\mathscr{F}_L)$ satisfies condition $(*)$. The theorem is thus proved. □

4 Representation for Algebraic L-Domains and BF-Domains

In this section, we discuss representations for algebraic L-domains and BF-domains. We first give a useful lemma.

Lemma 4.1 Let $(L,\leqslant)$ be a continuous domain and let B be a basis of L.

(1) If $(B,\leqslant)$ is a cusl, then $(L,\leqslant)$ is a bc-domain.

(2) If $(B,\leqslant)$ is an sL-cusl, then $(L,\leqslant)$ is an sL-domain.

(3) If $(B,\leqslant)$ is an L-cusl, then $(L,\leqslant)$ is an L-domain.

Proof (1) Let $(B,\leqslant)$ be a cusl. Since B has the least element $\perp$, obviously, $\perp$ is the least element of L. For any $x,y,z \in L$ satisfying $x,y \leqslant z$, we show that for all $a \in \Downarrow x \cap B$ and $b \in \Downarrow y \cap B$, $a \vee_B b$ exists. By $x,y \leqslant z$, we have $a \ll z, b \ll z$. Since B is a basis, there is $c \in \Downarrow z \cap B$ such that $a,b \leqslant c$. That $a \vee_B b$ exists by that $(B,\leqslant)$ is a cusl. Similar to the proof of Lemma 3.14, we have that $x \vee y = \bigvee\{a \vee_B b \mid a \in \Downarrow x \cap B, b \in \Downarrow y \cap B\}$ and L is a cusl. Noticing that L is a dcpo, we have that L is a bc-domain.

(2) Let $(B,\leqslant)$ be an sL-cusl. For any $x,y,z \in L$, let $x \leqslant z$ and $y \leqslant z$. Set

$$D = \{a \vee_{c(B)} b \mid a \in \Downarrow x \cap B, b \in \Downarrow y \cap B, c \in \Downarrow z \cap B, a,b \in \downarrow c\},$$

where $a \vee_{c(B)} b$ denote the supremum of a and b in $\downarrow c \cap B$. Let $a_i \in \Downarrow x \cap B, b_i \in \Downarrow y \cap B, c_i \in \Downarrow z \cap B$ and $a_i, b_i \in \downarrow c_i (i=1,2)$. Then $a_1 \vee_{c_1(B)} b_1$, $a_2 \vee_{c_2(B)} b_2 \in D$. Since B is a basis, there exists $a_3 \in \Downarrow x \cap B$ and $b_3 \in \Downarrow y \cap B$ such that $a_i \leqslant a_3, b_i \leqslant b_3$ $(i=1,2)$. Clearly, we have $\{a_3,b_3,c_1,c_2\} \subseteq \Downarrow z \cap B$. So, there exists $c_3 \in \Downarrow z \cap B$ such that $\{a_3,b_3,c_1,c_2\} \subseteq \downarrow c_3$. It follows from $a_i \leqslant a_3, b_i \leqslant b_3$ that $a_i \vee_{c_3(B)} b_i \leqslant a_3 \vee_{c_3(B)} b_3 (i=1,2)$. By Lemma 2.3 and $\{c_1,c_2\} \subseteq \downarrow c_3$, we know that $a_1 \vee_{c_1(B)} b_1 = a_1 \vee_{c_3(B)} b_1$ and $a_2 \vee_{c_2(B)} b_2 = a_1 \vee_{c_3(B)} b_1$. Hence, $a_1 \vee_{c_1(B)} b_1 \leqslant a_3 \vee_{c_3(B)} b_3$ and $a_2 \vee_{c_2(B)} b_2 \leqslant a_3 \vee_{c_3(B)} b_3$, this shows that D is directed. Thus, $\bigvee D$ exists. It is obvious that $x,y \leqslant \bigvee D \leqslant z$. Let $t \in L$ with $x,y \leqslant t \leqslant z$. For any $a \in \Downarrow x \cap B, b \in \Downarrow y \cap B, c \in \Downarrow z \cap B$ with $a,b \in \downarrow c$, we have $a \vee_{c(B)} b \in D$ and $a,b \in \Downarrow t \cap B$. Since B is a basis, there exists $c' \in \Downarrow t \cap B$ such that $a,b \in \downarrow c'$. Obviously, $a \vee_{c'(B)} b \in D$ and $a \vee_{c'(B)} b \leqslant t$. Since $c,c' \in \Downarrow z \cap B$, we know that there exists $d \in \Downarrow z \cap B$ such that $c,c' \leqslant d$. By Lemma 2.3, we have $a \vee_{c'(B)} b = a \vee_{d(B)} b = a \vee_{c(B)} b$. Thus $a \vee_{c(B)} b \leqslant t$. By arbitrariness of $a \vee_{c(B)} b \in D$, we have $\bigvee D \leqslant t$, showing that $x \vee_z y = \bigvee D$. Hence, $(L, \leqslant)$ is an sL-domain.

(3) Let $x \in L$ and $a \in \mathrel{\downarrow\!\!\downarrow} x \cap B$. We use t_a to denote the least element in $\downarrow a \cap B$. To prove t_a is the least element in $\downarrow x$, let $y \in \downarrow x$. Then there exists $b \in \mathrel{\downarrow\!\!\downarrow} y \cap B \subseteq \mathrel{\downarrow\!\!\downarrow} x \cap B$. By $a, b \in \mathrel{\downarrow\!\!\downarrow} x \cap B$, there exists $c \in \mathrel{\downarrow\!\!\downarrow} x \cap B$ such that $a, b \in \downarrow c$, It is obvious that $t_a = t_c = t_b \leqslant y$. Hence, $t_a \leqslant y$, showing that t_a is the least element in $\downarrow x$. By Part (2), we know that $(L, \leqslant)$ is an L-domain. □

Theorem 4.2 (Representation Theorem V: for algebraic sL-domains) Let $(X, \mathscr{C}, \mathcal{F})$ be an F-closure space. If $(\{\overline{F} \mid F \in \mathcal{F}\}, \subseteq)$ is an sL-cusl, then $(\mathfrak{C}(X, \mathscr{C}, \mathcal{F}), \subseteq)$ is an algebraic sL-domain. Conversely, if $(K(L), \leqslant)$ is an algebraic sL-domain, then the induced F-closure space $(K(L), \mathscr{C}_L, \mathcal{F}_L)$ satisfies that $(\{\overline{F} \mid F \in \mathcal{F}_L\}, \subseteq)$ is an sL-cusl.

Proof The first half of this theorem follows by Corollary 3.9 and Lemma 4.1(2). Next we prove the second half of this theorem.

Let $a \in K(L)$ and $x, y \in \downarrow a \cap K(L)$. If $D \subseteq L$ is a directed set and $x \vee_a y \leqslant \bigvee D = t$. By Lemma 2.3, we have $x \vee_a y = x \vee_t y$. It follows from $x, y \in K(L)$ and $x \vee_a y \leqslant \bigvee D$ that there exist $d_1, d_2 \in D$ such that $x \leqslant d_1$ and $y \leqslant d_2$. By the directedness of *D*, there is $d_3 \in D$ such that $d_1, d_2 \leqslant d_3$. By Lemma 2.3 and $d_3 \leqslant t$, we have $x \vee_{d_3} y = x \vee_t y$. Notice that $x \vee_a y = x \vee_t y$. Therefore $x \vee_a y = x \vee_{d_3} y \leqslant d_3$, showing that $x \vee_a y \in \downarrow a \cap K(L)$. Thus $x \vee_a y$ is the supremum of x, y in $\downarrow a \cap K(L)$, showing that $\downarrow a \cap K(L)$ is a sup-semilattice and $(K(L), \leqslant)$ is an sL-cusl. It follows from

$$(\{\overline{F} \mid F \in \mathcal{F}_L\}, \subseteq) = (\{\downarrow a \cap K(L) \mid a \in K(L)\}, \subseteq) \cong (K(L), \leqslant),$$

that $(\{\overline{F} \mid F \in \mathcal{F}_L\}, \subseteq)$ is an sL-cusl. □

An L-domain is a special sL-domain. Based on representations for algebraic sL-domains, the following theorem gives a representation for algebraic L-domains.

Theorem 4.3 (Representation Theorem VI: for algebraic L-domains) Let $(X, \mathscr{C}, \mathcal{F})$ be an F-closure space. If $(\{\overline{F} \mid F \in \mathcal{F}\}, \subseteq)$ is an L-cusl, then $(\mathfrak{C}(X, \mathscr{C}, \mathcal{F}), \subseteq)$ is an algebraic L-domain. Conversely, if $(L, \leqslant)$ is an algebraic L-domain, then the induced F-closure space $(K(L), \mathscr{C}_L, \mathcal{F}_L)$ satisfies that $(\{\overline{F} \mid F \in \mathcal{F}_L\}, \subseteq)$ is an L-cusl.

Proof The first half of this theorem follows by Corollary 3.9 and Lemma 4.1(3). The second half of this theorem follows from Lemma 2.6 and the proof of Theorem 4.2. □

A bc-domain is a special L-domain. A Scott domains are precisely algebraic bc-domains. The following theorem gives a representation for Scott domains.

Theorem 4.4 (Representation Theorem VII: for Scott domains) Let $(X, \mathscr{C}, \mathcal{F})$ be an F-closure space. If $(\{\overline{F} \mid F \in \mathcal{F}\}, \subseteq)$ is a cusl, then $(\mathfrak{C}(X, \mathscr{C}, \mathcal{F}), \subseteq)$ is a Scott domain. Conversely, if $(L, \leqslant)$ is a Scott domain, then the induced F-closure space $(K(L), \mathscr{C}_L, \mathcal{F}_L)$ satisfies that $(\{\overline{F} \mid F \in \mathcal{F}_L\}, \subseteq)$ is a cusl.

Proof The first half of this theorem follows by Corollary 3.9 and Lemma 4.1(1).

To prove the second half of this theorem, let $(L, \leqslant)$ be a Scott domain. Clearly, the least element of *L* is also the least element of $K(L)$. If $a, b, c \in K(L)$ with $a \leqslant c$ and $b \leqslant c$, then $a \vee b$ exists. It follows from $a \ll a, b \ll b$ and Lemma 2.4 that $a \vee b \ll a \vee b$, showing that $a \vee b \in K(L)$ and $a \vee_{K(L)} b = a \vee b$, where $a \vee_{K(L)} b$ denote the supremum of a, b in $(K(L), \leqslant)$. Thus $(K(L), \leqslant)$ is a cusl. It follows from

$(\{\overline{F} \mid F \in \mathcal{F}_L\}, \subseteq) \cong (K(L), \leqslant)$ that $(\{\overline{F} \mid F \in \mathcal{F}_L\}, \subseteq)$ is a cusl. □

To give representations of BF-domains, we need a concept of bifinite F-closure spaces.

Definition 4.5 A bifinite F-closure space is an F-closure space $(X, \mathscr{C}, \mathcal{F})$ satisfying that for all $K \subseteq_{fin} X$, there is a finite family $\mathcal{M}_K \subseteq_{fin} \mathcal{F}$ such that

(BF 1) $\mathcal{P}(K) \cap \mathcal{F} \subseteq \mathcal{M}_K$;

(BF 2) $(\forall F \in \mathcal{F})(\mathcal{G} \subseteq \mathcal{M}_K, \bigcup \mathcal{G} \subseteq \overline{F}) \Longrightarrow (\exists M \in \mathcal{M}_K)(\bigcup \mathcal{G} \subseteq \overline{M} \subseteq \overline{F})$.

Remark 4.6 For a bifinite F-closure space $(X, \mathscr{C}, \mathcal{F})$, we have

(1) for the same $K \subseteq_{fin} X$, we may have more than one finite family of $\mathcal{F}$ satisfying conditions (BF 1) and (BF 2), however, we can use the axiom of choice for all $K \subseteq_{fin} X$ to select a fixed one $\mathcal{M}_K \subseteq_{fin} \mathcal{F}$ satisfying conditions (BF 1) and (BF 2).

(2) for all $K \subseteq_{fin} X$, set $\mathcal{G} = \emptyset$, then by (BF 2) in Definition 4.5, we have that $\mathcal{M}_K \neq \emptyset$.

Theorem 4.7 If $(X, \mathscr{C}, \mathcal{F})$ is a bifinite F-closure space, then $(\mathfrak{C}(X, \mathscr{C}, \mathcal{F}), \subseteq)$ is a BF-domain.

Proof Clearly, $(\mathfrak{C}(X, \mathscr{C}, \mathcal{F}), \subseteq)$ is an algebraic domain. To show that $(\mathfrak{C}(X, \mathscr{C}, \mathcal{F}), \subseteq)$ is a BF-domain, we divide the proof into several steps by Lemma 2.10.

Step 1. Set $\mathcal{D} = \{K \subseteq_{fin} X \mid \exists F \in \mathcal{F} \text{ s.t. } F \subseteq K\}$. Then in set-theoretic inclusion order, $\mathcal{D}$ is clearly a directed family.

For all $K \in \mathcal{D}$, define $\delta_K : \mathfrak{C}(X, \mathscr{C}, \mathcal{F}) \longrightarrow \mathfrak{C}(X, \mathscr{C}, \mathcal{F})$ such that for all $E \in \mathfrak{C}(X, \mathscr{C}, \mathcal{F})$, $\delta_K(E) = \bigcup\{\overline{M} \mid M \in \mathcal{M}_K \text{ and } M \subseteq E\}$, where $\mathcal{M}_K$ is stated in Remark 4.6(2).

Step 2. Assert that for all $K \in \mathcal{D}$, δ_K is well defined and has finite range.

Let $E \in \mathfrak{C}(X, \mathscr{C}, \mathcal{F})$. By the finiteness of $\mathcal{M}_K$ and $\mathcal{M}_K \subseteq_{fin} \mathcal{F}$, we have $\bigcup\{M \mid M \in \mathcal{M}_K \text{ and } M \subseteq E\} \subseteq_{fin} E$. It follows from $E \in \mathfrak{C}(X, \mathscr{C}, \mathcal{F})$ that there exists $F \in \mathcal{F}$ such that $\bigcup\{M \mid M \in \mathcal{M}_K \text{ and } M \subseteq E\} \subseteq \overline{F} \subseteq E$. It follows from $\{M \mid M \in \mathcal{M}_K \text{ and } M \subseteq E\} \subseteq \mathcal{M}_K$ and (BF 2) that there is $M^* \in \mathcal{M}_K$ such that $\bigcup\{M \mid M \in \mathcal{M}_K \text{ and } M \subseteq E\} \subseteq \overline{M^*} \subseteq \overline{F} \subseteq E$. By Lemma 2.1, we see that $M^* \subseteq E$ and $\overline{M^*}$ is the greatest element in $\{\overline{M} \mid M \in \mathcal{M}_K \text{ and } M \subseteq E\}$ equipped with set-theoretic inclusion order. Hence $\delta_K(E) = \overline{M^*} \in \mathfrak{C}(X, \mathscr{C}, \mathcal{F})$, showing that δ_K is well defined. It follows from the finiteness of $\mathcal{M}_K$ and $M^* \in \mathcal{M}_K$ that δ_K has finite range.

Step 3. Assert that for all $K \in \mathcal{D}$, δ_K is Scott continuous.

Obviously, δ_K is order-preserving. Let $\{E_i\}_{i \in I} \subseteq (\mathfrak{C}(X, \mathscr{C}, \mathcal{F}), \subseteq)$ be a directed family and $E = \bigvee_{i \in I} E_i = \bigcup_{i \in I} E_i \in \mathfrak{C}(X, \mathscr{C}, \mathcal{F})$. By the proof of Step 2, there is $M_E^* \in \mathcal{M}_K$ and $M_E^* \subseteq E$ such that $\delta_K(E) = \overline{M_E^*}$. Since $M_E^* \subseteq_{fin} \bigcup_{i \in I} E_i = E$, there exists $j \in I$ such that $M_E^* \subseteq E_j$. So, $\delta_K(E_j) = \bigcup\{\overline{M} \mid M \in \mathcal{M}_K \text{ and } M \subseteq E_j\} \supseteq \overline{M_E^*} = \delta_K(E)$. Therefore $\delta_K(E) \subseteq \delta_K(E_j) \subseteq \bigcup_{i \in I} \delta_K(E_i) \subseteq \delta_K(E)$. Thus $\delta_K(\bigcup_{i \in I} E_i) = \bigcup_{i \in I} \delta_K(E_i)$, showing that δ_K is Scott continuous.

Step 4. Assert that $\{\delta_K\}_{K \in \mathcal{D}}$ is an approximate identity on $(\mathfrak{C}(X, \mathscr{C}, \mathcal{F}), \subseteq)$.

To show $\{\delta_K\}_{K \in \mathcal{D}}$ is directed, let $K_1, K_2 \in \mathcal{D}$ and $K = K_1 \cup K_2 \cup \bigcup(\mathcal{M}_{K_1} \cup \mathcal{M}_{K_2})$. Then $K \in \mathcal{D}$ and $\mathcal{M}_{K_1} \cup \mathcal{M}_{K_2} \subseteq \mathcal{P}(K) \cap \mathcal{F} \subseteq \mathcal{M}_K$. For all $E \in \mathfrak{C}(X, \mathscr{C}, \mathcal{F})$, we have that

$$\delta_{K_i}(E) = \cup\{\overline{M} \mid M \in \mathcal{M}_{K_i} \text{ and } M \subseteq E\} \subseteq \cup\{\overline{M} \mid M \in \mathcal{M}_K \text{ and } M \subseteq E\} = \delta_K(E) (i = 1, 2),$$

showing that $\{\delta_K\}_{K \in \mathcal{D}}$ is directed.

Let $F \in \mathcal{F}$ and $F \subseteq E \in \mathfrak{C}(X, \mathscr{C}, \mathcal{F})$. Then by (BF 1), there exists $\mathcal{M}_F \subseteq_{fin} \mathcal{F}$ satisfying $\mathcal{P}(F) \cap \mathcal{F} \subseteq \mathcal{M}_F$. It follows from $F \in \mathcal{P}(F) \cap \mathcal{F} \subseteq \mathcal{M}_F$ that $\overline{F} \subseteq \bigcup\{\overline{M} \mid M \in \mathcal{M}_F \text{ and } M \subseteq E\} = \delta_F(E)$. Noticing that $F \in \mathcal{F} \subseteq \mathcal{D}$, we have $\overline{F} \subseteq \bigvee_{K \in \mathcal{D}} \delta_K(E)$. By Proposition 3.6(2) and the arbitrariness F, we have $E = \{\overline{F} \mid F \in \mathcal{F}, F \subseteq E\} \subseteq \bigvee_{K \in \mathcal{D}} \delta_K(E)$. Obviously, $\bigvee_{K \in \mathcal{D}} \delta_K(E) \subseteq E$. Thus, $\bigvee_{K \in \mathcal{D}} \delta_K(E) = E$. This shows that $\bigvee_{K \in \mathcal{D}} \delta_K = id_{\mathfrak{C}(X, \mathscr{C}, \mathcal{F})}$.

Summing up Step 1 to Step 4, by Lemma 2.10, we have that $(\mathfrak{C}(X, \mathscr{C}, \mathcal{F}), \subseteq)$ is a BF-domain. □

Theorem 4.8 Let L be a BF-domain and let $(K(L), \mathscr{C}_L, \mathcal{F}_L)$ be the induced F-closure space. Then $(K(L), \mathscr{C}_L, \mathcal{F}_L)$ is a bifinite F-closure space.

Proof By Lemma 2.10, there is an approximate identity $\{\delta_i\}_{i \in I}$ for L consisting of kernel operators with finite range. For all $i \in I$, we use $\mathrm{Im}\,(\delta_i)$ to denote the range of δ_i. For any $m \in \mathrm{Im}\,(\delta_i)$, by Lemma 2.8 (2), we have that $m = \delta_i(m) \ll m$ and m is compact, showing that $\mathrm{Im}\,(\delta_i) \subseteq K(L)$.

For $H \subseteq_{fin} K(L)$ and $H \neq \emptyset$, it follows from $\bigvee_{i \in I} \delta_i(a) = a$ that there is $i_a \in I$ such that $a \leqslant \delta_{i_a}(a)$ for all $a \in H \subseteq K(L)$. Clearly, $\delta_{i_a}(a) \leqslant a$ and $\delta_{i_a}(a) = a$. For $\{\delta_{i_a} \mid a \in H\} \subseteq_{fin} \{\delta_i\}_{i \in I}$, it follows from the directedness of $\{\delta_i\}_{i \in I}$ that there exists $j \in I$ such that $\delta_{i_a} \leqslant \delta_j$ for all $a \in H$. Thus for all $a \in H$, we have that $a \geqslant \delta_j(a) \geqslant \delta_{i_a}(a) = a$, and $a = \delta_j(a)$, showing that $H \subseteq \mathrm{Im}(\delta_j)$. Set $\mathcal{M}_H = \mathcal{P}(\mathrm{Im}(\delta_j)) \cap \mathcal{F}_L$. It follows from $H \subseteq \mathrm{Im}(\delta_j)$ that $\mathcal{P}(H) \cap \mathcal{F}_L \subseteq \mathcal{M}_H$, showing that $\mathcal{M}_H$ satisfies (BF 1). To show $\mathcal{M}_H$ satisfies (BF 2), let $F \in \mathcal{F}$ and $\mathcal{G} \subseteq \mathcal{M}_H$ satisfying $\bigcup \mathcal{G} \subseteq \overline{F} = \downarrow c_F \cap K(L)$. Take $M = \{\delta_j(c_F)\} \subseteq \mathrm{Im}(\delta_j)$. Clearly, $M \in \mathcal{M}_H$. It follows from $\delta_j(c_F) \leqslant c_F$ that $\overline{M} = \downarrow \delta_j(c_F) \cap K(L) \subseteq \downarrow c_F \cap K(L) = \overline{F}$. For all $g \in \bigcup \mathcal{G}$, notice that $\bigcup \mathcal{G} \subseteq \mathrm{Im}(\delta_j)$, we have $\delta_j(g) = g$. It follows from $\bigcup \mathcal{G} \subseteq \overline{F} = \downarrow c_F \cap K(L)$ that $g \leqslant c_F$. Thus $g = \delta_j(g) \leqslant \delta_j(c_F)$, showing that $\bigcup \mathcal{G} \subseteq \downarrow \delta_j(c_F) \cap K(L) = \overline{M}$. Thus, we obtain that $\bigcup \mathcal{G} \subseteq \overline{M} \subseteq \overline{F}$, showing that $\mathcal{M}_H$ satisfies (BF 2).

For $H \subseteq_{fin} K(L)$ and $H = \emptyset$, we have $\mathcal{P}(H) \cap \mathcal{F}_L = \emptyset$. Let $\mathcal{M}_\emptyset = \mathcal{P}(\mathrm{Im}(\delta_i)) \cap \mathcal{F}_L$, for any $i \in I$. Obviously, $\mathcal{M}_\emptyset \subseteq_{fin} \mathcal{F}_L$ and $\mathcal{M}_\emptyset$ satisfies (BF 1). The checking of (BF 2) is similar to the case of $H \neq \emptyset$. □

Theorem 4.9 (Representation Theorem VIII: for BF-domains) A poset $(L, \leqslant)$ is a BF-domain iff there is a bifinite F-closure space $(X, \mathscr{C}, \mathcal{F})$ such that $(\mathfrak{C}(X, \mathscr{C}, \mathcal{F}), \subseteq) \cong (L, \leqslant)$.

Proof $\Leftarrow$: Follows from Theorem 4.7.

$\Rightarrow$: Let $(L, \leqslant)$ be a BF-domain, $(K(L), \mathscr{C}_L, \mathcal{F}_L)$ the induced F-closure space. It follows from Theorem 4.8 that $(K(L), \mathscr{C}_L, \mathcal{F}_L)$ is a bifinite F-closure space and $(\mathfrak{C}(K(L), \mathscr{C}_L, \mathcal{F}_L), \subseteq) \cong (L, \leqslant)$.

□

5 Direct Approach to Representing Algebraic Domains

As can be seen from the previous sections, F-closure spaces and F-augmented closure spaces are triples, which are different from the form of classical closure spaces. As is well known, a usual closure space is a pair $(X, \mathscr{C})$. In Ref. [13], Wu et al. used algebraic closure spaces rather than usual closure spaces to give representations for algebraic domains. In this section, we discuss direct approaches to representing algebraic domains by classical closure spaces.

Definition 5.1 Let $(X,\mathscr{C})$ be a closure space and $E\subseteq X$. If for any $K\subseteq_{fin} E$, there always exists $x\in X$ such that $K\subseteq \overline{x}\subseteq E$, then E is said to be FinSet-bounded. The collection of all FinSet-bounded sets of $(X,\mathscr{C})$ is denoted by $\mathfrak{S}(X,\mathscr{C})$.

Remark 5.2 (1) Given a closure space $(X,\mathscr{C})$, let $\mathscr{F}=\{x\mid x\in X\}$. It is easy to see $\mathfrak{S}(X,\mathscr{C})=\mathfrak{C}(X,\mathscr{C},\mathscr{F})$. So, closure spaces can be seen as special F-closure spaces.

(2) In Ref. [13], the definition of FinSet-bounded sets is corresponding to the case of algebraic closure spaces.

Theorem 5.3 (1) Let $(X,\mathscr{C})$ be a closure space. Then $(\mathfrak{S}(X,\mathscr{C}),\subseteq)$ is an algebraic domain.

(2) Let $(L,\leqslant)$ be an algebraic domain and let $\mathscr{C}_L$ be the family of all lower sets of $(K(L),\leqslant)$. Then $\mathfrak{S}(K(L),\mathscr{C}_L)=\{\downarrow x\cap K(L)\mid x\in K(L)\}$.

Proof (1) The proof follows directly from Theorem 3.10 and Remark 5.2.

(2) The proof is similar to the proof of Theorem 3.11. □

In the following, the closure space $(K(L),\mathscr{C}_L)$ is called the induced closure space of $(L,\leqslant)$.

Theorem 5.4 (Representation Theorem I': for algebraic domains) A dcpo $(L,\leqslant)$ is an algebraic domain iff there exists some closure space $(X,\mathscr{C})$ such that $(L,\leqslant)\cong(\mathfrak{S}(X,\mathscr{C}),\subseteq)$.

Proof The proof follows directly from Theorem 5.3. □

Definition 5.5[26] A non-empty family $\mathscr{C}$ of subsets of a set X is called a locally algebraic intersection structure if

(L1) for every directed family $\{A_i\in\mathscr{C}\mid i\in I\}$, one has $\bigcup_{i\in I}A_i\in\mathscr{C}$; and

(L2) for every $C\in\mathscr{C}$ and non-empty family $\{A_j\in\mathscr{C}\mid A_j\subseteq C, j\in J\}$, $\bigcap_{j\in J}A_j\in\mathscr{C}$.

Remark 5.6[26] (1) Let $\mathscr{C}$ be a locally algebraic intersection structure on a set X. Then the dcpo $(\mathscr{C},\subseteq)$ is an algebraic L-domain.

(2) Let $(L,\leqslant)$ be an algebraic L-domain. Then there is a locally algebraic intersection structure $\mathscr{C}$ such that $(\mathscr{C},\subseteq)\cong(L,\leqslant)$.

Proposition 5.7 Let $(X,\mathscr{C})$ be a closure space. If $(\{\overline{x}\mid x\in X\},\subseteq)$ is an L-cusl, then $\mathfrak{S}(X,\mathscr{C})$ is a locally algebraic intersection structure.

Proof It follows from Proposition 3.5 that $\mathfrak{S}(X,\mathscr{C})$ satisfies (L1). Next, we prove that $\mathfrak{S}(X,\mathscr{C})$ satisfies (L2). For any $C\in\mathfrak{S}(X,\mathscr{C})$ and non-empty family $\{A_j\in\mathfrak{S}(X,\mathscr{C})\mid A_j\subseteq C, j\in J\}$, we will show that $\bigcap_{j\in J}A_j\in\mathfrak{S}(X,\mathscr{C})$. Let $K\subseteq_{fin}\bigcap_{j\in J}A_j$. For $i\in J$, then $K\subseteq_{fin}A_i$. So, there exists $x_i\in X$ such that $K\subseteq\overline{x_i}\subseteq A_i$. Since $(\{\overline{x}\mid x\in X\},\subseteq)$ is an L-cusl, the supremum of $\{\overline{k}\mid k\in K\}$ in $(\{\overline{x}\mid \overline{x}\subseteq\overline{x_i}, x\in X\},\subseteq)$ exists, denoted by $\bigvee_{\overline{x_i}}\{\overline{k}\mid k\in K\}$. For $j\in J$, then exists $x_j\in X$ such that $K\subseteq\overline{x_j}\subseteq A_j$. Therefore the supremum of $\{\overline{k}\mid k\in K\}$ in $(\{\overline{x}\mid\overline{x}\subseteq\overline{x_j}, x\in X\},\subseteq)$ exists, denoted by $\bigvee_{\overline{x_j}}\{\overline{k}\mid k\in K\}$. Since $\{x_i,x_j\}\subseteq C$, there exists $x_t\in X$ such that $\{x_i,x_j\}\subseteq\overline{x_t}\subseteq C$. It follows from Lemma 2.3 that $\bigvee_{\overline{x_i}}\{\overline{k}\mid k\in K\}=\bigvee_{\overline{x_t}}\{\overline{k}\mid k\in K\}$ and $\bigvee_{\overline{x_j}}\{\overline{k}\mid k\in K\}=\bigvee_{\overline{x_t}}\{\overline{k}\mid k\in K\}$. This shows that $\bigvee_{\overline{x_i}}\{\overline{k}\mid k\in K\}=\bigvee_{\overline{x_j}}\{\overline{k}\mid k\in K\}$. Thus $\bigvee_{\overline{x_i}}\{\overline{k}\mid k\in K\}\subseteq\overline{x_j}\subseteq A_j$. By the arbitrariness of j, we have $K\subseteq\bigvee_{\overline{x_i}}\{\overline{k}\mid k\in K\}\subseteq\bigcap_{j\in J}A_j$. Thus $\bigcap_{j\in J}A_j\in\mathfrak{S}(X,\mathscr{C})$. □

Theorem 5.8 (Representation Theorem VI': for algebraic L-domains) Let $(X,\mathscr{C})$ be a closure space. If

$(\{\overline{x} \mid x \in X\}, \subseteq)$ is an L-cusl, then $(\mathfrak{S}(X,\mathscr{C}), \subseteq)$ is an algebraic L-domain. Conversely, if $(L, \leqslant)$ is an algebraic L-domain, then the induced closure space $(K(L), \mathscr{C}_L)$ satisfies that $(\{\overline{x} \mid x \in K(L)\}, \subseteq)$ is an L-cusl.

Proof The proof follows directly from Theorem 4.3 and Remark 5.2. □

Definition 5.9[10] A non-empty family $\mathscr{C}$ of subsets of a set X is called an algebraic intersection structure if

(A1) for every directed family $\{A_i \in \mathscr{C} \mid i \in I\}$, one has $\bigcup_{i \in I} A_i \in \mathscr{C}$; and

(A2) for every non-empty family $\{A_j \in \mathscr{C} \mid j \in J\}$, the intersection $\bigcap_{j \in J} A_j \in \mathscr{C}$.

Remark 5.10[10] (1) Let $\mathscr{C}$ be an algebraic intersection structure on a set X. Then the dcpo $(\mathscr{C}, \subseteq)$ is a Scott domain.

(2) Let $(L, \leqslant)$ be a Scott domain. Then there is an algebraic intersection structure $\mathscr{C}$ such that $(\mathscr{C}, \subseteq) \cong (L, \leqslant)$.

Proposition 5.11 Let $(X, \mathscr{C})$ be a closure space. If $(\{\overline{x} \mid x \in X\}, \subseteq)$ is a cusl, then $\mathfrak{S}(X,\mathscr{C})$ is an algebraic intersection structure.

Proof It suffices to prove that $\mathfrak{S}(X,\mathscr{C})$ satisfies (A2). For any non-empty family $\{A_j \mid j \in J\} \subseteq \mathfrak{S}(X,\mathscr{C})$, we will show that $\bigcap_{j \in J} A_j \in \mathfrak{S}(X,\mathscr{C})$. Let $K \subseteq_{fin} \bigcap_{j \in J} A_j$. Since $(\{\overline{x} \mid x \in X\}, \subseteq)$ is a cusl, the supremum of $\{\overline{k} \mid k \in K\}$ in $(\{\overline{x} \mid x \in X\}, \subseteq)$ exists, denoted by $\bigvee\{\overline{k} \mid k \in K\}$. Clearly, $K \subseteq \bigvee\{\overline{k} \mid k \in K\} \subseteq \bigcap_{j \in J} A_j$, showing that $\bigcap_{j \in J} A_j \in \mathfrak{S}(X,\mathscr{C})$. □

Theorem 5.12 (Representation Theorem VII': for Scott domains) Let $(X, \mathscr{C})$ be a closure space. If $(\{\overline{x} \mid x \in X\}, \subseteq)$ is a cusl, then $(\mathfrak{S}(X,\mathscr{C}), \subseteq)$ is a Scott domain. Conversely, if $(L, \leqslant)$ is a Scott domain, then the induced closure space $(K(L), \mathscr{C}_L)$ satisfies that $(\{\overline{x} \mid x \in K(L)\}, \subseteq)$ is a cusl.

Proof The proof follows directly from Theorem 4.4 and Remark 5.2. □

Definition 5.13[10] A non-empty family $\mathscr{C}$ of subsets of a set X is called a topped algebraic intersection structure if

(T1) for every directed family $\{A_i \in \mathscr{C} \mid i \in I\}$, one has $\bigcup_{i \in I} A_i \in \mathscr{C}$;

(T2) for every non-empty family $\{A_j \in \mathscr{C} \mid j \in J\}$, the intersection $\bigcap_{j \in J} A_j \in \mathscr{C}$; and

(T3) $X \in \mathscr{C}$.

Remark 5.14[10] (1) Let $\mathscr{C}$ be a topped algebraic intersection structure on a set X. Then the dcpo $(\mathscr{C}, \subseteq)$ is an algebraic lattice.

(2) Let $(L, \leqslant)$ be an algebraic lattice. Then there is a topped algebraic intersection structure $\mathscr{C}$ such that $(\mathscr{C}, \subseteq) \cong (L, \leqslant)$.

Proposition 5.15 Let $(X, \mathscr{C})$ be a closure space. If $(\{\overline{x} \mid x \in X\}, \subseteq)$ is a sup-semilattice with a least element, then $\mathfrak{S}(X,\mathscr{C})$ is a topped algebraic intersection structure.

Proof Similar to the proof of Proposition 5.11, it follows that $\mathfrak{S}(X,\mathscr{C})$ satisfies (T2). Next, it suffices to prove that $X \in \mathfrak{S}(X,\mathscr{C})$. Since $(\{\overline{x} \mid x \in X\}, \subseteq)$ is a sup-semilattice, we see that the set $\{\overline{x} \mid x \in X\}$ is directed. It follows from $X = \bigcup\{\overline{x} \mid x \in X\}$ and Proposition 3.6 that $X \in \mathfrak{S}(X,\mathscr{C})$. □

Theorem 5.16 (Representation Theorem III': for algebraic lattices) Let $(X, \mathscr{C})$ be a closure space. If

$(\{\overline{x} \mid x \in X\}, \subseteq)$ is a sup-semilattice with a least element, then $(\mathfrak{S}(X,\mathscr{C}), \subseteq)$ is an algebraic lattice. Conversely, if $(L, \leqslant)$ is an algebraic lattice, then the induced closure space $(K(L), \mathscr{C}_L)$ satisfies that $(\{\overline{x} \mid x \in X\}, \subseteq)$ is a sup-semilattice with a least element.

Proof The proof follows directly from Theorem 3.15 and Remark 5.2. □

Theorem 5.17 (Representation Theorem VIII′: for BF-domains) Let $(X,\mathscr{C})$ be a closure space. If $(X,\mathscr{C})$ satisfies that for all $K \subseteq_{fin} X$, there is a finite set $M_K \subseteq_{fin} X$ such that

($\dagger$1) $K \subseteq M_K$;

($\dagger$2) $(\forall x \in X)(G \subseteq M_K \cap \overline{x}) \Longrightarrow (\exists m \in M_K)(G \subseteq \overline{m} \subseteq \overline{x})$,

then $(\mathfrak{S}(X,\mathscr{C}), \subseteq)$ is a BF-domain.

Conversely, if $(L, \leqslant)$ is a BF-domain, then the induced closure space $(K(L), \mathscr{C}_L)$ satisfies that for all $K \subseteq_{fin} K(L)$, there is a finite set $M_K \subseteq_{fin} K(L)$ such that conditions ($\dagger$1) and ($\dagger$2) hold.

Proof Follows directly from Remark 5.2 and the proofs of Theorems 4.7 and 4.8. □

6 F-Relation and Scott Continuous Map

In order to investigate more relationship between F-closure spaces and algebraic domains, in this section, we introduce F-relations between two F-closure spaces.

Definition 6.1 Let $(X_1, \mathscr{C}_1, \mathcal{F}_1)$ and $(X_2, \mathscr{C}_2, \mathcal{F}_2)$ be two F-closure spaces. Then a binary relation $\Theta \subseteq \mathcal{F}_1 \times \mathcal{F}_2$ is called an F-relation, if it satisfies

(1) for all $F \in \mathcal{F}_1$, there is $G \in \mathcal{F}_2$ such that $F\Theta G$;

(2) for all $F, F' \in \mathcal{F}_1$, $G, G' \in \mathcal{F}_2$, if $F \subseteq \overline{F'}$, $G' \subseteq \overline{G}$, and $F\Theta G$, then $F'\Theta G'$; and

(3) for all $F \in \mathcal{F}_1$, $G_1, G_2 \in \mathcal{F}_2$, if $F\Theta G_1$ and $F\Theta G_2$, then there is $G_3 \in \mathcal{F}_2$ such that $G_1 \cup G_2 \subseteq \overline{G_3}$ and $F\Theta G_3$.

Since closure spaces can be seen as special F-closure spaces, we give the following definition.

Definition 6.2 Let $(X_1, \mathscr{C}_1)$ and $(X_2, \mathscr{C}_2)$ be two closure spaces. Then the relation $\Theta \subseteq X_1 \times X_2$ is called an approximable relation, if Θ satisfies the following conditions:

(1) for all $x \in X_1$, there is $y \in X_2$ such that $x\Theta y$;

(2) for all $x, x' \in X_1$, $y, y' \in X_2$, if $x \in \overline{x'}$, $y' \in \overline{y}$, and $x\Theta y$, then $x'\Theta y'$;

(3) for all $x \in X_1$, $y_1, y_2 \in X_2$, if $x\Theta y_1$ and $x\Theta y_2$, then there is $y_3 \in X_2$ such that $\{y_1, y_2\} \subseteq \overline{y_3}$ and $x\Theta y_3$.

Proposition 6.3 Let Θ be an F-relation between F-closure spaces $(X_1, \mathscr{C}_1, \mathcal{F}_1)$ and $(X_2, \mathscr{C}_2, \mathcal{F}_2)$. Then following statements are equivalent:

(1) $F\Theta G$;

(2) There exists $F' \in \mathcal{F}_1$ such that $F' \subseteq \overline{F}$ and $F'\Theta G$;

(3) There exists $G' \in \mathcal{F}_2$ such that $F\Theta G'$ and $G \subseteq \overline{G'}$;

(4) There exists $F' \in \mathcal{F}_1$ and $G' \in \mathcal{F}_2$ such that $F' \subseteq \overline{F}$, $G \subseteq \overline{G'}$ and $F'\Theta G'$.

Proof Follows directly from Lemma 2.1(1), (2) and Definition 6.1. □

Definition 6.4 Let $(X, \mathscr{C}, \mathcal{F})$ be an F-closure space and

$$\mathrm{Id}_{(X,\mathscr{C},\mathcal{F})} = \{(F,G) \mid F,G \in \mathcal{F}, G \subseteq \overline{F}\} \subseteq \mathcal{F} \times \mathcal{F}.$$

Then it is easy to check that $\mathrm{Id}_{(X,\mathscr{C},\mathcal{F})}$ is an F-relation from $(X,\mathscr{C},\mathcal{F})$ to itself. $\mathrm{Id}_{(X,\mathscr{C},\mathcal{F})}$ is called the identity F-relation on $(X,\mathscr{C},\mathcal{F})$.

Proposition 6.5 Let Θ be an F-relation from $(X_1,\mathscr{C}_1,\mathcal{F}_1)$ to $(X_2,\mathscr{C}_2,\mathcal{F}_2)$ and $E \in \mathfrak{C}(X_1,\mathscr{C}_1,\mathcal{F}_1)$. Then the family $\mathcal{D} = \{\overline{G} \mid F \in \mathcal{F}_1, F \subseteq E, G \in \mathcal{F}_2 \text{ and } F\Theta G\}$ is directed.

Proof It is easy to see by Definition 3.4 and the condition (1) in Definition 6.1 that $\mathcal{D} \neq \emptyset$. Let $X_1, X_2 \in \mathcal{D}$. Then there are $F_i \in \mathcal{F}_1$ and $G_i \in \mathcal{F}_2$ such that $F_i \subseteq E$, $F_i \Theta G_i$ and $X_i = \overline{G_i}$ $(i=1,2)$. It follows from Definition 3.4 that there exists $F_3 \in \mathcal{F}_1$ such that $F_1 \cup F_2 \subseteq \overline{F_3} \subseteq E$. It follows from the condition (2) in Proposition 6.3 that $F_3 \Theta G_1$ and $F_3 \Theta G_2$. By the condition (3) in Definition 6.1, there exists $G_3 \in \mathcal{F}_2$ such that $G_1 \cup G_2 \subseteq \overline{G_3}$ and $F_3 \Theta G_3$. By Lemma 2.1, we have $X_1 \cup X_2 \subseteq \overline{G_3}$, showing that $\mathcal{D} = \{\overline{G} \mid F \in \mathcal{F}_1, F \subseteq E, G \in \mathcal{F}_2 \text{ and } F\Theta G\}$ is directed. □

The following results shows that there is a one-to-one correspondence between the set of all Scott continuous maps from $\mathfrak{C}(X_1,\mathscr{C}_1,\mathcal{F}_1)$ to $\mathfrak{C}(X_2,\mathscr{C}_2,\mathcal{F}_2)$ and that of all F-relations from $(X_1,\mathscr{C}_1,\mathcal{F}_1)$ to $(X_2,\mathscr{C}_2,\mathcal{F}_2)$.

Theorem 6.6 (1) Let Θ be an F-relation from $(X_1,\mathscr{C}_1,\mathcal{F}_1)$ to $(X_2,\mathscr{C}_2,\mathcal{F}_2)$. Define a map $f_\Theta : \mathfrak{C}(X_1,\mathscr{C}_1,\mathcal{F}_1) \longrightarrow \mathfrak{C}(X_2,\mathscr{C}_2,\mathcal{F}_2)$ such that for all $E \in \mathfrak{C}(X_1,\mathscr{C}_1,\mathcal{F}_1)$,

$$f_\Theta(E) = \bigcup\{\overline{G} \mid F \in \mathcal{F}_1, F \subseteq E, G \in \mathcal{F}_2 \text{ and } F\Theta G\}.$$

Then f_Θ is a Scott continuous map.

(2) Let $f: \mathfrak{C}(X_1,\mathscr{C}_1,\mathcal{F}_1) \longrightarrow \mathfrak{C}(X_2,\mathscr{C}_2,\mathcal{F}_2)$ be a Scott continuous map. Define $\Theta_f \subseteq \mathcal{F}_1 \times \mathcal{F}_2$ such that

$$\forall F \in \mathcal{F}_1, G \in \mathcal{F}_2, F\Theta_f G \Leftrightarrow G \subseteq f(\overline{F}).$$

Then Θ_f is an F-relation from $(X_1,\mathscr{C}_1,\mathcal{F}_1)$ to $(X_2,\mathscr{C}_2,\mathcal{F}_2)$.

(3) Let $f: \mathfrak{C}(X_1,\mathscr{C}_1,\mathcal{F}_1) \longrightarrow \mathfrak{C}(X_2,\mathscr{C}_2,\mathcal{F}_2)$ be a Scott continuous map, Θ be an F-relation from $(X_1,\mathscr{C}_1,\mathcal{F}_1)$ to $(X_2,\mathscr{C}_2,\mathcal{F}_2)$. Then $\Theta_{f_\Theta} = \Theta$ and $f_{\Theta_f} = f$.

Proof (1) Clearly, f_Θ is order-preserving. To prove that f_Θ is Scott continuous, by Proposition 3.5(3), it suffices to prove that for any directed family $\{E_i\}_{i\in I} \subseteq \mathfrak{C}(X_1,\mathscr{C}_1,\mathcal{F}_1)$, one has $f_\Theta(\bigcup_{i\in I} E_i) = \bigcup_{i\in I} f_\Theta(E_i)$. In fact,

$$\begin{aligned} f_\Theta(\bigcup_{i\in I} E_i) =& \bigcup\{\overline{G} \mid F \in \mathcal{F}_1, F \subseteq \bigcup_{i\in I} E_i, G \in \mathcal{F}_2 \text{ and } F\Theta G\} = \\ & \bigcup_{i\in I}\bigcup\{\overline{G} \mid F \in \mathcal{F}_1, F \subseteq E_i, G \in \mathcal{F}_2 \text{ and } F\Theta G\} \\ & (\text{by directedness of } \{E_i\}_{i\in I}) = \\ & \bigcup_{i\in I} f_\Theta(E_i). \end{aligned}$$

(2) It follows from Definition 3.4 that Θ_f satisfies the condition (1) in Definition 6.1.

Let $F_1, F_2 \in \mathcal{F}_1$, $G_1, G_2 \in \mathcal{F}_2$. If $F_1 \subseteq \overline{F_2}$, $F_1 \Theta_f G_1$ and $G_2 \subseteq \overline{G_1}$, then by the definition of Θ_f, we have

$G_1 \subseteq f(\overline{F_1}) \subseteq f(\overline{F_2})$. By $f(\overline{F_2}) \in \mathfrak{C}(X_2, \mathscr{C}_2, \mathcal{F}_2)$ and Proposition 3.5(2), we have $\overline{G_1} \subseteq f(\overline{F_2})$. Since $G_2 \subseteq \overline{G_1}$, we have $G_2 \subseteq f(\overline{F_2})$. Thus $F_2 \Theta_f G_2$, showing that Θ_f satisfies the condition (2) in Definition 6.1.

Let $F \in \mathcal{F}_1$ and $G_1, G_2 \in \mathcal{F}_2$. If $F\Theta_f G_1$ and $F\Theta_f G_2$, then $G_1 \cup G_2 \subseteq_{fin} f(\overline{F})$. It follows from $f(\overline{F}) \in \mathfrak{C}(X_2, \mathscr{C}_2, \mathcal{F}_2)$ that there exists $G_3 \in \mathcal{F}_2$ such that $G_1 \cup G_2 \subseteq \overline{G_3} \subseteq f(\overline{F})$. Thus $G_3 \subseteq \overline{G_3} \subseteq f(\overline{F})$. Therefore $F\Theta_f G_3$, showing that Θ_f satisfies the condition (3) in Definition 6.1.

To sum up, Θ_f is an F-relation from $(X_1, \mathscr{C}_1, \mathcal{F}_1)$ to $(X_2, \mathscr{C}_2, \mathcal{F}_2)$.

(3) Let $F \in \mathcal{F}_1, G \in \mathcal{F}_2$. We have

$$
\begin{aligned}
(F, G) \in \Theta_{f_\Theta} \Leftrightarrow & G \subseteq f_\Theta(\overline{F}) \Leftrightarrow \\
& \exists F' \in \mathcal{F}_1, G' \in \mathcal{F}_2 \text{ such that } G \subseteq \overline{G'}, F' \subseteq \overline{F} \text{ and } (F', G') \in \Theta \\
& (\text{by Propositions 6.5, 6.3 and the finiteness of } G \in \mathcal{F}_2) \Leftrightarrow \\
& (F, G) \in \Theta \ \ (\text{by Proposition 6.3}).
\end{aligned}
$$

Thus $\Theta_{f_\Theta} = \Theta$.

Let $E \in \mathfrak{C}(X_1, \mathscr{C}_1, \mathcal{F}_1)$,

$$
\begin{aligned}
f_{\Theta_f}(E) = & \bigcup \{\overline{G} \mid F \in \mathcal{F}_1, G \in \mathcal{F}_2, F \subseteq E \text{ and } F\Theta_f G\} = \\
& \bigcup \{\overline{G} \mid F \in \mathcal{F}_1, G \in \mathcal{F}_2, F \subseteq E \text{ and } G \subseteq f(\overline{F})\} = \\
& \bigcup \{\overline{G} \mid G \in \mathcal{F}_2 \text{ and } G \subseteq f(E)\} \ \ (\text{by Scott continuity}) = \\
& f(E) \ \ (\text{by Proposition 3.6(2) and } f(E) \in \mathfrak{C}(X_2, \mathscr{C}_2, \mathcal{F}_2)).
\end{aligned}
$$

Thus $f_{\Theta_f} = f$. □

Next, inspired by the ideals and methods in[15], we give another representation for BF-domains by F-relations. First of all, we give a definition as follows.

Definition 6.7 An F-closure space $(X, \mathscr{C}, \mathcal{F})$ is called a weak bifinite F-closure space if there exists a directed family $\{\Theta_i\}_{i \in I}$ of F-relations on $(X, \mathscr{C}, \mathcal{F})$ satisfying the following conditions:

(FS 1) $\bigcup_{i \in I} \Theta_i = \mathrm{Id}_{(X, \mathscr{C}, \mathcal{F})}$;

(FS 2) For all $\Theta_i \ (i \in I)$, there is $\mathcal{M}_i \subseteq_{fin} \mathcal{F}$ such that for every $F \in \mathcal{F}$, there exists $M \in \mathcal{M}_i$ satisfying

$$\forall G \in \mathcal{F}, F\Theta_i G \Rightarrow G \subseteq \overline{M} \subseteq \overline{F}.$$

Theorem 6.8 For a weak bifinite F-closure space $(X, \mathscr{C}, \mathcal{F})$, $(\mathfrak{C}(X, \mathscr{C}, \mathcal{F}), \subseteq)$ is a BF-domain.

Proof Clearly, $(\mathfrak{C}(X, \mathscr{C}, \mathcal{F}), \subseteq)$ is an algebraic domain. Next we prove that there is an approximate identity for $(\mathfrak{C}(X, \mathscr{C}, \mathcal{F}), \subseteq)$ consisting of finitely separating maps. Let $\{\Theta_i\}_{i \in I}$ be a directed family of F-relations on $(X, \mathscr{C}, \mathcal{F})$ satisfying the condition (FS 1) and (FS 2) in Definition 6.7. For all $i \in I$, the map f_{Θ_i} is a Scott continuous maps on $(\mathfrak{C}(X, \mathscr{C}, \mathcal{F}), \subseteq)$ by Theorem 6.6(1). Let $\Theta_j, \Theta_k \in \{\Theta_i\}_{i \in I}$ and $\Theta_j \subseteq \Theta_k$. Notice that $f_{\Theta_i}(E) = \bigcup \{\overline{G} \mid F, G \in \mathcal{F}, F \subseteq E \text{ and } F\Theta_i G\} \ \ (i \in I)$ for all $E \in \mathfrak{C}(X, \mathscr{C}, \mathcal{F})$. Then we have $f_{\Theta_j}(E) \subseteq f_{\Theta_k}(E)$ for all $E \in \mathfrak{C}(X, \mathscr{C}, \mathcal{F})$, showing that the family $\{f_{\Theta_i}\}_{i \in I}$ is directed by the directedness of $\{\Theta_i\}_{i \in I}$. For all $E \in \mathfrak{C}(X, \mathscr{C}, \mathcal{F})$, we have

$$
\begin{aligned}
(\bigvee_{i\in I} f_{\Theta_i})(E) = & \bigvee_{i\in I}(f_{\Theta_i}(E))(\text{by Lemma II}-2.5 \text{ in Ref. } [2]) = \\
& \bigcup_{i\in I}(f_{\Theta_i}(E)) \ (\text{by Lemma } 3.5(3)) = \\
& \bigcup_{i\in I}(\bigcup\{\overline{G} \mid F,G\in\mathcal{F}, F\subseteq E \text{ and } F\Theta_i G\}) = \\
& \bigcup\{\overline{G} \mid F,G\in\mathcal{F}, F\subseteq E \text{ and } (F,G)\in\bigcup_{i\in I}\Theta_i\} = \\
& \bigcup\{\overline{G} \mid F,G\in\mathcal{F}, F\subseteq E \text{ and } (F,G)\in \mathrm{Id}_{(X,\mathscr{C},\mathcal{F})}\} \ (\text{by (FS 1)}) = \\
& \bigcup\{\overline{G} \mid F,G\in\mathcal{F}, F\subseteq E \text{ and } G\subseteq\overline{F}\} = \\
& \bigcup\{\overline{F} \mid F\in\mathcal{F} \text{ and } F\subseteq E\}(\text{by Propositions 3.5(1) and 3.6(2)}) = \\
& E \ (\text{by Proposition } 3.6(2)).
\end{aligned}
$$

This shows that $\bigvee_{i\in I} f_{\Theta_i} = id_{\mathfrak{C}(X,\mathscr{C},\mathcal{F})}$.

Next we verify that for all $i\in I$, f_{Θ_i} is finitely separating. By the condition (FS 2) in Definition 6.7, we have a finite subfamily $\mathcal{M}_i$ of $\mathcal{F}$ such that for all $F\in\mathcal{F}$, there exists $N\in\mathcal{M}_i$ satisfying

$$\forall G\in\mathcal{F}, F\Theta_i G \Rightarrow G\subseteq\overline{N}\subseteq\overline{F}.$$

For $M\in\mathcal{M}_i$ and $E\in\mathfrak{C}(X,\mathscr{C},\mathcal{F})$, set

$$\mathcal{D}_M = \{\overline{F} \mid (F\in\mathcal{F}, F\subseteq E) \text{ and } (\forall G\in\mathcal{F}, F\Theta_i G\Rightarrow G\subseteq\overline{M}\subseteq\overline{F}\}.$$

Then we have $\bigcup_{M\in\mathcal{M}_i}\mathcal{D}_M = \{\overline{F} \mid F\in\mathcal{F} \text{ and } F\subseteq E\}$. By by Proposition 3.6(2), family $\{\overline{F} \mid F\in\mathcal{F} \text{ and } F\subseteq E\}$ is directed. Since $\mathcal{M}_i$ is finite, by Lemma 2.2, there exists $M_0\in\mathcal{M}_i$ such that $\mathcal{D}_{M_0}$ is a cofinal subfamily of $\{\overline{F} \mid F\in\mathcal{F} \text{ and } F\subseteq E\}$ and $\bigcup\mathcal{D}_{M_0} = \bigcup\{\overline{F} \mid F\in\mathcal{F} \text{ and } F\subseteq E\}$. It follows from Proposition 3.6(2) that $\bigcup\mathcal{D}_{M_0} = E$. Set $\overline{\mathcal{M}_i} = \{\overline{M} \mid M\in\mathcal{M}_i\}$. For $F,G\in\mathcal{F}$ with $F\subseteq E$ and $F\Theta_i G$, by finiteness of F, directedness of $\mathcal{D}_{M_0}$ and that $\bigcup\mathcal{D}_{M_0} = E$, we can find some $F_0\in\mathcal{F}$ satisfying $F_0\subseteq E$ and,

$$\forall G\in\mathcal{F}, F_0\Theta_i G\Rightarrow G\subseteq\overline{M_0}\subseteq\overline{F_0}.$$

(hence $\overline{F_0}\in\mathcal{D}_{M_0}$) such that $F\subseteq\overline{R}(F_0)$. It follows from $F\subseteq\overline{F_0}$ and $F\Theta_i G$ that $F_0\Theta_i G$ by Propositon 6.3. Thus $G\subseteq\overline{M_0}\subseteq\overline{F_0}$ and $\overline{G}\subseteq\overline{M_0}$. Since

$$f_{\Theta_i}(E) = \bigcup\{\overline{G} \mid F,G\in\mathcal{F}, F\subseteq E \text{ and } F\Theta_i G\},$$

we have $f_{\Theta_i}(E)\subseteq\overline{M_0}\subseteq E$ and $\overline{M_0}\in\overline{\mathcal{M}_i}$. By finiteness of $\overline{\mathcal{M}_i}$, we see that f_{Θ_i} is finitely separating.

To sum up, it is proved that $(\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)$ is a BF-domain. □

Proposition 6.9 A bifinite F-closure space is a weak bifinite F-closure space.

Proof Let $(X,\mathscr{C},\mathcal{F})$ be a bifinite F-closure space. Then $\mathcal{D} = \{K\subseteq_{fin} X \mid \exists F\in\mathcal{F} \text{ such that } F\subseteq K\}$ defined in the proof of Theorem 4.7 is a directed family. For all $K\in\mathcal{D}$, define a binary relation Θ_K on $\mathcal{F}$ such that

$$\forall F,G\in\mathcal{F}, (F,G)\in\Theta_K \Longleftrightarrow \exists M\in\mathcal{M}_K \text{ such that } G\subseteq\overline{M}\subseteq\overline{F},$$

where $\mathcal{M}_K$ is stated in Remark 4.6(2).

To show the proposition, it suffices by Definition 6.7 to show that the family $\{\Theta_K\}_{K\in\mathcal{D}}$ is a directed family of F-relations on $(X,\mathscr{C},\mathcal{F})$ satisfying (BF 1) and (BF 2) in Definition 6.7. To this end, we divide the proof into several steps.

Step 1. Show that for all $K\in\mathcal{D}$, Θ_K is an F-relation.

It is a routine work to check Θ_K is an F-relation on $(X,\mathscr{C},\mathcal{F})$.

Step 2. It follows from Step 4 in the proof of Theorem 4.7 that $(\{\mathcal{M}_K\}_{K\in\mathcal{D}},\subseteq)$ is directed. Thus $\{\Theta_K\}_{K\in\mathcal{D}}$ is directed.

Step 3. Check the condition (FS 1) in Definition 6.7 for $\{\Theta_K\}_{K\in\mathcal{D}}$.

Clearly, for all $K\in\mathcal{D}$, $\Theta_K\subseteq \mathrm{Id}_{(X,\mathscr{C},\mathcal{F})}$. Conversely, let $F,G\in\mathcal{F}$. If $(F,G)\in\mathrm{Id}_{(X,\mathscr{C},\mathcal{F})}$, then $G\subseteq\overline{F}$. By the proof of Theorem 4.7, we have an approximate identity $\{\delta_K\}_{K\in\mathcal{D}}$ on $\mathfrak{C}(X,\mathscr{C},\mathcal{F})$, where for all $E\in\mathfrak{C}(X,\mathscr{C},\mathcal{F})$, $\delta_K(E)=\bigcup\{\overline{M}\mid M\in\mathcal{M}_K \text{ and } M\subseteq E\}$. Thus $\bigvee_{K\in\mathcal{D}}\delta_K(\overline{F})=\overline{F}$. Since $\overline{F}$ is a compact element of $(\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)$, there exists $J\in\mathcal{D}$ such that $\delta_J(\overline{F})=\overline{F}$. Thus $G\subseteq\delta_J(\overline{F})$. By Step 2 of Theorem 4.7, there exists $M_F^*\in\mathcal{M}_J$ such that $\delta_J(\overline{F})=\overline{M_F^*}\subseteq\overline{F}$. Hence, $G\subseteq\overline{M_F^*}\subseteq\overline{F}$, showing that $(F,G)\in\Theta_J\subseteq\bigcup_{K\in\mathcal{D}}\Theta_K$ and $\bigcup_{K\in\mathcal{D}}\Theta_K=\mathrm{Id}_{(X,\mathscr{C},\mathcal{F})}$. This shows that $\{\Theta_K\}_{K\in\mathcal{D}}$ satisfies the condition (FS 1).

Step 4. Check that $\{\Theta_K\}_{K\in\mathcal{D}}$ satisfies the condition (FS 2) in Definition 6.7.

For all $K\in\mathcal{D}$, let $\mathcal{M}_K\subseteq_{fin}\mathcal{F}$ be the one stated in Remark 4.6(2). Let $F\in\mathcal{F}$. By the proof of Theorem 4.7, there is an $M^*\in\mathcal{M}_K$ such that $\overline{M^*}$ is the greatest element in $\{\overline{M}\mid M\in\mathcal{M}_K \text{ and } M\subseteq\overline{F}\}$. If $G\in\mathcal{F}$ and $(F,G)\in\Theta_K$, then there is $M\in\mathcal{M}_K$ such that $G\subseteq\overline{M}\subseteq\overline{F}$. Noticing that $\overline{M^*}$ is the greatest element in $\{\overline{M}\mid M\in\mathcal{M}_K \text{ and } M\subseteq\overline{F}\}$, we have $G\subseteq\overline{M^*}\subseteq\overline{F}$, showing that $\{\Theta_K\}_{K\in\mathcal{D}}$ satisfies the condition (FS 2).

To sum up, $(X,\mathscr{C},\mathcal{F})$ is a weak bifinite F-closure space. □

It is natural to ask that weak bifinite F-closure spaces are bifinite F-closure spaces? We leave this problem as an open question to interested readers. However, as the following theorem shows, we can also give representations for BF-domains by weak bifinite F-closure spaces.

Theorem 6.10 (Representation Theorem VIII': for BF-domains) A poset $(L,\leqslant)$ is a BF-domain iff there is a weak bifinite F-closure space $(X,\mathscr{C},\mathcal{F})$ such that $(\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)\cong(L,\leqslant)$.

Proof $\Leftarrow$: Follows from Theorem 6.8.

$\Rightarrow$: Let $(L,\leqslant)$ a BF-domain, $(K(L),\mathscr{C}_L,\mathcal{F}_L)$ the induced F-closure space. By Theorem 3.12, it suffices to show that $(K(L),\mathscr{C}_L,\mathcal{F}_L)$ is a weak bifinite F-closure space. It follows from Theorem 4.8 that $(K(L),\mathscr{C}_L,\mathcal{F}_L)$ is a bifinite F-closure space, hence a weak bifinite F-closure space. □

7 Categorical Equivalence between Related Categories

In this section, we establish a categorical equivalence between the category of F-closure spaces and that of algebraic domains. For some basic notions and results in category theory, please refer to Ref. [27].

Definition 7.1 Let $(X_1,\mathscr{C}_1,\mathcal{F}_1)$, $(X_2,\mathscr{C}_2,\mathcal{F}_2)$ and $(X_3,\mathscr{C}_3,\mathcal{F}_3)$ be three F-closure spaces, and let $\Theta\subseteq\mathcal{F}_1\times\mathcal{F}_2$ and $\Upsilon\subseteq\mathcal{F}_2\times\mathcal{F}_3$ be two F-relations. Then the composition $\Upsilon\circ\Theta\subseteq\mathcal{F}_1\times\mathcal{F}_3$ of Υ and Θ is define by that for any $F_1\in\mathcal{F}_1, F_3\in\mathcal{F}_3$, $(F_1,F_3)\in\Upsilon\circ\Theta$ iff there exists $F_2\in\mathcal{F}_2$ satisfying $(F_1,F_2)\in\Theta$

and $(F_2,F_3)\in\Upsilon$.

Proposition 7.2 Let $(X_1,\mathscr{C}_1,\mathcal{F}_1)$, $(X_2,\mathscr{C}_2,\mathcal{F}_2)$ and $(X_3,\mathscr{C}_3,\mathcal{F}_3)$ be F-closure spaces, and let $\Theta\subseteq\mathcal{F}_1\times\mathcal{F}_2$ and $\Upsilon\subseteq\mathcal{F}_2\times\mathcal{F}_3$ be F-relations. Then the composition $\Upsilon\circ\Theta$ is an F-relation from $(X_1,\mathscr{C}_1,\mathcal{F}_1)$ to $(X_3,\mathscr{C}_3,\mathcal{F}_3)$.

Proof Since the composition of F-relation "$\circ$" is precisely the composition of binary relation, we have that $\Upsilon\circ\Theta$ satisfies the condition (1) in Definition 6.1.

To check that $\Upsilon\circ\Theta$ satisfies the condition (2) in Definition 6.1, let $F_1,F_2\in\mathcal{F}_1$ and $H_1,H_2\in\mathcal{F}_3$,

$$\begin{aligned}&F_1\subseteq\overline{F_2},H_2\subseteq\overline{H_1}\text{ and }(F_1,H_1)\in\Upsilon\circ\Theta\Rightarrow\\&\exists G\in\mathcal{F}_2\text{ such that }F_1\subseteq\overline{F_2},H_2\subseteq\overline{H_1},F_1\Theta G\text{ and }G\Upsilon H_1\Rightarrow\\&\exists G\in\mathcal{F}_2\text{ such that }F_2\Theta G,G\Upsilon H_2\ \text{ (by Proposition 6.3)}\Rightarrow\\&\exists(F_2,H_2)\in\Upsilon\circ\Theta.\end{aligned}$$

This shows that $\Upsilon\circ\Theta$ satisfies the condition (2) in Definition 6.1.

To check that $\Upsilon\circ\Theta$ satisfies the condition (3) in Definition 6.1, let $F\in\mathcal{F}_1$, $H_1,H_2\in\mathcal{F}_3$.

$$\begin{aligned}&(F,H_1)\in\Upsilon\circ\Theta\text{ and }(F,H_2)\in\Upsilon\circ\Theta\Rightarrow\\&\exists G_1,G_2\in\mathcal{F}_2\text{ such that }F\Theta G_1,F\Theta G_2;G_1\Upsilon H_1,G_2\Upsilon H_2\Rightarrow\\&\exists G_3\in\mathcal{F}_2\text{ such that }G_1\cup G_2\subseteq\overline{G_3},F\Theta G_3;G_1\Upsilon H_1,G_2\Upsilon H_2\\&(\text{by }\Theta\text{ satisfying (3) in Definition 6.1})\Rightarrow\\&G_3\Upsilon H_1,G_3\Upsilon H_2\text{ and }F\Theta G_3\ \text{ (by Proposition 6.3)}\Rightarrow\\&\exists H_3\in\mathcal{F}_3\text{ such that }H_1\cup H_2\subseteq\overline{H_3},G_3\Upsilon H_3\text{ and }F\Theta G_3\\&(\text{by }\Upsilon\text{ satisfying (3) in Definition 6.1})\Rightarrow\\&\exists H_3\in\mathcal{F}_3\text{ such that }H_1\cup H_2\subseteq\overline{H_3}\text{ and }(F,H_3)\in\Upsilon\circ\Theta.\end{aligned}$$

To sum up, $\Upsilon\circ\Theta$ is an F-relation from $(X_1,\mathscr{C}_1,\mathcal{F}_1)$ to $(X_3,\mathscr{C}_3,\mathcal{F}_3)$. □

Proposition 7.3 Let Θ be an F-relation from F-closure space $(X_1,\mathscr{C}_1,\mathcal{F}_1)$ to $(X_2,\mathscr{C}_2,\mathcal{F}_2)$. Then

$$\Theta\circ\mathrm{Id}_{(X_1,\mathscr{C}_1,\mathcal{F}_1)}=\mathrm{Id}_{(X_2,\mathscr{C}_2,\mathcal{F}_2)}\circ\Theta=\Theta,$$

where the identity F-relation $\mathrm{Id}_{(X_1,\mathscr{C}_1,\mathcal{F}_1)}$ is defined in Definition 6.4.

Proof For all $F\in\mathcal{F}_1,G\in\mathcal{F}_2$, we have

$$\begin{aligned}(F,G)\in\Theta\Leftrightarrow&\exists G'\in\mathcal{F}_2\ s.t.\ F\Theta G',G\subseteq\overline{G'}\ \text{(by Proposition 6.3)}\Leftrightarrow\\&\exists G'\in\mathcal{F}_2\ s.t.\ F\Theta G',(G',G)\in\mathrm{Id}_{(X_2,\mathscr{C}_2,\mathcal{F}_2)}\Leftrightarrow\\&(F,G)\in\mathrm{Id}_{(X_2,\mathscr{C}_2,\mathcal{F}_2)}\circ\Theta.\end{aligned}$$

This shows that $\mathrm{Id}_{(X_2,\mathscr{C}_2,\mathcal{F}_2)}\circ\Theta=\Theta$. Similarly, we have $\Theta\circ\mathrm{Id}_{(X_1,\mathscr{C}_1,\mathcal{F}_1)}=\Theta$. □

By Propositions 7.2, 7.3, F-closure spaces as objects and F-relations as morphisms form a category, denoted by F-CLS. Similarly, closure spaces as objects and approximable relations as morphisms can also form a category, denoted by CLS. In this paper, we use AL-DOM to denote the category of algebraic domains with Scott continuous maps.

For a category C, it is customary to denote the class of objects of C by $ob(C)$ and denote the class of morphisms of C by $Mor(C)$.

Lemma 7.4[27] Let $\mathcal{C},\mathcal{D}$ be two categories. If there is a functor $\Phi:\mathcal{C}\longrightarrow\mathcal{D}$ such that

(1) Φ is full, namely, for all $A,B\in ob(\mathcal{C})$, $g\in Mor_{\mathcal{D}}(\Phi(A),\Phi(B))$, there is $f\in Mor_{\mathcal{C}}(A,B)$ such that $\Phi(f)=g$;

(2) Φ is faithful, namely, for all $A,B\in ob(\mathcal{C})$, $f,g\in Mor_{\mathcal{C}}(A,B)$, if $f\neq g$, then $\Phi(f)\neq\Phi(g)$;

(3) for all $B\in ob(\mathcal{D})$, there is $A\in ob(\mathcal{C})$ such that $\Phi(A)\cong B$,

then $\mathcal{C}$ and $\mathcal{D}$ are equivalent.

Theorem 7.5 F-CLS is equivalent to AL-DOM.

Proof Define Ψ: F-CLS$\rightarrow$ AL-DOM such that for all $(X,\mathscr{C},\mathcal{F})\in ob$(F-CLS), $\Psi((X,\mathscr{C},\mathcal{F}))=(\mathfrak{C}(X,\mathscr{C},\mathcal{F}),\subseteq)\in ob($AL-DOM); for all $\Theta\in Mor$(F-CLS), $\Psi(\Theta)=f_\Theta\in Mor$(AL-DOM).

Give an F-closure space, for any $E\in\mathfrak{C}(X,\mathscr{C},\mathcal{F})$, we have

$$\begin{aligned}\Psi(\mathrm{Id}_{(X,\mathscr{C},\mathcal{F})})(E)=&f_{\mathrm{Id}_{(X,\mathscr{C},\mathcal{F})}}(E)=\\ &\bigcup\{\overline{G}\mid F,G\in\mathcal{F},F\subseteq E \text{ and } (F,G)\in\mathrm{Id}_{(X,\mathscr{C},\mathcal{F})}\}=\\ &\bigcup\{\overline{G}\mid F,G\in\mathcal{F},F\subseteq E \text{ and } G\subseteq\overline{F}\}=\\ &\bigcup\{\overline{F}\mid F\in\mathcal{F}, \text{ and } F\subseteq E\}=\\ &E\ \ (\text{by } E\in\mathfrak{C}(X,\mathscr{C},\mathcal{F}) \text{ and Proposition 3.6})=\\ &id_{\mathfrak{C}(X,\mathscr{C},\mathcal{F})}(E).\end{aligned}$$

This shows that $\Psi(\mathrm{Id}_{(X,\mathscr{C},\mathcal{F})})=id_{\mathfrak{C}(X,\mathscr{C},\mathcal{F})}$.

For F-closure spaces $(X_1,\mathscr{C}_1,\mathcal{F}_1)$, $(X_2,\mathscr{C}_2,\mathcal{F}_2)$ and $(X_3,\mathscr{C}_3,\mathcal{F}_3)$, let $\Theta\subseteq\mathcal{F}_1\times\mathcal{F}_2$ and $\Upsilon\subseteq\mathcal{F}_2\times\mathcal{F}_3$ are F-relations, we have

$$\begin{aligned}&\Psi(\Upsilon)\circ\Psi(\Theta)(E)=f_\Upsilon(f_\Theta(E))=\\ &\bigcup\{\overline{G}\mid F\subseteq f_\Theta(E),F\in\mathcal{F}_2,G\in\mathcal{F}_3 \text{ and } F\Upsilon G\}=\\ &\bigcup\{\overline{G}\mid F_1\in\mathcal{F}_1,F_1\subseteq E,G_1\in\mathcal{F}_2,F_1\Theta G_1,F\subseteq\overline{G_1},F\in\mathcal{F}_2,G\in\mathcal{F}_3 \text{ and } F\Upsilon G\}\\ &(\text{by the definiton of } f_\Theta(E), \text{ Proposition 6.5 and finiteness of members in } \mathcal{F}_2)=\\ &\bigcup\{\overline{G}\mid F_1\in\mathcal{F}_1,F_1\subseteq E,G_1\in\mathcal{F}_2,F_1\Theta G_1,G\in\mathcal{F}_3 \text{ and } G_1\Upsilon G\}\\ &(\text{by } F\subseteq\overline{G_1},F\Upsilon G, \text{ and Proposition 6.3})=\\ &\bigcup\{\overline{G}\mid F_1\in\mathcal{F}_1,F_1\subseteq E,G\in\mathcal{F}_3 \text{ and } (F_1,G)\in\Upsilon\circ\Theta\}\ \ (\text{by } F_1\Theta G_1 \text{ and } G_1\Upsilon G)=\\ &f_{\Upsilon\circ\Theta}(E)=\Psi(\Upsilon\circ\Theta)(E).\end{aligned}$$

This shows that $\Psi(\Upsilon)\circ\Psi(\Theta)=\Psi(\Upsilon\circ\Theta)$, and thus Ψ is a functor.

To show that F-CLS is equivalent to AL-DOM, it suffices to check that Ψ satisfies the three conditions in Lemma 7.4.

Let $(X_1,\mathscr{C}_1,\mathcal{F}_1)$ and $(X_2,\mathscr{C}_2,\mathcal{F}_2)$ be two F-closure spaces. It follows from Theorem 6.6(3) that Ψ is faithful. Let $f:\mathfrak{C}(X_1,\mathscr{C}_1,\mathcal{F}_1)\longrightarrow\mathfrak{C}(X_2,\mathscr{C}_2,\mathcal{F}_2)$ be a Scott continuous map, by Theorem 6.6(3), there is $\Theta_f\in Mor$(F-CLS) such that $\Psi(\Theta_f)=f_{\Theta_f}=f$, showing that Ψ is full. It follows from Theorem 3.12 that Ψ satisfies the condition (3) in Lemma 7.4. Thus, F-CLS is equivalent to AL-DOM. □

Similarly, we can also establish a categorical equivalence between CLS and AL-DOM, which builds a direct bridge between classical closure spaces and algebraic domains. Following this ideal, the categorical equivalences between some special F-closure spaces and subclasses of algebraic domains can also be directly obtained. We leave these details to the interested readers.

8 Conclusion

This paper generalized the notion of F-augmented closure spaces to that of F-closure spaces. The family of all F-closed sets of F-closure spaces ordered by inclusion was used to form an algebraic domain and, conversely, all the algebraic domains can be generated in this way. It was also shown that the collection of F-closed sets is a tool of building bridges between some special F-closure spaces and subclasses of algebraic domains. Being different from the method in Ref. [15], a skillful set-theoretic method without using morphisms to represent BF-domains was given. We also constructed a categorical equivalence between the category of F-closure spaces with F-relations and that of algebraic domains with Scott continuous maps. Following this idea, we gave a representation of various algebraic domains in terms of classical closure spaces and constructed a categorical equivalence between the category of closure spaces with approximable relations and that of algebraic domains with Scott continuous maps, establishing a direct connection between classical closure spaces and algebraic domains.

The work in this paper enriched the links between closure spaces and domain theory. In the future, we will consider extending these links to fuzzy settings, that is to say, representing for fuzzy algebraic domains via fuzzy closure spaces will be a future topic to work with.

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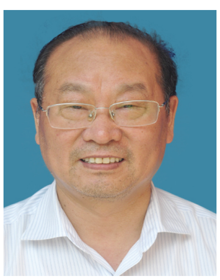

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