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New Construction of Quadriphase Golay Complementary Pairs

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Abstract — Based on an arbitrarily-chosen binary Golay complementary pair (BGCP) ($\boldsymbol{c}$,$\boldsymbol{d}$) of even length N, first of all, construct quadriphase sequences $\boldsymbol{a}$ and $\boldsymbol{b}$ of length N by weighting addition and difference of the aforementioned pair with different weights, respectively. Secondly, new quadriphase sequence $\boldsymbol{u}$ is given by interleaving three sequences $\boldsymbol{d}$, $\boldsymbol{a}$, and $-\boldsymbol{c}$, and similarly, the sequence $\boldsymbol{v}$ is acquired from three sequences $\boldsymbol{d}$, $\boldsymbol{b}$, and $\boldsymbol{c}$. Thus, the resultant pair ($\boldsymbol{u}$,$\boldsymbol{v}$) is the quadriphase Golay complementary pair (QGCP) of length $3N$. The QGCPs play a fairly important role in communications, radar, and so on.

Key words — Golay complementary pair (GCP), Binary GCP, Quadriphase GCP, Interleaving function.

I. Introduction

A Golay complementary pair (GCP) consists of two equal-length sequences whose aperiodic autocorrelation functions (AACFs) sum to zero for all non-zero time shifts [1]. Exactly due to impulse-like AACF sums of GCPs, GCPs can be applied to optical spectroscopy [2], pulsed radar for range detection [3], radar clutter rejection [4], improvement of signal detection range [5] or resolution [6] in radar systems, digital watermarking [7], and ultrasonic imaging [8]. In particular, GCPs play fairly important roles in MIMO CDMA wireless communications, which is in detail surveyed by [9]. In the applications to multi-carrier communication systems (MC-CSs), in addition, polyphase GCPs have peak-to-mean envelope power ratio (PMEPR) upper bound as low as 2 [10].

GCPs over constellations $\{\pm 1\}$ and $\{\pm 1, \pm \mathrm{j}\}$ ($\mathrm{j}^2 = -1$) are referred to as binary and quadriphase GCPs (BGCPs/QGCPs), respectively. In contrast to BGCPs, lengths of QGCPs are more flexible. BGCPs only have lengths in the form of $2^{\alpha}10^{\beta}26^{\gamma}$ (α, β, and γ are non-negative integers), whereas, QGCPs have more lengths. By a computer exhaustive searching, lengths of known QGCPs include [11]

$$2, 3, 4, 5, 6, 8, 10, 11, 12, 13, 16, 18, 20, 22, 24, 26, \ldots$$

Moreover, when QGCPs are generalized to quadriphase Golay complementary sequence sets (QGCSSs) [12]–[14], more lengths appear. However, the corresponding PMEPR is bounded by set size of QGCSSs [13].

To date, there have been a large number of available BGCPs [1]. On the contrary, the constructions of QGCPs are challenging. In 1978 and 1980, by recursive methods, Sivaswamy [15] and Frank [12] respectively presented QGCPs with lengths in the form of 3×2^k, in which k is the recursive times. In 1994, Craigen [16] claimed "twice the product of complex Golay number is a complex Golay number", and later, by using Hall polynomials, the aforementioned claim "twice" was further expanded to "any even Golay number times" in 2002 [17]. Hence, the resultant QGCPs have lengths in the form of $3 \times 2^{\alpha}10^{\beta}26^{\gamma}$. In 1999, based on standard generalized Boolean functions (GBFs) with m indeterminates, Davis and Jedwab presented a family of 2^h-phase GCPs, called Golay-Davis-Jedwab complementary sequences (GDJ-CSs), with length 2^m and family size $2^{h(m+1)}m!/2$ (h and m are integers, and $h \geq 1$, $m \geq 2$), therein, GDJ-CSs are quadriphase whenever $h = 2$ [10]. Via a computer exhaustive searching, in 2005, Li and Chu found non-GDJ-CSs of length 2^4 [18], whose existence was explained by Fiedler and Jedwab in 2006 [19]. On basis of non-standard GBFs with $(n+3)$ indeterminates (integer $n \geq 1$), in 2008, Fiedler *et al.* proposed a

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framework for the construction of 2^h-phase GCPs of lengths in the form of 2^{n+3}, including QGCPs as a special case [20], and further investigated GCPs under a view of a GCP as the "projection" of a multi-dimensional Golay array pair [21]. In 2011, Gibson and Jedwab explained the origin of all QGCPs of even length at most 26 and QGCPs of length 3×2^m from three-stage construction [11], and explored the constructions of odd-length QGCPs by using Barker sequences of the same lengths [22], herein, a given question: "Does there exist a QGCP of odd length greater than 13?" is still open. It is worth noting that the QGCPs produced by the above systematic constructions have power-of-two lengths. Via weighted linear combination of a known BGCP and its mate, in 2016, Li *et al.* proposed a construction to convert BGCPs into QGCPs with the length unaltered [23]. Recently, Zeng *et al.* constructed a QGCP from a known BGCP and the reversals of its sequences by aid of interleaving technique in 2020, therein, the length of resultant pair is twice as many as the one of employed pair [24].

In this letter, based on an arbitrarily-chosen BGCP of length in the form of $N=2^\alpha 10^\beta 26^\gamma$, a quadriphase sequence pair of the same length is firstly constructed by weighting addition and difference of the aforementioned BGCP. Then, a QGCP is produced by interleaving two pairs referred to above properly. The resultant QGCPs have lengths in the form of $3\times 2^\alpha 10^\beta 26^\gamma$. The simulation results by a computer show that the proposed QGCPs have lower PMEPRs, which is advantageous to the control of PMEPR in an multi-carrier communication system.

II. Preliminaries

Let $\boldsymbol{a}=\big(a(0),a(1),\dots,a(N-1)\big)$ and $\boldsymbol{b}=\big(b(0),b(1),\dots,b(N-1)\big)$ be two N-length sequences. For time shift $0\le\tau\le N-1$, we define their correlation function as

$$\rho_{\boldsymbol{a},\boldsymbol{b}}(\tau)=\sum_{i=0}^{N-1-\tau}a(i)b^*(i+\tau) \tag{1}$$

where superscript * stands for complex conjugate. The function $\rho_{\boldsymbol{a},\boldsymbol{b}}(\tau)$ is referred to as an aperiodic crosscorrelation function (ACCF) whenever $\boldsymbol{a}\neq\boldsymbol{b}$, otherwise is said to be an AACF and is denoted as $\rho_{\boldsymbol{a}}(\tau)$ for short.

For two N-length sequences $\boldsymbol{a}$ and $\boldsymbol{b}$, this pair $(\boldsymbol{a},\boldsymbol{b})$ is referred to as a GCP if and only if

$$\rho_{\boldsymbol{a}}(\tau)+\rho_{\boldsymbol{b}}(\tau)=0\ (1\le\forall\ \tau\le N-1) \tag{2}$$

For three N-length sequences $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, an interleaving sequence $I[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]$ of length $3N$ is given by

$$\begin{aligned}I[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]=\big(&a(0),b(0),c(0),a(1),b(1),c(1),\\&\dots,a(N-1),b(N-1),c(N-1)\big)\end{aligned} \tag{3}$$

Lemma 1 For three N-length sequences $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, the resultant interleaving sequence satisfies

$$\rho_{I[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]}(\tau)=\begin{cases}\rho_{\boldsymbol{a}}(\tau_0)+\rho_{\boldsymbol{b}}(\tau_0)+\rho_{\boldsymbol{c}}(\tau_0),\\ \quad\tau=3\tau_0\qquad(0\le\tau_0\le N-1)\\ \rho_{\boldsymbol{a},\boldsymbol{b}}(\tau_0)+\rho_{\boldsymbol{b},\boldsymbol{c}}(\tau_0)+\rho_{\boldsymbol{c},\boldsymbol{a}}(\tau_0+1),\\ \quad\tau=3\tau_0+1\quad(0\le\tau_0\le N-2)\\ \rho_{\boldsymbol{a},\boldsymbol{c}}(\tau_0)+\rho_{\boldsymbol{b},\boldsymbol{a}}(\tau_0+1)+\rho_{\boldsymbol{c},\boldsymbol{b}}(\tau_0+1),\\ \quad\tau=3\tau_0+2\quad(0\le\tau_0\le N-2)\\ \rho_{\boldsymbol{a},\boldsymbol{b}}(N-1)+\rho_{\boldsymbol{b},\boldsymbol{c}}(N-1),\\ \quad\tau=3\tau_0+1\quad(\tau_0=N-1)\\ \rho_{\boldsymbol{a},\boldsymbol{c}}(N-1),\\ \quad\tau=3\tau_0+2\quad(\tau_0=N-1)\end{cases} \tag{4}$$

Proof By formula (1), Lemma 1 follows immediately.

III. Novel Construction

Theorem 1 Consider a BGCP $(\boldsymbol{c},\boldsymbol{d})$ of length $N=2^\alpha 10^\beta 26^\gamma$. Construct a N-length quadriphase sequence pair $(\boldsymbol{a},\boldsymbol{b})$ in equations (5) and (6).

$$\begin{aligned}\boldsymbol{a}=\bigg(&\frac{c(0)+d(0)}{2}+\mathrm{j}\frac{d(0)-c(0)}{2},\\&\frac{c(1)+d(1)}{2}+\mathrm{j}\frac{d(1)-c(1)}{2},\dots,\\&\frac{c(N-1)+d(N-1)}{2}+\mathrm{j}\frac{d(N-1)-c(N-1)}{2}\bigg)\end{aligned} \tag{5}$$

$$\begin{aligned}\boldsymbol{b}=\bigg(&-\mathrm{j}\frac{c(0)+d(0)}{2}-\frac{d(0)-c(0)}{2},\\&-\mathrm{j}\frac{c(1)+d(1)}{2}-\frac{d(1)-c(1)}{2},\dots,\\&-\mathrm{j}\frac{c(N-1)+d(N-1)}{2}-\frac{d(N-1)-c(N-1)}{2}\bigg)\end{aligned} \tag{6}$$

By making use of interleaving function in (3), quadriphase sequence pair $(\boldsymbol{u},\boldsymbol{v})$ can be deduced from

$$\begin{aligned}\boldsymbol{u}&=I[\boldsymbol{d},\boldsymbol{a},-\boldsymbol{c}]\\ \boldsymbol{v}&=I[\boldsymbol{d},\boldsymbol{b},\boldsymbol{c}]\end{aligned} \tag{7}$$

where $-\boldsymbol{c}=\big(-c(0),-c(1),\dots,-c(N-1)\big)$. Then, the resultant pair $(\boldsymbol{u},\boldsymbol{v})$ is the QGCP of length in the form of $3\times 2^\alpha 10^\beta 26^\gamma$.

Proof In order to derive this theorem easily, time shifts $0\le\tau\le 3N-1$ are divided into five sub-sets shown below.

$$\begin{aligned}
&\text{Range}\,1: \tau = 3\tau_0 && (0\le\tau_0\le N-1).\\
&\text{Range}\,2: \tau = 3\tau_0+1 && (0\le\tau_0\le N-2).\\
&\text{Range}\,3: \tau = 3\tau_0+2 && (0\le\tau_0\le N-2).\\
&\text{Range}\,4: \tau = 3\tau_0+1 && (\tau_0 = N-1).\\
&\text{Range}\,5: \tau = 3\tau_0+2 && (\tau_0 = N-1).
\end{aligned}$$

Then, we calculate sums of AACFs of the pair $(\boldsymbol{u},\boldsymbol{v})$ by the divided ranges and Lemma 1, respectively.

Range 1 $\tau = 3\tau_0$ $(0\le\tau_0\le N-1)$.

According to Lemma 1, we have

$$\begin{aligned}
\rho_{\boldsymbol{u}}(\tau) &= \rho_{\boldsymbol{d}}(\tau_0)+\rho_{\boldsymbol{a}}(\tau_0)+\rho_{-\boldsymbol{c}}(\tau_0)\\
\rho_{\boldsymbol{v}}(\tau) &= \rho_{\boldsymbol{d}}(\tau_0)+\rho_{\boldsymbol{b}}(\tau_0)+\rho_{\boldsymbol{c}}(\tau_0)
\end{aligned} \tag{8}$$

Therefore, it is sufficient to calculate the AACFs of sequences $\boldsymbol{a}$ and $\boldsymbol{b}$ for obtaining the sums in Range 1, due to the fact that $\boldsymbol{c}$ and $\boldsymbol{d}$ are the BGCP. According to equation (1), we have

$$\begin{aligned}
\rho_{\boldsymbol{a}}(\tau_0) = &\sum_{k=0}^{N-1-\tau_0}\Big[\frac{c(k)+d(k)}{2}+\mathrm{j}\frac{d(k)-c(k)}{2}\Big]\\
&\cdot\Big[\frac{c(k+\tau_0)+d(k+\tau_0)}{2}+\mathrm{j}\frac{d(k+\tau_0)-c(k+\tau_0)}{2}\Big]^*\\
= &\frac{1}{2}\sum_{k=0}^{N-1-\tau_0}\Big\{\Big[c(k)c(k+\tau_0)+d(k)d(k+\tau_0)\Big]\\
&+\mathrm{j}\Big[d(k)c(k+\tau_0)-c(k)d(k+\tau_0)\Big]\Big\}\\
= &\frac{1}{2}\Big[\rho_{\boldsymbol{c}}(\tau_0)+\rho_{\boldsymbol{d}}(\tau_0)\Big]+\frac{\mathrm{j}}{2}\Big[\rho_{\boldsymbol{d},\boldsymbol{c}}(\tau_0)-\rho_{\boldsymbol{c},\boldsymbol{d}}(\tau_0)\Big]
\end{aligned} \tag{9}$$

Similarly, we have

$$\rho_{\boldsymbol{b}}(\tau_0) = \frac{1}{2}\Big[\rho_{\boldsymbol{c}}(\tau_0)+\rho_{\boldsymbol{d}}(\tau_0)\Big]+\frac{\mathrm{j}}{2}\Big[\rho_{\boldsymbol{c},\boldsymbol{d}}(\tau_0)-\rho_{\boldsymbol{d},\boldsymbol{c}}(\tau_0)\Big] \tag{10}$$

Notice that $\rho_{-\boldsymbol{c}}(\tau_0)=\rho_{\boldsymbol{c}}(\tau_0)$. Consequently, the sum of AACFs of the pair $(\boldsymbol{u},\boldsymbol{v})$ is given by

$$\begin{aligned}
\rho_{\boldsymbol{u}}(\tau)+\rho_{\boldsymbol{v}}(\tau) &= 3\Big[\rho_{\boldsymbol{c}}(\tau_0)+\rho_{\boldsymbol{d}}(\tau_0)\Big]\\
&= \begin{cases} 3N, & \tau_0 = 0\\ 0, & \text{otherwise}\end{cases}
\end{aligned} \tag{11}$$

Range 2 $\tau = 3\tau_0+1$ $(0\le\tau_0\le N-2)$. Notice that we have

$$\begin{aligned}
\rho_{\boldsymbol{d},\boldsymbol{a}}(\tau_0) = &\sum_{k=0}^{N-1-\tau_0} d(k)\Big[\frac{c(k+\tau_0)+d(k+\tau_0)}{2}\\
&+\mathrm{j}\frac{d(k+\tau_0)-c(k+\tau_0)}{2}\Big]^*\\
= &\frac{1}{2}\Big[\rho_{\boldsymbol{d},\boldsymbol{c}}(\tau_0)+\rho_{\boldsymbol{d}}(\tau_0)-\mathrm{j}\rho_{\boldsymbol{d}}(\tau_0)+\mathrm{j}\rho_{\boldsymbol{d},\boldsymbol{c}}(\tau_0)\Big]
\end{aligned} \tag{12}$$

$$\begin{aligned}
\rho_{\boldsymbol{a},-\boldsymbol{c}}(\tau_0) = &-\sum_{k=0}^{N-1-\tau_0}\Big[\frac{c(k)+d(k)}{2}+\mathrm{j}\frac{d(k)-c(k)}{2}\Big]c(k+\tau_0)\\
= &\frac{1}{2}\Big[-\rho_{\boldsymbol{c}}(\tau_0)-\rho_{\boldsymbol{d},\boldsymbol{c}}(\tau_0)-\mathrm{j}\rho_{\boldsymbol{d},\boldsymbol{c}}(\tau_0)+\mathrm{j}\rho_{\boldsymbol{c}}(\tau_0)\Big]
\end{aligned} \tag{13}$$

$$\begin{aligned}
\rho_{\boldsymbol{d},\boldsymbol{b}}(\tau_0) = &\sum_{k=0}^{N-1-\tau_0} d(k)\Big[-\mathrm{j}\frac{c(k+\tau_0)+d(k+\tau_0)}{2}\\
&-\frac{d(k+\tau_0)-c(k+\tau_0)}{2}\Big]^*\\
= &\frac{1}{2}\Big[\mathrm{j}\rho_{\boldsymbol{d},\boldsymbol{c}}(\tau_0)+\mathrm{j}\rho_{\boldsymbol{d}}(\tau_0)-\rho_{\boldsymbol{d}}(\tau_0)+\rho_{\boldsymbol{d},\boldsymbol{c}}(\tau_0)\Big]
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
\rho_{\boldsymbol{b},\boldsymbol{c}}(\tau_0) = &\sum_{k=0}^{N-1-\tau_0}\Big[-\mathrm{j}\frac{c(k)+d(k)}{2}-\frac{d(k)-c(k)}{2}\Big]c(k+\tau_0)\\
= &\frac{1}{2}\Big[-\mathrm{j}\rho_{\boldsymbol{c}}(\tau_0)-\mathrm{j}\rho_{\boldsymbol{d},\boldsymbol{c}}(\tau_0)-\rho_{\boldsymbol{d},\boldsymbol{c}}(\tau_0)+\rho_{\boldsymbol{c}}(\tau_0)\Big]
\end{aligned} \tag{15}$$

From (12)–(15), $\rho_{-\boldsymbol{c},\boldsymbol{d}}(\tau_0+1)=-\rho_{\boldsymbol{c},\boldsymbol{d}}(\tau_0+1)$, and Lemma 1, we can come to the following conclusion:

$$\begin{aligned}
\rho_{\boldsymbol{u}}(\tau)+\rho_{\boldsymbol{v}}(\tau) = &\Big[\rho_{\boldsymbol{d},\boldsymbol{a}}(\tau_0)+\rho_{\boldsymbol{a},-\boldsymbol{c}}(\tau_0)+\rho_{-\boldsymbol{c},\boldsymbol{d}}(\tau_0+1)\Big]\\
&+\Big[\rho_{\boldsymbol{d},\boldsymbol{b}}(\tau_0)+\rho_{\boldsymbol{b},\boldsymbol{c}}(\tau_0)+\rho_{\boldsymbol{c},\boldsymbol{d}}(\tau_0+1)\Big] = 0
\end{aligned} \tag{16}$$

Range 3 $\tau = 3\tau_0+2$ $(0\le\tau_0\le N-2)$.

Similarly, we firstly calculate

$$\begin{aligned}
\rho_{\boldsymbol{a},\boldsymbol{d}}(\tau_0+1) = &\sum_{k=0}^{N-1-(\tau_0+1)}\Big[\frac{c(k)+d(k)}{2}+\mathrm{j}\frac{d(k)-c(k)}{2}\Big]\\
&\cdot d(k+\tau_0+1)\\
= &\frac{1}{2}\Big[\rho_{\boldsymbol{c},\boldsymbol{d}}(\tau_0+1)+\rho_{\boldsymbol{d}}(\tau_0+1)+\mathrm{j}\rho_{\boldsymbol{d}}(\tau_0+1)\\
&-\mathrm{j}\rho_{\boldsymbol{c},\boldsymbol{d}}(\tau_0+1)\Big]
\end{aligned} \tag{17}$$

$$\begin{aligned}
&\rho_{-\boldsymbol{c},\boldsymbol{a}}(\tau_0+1)\\
&= -\sum_{k=0}^{N-1-(\tau_0+1)} c(k)\Big[\frac{c(k+\tau_0+1)+d(k+\tau_0+1)}{2}\\
&\quad+\mathrm{j}\frac{d(k+\tau_0+1)-c(k+\tau_0+1)}{2}\Big]^*\\
&= \frac{1}{2}\Big[-\rho_{\boldsymbol{c}}(\tau_0+1)-\rho_{\boldsymbol{c},\boldsymbol{d}}(\tau_0+1)+\mathrm{j}\rho_{\boldsymbol{c},\boldsymbol{d}}(\tau_0+1)\\
&\quad-\mathrm{j}\rho_{\boldsymbol{c}}(\tau_0+1)\Big]
\end{aligned} \tag{18}$$

$$\rho_{\boldsymbol{b},\boldsymbol{d}}(\tau_0+1)=\sum_{k=0}^{N-1-(\tau_0+1)}\Big[-\mathrm{j}\frac{c(k)+d(k}{2}-\frac{d(k)-c(k)}{2}\Big]d(k+\tau_0+1)$$
$$=\frac{1}{2}\Big[-\mathrm{j}\rho_{\boldsymbol{c},\boldsymbol{d}}(\tau_0+1)-\mathrm{j}\rho_{\boldsymbol{d}}(\tau_0+1)-\rho_{\boldsymbol{d}}(\tau_0+1)+\rho_{\boldsymbol{c},\boldsymbol{d}}(\tau_0+1)\Big] \tag{19}$$

and

$$\rho_{\boldsymbol{c},\boldsymbol{b}}(\tau_0+1)=\sum_{k=0}^{N-1-(\tau_0+1)}c(k)\Big[-\mathrm{j}\frac{c(k+\tau_0+1)+d(k+\tau_0+1)}{2}-\frac{d(k+\tau_0+1)-c(k+\tau_0+1)}{2}\Big]^*$$
$$=\frac{1}{2}\Big[\mathrm{j}\rho_{\boldsymbol{c}}(\tau_0+1)+\mathrm{j}\rho_{\boldsymbol{c},\boldsymbol{d}}(\tau_0+1)-\rho_{\boldsymbol{c},\boldsymbol{d}}(\tau_0+1)+\rho_{\boldsymbol{c}}(\tau_0+1)\Big] \tag{20}$$

From (17)–(20), $\rho_{\boldsymbol{d},-\boldsymbol{c}}(\tau_0)=-\rho_{\boldsymbol{d},\boldsymbol{c}}(\tau_0)$, and Lemma 1, we can draw the conclusion below.

$$\rho_{\boldsymbol{u}}(\tau)+\rho_{\boldsymbol{v}}(\tau)=\Big[\rho_{\boldsymbol{d},-\boldsymbol{c}}(\tau_0)+\rho_{\boldsymbol{a},\boldsymbol{d}}(\tau_0+1)+\rho_{-\boldsymbol{c},\boldsymbol{a}}(\tau_0+1)\Big]+\Big[\rho_{\boldsymbol{d},\boldsymbol{c}}(\tau_0)+\rho_{\boldsymbol{b},\boldsymbol{d}}(\tau_0+1)+\rho_{\boldsymbol{c},\boldsymbol{d}}(\tau_0+1)\Big]=0 \tag{21}$$

Range 4 $\tau=3\tau_0+1$ $(\tau_0=N-1)$.

According to the definition of correlation functions in (1), we easily calculate the following required correlation functions for obtaining the sums in Range 4.

$$\rho_{\boldsymbol{d},\boldsymbol{a}}(N-1)=d(0)\Big[\frac{c(N-1)+d(N-1)}{2}+\mathrm{j}\frac{d(N-1)-c(N-1)}{2}\Big]^*$$
$$=\frac{1}{2}\Big[\rho_{\boldsymbol{d},\boldsymbol{c}}(N-1)+\rho_{\boldsymbol{d}}(N-1)-\mathrm{j}\rho_{\boldsymbol{d}}(N-1)+\mathrm{j}\rho_{\boldsymbol{d},\boldsymbol{c}}(N-1)\Big] \tag{22}$$

$$\rho_{\boldsymbol{a},-\boldsymbol{c}}(N-1)=-\Big[\frac{c(0)+d(0)}{2}+\mathrm{j}\frac{d(0)-c(0)}{2}\Big]c(N-1)$$
$$=\frac{1}{2}\Big[-\rho_{\boldsymbol{c}}(N-1)-\rho_{\boldsymbol{d},\boldsymbol{c}}(N-1)-\mathrm{j}\rho_{\boldsymbol{d},\boldsymbol{c}}(N-1)+\mathrm{j}\rho_{\boldsymbol{c}}(N-1)\Big] \tag{23}$$

$$\rho_{\boldsymbol{d},\boldsymbol{b}}(N-1)=d(0)\Big[-\mathrm{j}\frac{c(N-1)+d(N-1)}{2}-\frac{d(N-1)-c(N-1)}{2}\Big]^*$$
$$=\frac{1}{2}\Big[\mathrm{j}\rho_{\boldsymbol{d},\boldsymbol{c}}(N-1)+\mathrm{j}\rho_{\boldsymbol{d}}(N-1)-\rho_{\boldsymbol{d}}(N-1)+\rho_{\boldsymbol{d},\boldsymbol{c}}(N-1)\Big] \tag{24}$$

and

$$\rho_{\boldsymbol{b},\boldsymbol{c}}(N-1)=\Big[-\mathrm{j}\frac{c(0)+d(0)}{2}-\frac{d(0)-c(0)}{2}\Big]c(N-1)$$
$$=\frac{1}{2}\Big[-\mathrm{j}\rho_{\boldsymbol{c}}(N-1)-\mathrm{j}\rho_{\boldsymbol{d},\boldsymbol{c}}(N-1)-\rho_{\boldsymbol{d},\boldsymbol{c}}(N-1)+\rho_{\boldsymbol{c}}(N-1)\Big] \tag{25}$$

From Lemma 1, as a result, we have

$$\rho_{\boldsymbol{u}}(\tau)+\rho_{\boldsymbol{v}}(\tau)=\Big[\rho_{\boldsymbol{d},\boldsymbol{a}}(N-1)+\rho_{\boldsymbol{a},-\boldsymbol{c}}(N-1)\Big]+\Big[\rho_{\boldsymbol{d},\boldsymbol{b}}(N-1)+\rho_{\boldsymbol{b},\boldsymbol{c}}(N-1)\Big]=0 \tag{26}$$

Range 5 $\tau=3\tau_0+2$ $(\tau_0=N-1)$.

According to Lemma 1, apparently, we have

$$\rho_{\boldsymbol{u}}(\tau)+\rho_{\boldsymbol{v}}(\tau)=\rho_{\boldsymbol{d},-\boldsymbol{c}}(N-1)+\rho_{\boldsymbol{d},\boldsymbol{c}}(N-1)=0 \tag{27}$$

Summarizing the above, this theorem follows immediately.

Based on kernels of lengths 2, 10 and 26, Turyn's construction [1] can produce a BGCP of any length in the form of $2^\alpha 10^\beta 26^\gamma$. For an arbitrarily-given length in the form of $3\times 2^\alpha 10^\beta 26^\gamma$, hence, Theorem 1 can produce a QGCP with such length. Table 1 gives QGCPs, produced by [16] and this letter, of lengths up to 96 one for each length, where 0, 1, 2, and 3 stand for 1, j, -1, and $-$j, respectively, and 0_r denotes r-run of length r. Table 2 compares existing known systematic constructions for producing QGCPs.

Remarks Here are some brief reviews on the closely-relevant references, therein, the relevant constructions are equivalently and briefly re-written by mathematical symbols employed by this letter.

1) On references [16] and [17].

Let $(\boldsymbol{a},\boldsymbol{b})$ and $(\boldsymbol{c},\boldsymbol{d})$ be two complex Golay complementary pairs (CGCPs) of lengths N_1 and N_2, respectively. Craigen [16] presented the following construction so as to produce CGCP $(\boldsymbol{u},\boldsymbol{v})$ of length N_1N_2.

$$(\boldsymbol{u},\boldsymbol{v})=(\boldsymbol{a}\otimes\boldsymbol{c}||\boldsymbol{b}\otimes\overleftarrow{\boldsymbol{d}}^*,\boldsymbol{a}\otimes\boldsymbol{d}||-\boldsymbol{b}\otimes\overleftarrow{\boldsymbol{c}}^*) \tag{28}$$

where $\otimes$, $||$, and $\overleftarrow{\boldsymbol{x}}$ stand for Kronecker product, horizontal concatenation, and reversal of a sequence $\boldsymbol{x}$, re-

Table 1. The QGCPs of lengths up to 96 from [16] and this letter (Th.1 means Theorem 1)

N	Ref.	QGCPs	PMEPR	$\mathrm{sum}_{\tau=0}^{N-1}$
6	Th.1	(002232,030200)	1.9825	$(12, 0_5)$
	[16]	(010133,113212)	1.9825	
12	Th.1	(002220232232,030212200200)	1.9874	$(24, 0_{11})$
	[16]	(002002030212,010010022200)	1.9893	
24	Th.1	(002220220220232010232232,030212212212200022200200)	1.9976	$(48, 0_{23})$
	[16]	(002220002002030212212212,010232010010022200200200)	1.9990	
30	Th.1	(002002220002010002010010232232,030030212030022030022022200200)	1.9877	$(60, 0_{29})$
	[16]	(010232303303031330223111133200,113331002002002212101030212323)	1.9924	
48	Th.1	(002232220010002232002232002010220232220232220232, 030200212022030200030200030022212200212200212200)	1.9873	$(96, 0_{47})$
	[16]	(010232232232010232010010133311311311311133311311, 113331331331113331113113212030030030030212030030)	1.9989	
60	Th.1	([002232220010220010220010220232220010220232002010002010220232, 030200212022212022212022212200212022212200030022030022212200)	1.9843	$(120, 0_{59})$
	[16]	(022022200022200022200200022022010010232010010010010010232232, 030030212030212030212212030030002002220002002002002002220220)	1.9957	
78	Th.1	(002002002002220002002220220002220002010002010010232010232232232010010232232232, 030030030030212030030212212030212030022030022022200022200200200022022200200200)	1.9951	$(156, 0_{77})$
	[16]	(002002002113220002002331002220002331113030101212212212101212030030323212030323, 010010010121232010010303010232010303121022133200200200133200022022311200022311)	1.9987	
96	Th.1	(002220220220220002220220220002002002220002220220232010010010010232010010232010010 0010232010232232,030212212212212030212212212030030030212030212212200022022022 022200022022200022022022200022200200)	1.9981	$(192, 0_{95})$
	[16]	(002220220220220002220220002220220220002220002002030212212212212030212212212120300 30030212030212212,010232232232232010232232010232232232010232010010022200200200 200022200200200022022022200022200200)	1.9990	

Table 2. The comparison between QGCPs from existing known systematic constructions.

Ref.	Length	Seeds
[10]	2^m	Standard GBFs
[20]	2^{n+3}	Non-standard GBFs
[12]	3×2^k	QGCPs
[15]		
[11]	3×2^m	QGCP+GBFs
[24]	$2 \times 2^\alpha 10^\beta 26^\gamma$	BGCPs
[23]	$2^\alpha 10^\beta 26^\gamma$	
[16],[17]	$3 \times 2^\alpha 10^\beta 26^\gamma$	
Theorem 1		

spectively. Hence, Craigen claimed that “twice the product of complex Golay number is a complex Golay number”. Later, by using Hall polynomials, the aforementioned claim “twice” was further expanded to “any even Golay number times” in [17]. Consequently, the resultant pairs have lengths in the form of $3 \times 2^\alpha 10^\beta 26^\gamma$, when one of the pairs $(\boldsymbol{a}, \boldsymbol{b})$ and $(\boldsymbol{c}, \boldsymbol{d})$ is QGCP of length 3, and the other is BGCP of length in the form of $2^\alpha 10^\beta 26^\gamma$.

Example 1 Consider QGCP of length 24. Take on the QGCP $(\boldsymbol{a}, \boldsymbol{b}) = (002, 010)$ of length 3 and the BGCP $(\boldsymbol{c}, \boldsymbol{d}) = (0200, 0222)$ of length 4, i.e., $(\alpha, \beta, \gamma) = (2, 0, 0)$. Then, the resultant QGCP $(\boldsymbol{u}, \boldsymbol{v})$ of length 24 in reference [16] is

$$\boldsymbol{u} = (020002002022222033312220)$$
$$\boldsymbol{v} = (022202222000220233132202)$$

In this letter, take on the BGCP $(\boldsymbol{c}, \boldsymbol{d}) = (02220200, 02222022)$ of length 8, i.e., $(\alpha, \beta, \gamma) = (3, 0, 0)$. As a consequence, the resultant QGCP $(\boldsymbol{u}, \boldsymbol{v})$ of length 24 is

$$\boldsymbol{u} = (002220220220232010232232)$$
$$\boldsymbol{v} = (030212212212200022200200)$$

Although the constructions in [16], [17] and this letter produce QGCPs with the same lengths, hence, both are different at all, no matter in both their constructions and numerical examples.

2) On References [12], [15], and [11].

These three references produce QGCPs with lengths in the form of 3×2^k or 3×2^m, therein, references [12] and [15] employed recursive technique, and reference [11] made use of three-stage construction so as to produce QGCPs from GBFs constructed by a known QGCP. Here, only the construction in [15] is given below.

Let $(\boldsymbol{S}_1, \boldsymbol{C}_1)$ be a known multiphase GCP of length N. Reference [15] presented the recursive construction below.

$$(\boldsymbol{S}_{k+1}, \boldsymbol{C}_{k+1}) = (\boldsymbol{S}_k || \boldsymbol{C}_k, \boldsymbol{S}_k || -\boldsymbol{C}_k) \tag{29}$$

As a result, the resultant multiphase GCP $(\boldsymbol{S}_{k+1}, \boldsymbol{C}_{k+1})$ has length in the form of $N \times 2^k$. In particular, when the initial QGCP of length 3 is employed, the resulting pair $(\boldsymbol{S}_{k+1}, \boldsymbol{C}_{k+1})$ is the QGCP of length in the form of 3×2^k.

Example 2 Consider QGCP of length 12. Take on the initial QGCP $(\boldsymbol{S}_1, \boldsymbol{C}_1) = (002, 010)$ of length 3, and $k = 2$. Hence, the construction in Reference [15] produces the length-12 QGCP $(\boldsymbol{S}_2, \boldsymbol{C}_2) =$(002010002232, 002010220010).

In this letter, take on BGCP $(\boldsymbol{c}, \boldsymbol{d}) = (0200, 0222)$ of length 4, i.e., $(\alpha, \beta, \gamma) = (2, 0, 0)$. Hence, the resultant QGCP $(\boldsymbol{u}, \boldsymbol{v})$ of length 12 is $(\boldsymbol{u}, \boldsymbol{v}) = (002220232232$, $030212200200)$.

Incidentally, References [12], [15], and [11] cannot produce QGCPs of length in the form of $3 \times 2^{\alpha} 10^{\beta} 26^{\gamma}$ whenever $\beta + \gamma \geq 1$.

IV. PMEPRs of the Resultant QGCPs

GCPs can be used to control the PMEPR of an MC-CS. Consider an MC-CS of N subcarriers where the frequency of the i-th subcarrier is given by $f_i = f_0 + i\Delta f$ $(0 \leq i \leq N-1)$, where f_0 and Δf stand for the carrier frequency and the subcarrier spacing, respectively. For this MC system, at every symbol duration $0 \leq t < 1/\Delta f$ the transmitted signal, associated with a binary or quadriphase sequence $\boldsymbol{a}$ of length N, can be represented as the real part of complex signal $s_{\boldsymbol{a}}(t)$ given below [10], [25]–[26].

$$s_{\boldsymbol{a}}(t) = \sum_{k=0}^{N-1} a(k) e^{2\pi j f_k t} \tag{30}$$

The PMEPR of transmitted signal $s_{\boldsymbol{a}}(t)$ is defined as [10], [25]–[26]

$$\mathrm{PMEPR}(a) = \frac{1}{N} \sup\nolimits_{0 \leq t < 1/\Delta f} |s_{\boldsymbol{a}}(t)|^2 \tag{31}$$

Theorem 2 [10] Let $(\boldsymbol{a}, \boldsymbol{b})$ be a length-N BGCP or QGCP. Hence, the PMEPR of transmitted signal $s_{\boldsymbol{a}}(t)$ is upper bounded by

$$\mathrm{PMEPR}(a) \leq 2 \tag{32}$$

For an MC system, the lower the PMEPR, the better the performance of this system. In order to obtain the PMEPRs of all the QGCPs in Table 1, the simulation model in Example 3 in [27] is employed by this letter. Take on $f_0 = 0$ and $\Delta f = 1$. After calculation, the PMEPRs of the sequence $\boldsymbol{u}$'s of all the QGCPs in Table 1 are listed in Table 1, and Fig.1 visually depicts the curves of these PMEPRs. Apparently, all the results is perfectly in accord with the conclusions of Theorem 2. The proposed QGCPs in Table 1 have lower PMEPR values.

V. Conclusions

Based on BGCPs and interleaving technique, this letter presents a new systematic construction so as to produce QGCPs with the lengths in the form of $3 \times 2^{\alpha} 10^{\beta} 26^{\gamma}$, and the computer simulation suggests that the resultant QGCPs have lower PMEPRs. In the future, more flexible lengths need to be explored.

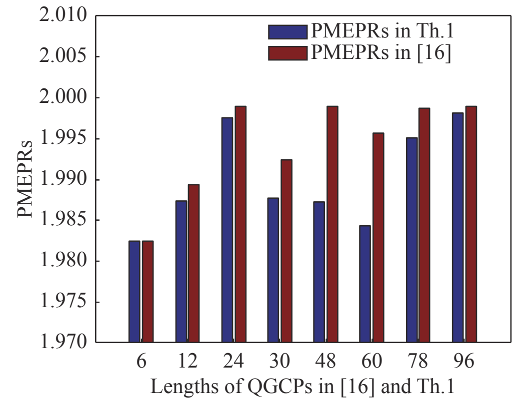

Fig. 1. PMEPR histogram of the QGCPs in Table 1.

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