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Asymptotically Optimal Golay-ZCZ Sequence Sets with Flexible Length

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Abstract — Zero correlation zone (ZCZ) sequences and Golay complementary sequences are two kinds of sequences with different preferable correlation properties. Golay-ZCZ sequences are special kinds of complementary sequences which also possess a large ZCZ and are good candidates for pilots in OFDM systems. Known Golay-ZCZ sequences reported in the literature have a limitation in the length which is the form of a power of 2. In this paper, we propose two constructions of Golay-ZCZ sequence sets with new parameters which generalize the constructions of Gong *et al.* (*IEEE Trans. Commun.*, 61(9), 2013) and Chen *et al.* (*IEEE Trans. Commun.*, 66(11), 2018). Notably, one of the two constructions generates optimal binary Golay-ZCZ sequences, while the other generates asymptotically optimal polyphase Golay-ZCZ sequences as the number of sequences increases. We also show, through numerical simulations, the applicability of the proposed Golay-ZCZ sequences in inter-symbol interference channel estimation. Interestingly, in certain application scenarios, the proposed Golay-ZCZ sequences performs better as compared to the existing state-of-the-art sequences.

Key words — Aperiodic correlation, Binary sequence, Complete complementary codes, Golay complementary sequence, Golay-ZCZ sequences, Inter-symbol interference channel estimation, Zero correlation zone.

I. Introduction

Golay complementary sets (GCSs) and zero correlation zone (ZCZ) sequence sets are two kinds of sequence sets with different desirable correlation properties. GCS are sequence sets have zero aperiodic autocorrelation sums (AACS) at all non-zero time shifts [1], whereas ZCZ sequence sets have zero correlation zone within a certain time-shift [2]. Due to its favourable correlation properties GCSs or ZCZ sequence sets have been widely used to reduce peak average power ratio (PAPR) in orthogonal frequency division multiplexing systems [3], [4]. However, the sequences own periodic autocorrelation plays an important role in some applications like synchronization and detection of the signal.

Working in this direction, Gong *et al.* [5] investigated the periodic autocorrelation behaviour of a single Golay sequence in 2013. To be more specific, Gong *et al.* presented two constructions of Golay sequences of length 2^m, using generalized Boolean functions (GBF), each displaying a periodic zero autocorrelation zone (ZACZ) of 2^{m-2}, and 2^{m-3}, respectively, around the in-phase position [5]. In 2018, Chen *et al.* [6] studied the zero cross-correlation zone among the Golay sequences and proposed Golay-ZCZ sequence sets. Golay-ZCZ sequence sets are sequence sets having periodic ZACZ for each sequence, periodic zero cross-correlation zone (ZCCZ) for any two sequences and also the aperiodic autocorrelation sum is zero for all non-zero time shifts. Specifically, using GBFs, Chen *et al.* [6] gave a systematic construction of Golay-ZCZ sequence set which consists of 2^k sequences with length 2^m, and $\min\{\mathrm{ZACZ}, \mathrm{ZCCZ}\}$ is 2^{m-k-1}.

In [5], the authors discussed the application of Golay sequences with large ZACZ for inter-symbol interference (ISI) channel estimation. Using Golay sequences with large ZACZ as channel estimation sequences (CES), the authors analysed the performance of Golay-sequence-aided channel estimation in terms of the error variance and the classical Cramer-Rao lower

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bound (CRLB). It was shown in [5] that when the channel impulse response (CIR) is within the ZACZ width then the variance of the Golay sequences attains the CRLB. Recently in 2021, Yu [7] demonstrated that sequence sets having low coherence of the spreading matrix along with low PAPR are suitable as pilot sequences for uplink grant-free non-orthogonal multiple access (NOMA). The work of [7] depicts that Golay-ZCZ sequences can be suitably used as pilot sequences for uplink grant-free NOMA.

Inspired by the works of Gong *et al.* [5] and Chen *et al.* [6] and by the applicability of the Golay-ZCZ sequences as pilot sequences for uplink grant-free NOMA and channel estimation, we propose Golay-ZCZ sequence sets with new lengths. Note that, the lengths of the Golay complementary pairs (GCPs) with large ZCZs discussed in the works of Gong *et al.* [5] and Chen *et al.* [6] are all in the powers of two. To the best of the authors' knowledge, the problem of investigating the individual periodic autocorrelations of the GCPs as well as the periodic cross-correlations of the pairs when the length of the GCPs are non-power-of-two, remains largely open.

An overview of the previous works, which considers the periodic ZACZ of the individual sequences and ZCCZ of a GCP, is given in Table 1. We have proposed two constructions which generate Golay-ZCZ sequence sets consisting of sequences with more flexible lengths. To be more specific, assuming that Golay complementary pairs of length N exist, we have proposed a systematic construction of Golay-ZCZ sequences of length $4N$, having ZCZ width of $N+1$.

Table 1. Golay sequences with periodic ZACZ and ZCCZ

Ref.	Length	Set size	ZACZ width	ZCCZ width	Based on
[5]	2^m	2	2^{m-2} or 2^{m-3}	Not discussed	GBF
[6]	2^m	2^k	2^{m-k-1}	2^{m-k-1}	GBF
Theorem 1	$4N$	2	N	N	GCP of length N
Theorem 2	M^2N	M	$(M-1)N$	$(M-1)N$	(M,M,N)-CCC

To increase the ZCZ width for the parameters of Golay-ZCZ sequence sets reported in [6], and also to make the parameters more flexible, we proposed another construction of Golay-ZCZ sequences consisting of sequences of length M^2N having ZCZ length $(M-1)N$ from an (M,M,N) complete complementary code (CCC). Interestingly, the resultant Golay-ZCZ sequences derived from the CCC are asymptotically optimal with respect to Tang-Fan-Matsufuji bound [8] as the number of sequences increases, when polyphase sequences are considered. To increase the availability of the CCCs to design Golay-ZCZ sequences of more flexible lengths we have also proposed a new iterative construction of CCCs based on Kronecker product. A brief overview of all the parameters of CCCs proposed till date can be found in Table 2 (see [9]–[24]). We have also provided simulation results discussing the application of the proposed sequences in ISI channel estimation. Interestingly, the proposed Golay-ZCZ sequences performs efficiently in certain application scenarios.

Table 2. Parameters of CCCs

Ref.	Parameters	Constraints	Construction based on
[9]	(M,M,M^N)	$N\geq 2$	Unitary matrices
[10]	$(2^{N-r},2^{N-r},2^N)$	$r=1,2,\ldots,N-1$	Golay-paired Hadamard matrices
[11]	(M,M,MN)	$N\leq M$	Unitary matrices
[12]	$(M,M,MN/P)$	$N,P\leq M$	Unitary matrices
[13]	$(M,M,2^mM)$	$m\geq 1$	Unitary matrices
[14]	$(2^{k+1},2^{k+1},2^m)$	$m,k\geq 1,\ k=m-1$	GBF
[15]	$(2N,2N,l)$	$l>1,\ N\geq 1$	Unitary matrices
[16]	$(2^m,2^m,2^{mN})$	$m\geq 1$	Equivalent Hadamard matrices
[17]	$(2^m,2^m,2^k)$	$m,k\geq 1$ with $k\geq m$	GBF
[18]	$(2^m,2^m,2^k)$	$m,k\geq 1$	GBF
[19]–[21]	(M,M,M^N)	$N\geq 1$	Paraunitary (PU) matrices
[21], [22]	(M,M,P^N)	$N\geq 1,\ P\|M$	PU matrices
[23]	(M_1M_2,M_1M_2,N_1N_2)	(M_1,M_1,N_1)-CCC and (M_2,M_2,N_2)-CCC exists	Kronecker product
[24]	$(M,M,\mathbb{L})$	$M=\sum_{p=0}^{P-1}M_p,\ \mathbb{L}=\{LM_pL_p\}_{p=0}^{P-1}$	PU matrices
Proposed	(M,M,MN_1N_2)	(M,M,N_1)-CCC and (M,M,N_2)-CCC exists	Kronecker product

The rest of the paper is organized as follows. In Section II, some useful notations and preliminaries are recalled. In Section III, a systematic construction of Golay-ZCZ sequence pairs with flexible lengths is proposed. In Section IV, a systematic construction of Golay-ZCZ sequence sets is proposed based on existing CCCs. Also in this section, we have proposed an iterative construction of CCCs to increase the flexibility of the parameters of the proposed Golay-ZCZ sequences. In Section V, we discuss the applicability of the proposed sequences in ISI channel estimation through numerical simulations. The optimality of the proposed Golay-ZCZ sequence sets is discussed in Section VI. In Section VII, we have discussed the novelty of the proposed constructions as compared to previous works. Finally, we conclude the paper in Section VIII.

II. Preliminaries

Before we begin, let us define the following notations:

- 0_L denotes the all-zero vector of length-L.
- $\overleftarrow{\boldsymbol{a}}$ denotes the reverse of the sequence $\boldsymbol{a}$.
- x^* denotes the complex conjugate of x.
- $\boldsymbol{a}||\boldsymbol{b}$ denotes the concatenation of the sequences $\boldsymbol{a}$ and $\boldsymbol{b}$.
- '$\boldsymbol{a}\cdot\boldsymbol{b}$' denotes the 'inner product' of two sequences $\boldsymbol{a}$ and $\boldsymbol{b}$.
- $\langle x\rangle_M$ denotes x mod M.
- $\boldsymbol{x}\otimes\boldsymbol{y}=[x_0\boldsymbol{y},x_1\boldsymbol{y},\dots,x_{N_1-1}\boldsymbol{y}]$, denotes the Kronecker product of the sequences $\boldsymbol{x}$ and $\boldsymbol{y}$.

Definition 1 [5] Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two length N sequences. The periodic cross-correlation function (PCCF) of $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as

$$R_{\boldsymbol{a},\boldsymbol{b}}(\tau):=\begin{cases}\sum_{k=0}^{N-1}a_kb^*_{\langle k+\tau\rangle_N}, & 0\le\tau\le N-1\\ \sum_{k=0}^{N-1}a_{\langle k-\tau\rangle_N}b^*_k, & -(N-1)\le\tau\le-1\end{cases}$$

When $\boldsymbol{a}=\boldsymbol{b}$, $R_{\boldsymbol{a},\boldsymbol{b}}(\tau)$ is called periodic auto-correlation function (PACF) of $\boldsymbol{a}$ and is denoted as $R_{\boldsymbol{a}}(\tau)$.

Definition 2 [3] Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two length N sequences. The aperiodic cross-correlation function (ACCF) of $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as

$$C_{\boldsymbol{a},\boldsymbol{b}}(\tau):=\begin{cases}\sum_{k=0}^{N-1-\tau}a_kb^*_{k+\tau}, & 0\le\tau\le N-1\\ \sum_{k=0}^{N-1+\tau}a_{k-\tau}b^*_k, & -(N-1)\le\tau\le-1\end{cases}$$

When $\boldsymbol{a}=\boldsymbol{b}$, $C_{\boldsymbol{a},\boldsymbol{b}}(\tau)$ is called aperiodic auto-correlation function (AACF) of $\boldsymbol{a}$ and is denoted as $C_{\boldsymbol{a}}(\tau)$.

Definition 3 [3] Let $(\boldsymbol{a},\boldsymbol{b})$ be a sequence pair of length N. Then $(\boldsymbol{a},\boldsymbol{b})$ is called a Golay complementary pair (GCP) if

$$C_{\boldsymbol{a}}(\tau)+C_{\boldsymbol{b}}(\tau)=0,\quad\text{for all }\tau\neq0$$

Definition 4 [3] Let $(\boldsymbol{a},\boldsymbol{b})$ be a GCP of length N. A sequence pair $(\boldsymbol{c},\boldsymbol{d})$ is called one of the complementary mates of $(\boldsymbol{a},\boldsymbol{b})$, if

$$C_{\boldsymbol{a},\boldsymbol{c}}(\tau)+C_{\boldsymbol{b},\boldsymbol{d}}(\tau)=0,\ \text{for all }\tau$$

In this paper, we have considered $(\boldsymbol{c},\boldsymbol{d})=(\overleftarrow{\boldsymbol{b}}^*,-\overleftarrow{\boldsymbol{a}}^*)$.

Definition 5 [8] A set $\mathcal{C}=\{\boldsymbol{c}_0,\boldsymbol{c}_1,\dots,\boldsymbol{c}_{M-1}\}$ consisting of M sequences of length N is said to be an (M,N,Z)-ZCZ sequence set if it satisfies

$$R_{\boldsymbol{c}_i}(\tau)=0,\ \text{for }1\le|\tau|\le Z,\ 0\le i\le M-1$$
$$R_{\boldsymbol{c}_i,\boldsymbol{c}_j}(\tau)=0,\ \text{for }|\tau|\le Z,\ 0\le i\neq j\le M-1$$

Definition 6 [5], [6] An (M,N,Z)-ZCZ sequence set becomes an (M,N,Z)-Golay-ZCZ sequence set if it satisfies

$$\text{C1}:\sum_{i=0}^{M-1}C_{\boldsymbol{c}_i}(\tau)=0,\quad\text{for all }\tau\neq0$$
$$\text{C2}:R_{\boldsymbol{c}_i}(\tau)=0,\ \text{for }1\le|\tau|\le Z,\ 0\le i\le M-1$$
$$\text{C3}:R_{\boldsymbol{c}_i,\boldsymbol{c}_j}(\tau)=0,\ \text{for }|\tau|\le Z,\ 0\le i\neq j\le M-1$$

Definition 7 [9] Let $\mathcal{C}$ be a $P\times N$ matrix, consisting of P sequences of length N, as follows:

$$\mathcal{C}=\begin{bmatrix}\boldsymbol{c}_0\\ \boldsymbol{c}_1\\ \vdots\\ \boldsymbol{c}_{P-1}\end{bmatrix}_{P\times N}=\begin{bmatrix}c_{0,0} & c_{0,1} & \cdots & c_{0,N-1}\\ c_{1,0} & c_{1,1} & \cdots & c_{1,N-1}\\ \vdots & \vdots & \ddots & \vdots\\ c_{P-1,0} & c_{P-1,1} & \cdots & c_{P-1,N-1}\end{bmatrix}_{P\times N}$$

Then $\mathcal{C}$ is called a complementary sequence set (CS) of size P if

$$C_{\boldsymbol{c}_0}(\tau)+C_{\boldsymbol{c}_1}(\tau)+\cdots+C_{\boldsymbol{c}_{P-1}}(\tau)=\begin{cases}PN, & \text{if }\tau=0\\ 0, & \text{if }0<\tau<N\end{cases}$$

Definition 8 [9] Consider $\mathfrak{C}=\{\mathcal{C}^0,\mathcal{C}^1,\dots,\mathcal{C}^{M-1}\}$, which consists of M CSs $\mathcal{C}^k$, $0\le k<M$, each having M sequences of length N, i.e.,

$$\mathcal{C}^k=\begin{bmatrix}\boldsymbol{c}_0^k\\ \boldsymbol{c}_1^k\\ \vdots\\ \boldsymbol{c}_{M-1}^k\end{bmatrix}_{M\times N},\ 0\le k\le M-1\tag{1}$$

where $\boldsymbol{c}_m^k$ is the m-th sequence of length N and is expressed as $\boldsymbol{c}_m^k = (c_{m,0}^k, c_{m,1}^k, \ldots, c_{m,N-1}^k)$, $0 \le m \le M-1$. The set $\mathfrak{C}$ is called an (M, M, N) complementary code (CCC) if for any $\mathcal{C}^{k_1}, \mathcal{C}^{k_2} \in \mathfrak{C}$, $0 \le k_1, k_2 \le M-1$, $0 \le \tau \le N-1$, $k_1 = k_2$ or $0 < \tau \le N-1$, $k_1 \ne k_2$,

$$|C_{\mathcal{C}^{k_1},\mathcal{C}^{k_2}}(\tau)| = \left|\sum_{m=0}^{M-1} C_{\boldsymbol{c}_m^{k_1},\boldsymbol{c}_m^{k_2}}(\tau)\right| = 0$$

where M denotes the set size and the number of sequences in each sequence set, and N the length of constituent sequences of $\mathfrak{C}$.

Let $\mathcal{C}$ be an $M \times N$ matrix given by

$$\mathcal{C} = \begin{bmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,N-1} \\ c_{1,0} & c_{1,1} & \cdots & c_{1,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{M-1,0} & c_{M-1,1} & \cdots & c_{M-1,N-1} \end{bmatrix}_{M\times N}$$

Then let us define a transformation $L_s^r(\mathcal{C})$ as follows:

$$L_s^r(\mathcal{C}) = \begin{bmatrix} c_{r,s} & c_{r,s+1} & \cdots & c_{r,N-1} & c_{r+1,0} & c_{r+1,1} & \cdots & c_{r+1,s-1} \\ c_{r+1,s} & c_{r+1,s+1} & \cdots & c_{r+1,N-1} & c_{r+2,0} & c_{r+2,1} & \cdots & c_{r+2,s-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{M-2,s} & c_{M-2,s+1} & \cdots & c_{M-2,N-1} & c_{M-1,0} & c_{M-1,1} & \cdots & c_{M-1,s-1} \\ c_{M-1,s} & c_{M-1,s+1} & \cdots & c_{M-1,N-1} & c_{0,0} & c_{0,1} & \cdots & c_{0,s-1} \\ c_{0,s} & c_{0,s+1} & \cdots & c_{0,N-1} & c_{1,0} & c_{1,1} & \cdots & c_{1,s-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{r-1,s} & c_{r-1,s+1} & \cdots & c_{r-1,N-1} & c_{r,0} & c_{r,1} & \cdots & c_{r,s-1} \end{bmatrix}_{M\times N}$$
$$= \left[\; T^r(\boldsymbol{c}_{:,s}) \;\; T^r(\boldsymbol{c}_{:,s+1}) \;\; \ldots \;\; T^r(\boldsymbol{c}_{:,N-1}) \;\; T^{r+1}(\boldsymbol{c}_{:,0}) \;\; T^{r+1}(\boldsymbol{c}_{:,1}) \;\; \ldots \;\; T^{r+1}(\boldsymbol{c}_{:,s-1}) \;\right]$$

where $\boldsymbol{c}_{:,s}$ denotes the s-th column of $\mathcal{C}$, and $T^r(\boldsymbol{c}_{:,s})$ denotes the r-step up-shift operator.

Definition 9 [22] Let $\boldsymbol{a} = (a_0, a_1, \ldots, a_{N-1})$ be a sequence of length N. Then the characteristic polynomial of $\boldsymbol{a}$ is defined as

$$\boldsymbol{a}(x) = a_0 + a_1 x + \cdots + a_{N-1}x^{N-1}$$

and the corresponding complex conjugate is given by

$$\boldsymbol{a}^*(x) = a_0^* + a_1^* x + \cdots + a_{N-1}^* x^{N-1}$$

Definition 10 [22] Let $\boldsymbol{a}(x)$ and $\boldsymbol{b}(x)$ be the characteristic polynomial of the length N sequence $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. Then the ACCF is defined as

$$C_{\boldsymbol{a},\boldsymbol{b}}(x) = \sum_{\tau=0}^{N-1} C_{\boldsymbol{a},\boldsymbol{b}}(\tau)x^\tau = \boldsymbol{a}(x^{-1})\boldsymbol{b}^*(x)$$

Definition 11 [22] A pair of sequences $(\boldsymbol{a}, \boldsymbol{b})$ is said to be a GCP of length N if

$$C_{\boldsymbol{a}}(x) + C_{\boldsymbol{b}}(x) = 2N$$

Definition 12 [22] Consider $\mathfrak{C} = \{\mathcal{C}^0, \mathcal{C}^1, \ldots, \mathcal{C}^{M-1}\}$, which consists of M CSs $\mathcal{C}^k$, $0 \le k < M$, each having M sequences of length N, as described in equation (1). Then $\mathfrak{C}$ is a CCC if

$$C_{\mathcal{C}^i,\mathcal{C}^k}(x) = \sum_{j=0}^{M-1} C_{\boldsymbol{c}_j^i,\boldsymbol{c}_j^k}(x) = MN\delta(i-k)$$

where δ is Kronecker delta function.

III. New Construction of Golay Complementary Pair with ZCZ

In [5] and [6], the authors considered complementary sequences whose length is only of the form 2^m. In this section, we considered GCPs of more flexible lengths and proposed Golay-ZCZ sequences of new lengths. Before introducing the construction, we need the following lemma.

Lemma 1 Let $(\boldsymbol{a}, \boldsymbol{b})$ be a GCP of length N and $(\boldsymbol{c}, \boldsymbol{d})$ be its complementary mate. Then

$$C^*_{\boldsymbol{a},\boldsymbol{b}}(\tau) + C^*_{\boldsymbol{c},\boldsymbol{d}}(\tau) = 0$$

Proof From the properties of aperiodic cross-correlation, for any sequence $\boldsymbol{x}$ and $\boldsymbol{y}$, we have $C_{\boldsymbol{x},\boldsymbol{y}}(\tau) = C_{\overleftarrow{\boldsymbol{y}}^*,\overleftarrow{\boldsymbol{x}}^*}(\tau)$ and $C_{-\boldsymbol{x},\boldsymbol{y}}(\tau) = -C_{\boldsymbol{x},\boldsymbol{y}}(\tau)$. Since $(\boldsymbol{c}, \boldsymbol{d}) = (\overleftarrow{\boldsymbol{b}}^*, -\overleftarrow{\boldsymbol{a}}^*)$, hence $C_{\boldsymbol{c},\boldsymbol{d}}(\tau) = C_{\overleftarrow{\boldsymbol{d}}^*,\overleftarrow{\boldsymbol{c}}^*}(\tau) = C_{-\boldsymbol{a},\boldsymbol{b}}(\tau) = -C_{\boldsymbol{a},\boldsymbol{b}}(\tau)$. Hence the proof follows.

Motivated by the works of [25], using Lemma 1, we have the following theorem.

Theorem 1 Let $(\boldsymbol{a}, \boldsymbol{b})$ be a GCP of size N and $(\boldsymbol{c}, \boldsymbol{d})$ be a Golay complementary mate of $(\boldsymbol{a}, \boldsymbol{b})$. Define

$$\begin{aligned} \boldsymbol{p} &= (\; x_1\boldsymbol{a} \,||\, x_2\boldsymbol{b} \,||\, x_3\boldsymbol{a} \,||\, x_4\boldsymbol{b} \;) \\ \boldsymbol{q} &= (\; x_1\boldsymbol{c} \,||\, x_2\boldsymbol{d} \,||\, x_3\boldsymbol{c} \,||\, x_4\boldsymbol{d} \;) \end{aligned}$$

where $x_1, x_2, x_3, x_4 \in \{+1, -1\}$. Then $(\boldsymbol{p}, \boldsymbol{q})$ is a Golay-ZCZ sequence pair of length $4N$ with $Z_{\min} = N$, if x_1, x_2, x_3, x_4 satisfy the following condition:

$$x_1x_2 + x_3x_4 = 0$$

Proof Let us consider $0 \le \tau \le N-1$. Then we have

$$\begin{aligned}C_{\boldsymbol{p}}(\tau) =& 2\big(C_{\boldsymbol{a}}(\tau)+C_{\boldsymbol{b}}(\tau)\big)\\&+(x_1x_2+x_3x_4)C^*_{\boldsymbol{b},\boldsymbol{a}}(N-\tau)+x_2x_3C^*_{\boldsymbol{a},\boldsymbol{b}}(N-\tau)\\C_{\boldsymbol{q}}(\tau) =& 2\big(C_{\boldsymbol{c}}(\tau)+C_{\boldsymbol{d}}(\tau)\big)\\&+(x_1x_2+x_3x_4)C^*_{\boldsymbol{d},\boldsymbol{c}}(N-\tau)+x_2x_3C^*_{\boldsymbol{c},\boldsymbol{d}}(N-\tau)\end{aligned}$$

Hence, for $0\le\tau\le N-1$, we have

$$C_{\boldsymbol{p}}(\tau)+C_{\boldsymbol{q}}(\tau)=4\big(C_{\boldsymbol{a}}(\tau)+C_{\boldsymbol{b}}(\tau)\big)$$

Consider $N\le\tau\le 2N-1$. Then one has

$$\begin{aligned}C_{\boldsymbol{p}}(\tau) =& (x_1x_2+x_3x_4)C_{\boldsymbol{a},\boldsymbol{b}}(\tau-N)+x_2x_3C_{\boldsymbol{b},\boldsymbol{a}}(\tau-N)\\&+x_1x_3C^*_{\boldsymbol{a}}(2N-\tau)+x_2x_4C^*_{\boldsymbol{b}}(2N-\tau)\\C_{\boldsymbol{q}}(\tau) =& (x_1x_2+x_3x_4)C_{\boldsymbol{c},\boldsymbol{d}}(\tau-N)+x_2x_3C_{\boldsymbol{d},\boldsymbol{c}}(\tau-N)\\&+x_1x_3C^*_{\boldsymbol{c}}(2N-\tau)+x_2x_4C^*_{\boldsymbol{d}}(2N-\tau)\end{aligned}$$

Hence, for $N\le\tau\le 2N-1$, we have

$$C_{\boldsymbol{p}}(\tau)+C_{\boldsymbol{q}}(\tau)=0$$

Consider $2N\le\tau\le 3N-1$. Then we have

$$\begin{aligned}C_{\boldsymbol{p}}(\tau) =& x_1x_3C_{\boldsymbol{a}}(\tau-2N)+x_2x_4C_{\boldsymbol{b}}(\tau-2N)\\&+x_1x_4C^*_{\boldsymbol{b},\boldsymbol{a}}(3N-\tau)\\C_{\boldsymbol{q}}(\tau) =& x_1x_3C_{\boldsymbol{c}}(\tau-2N)+x_2x_4C_{\boldsymbol{d}}(\tau-2N)\\&+x_1x_4C^*_{\boldsymbol{d},\boldsymbol{c}}(3N-\tau)\end{aligned}$$

Hence, for $2N\le\tau\le 3N-1$, we have

$$C_{\boldsymbol{p}}(\tau)+C_{\boldsymbol{q}}(\tau)=0$$

Consider $3N\le\tau\le 4N-1$. Then we have

$$\begin{aligned}C_{\boldsymbol{p}}(\tau)=x_1x_4C_{\boldsymbol{a},\boldsymbol{b}}(\tau-3N)\\C_{\boldsymbol{q}}(\tau)=x_1x_4C_{\boldsymbol{c},\boldsymbol{d}}(\tau-3N)\end{aligned}$$

Hence, for $3N\le\tau\le 4N-1$, we have

$$C_{\boldsymbol{p}}(\tau)+C_{\boldsymbol{q}}(\tau)=0$$

Hence, $(\boldsymbol{p},\boldsymbol{q})$ is a complementary pair of length $4N$.

Next, we have to prove the other two conditions of Definition 7. Consider $1\le\tau\le N$, then we have

$$\begin{aligned}R_{\boldsymbol{p}}(\tau)=&\sum_{k=0}^{4N-1}p_kp^*_{k+\tau}\\=&\sum_{k=0}^{N-1-\tau}a_ka^*_{k+\tau}+\sum_{k=N-\tau}^{N-1}x_1a_kx_2^*b^*_{k-(N-\tau)}\\&+\sum_{k=0}^{N-1-\tau}b_kb^*_{k+\tau}+\sum_{k=N-\tau}^{N-1}x_2b_kx_3^*a^*_{k-(N-\tau)}\\&+\sum_{k=0}^{N-1-\tau}a_ka^*_{k+\tau}+\sum_{k=N-\tau}^{N-1}x_3a_kx_4^*b^*_{k-(N-\tau)}\\&+\sum_{k=0}^{N-1-\tau}b_kb^*_{k+\tau}+\sum_{k=N-\tau}^{N-1}x_4b_kx_1^*a^*_{k-(N-\tau)}\\=&2\big(C_{\boldsymbol{a}}(\tau)+C_{\boldsymbol{b}}(\tau)\big)\end{aligned}$$

Similarly for $1\le\tau\le N$, we have

$$\begin{aligned}R_{\boldsymbol{q}}(\tau)=2\big(C_{\boldsymbol{c}}(\tau)+C_{\boldsymbol{d}}(\tau)\big)\\R_{\boldsymbol{p},\boldsymbol{q}}(\tau)=2\big(C_{\boldsymbol{a},\boldsymbol{c}}(\tau)+C_{\boldsymbol{b},\boldsymbol{d}}(\tau)\big)\end{aligned}$$

Since $(\boldsymbol{a},\boldsymbol{b})$ is a GCP and $(\boldsymbol{c},\boldsymbol{d})$ is one of the complementary mates of $(\boldsymbol{a},\boldsymbol{b})$, $R_{\boldsymbol{p}}(\tau)=R_{\boldsymbol{q}}(\tau)=0$, for all $1\le\tau\le N$, and $R_{\boldsymbol{p},\boldsymbol{q}}(\tau)=0$, for all $0\le\tau\le N$.

Therefore, $(\boldsymbol{p},\boldsymbol{q})$ is a $(2,4N,N)$-Golay-ZCZ sequence pair, which consists of sequences of length $4N$ with $Z_{\min}=N$.

Example 1 Let $(\boldsymbol{a},\boldsymbol{b})$ be a binary GCP of length 10, given by $\boldsymbol{a}=(1,1,-1,1,1,1,1,1,-1,-1)$, $\boldsymbol{b}=(1,1,-1,1,-1,1,-1,-1,1,1)$. Then $(\boldsymbol{c},\boldsymbol{d})=(\overleftarrow{\boldsymbol{b}}^*,-\overleftarrow{\boldsymbol{a}}^*)$ is a Golay mate of $(\boldsymbol{a},\boldsymbol{b})$. Define

$$\begin{aligned}\boldsymbol{p}&=\boldsymbol{a}\parallel\boldsymbol{b}\parallel\boldsymbol{a}\parallel-\boldsymbol{b}\\\boldsymbol{q}&=\boldsymbol{c}\parallel\boldsymbol{d}\parallel\boldsymbol{c}\parallel-\boldsymbol{d}\end{aligned}$$

Then $(\boldsymbol{p},\boldsymbol{q})$ is $(2,40,10)$-Golay-ZCZ sequence pair, because

$$\begin{aligned}\big(R_{\boldsymbol{p}}(\tau)\big)_{\tau=0}^{39}=&(40,0_{10},-4,-8,4,8,-4,0,4,0,12,0,\\&12,0,4,0,-4,8,4,-8,-4,0_{10})\\\big(R_{\boldsymbol{q}}(\tau)\big)_{\tau=0}^{39}=&(40,0_{10},4,8,-4,-8,4,0,-4,0,-12,0,\\&-12,0,-4,0,4,-8,-4,8,4,0_{10})\\\big(R_{\boldsymbol{p},\boldsymbol{q}}(\tau)\big)_{\tau=0}^{39}=&(0_{11},-4,-8,4,16,4,0,4,-8,-4,0,4,\\&-8,12,0,12,0,-4,8,4,0_{10})\end{aligned}$$

Example 2 Let $\boldsymbol{a}=(1,i,-i,-1,i),\boldsymbol{b}=(1,1,1,i,-i)$, then $(\boldsymbol{a},\boldsymbol{b})$ be a quadriphase GCP of length 5. Then $(\boldsymbol{c},\boldsymbol{d})=(\overleftarrow{\boldsymbol{b}}^*,-\overleftarrow{\boldsymbol{a}}^*)$ is a Golay mate of $(\boldsymbol{a},\boldsymbol{b})$. Define

$$\begin{aligned}\boldsymbol{p}&=\boldsymbol{a}\parallel\boldsymbol{b}\parallel\boldsymbol{a}\parallel-\boldsymbol{b}\\\boldsymbol{q}&=\boldsymbol{c}\parallel\boldsymbol{d}\parallel\boldsymbol{c}\parallel-\boldsymbol{d}\end{aligned}$$

Then $(\boldsymbol{p},\boldsymbol{q})$ is a $(2,20,5)$-Golay-ZCZ sequence pair. The periodic correlation magnitudes of $(2,20,5)$-Golay-ZCZ sequence pair $(\boldsymbol{p},\boldsymbol{q})$ are shown in Fig.1.

Remark 1 Please note that binary Golay-ZCZ sequences of length 40 and quadriphase Golay-ZCZ sequences of length 20 has not been previously reported in the literature. By considering a GCP $(\boldsymbol{a},\boldsymbol{b})$ of length 2^m in Theorem 1, we can construct $(\boldsymbol{p},\boldsymbol{q})$ of length 2^{m+2} and the resultant Golay-ZCZ sequences will have the parameters equivalent to the Golay-ZCZ sequences reported in [6].

IV. New Construction of Complementary Sets with ZCZ

The resultant sequence sets of the constructions reported in [6] have the property that, as the number of sequences of the Golay-ZCZ sequence set increases, the ZCZ width decreases. Also, the resultant sequence sets

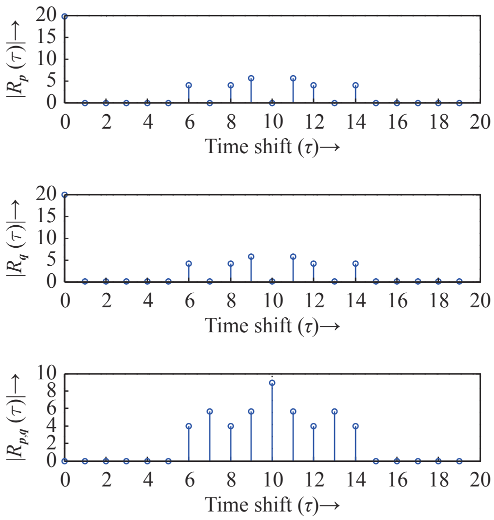

Fig. 1. A glimpse of the periodic correlations of the proposed Golay-ZCZ sequence pair given in Example 2.

in [6] are not optimal for polyphase cases. These problems are taken care of in the following construction. In this section, we propose a new construction of Golay-ZCZ sequence sets based on CCCs, which are asymptotically optimal. Before we proceed further, we reveal a nice property of CCC.

Property 1 Let $\mathfrak{C}=\left\{\mathcal{C}^0,\mathcal{C}^1,\ldots,\mathcal{C}^{M-1}\right\}$ be an (M,M,N)-CCC, consisting of M CSs $\mathcal{C}^k$, $0\le k<M$, each having M sequences of length N. Let $\mathcal{D}$ be a sequence set of order $M\times MN$ defined as follows:

$$\mathcal{D}=\begin{bmatrix} \boldsymbol{c}_0^0 & \boldsymbol{c}_1^0 & \cdots & \boldsymbol{c}_{M-1}^0 \\ \boldsymbol{c}_0^1 & \boldsymbol{c}_1^1 & \cdots & \boldsymbol{c}_{M-1}^1 \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{c}_0^{M-1} & \boldsymbol{c}_1^{M-1} & \cdots & \boldsymbol{c}_{M-1}^{M-1} \end{bmatrix}_{M\times MN}$$

where the i-th row is generated from $\mathcal{C}^i$, by appending the rows of $\mathcal{C}^i$ to the right of one another. for $0\le i\le M-1$, let

$$\mathcal{D}^i=\begin{bmatrix} \boldsymbol{c}_i^0 \\ \boldsymbol{c}_i^1 \\ \vdots \\ \boldsymbol{c}_i^{M-1} \end{bmatrix}$$

Then $\mathfrak{D}=\{\mathcal{D}^0,\mathcal{D}^1,\ldots,\mathcal{D}^{M-1}\}$ is also an (M,M,N)-CCC.

Proof Let $\boldsymbol{c}_i^k(x)$ denote characteristic polynomial of sequence $\boldsymbol{c}_i^k$. As $\mathcal{C}^k=\{\boldsymbol{c}_0^k,\boldsymbol{c}_1^k,\ldots,\boldsymbol{c}_{M-1}^k\}$ is a CS, then we have

$$\sum_{i=0}^{M-1}\boldsymbol{c}_i^k(x)\left(\boldsymbol{c}_i^k(x^{-1})\right)^*=MN,\ \text{for any } k$$

As $\mathfrak{C}=\left\{\mathcal{C}^0,\mathcal{C}^1,\ldots,\mathcal{C}^{M-1}\right\}$ is an (M,M,N)-CCC, we have

$$\sum_{i=0}^{M-1}\boldsymbol{c}_i^{k_1}(x)\left(\boldsymbol{c}_i^{k_2}(x^{-1})\right)^*=0,\ \text{if } k_1\ne k_2$$

By matrix representation, we have

$$\begin{pmatrix} \boldsymbol{c}_0^0(x) & \cdots & \boldsymbol{c}_{M-1}^0(x) \\ \vdots & \ddots & \vdots \\ \boldsymbol{c}_0^{M-1}(x) & \cdots & \boldsymbol{c}_{M-1}^{M-1}(x) \end{pmatrix}\begin{pmatrix} \boldsymbol{c}_0^0(x^{-1}) & \cdots & \boldsymbol{c}_0^{M-1}(x^{-1}) \\ \vdots & \ddots & \vdots \\ \boldsymbol{c}_{M-1}^0(x^{-1}) & \cdots & \boldsymbol{c}_{M-1}^{M-1}(x^{-1}) \end{pmatrix}^*=\begin{pmatrix} MN & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & MN \end{pmatrix}$$

By the commutative law of matrix, we have

$$\begin{pmatrix} \boldsymbol{c}_0^0(x^{-1}) & \cdots & \boldsymbol{c}_0^{M-1}(x^{-1}) \\ \vdots & \ddots & \vdots \\ \boldsymbol{c}_{M-1}^0(x^{-1}) & \cdots & \boldsymbol{c}_{M-1}^{M-1}(x^{-1}) \end{pmatrix}^*\begin{pmatrix} \boldsymbol{c}_0^0(x) & \cdots & \boldsymbol{c}_{M-1}^0(x) \\ \vdots & \ddots & \vdots \\ \boldsymbol{c}_0^{M-1}(x) & \cdots & \boldsymbol{c}_{M-1}^{M-1}(x) \end{pmatrix}=\begin{pmatrix} MN & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & MN \end{pmatrix}$$

Let $y=x^{-1}$, we have

$$\begin{pmatrix} \boldsymbol{c}_0^0(y) & \cdots & \boldsymbol{c}_0^{M-1}(y) \\ \vdots & \ddots & \vdots \\ \boldsymbol{c}_{M-1}^0(y) & \cdots & \boldsymbol{c}_{M-1}^{M-1}(y) \end{pmatrix}^*\begin{pmatrix} \boldsymbol{c}_0^0(y^{-1}) & \cdots & \boldsymbol{c}_{M-1}^0(y^{-1}) \\ \vdots & \ddots & \vdots \\ \boldsymbol{c}_0^{M-1}(y^{-1}) & \cdots & \boldsymbol{c}_{M-1}^{M-1}(y^{-1}) \end{pmatrix}=\begin{pmatrix} MN & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & MN \end{pmatrix}$$

Taking conjugation on the above equation, we have

$$\begin{pmatrix} \boldsymbol{c}_0^0(y) & \cdots & \boldsymbol{c}_0^{M-1}(y) \\ \vdots & \ddots & \vdots \\ \boldsymbol{c}_{M-1}^0(y) & \cdots & \boldsymbol{c}_{M-1}^{M-1}(y) \end{pmatrix}\begin{pmatrix} \boldsymbol{c}_0^0(y^{-1}) & \cdots & \boldsymbol{c}_{M-1}^0(y^{-1}) \\ \vdots & \ddots & \vdots \\ \boldsymbol{c}_0^{M-1}(y^{-1}) & \cdots & \boldsymbol{c}_{M-1}^{M-1}(y^{-1}) \end{pmatrix}^*=\begin{pmatrix} MN & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & MN \end{pmatrix}$$

Hence proved.

Motivated by the work of [26], we have revisited the construction of ZCZ sequence sets, and propose the following theorem.

Theorem 2 Let $\mathfrak{C}=\{\mathcal{C}^0,\mathcal{C}^1,\ldots,\mathcal{C}^{M-1}\}$ be an (M,M,N)-CCC, and $\mathcal{F}$ be an IDFT matrix of order $M\times M$, where $f_{i,j}$ denotes the element of i-th row and j-th column of $\mathcal{F}$. Define $\mathcal{B}_k$ for $0\le k<M$, as follows:

$$\mathcal{B}_k=\begin{bmatrix} f_{0,0}\boldsymbol{c}_k^0 & f_{0,1}\boldsymbol{c}_k^1 & \cdots & f_{0,M-1}\boldsymbol{c}_k^{M-1} \\ f_{1,0}\boldsymbol{c}_k^0 & f_{1,1}\boldsymbol{c}_k^1 & \cdots & f_{1,M-1}\boldsymbol{c}_k^{M-1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{M-1,0}\boldsymbol{c}_k^0 & f_{M-1,1}\boldsymbol{c}_k^1 & \cdots & f_{M-1,M-1}\boldsymbol{c}_k^{M-1} \end{bmatrix}_{M\times MN}$$

Let $\tilde{\mathcal{B}}_k$ denote a sequence of length M^2N which is generated from $\mathcal{B}_k$, by appending the rows of $\mathcal{B}_k$ to the right of one another. Then the sequence set $\mathcal{D}=\{\tilde{\mathcal{B}}_0,\tilde{\mathcal{B}}_1,\ldots,\tilde{\mathcal{B}}_{M-1}\}$ is an $(M,M^2N,(M-1)N)$-Golay-ZCZ sequence set.

Proof First we prove that $\mathcal{D}$ is a CS. For $0<\tau\le N-1$, we have

$$\begin{aligned}\sum_{i=0}^{M-1} C_{\tilde{\mathcal{B}}_i}(\tau) &= \sum_{j=0}^{M-1}\left[\left(\sum_{k=0}^{M-1} f_{k,j}^2\right)\left(\sum_{k=0}^{M-1} C_{\boldsymbol{c}_k^j}(\tau)\right)\right] \\ &+ \sum_{j=0}^{M-2}\left[\left(\sum_{k=0}^{M-1} f_{k,j}f_{k,j+1}^*\right)\left(\sum_{k=0}^{M-1} C_{\boldsymbol{c}_k^j,\boldsymbol{c}_k^{j+1}}(\tau)\right)\right] \\ &+ \left(\sum_{k=0}^{M-2} f_{k,M-1}f_{k+1,0}^*\right)\left(\sum_{k=0}^{M-1} C_{\boldsymbol{c}_k^{M-1},\boldsymbol{c}_k^0}(\tau)\right)\end{aligned}$$

Since $\mathfrak{C}=\{\mathcal{C}^0,\mathcal{C}^1,\ldots,\mathcal{C}^{M-1}\}$ is a CCC, we have $\sum_{i=0}^{M-1} C_{\tilde{\mathcal{B}}_i}(\tau)=0$. Similarly, we can show for other values of τ that $\sum_{i=0}^{M-1} C_{\tilde{\mathcal{B}}_i}(\tau)=0$.

Next, we prove the other two conditions of Definition 7. For $0\le\tau<M^2N$, let $\tau=r'NM+s'$. Let us define

$$\mathcal{H}_{\mathcal{B}_{k_1},\mathcal{B}_{k_2}}=\mathcal{B}_{k_1}\odot(L_{s'}^{r'}(\mathcal{B}_{k_2}))^*$$

where $\odot$ denotes elementwise product of the matrices $\mathcal{B}_{k_1}$ and $L_{s'}^{r'}(\mathcal{B}_{k_2})$. Note that $\mathcal{H}_{\mathcal{B}_{k_1},\mathcal{B}_{k_2}}$ is a matrix of size $M\times N$. When $k_1=k_2$, we write $\mathcal{H}_{\mathcal{B}_{k_1}}$ instead of $\mathcal{H}_{\mathcal{B}_{k_1},\mathcal{B}_{k_2}}$. $\sum\mathcal{H}_{\mathcal{B}_{k_1},\mathcal{B}_{k_2}}$ denotes sum of all elements of $\mathcal{H}_{\mathcal{B}_{k_1},\mathcal{B}_{k_2}}$. To check the periodic autocorrelation of the constituent sequences of $\mathcal{D}$, for $0<\tau\le(M-1)N+1$ and $0\le k<M$, we have,

$$\begin{aligned}R_{\tilde{\mathcal{B}}_k}(\tau) &= \sum\mathcal{H}_{\mathcal{B}_k}=\mathcal{B}_k\odot(L_r^0(\mathcal{B}_k))^* \\ &= \sum_{i=0}^{M-1}(f_{:,i})\cdot(T^{\lfloor\frac{i+r}{M}\rfloor}(f_{:,\langle i+r\rangle_M})) \\ &\quad\cdot C_{\boldsymbol{c}_k^i,\boldsymbol{c}_k^{\langle i+r\rangle_M}}(\tau-rN) \\ &\quad+\sum_{i=0}^{M-1}(f_{:,i})\cdot(T^{\lfloor\frac{i+r+1}{M}\rfloor}(f_{:,\langle i+r+1\rangle_M})) \\ &\quad\cdot C_{\boldsymbol{c}_k^i,\boldsymbol{c}_k^{\langle i+r+1\rangle_M}}(\tau-(r+1)N)\end{aligned}$$

Since $\mathcal{F}$ is an IDFT matrix, using Property 1, we have

$$R_{\tilde{\mathcal{B}}_k}(\tau)=\begin{cases} M^2N, & \text{when } \tau=0 \\ 0, & \text{when } 1\le\tau\le(M-1)N \\ \text{non-zero}, & \text{when } (M-1)N<\tau\end{cases}$$

Similarly, to check the cross-correlation we have for $0\le k_1\ne k_2<M$,

$$\begin{aligned}R_{\tilde{\mathcal{B}}_{k_1},\tilde{\mathcal{B}}_{k_2}}(\tau) &= \sum\mathcal{H}_{\mathcal{B}_{k_1},\mathcal{B}_{k_2}} \\ &= \sum_{i=0}^{M-1}(f_{:,i})\cdot(T^{\lfloor\frac{i+r}{M}\rfloor}(f_{:,\langle i+r\rangle_M})) \\ &\quad\cdot C_{\boldsymbol{c}_{k_1}^i,\boldsymbol{c}_{k_2}^{\langle i+r\rangle_M}}(\tau-rN) \\ &\quad+\sum_{i=0}^{M-1}(f_{:,i})\cdot(T^{\lfloor\frac{i+r+1}{M}\rfloor}(f_{:,\langle i+r+1\rangle_M})) \\ &\quad\cdot C_{\boldsymbol{c}_{k_1}^i,\boldsymbol{c}_{k_2}^{\langle i+r+1\rangle_M}}(\tau-(r+1)N)\end{aligned}$$

Since $\mathfrak{C}$ is a CCC and $\mathcal{F}$ is an IDFT matrix, using Property 1, we have

$$R_{\tilde{\mathcal{B}}_{k_1},\tilde{\mathcal{B}}_{k_2}}(\tau)=\begin{cases} 0, & \text{when } \tau=0 \\ 0, & \text{when } 1\le\tau\le(M-1)N \\ \text{non-zero}, & \text{when } (M-1)N<\tau\end{cases}$$

Hence, $\mathcal{D}$ is an $(M,M^2N,(M-1)N)$-Golay-ZCZ sequence set of set size M, which consists of sequences of length M^2N having ZCZ width $(M-1)N$.

Example 3 Let $\mathfrak{C}=\{\mathcal{C}^0,\mathcal{C}^1,\mathcal{C}^2,\mathcal{C}^3\}$ be a $(4,4,4)$-CCC given by

$$\mathcal{C}^0=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1\end{bmatrix},\ \mathcal{C}^1=\begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1\end{bmatrix},$$

$$\mathcal{C}^2=\begin{bmatrix} -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1\end{bmatrix},\ \mathcal{C}^3=\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1\end{bmatrix},$$

and $\mathcal{F}$ be a 4×4 IDFT matrix. Construct $\mathcal{D}=\{\tilde{\mathcal{B}}_0,\tilde{\mathcal{B}}_1,\tilde{\mathcal{B}}_2,\tilde{\mathcal{B}}_3\}$ as per Construction 2. Then $\mathcal{D}$ is a $(4,64,12)$-Golay-ZCZ sequence set. A glimpse of the correlations of the sequence set $\mathcal{D}$ is shown in Fig.2.

Remark 2 In [6], the authors reported $(2^k,2^m,2^{m-k-1})$-Golay-ZCZ sequence sets. Considering $k=2$ and $m=6$, we get $(4,64,8)$-Golay-ZCZ sequence set. As we see in Example 3, the Golay-ZCZ sequence sets proposed in Theorem 2 have larger ZCZ widths, as compared to the results in [6].

Since Theorem 2 is based on CCCs, the availability of CCCs for various flexible lengths highly improves the outcome of the construction. In Table 2, we list down all the well-known construction of CCCs till date.

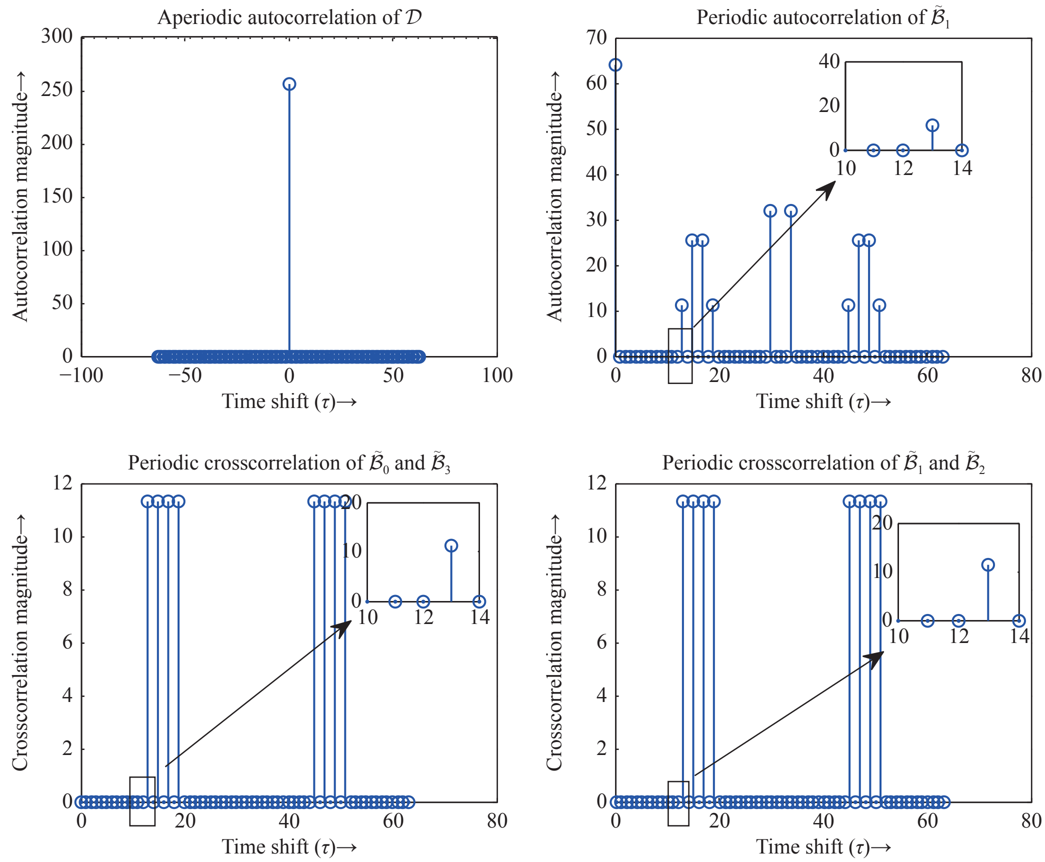

Fig. 2. A glimpse of the correlations of the sequence set $\mathcal{D}$, constructed in Example 3.

We also provide an iterative construction of CCC to improve the flexibility of the parameters. Before we proceed, we need the following lemma.

Lemma 2 Let $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ be two sequences of length N_1, $(\boldsymbol{y}_1, \boldsymbol{y}_2)$ be two sequences of length N_2. Then aperiodic correlation of $\boldsymbol{x}_1 \otimes \boldsymbol{y}_1$ between $\boldsymbol{x}_2 \otimes \boldsymbol{y}_2$ is given by [23]

$$\begin{aligned} C_{\boldsymbol{x}_1\otimes\boldsymbol{y}_1,\boldsymbol{x}_2\otimes\boldsymbol{y}_2}(\tau) = & C_{\boldsymbol{x}_1,\boldsymbol{x}_2}(k_1)C_{\boldsymbol{y}_1,\boldsymbol{y}_2}(k_2) \\ & + C_{\boldsymbol{x}_1,\boldsymbol{x}_2}(k_1+1)C_{\boldsymbol{y}_1,\boldsymbol{y}_2}(k_2-N) \end{aligned}$$

where $\tau = k_1 N_2 + k_2$.

Theorem 3 Let $\mathfrak{C} = \{\mathcal{C}^0, \mathcal{C}^1, \ldots, \mathcal{C}^{M-1}\}$ and $\mathfrak{D} = \{\mathcal{D}^0, \mathcal{D}^1, \ldots, \mathcal{D}^{M-1}\}$ be two CCCs with parameters (M, M, N_1) and (M, M, N_2), respectively. Then $\mathfrak{E} = \{\mathcal{E}^0, \mathcal{E}^1, \ldots, \mathcal{E}^{M-1}\}$, given by

$$\mathcal{E}^m = \begin{bmatrix} \boldsymbol{e}_0^m \\ \boldsymbol{e}_1^m \\ \vdots \\ \boldsymbol{e}_{M-1}^m \end{bmatrix}_{M\times MN_1N_2} = \begin{bmatrix} \boldsymbol{c}_0^m \otimes \boldsymbol{d}_0^0 & \boldsymbol{c}_1^m \otimes \boldsymbol{d}_1^0 & \cdots & \boldsymbol{c}_{M-1}^m \otimes \boldsymbol{d}_{M-1}^0 \\ \boldsymbol{c}_0^m \otimes \boldsymbol{d}_0^1 & \boldsymbol{c}_1^m \otimes \boldsymbol{d}_1^1 & \cdots & \boldsymbol{c}_{M-1}^m \otimes \boldsymbol{d}_{M-1}^1 \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{c}_0^m \otimes \boldsymbol{d}_0^{M-1} & \boldsymbol{c}_1^m \otimes \boldsymbol{d}_1^{M-1} & \cdots & \boldsymbol{c}_{M-1}^m \otimes \boldsymbol{d}_{M-1}^{M-1} \end{bmatrix}$$

is a CCC with parameters (M, M, MN_1N_2).

Proof First, we prove that for $0 \le m < M$, $\mathcal{E}^m$ is a CS of length MN_1N_2.

By Lemma 2, we have

$$\begin{aligned} \sum_{l=0}^{M-1} C_{\boldsymbol{c}_{n_1}^m\otimes\boldsymbol{d}_{n_1}^l,\boldsymbol{c}_{n_2}^m\otimes\boldsymbol{d}_{n_2}^l}(\tau_0) &= \sum_{l=0}^{M-1}\left(C_{\boldsymbol{c}_{n_1}^m,\boldsymbol{c}_{n_2}^m}(k_1)C_{\boldsymbol{d}_{n_1}^l,\boldsymbol{d}_{n_2}^l}(k_2) + C_{\boldsymbol{c}_{n_1}^m,\boldsymbol{c}_{n_2}^m}(k_1+1)C_{\boldsymbol{d}_{n_1}^l,\boldsymbol{d}_{n_2}^l}(k_2-N_2)\right) \\ &= C_{\boldsymbol{c}_{n_1}^m,\boldsymbol{c}_{n_2}^m}(k_1)\sum_{l=0}^{M-1} C_{\boldsymbol{d}_{n_1}^l,\boldsymbol{d}_{n_2}^l}(k_2) + C_{\boldsymbol{c}_{n_1}^m,\boldsymbol{c}_{n_2}^m}(k_1+1)\sum_{l=0}^{M-1} C_{\boldsymbol{d}_{n_1}^l,\boldsymbol{d}_{n_2}^l}(k_2-N_2) \\ &= \begin{cases} MN_2 C_{\boldsymbol{c}_{n_1}^m,\boldsymbol{c}_{n_2}^m}(k_1), & k_2 = 0, n_1 = n_2 \\ 0, & k_2 \neq 0, n_1 = n_2 \\ 0, & \text{for all } k_2, n_1 \neq n_2 \end{cases} \end{aligned}$$

where $\tau_0 = k_1 N_2 + k_2$. For $\mathcal{E}^m$, let us consider $\tau = rN_1N_2 + \tau_0$, then

$$\sum_{l=0}^{M-1} C_{\boldsymbol{e}_l^m}(\tau) = \sum_{l=0}^{M-1} \sum_{n_1=0}^{M-1-r} (C_{\boldsymbol{c}_{n_1}^m \otimes \boldsymbol{d}_{n_1}^l, \boldsymbol{c}_{n_2}^m \otimes \boldsymbol{d}_{n_2}^l}(\tau_0) + C_{\boldsymbol{c}_{n_1}^m \otimes \boldsymbol{d}_{n_1}^l, \boldsymbol{c}_{n_2+1}^m \otimes \boldsymbol{d}_{n_2+1}^l}(\tau_0 - N_1 N_2)) \tag{2}$$

where $n_2 = n_1 + r$.

Clearly, we have $\sum_{l=0}^{M-1} C_{e_l^m}(\tau) = 0$, if $r \geq 1$. Consider $r = 0$, then

$$\begin{aligned}
&\sum_{l=0}^{M-1} C_{\boldsymbol{e}_l^m}(\tau) \\
&= \sum_{l=0}^{M-1} C_{\boldsymbol{e}_l^m}(\tau_0) \\
&= \sum_{l=0}^{M-1} \sum_{n_1=n_2=0}^{M-1} C_{\boldsymbol{c}_{n_1}^m \otimes \boldsymbol{d}_{n_1}^l, \boldsymbol{c}_{n_2}^m \otimes \boldsymbol{d}_{n_2}^l}(\tau_0) \\
&= \begin{cases} \sum\limits_{n_1=n_2=0}^{M-1} M N_2 C_{\boldsymbol{c}_{n_1}^m, \boldsymbol{c}_{n_2}^m}(k_1), & k_2 = 0 \\ 0, & k_2 \neq 0 \end{cases} \\
&= \begin{cases} M^2 N_1 N_2, & k_1 = 0, k_2 = 0 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

Next, we prove that $\mathcal{E}^{m_1}$ and $\mathcal{E}^{m_2}$ are orthogonal if $m_1 \neq m_2$.

Similar to (2), we consider cross-correlation sum of $\mathcal{E}^{m_1}$ between $\mathcal{E}^{m_2}$, then we have

$$\begin{aligned}
&\sum_{l=0}^{M-1} C_{\boldsymbol{e}_l^{m_1}, \boldsymbol{e}_l^{m_2}}(\tau) \\
&= \sum_{l=0}^{M-1} \sum_{n_1=0}^{M-1-r} (C_{\boldsymbol{c}_{n_1}^{m_1} \otimes \boldsymbol{d}_{n_1}^l, \boldsymbol{c}_{n_2}^{m_2} \otimes \boldsymbol{d}_{n_2}^l}(\tau_0) \\
&\quad + C_{\boldsymbol{c}_{n_1}^{m_1} \otimes \boldsymbol{d}_{n_1}^l, \boldsymbol{c}_{n_2+1}^{m_2} \otimes \boldsymbol{d}_{n_2+1}^l}(\tau_0 - N_1 N_2))
\end{aligned}$$

where $n_2 = n_1 + r$.

Clearly, we have $\sum_{l=0}^{M-1} C_{e_l^m}(\tau) = 0,\ if\ r \geq 1$. Consider $r = 0$, then

$$\begin{aligned}
&\sum_{l=0}^{M-1} C_{\boldsymbol{e}_l^{m_1}, \boldsymbol{e}_l^{m_2}}(\tau) \\
&= \sum_{l=0}^{M-1} C_{\boldsymbol{e}_l^{m_1}, \boldsymbol{e}_l^{m_2}}(\tau_0) \\
&= \sum_{l=0}^{M-1} \sum_{n_1=n_2=0}^{M-1} C_{\boldsymbol{c}_{n_1}^{m_1} \otimes \boldsymbol{d}_{n_1}^l, \boldsymbol{c}_{n_2}^{m_2} \otimes \boldsymbol{d}_{n_2}^l}(\tau_0) \\
&= \begin{cases} \sum\limits_{n_1=n_2=0}^{M-1} M N_2 C_{\boldsymbol{c}_{n_1}^{m_1}, \boldsymbol{c}_{n_2}^{m_2}}(k_1), & k_2 = 0 \\ 0, & k_2 \neq 0 \end{cases} \\
&= 0
\end{aligned}$$

Hence $\mathfrak{E}=\{\mathcal{E}^0, \mathcal{E}^1, \ldots, \mathcal{E}^{M-1}\}$ is an (M, M, MN_1N_2)-CCC.

Remark 3 To increase the flexibility of the proposed construction in Theorem 2, we have also found binary CCCs with parameters $(4,4,3)$, $(4,4,5)$, $(4,4,7)$, $(4,4,11)$, and $(4,4,13)$ based on computer search, which can be used as seed CCCs of the proposed construction in Theorem 2. Considering $\mathfrak{C} = \{\mathcal{C}^0, \mathcal{C}^1, \mathcal{C}^2, \mathcal{C}^3\}$ as a $(4,4,N)$-CCC, the search results can be found in Table 3, where each element represents a power of (-1). The search results are important in itself, because in recent results [27]–[29], we observe that for a (K, M, N) mutually orthogonal complementary sequence set, through systematic construction, the maximum achievable K/M ratio is $1/2$, when N is not in the form of 2^m. However, for our case, although N is not in the form of power-of-two, since the sequence sets are CCC (i.e., $K = M$), the K/M ratio is 1. Moreover, for length up to 200 (i.e., $N \leq 200$), using the CCCs given in Table 3 as seed CCCs, using the results of [23] and Theorem 3, we can design binary $(4,4,N)$-CCCs for $N = 12$, 13, 20, 24, 28, 36, 40, 44, 48, 52, 56, 60, 72, 80, 84, 88, 96, 112, 120, 132, 140, 144, 156, 160, 168, 176, 192, 196, 200.

Table 3. Computer search results of (4, 4, *N*)-CCCs

N	$\mathcal{C}^0$	$\mathcal{C}^1$	$\mathcal{C}^2$	$\mathcal{C}^3$
3	000 001 001 010	010 001 110 111	011 000 101 100	011 010 000 011
5	00001 01100 01000 01011	00010 00101 01111 00110	00101 11101 00110 10000	01100 11110 01011 10111
7	0000001 0011010 0011010 0100011	0011101 0011010 1100101 1000000	0101100 0100011 1111110 1010011	0101100 0111111 0011101 0101100
11	01110110110 00111000101 00011010100 00000001101	00011010100 00000001101 10001001001 11000111010	10110000000 11010100111 10100011100 10010010001	01011100011 01101101110 10110000000 11010100111
13	0111011010100 0011101001101 0001100001001 0000000111010	0001100001001 0000000111010 1000100101011 1100010110010	0101110000000 0110111100111 1011001011100 1101010010001	0100110100011 0010101101110 0101110000000 0110111100111

V. Application of the Proposed Sequences in ISI Channel Estimation

Let us assume a channel model as follows:

$$y_k = \sum_{\tau=0}^{\infty} h_\tau x_{k-\tau} + w_k$$

where $\boldsymbol{h} = [h_0, h_1, \ldots, h_n]^{\mathrm{T}}$ are complex valued channel taps, $\boldsymbol{x} = [x_0, x_1, \ldots, x_n]^{\mathrm{T}}$ are transmitted symbols,

$\boldsymbol{y} = [y_0, y_1, \ldots, y_n]^{\mathrm{T}}$ are received symbols, and w_k is complex Gaussian variate with the variance σ^2. Consider the case where the ISI tap vector $\boldsymbol{h}$ is unknown to the receiver, and thus needs to be estimated. The actual number of taps is unknown, however, we consider a case where the receiver tries to estimate L taps. That is, the maximum multipath delay is less than L and the channel has the form $\boldsymbol{h} = [h_0, h_1, \ldots, h_{L-1}]^{\mathrm{T}}$. We will use least squares (LS) method for this estimation, which directly seeks to estimate the channel impulse response $\boldsymbol{h}$ in the time domain.

Describing the channel model in the form of matrix, we have

$$\boldsymbol{y} = \boldsymbol{h} * \boldsymbol{x} + \boldsymbol{w} = \boldsymbol{X}\boldsymbol{h} + \boldsymbol{w}$$

where $\boldsymbol{X}$ is a matrix of size $(N + L - 1) \times L$, given by

$$\boldsymbol{X} = \begin{pmatrix} x_0 & 0 & \cdots & 0 \\ x_1 & x_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{N-1} & x_{N-2} & \cdots & x_{N-L} \\ 0 & x_{N-1} & \cdots & x_{N+1-L} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{N-1} \end{pmatrix}_{N+L-1,L}$$

and $\boldsymbol{h}$ is a column vector of length L and $\boldsymbol{y}, \boldsymbol{w}$ are column vectors of length $N + L - 1$.

Let us consider $\boldsymbol{a}$ to be a training sequence. There are usually three ways to use training sequences.

1) Use it directly, i.e., $\boldsymbol{x} = \boldsymbol{a}$.

2) Use as cyclic prefix, i.e., $\boldsymbol{x} = [a_{N_0-L+1}, \ldots, a_{N_0-1}, \boldsymbol{a}^{\mathrm{T}}]^{\mathrm{T}}$. Note that $N = N_0 + L - 1$, and the length of $\boldsymbol{a}$ is equal to N_0.

3) Use as negative cyclic prefix, $\boldsymbol{x} = [-a_{N_0-L+1}, \ldots, -a_{N_0-1}, \boldsymbol{a}^{\mathrm{T}}]^{\mathrm{T}}$.

For the first method, i.e., when the sequence is used directly, the LS method indicates that the estimated value $\hat{\boldsymbol{h}}$ of $\boldsymbol{h}$ is given by

$$\hat{\boldsymbol{h}} = \left(\boldsymbol{X}^{\mathrm{H}}\boldsymbol{X}\right)^{-1} \boldsymbol{X}^{\mathrm{H}}\boldsymbol{y}$$

To obtain the optimal value, this requires that the inner product of any two columns of matrix $\boldsymbol{X}$ be as low as possible, i.e., $|C_{\boldsymbol{a}}(\tau)|, \tau = 1, 2, \ldots, L-1$ be as low as possible.

For the second method, i.e., when the sequence is used in the form of cyclic prefix, there is a cyclic matrix $\boldsymbol{A}$ as a submatrix of $\boldsymbol{X}$, and $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{pmatrix} a_0 & a_{N_0-1} & \cdots & a_{N_0+1-L} \\ a_1 & a_0 & \cdots & a_{N_0+2-L} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N_0-1} & a_{N_0-2} & \cdots & a_{N_0-L} \end{pmatrix}_{N_0,L} \quad (3)$$

Then LS method indicates that the estimated value $\hat{\boldsymbol{h}}$ is given by

$$\hat{\boldsymbol{h}} = \left(\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A}\right)^{-1} \boldsymbol{A}^{\mathrm{H}}\tilde{\boldsymbol{y}}$$

where $\tilde{\boldsymbol{y}} = [y_{L-1}, y_L, \ldots, y_{N_0+L-2}]^{\mathrm{T}}$. To get the optimal value, this requires that the inner product of any two columns of matrix $\boldsymbol{A}$ be as low as possible, i.e., $|R_{\boldsymbol{a}}(\tau)|, \tau = 1, 2, \ldots, L-1$ be as low as possible.

Theorem 4 Assume that the cyclic prefix length is greater than or equal to the maximum multipath delay. Then the mean square error (MSE) of LS method is give by

$$\mathrm{MSE} = \|(\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathrm{H}}\|_{\mathrm{F}}^2\sigma^2$$

where σ^2 is variance of Gaussian noise.

Proof Let $\boldsymbol{B} = (\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathrm{H}}$, then we have $\hat{\boldsymbol{h}} = \boldsymbol{B}\tilde{\boldsymbol{y}}$. From definition of MSE, we have

$$\begin{aligned} \mathrm{MSE} &= E(\|\hat{\boldsymbol{h}} - \boldsymbol{h}\|_2^2) \\ &= E(\|B(\boldsymbol{A}\boldsymbol{h} + \boldsymbol{w}) - \boldsymbol{h}\|_2^2) \\ &= E(\|\boldsymbol{B}\boldsymbol{w}\|_2^2) \\ &= E\left(\sum_{l=0}^{L-1}\left|\sum_{n=0}^{N_0-1}[B]_{l,n}w_n\right|^2\right) \\ &= E\left(\sum_{l=0}^{L-1}\sum_{n_1=0}^{N_0-1}\sum_{n_2=0}^{N_0-1}[B]_{l,n_1}w_{n_1}[B]_{l,n_2}^*w_{n_2}^*\right) \\ &= \sum_{l=0}^{L-1}\sum_{n_1=0}^{N_0-1}\sum_{n_2=0}^{N_0-1}[B]_{l,n_1}[B]_{l,n_2}^*E\left(w_{n_1}w_{n_2}^*\right) \end{aligned}$$

Since each w_n's are independent, we have

$$E\left(w_{n_1}w_{n_2}^*\right) = \begin{cases} E(|w_{n_1}|^2), & n_1 = n_2 \\ 0, & n_1 \neq n_2 \end{cases}$$

Hence, we have

$$\mathrm{MSE} = \sum_{l=0}^{L-1}\sum_{n=0}^{N_0-1}|[B]_{l,n}|^2E\left(|w_n|^2\right) = \|\boldsymbol{B}\|_F^2\sigma^2$$

Corollary 1 Let $\boldsymbol{A}$ be constructed as in (3). Then, the MSE of LS method is lower bounded by $L/N_0\sigma^2$.

Proof $\|(\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathrm{H}}\|_{\mathrm{F}}^2 \geq \|(N_0\boldsymbol{I}_L)^{-1}\boldsymbol{A}^{\mathrm{H}}\|_{\mathrm{F}}^2 = L/N_0$.

Remark 4 From Corollary 1, we can observe that the MSE of LS method meets the lower bound with equality if $\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A} = N_0\boldsymbol{I}_L$, which is equivalent to, sequence a, having ZCZ width of $L-1$. Since, all the Golay sequences proposed in Theorem 1 and Theorem 2 have a ZCZ width, the performance of MSE is optimal, with a condition that the maximum multipath delay is less than the ZCZ width.

1. SISO channel model simulation results

In this subsection, we will simulate our proposed Golay-ZCZ sequence aided channel estimation in MATLAB. We consider TDL-C channel model in 3GPP standard. The parameters of which are provided in Table 4. Doppler shift is not considered. The delay spread is 300 ns. The chip rate is 30.72 MHz.

Table 4. Experimental parameters of the TDL-C model

Tap #	Normalized delays	Power	Fading distribution
1	0	−4.4	Rayleigh
2	0.2099	−1.2	Rayleigh
3	0.2219	−3.5	Rayleigh
4	0.2329	−5.2	Rayleigh
5	0.2176	−2.5	Rayleigh
6	0.6366	0	Rayleigh
7	0.6448	−2.2	Rayleigh
8	0.6560	−3.9	Rayleigh
9	0.6584	−7.4	Rayleigh
10	0.7935	−7.1	Rayleigh
11	0.8213	−10.7	Rayleigh
12	0.9336	−11.1	Rayleigh
13	1.2285	−5.1	Rayleigh
14	1.3083	−6.8	Rayleigh
15	2.1704	−8.7	Rayleigh
16	2.7105	−13.2	Rayleigh
17	4.2589	−13.9	Rayleigh
18	4.6003	−13.9	Rayleigh
19	5.4902	−15.8	Rayleigh
20	5.6077	−17.1	Rayleigh
21	6.3065	−16	Rayleigh
22	6.6374	−15.7	Rayleigh
23	7.0427	−21.6	Rayleigh
24	8.6523	−22.8	Rayleigh

We assume a perfect synchronization during the simulation. The setup is as follows:

- The actual number of taps is 80, however, it is not known for the receiver. Hence, the receiver tries to estimate $L = 96$ taps.
- Sequence length of $\boldsymbol{a}$ is 384, and sequence length of $\boldsymbol{x}$ is 480.
- Four CES will be simulated. They are ZC sequence, ordinary Golay sequence, the proposed Golay sequence, and random sequence.
- The MSE will be computed based on 10^5 transmitted symbols.

The performances of MSE shown in Fig.3. It shows that the performances of ZC sequence and the proposed Golay-ZCZ sequence are optimal, both of which meet the bound $L/N_0\sigma^2$. The ordinary Golay sequence and random sequence have poor channel estimation performance because they haven't optimal correlation in desired region.

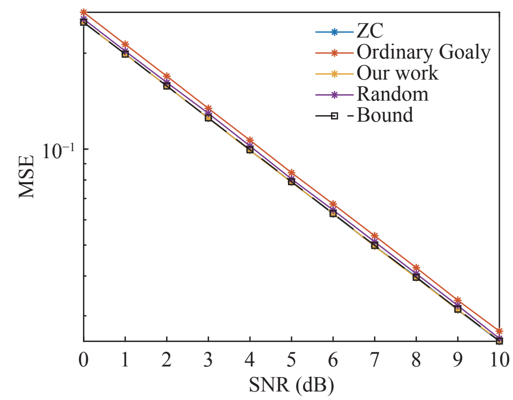

Fig. 3. MSE performances of 4 CES 384.

Next, we compare the channel estimation performance of Golay-ZCZ sequences in terms of flexible sequence length. We assume a perfect synchronization during the simulation. In this case also the receiver tries to estimate $L = 96$ taps. In this comparison, we have selected three Golay-ZCZ sequences of lengths 256, 384, and 512. The Golay sequence of length 256 and 512 can be generated by the methods proposed in [6] and [5]. The Golay-ZCZ sequence of length 384 is proposed this paper. Fig.4 shows that the performance of the Golay-ZCZ sequences of length 384 and 512 are optimal. However, in practical scenarios, a longer training sequence will give rise to a higher training overhead, therefore, selection of the training length is a trade-off between channel estimation performance and training overhead. This will improve the system performance. As the ZCZ width of the Golay-ZCZ sequence of length 256 is 64, its performance is not optimal.

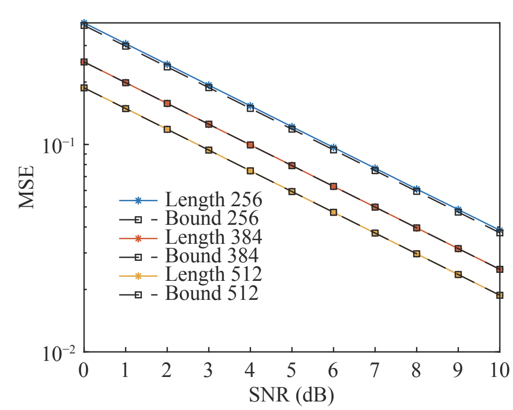

Fig. 4. MSE performances difference length Golay ZCZ sequences.

2. Simulation of ISI MIMO channel estimation

In this setup, let us consider the number of transmitting antennas be M and the number of receiving antennas be 1. Fig.5 shows that the transmission model of MIMO.

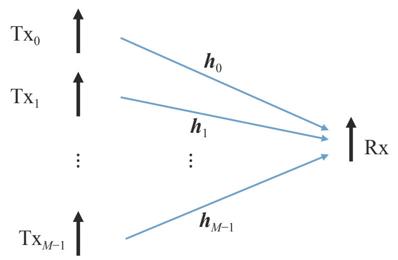

Fig. 5. Transmission model of MIMO.

Assume a channel model as follows:

$$\boldsymbol{y} = \sum_{i=0}^{M-1} \boldsymbol{h}_i * \boldsymbol{x}_i + \boldsymbol{w}$$

where $\boldsymbol{x}_i$'s are transmitted symbols, $\boldsymbol{h}_i$'s denote the channel and $\boldsymbol{w}$ is the additive white Gaussian noise. Similar to the case of SISO, we use cyclic prefix to transmit sequence $\boldsymbol{a}_i$. Then LS method indicates that the estimated value $\hat{\boldsymbol{h}}_i$ of $\boldsymbol{h}_i$ is given by

$$\hat{\boldsymbol{h}}_i = \left(\boldsymbol{A}_i^{\mathrm{H}} \boldsymbol{A}_i\right)^{-1} \boldsymbol{A}_i^{\mathrm{H}} \boldsymbol{y}$$

where $\boldsymbol{A}_i$ is cyclic matrix generated by $\boldsymbol{a}_i$. Similarly, the MSE of $\boldsymbol{h}_i$ is given by

$$\mathrm{MSE}_i = E(\|\hat{\boldsymbol{h}}_i - \boldsymbol{h}_i\|_2^2) = E\left(\left\|\sum_{k\neq i} \left(\boldsymbol{A}_k^{\mathrm{H}} \boldsymbol{A}_k\right)^{-1} \boldsymbol{A}_k^{\mathrm{H}} \boldsymbol{A}_i h_i + \left(\boldsymbol{A}_i^{\mathrm{H}} \boldsymbol{A}_i\right)^{-1} \boldsymbol{A}_i^{\mathrm{H}} \boldsymbol{w}\right\|_2^2\right)$$

The MSE of each MSE_i is low bounded by $L/N_0\sigma^2$.

We consider two transmission antennas and the other parameters are the same as SISO. We have considered Zadoff-Chu sequences, ordinary Golay sequences, our proposed sequences and random sequences, which are generated "on-the-fly." Figs.6 and 7 show the MSE performances of $\boldsymbol{h}_0, \boldsymbol{h}_1$ on MIMO, respectively. It shows that the proposed Golay-ZCZ sequence achieves the lower bound.

Next, we compare the channel estimation performance of Golay-ZCZ sequences in terms of flexible sequence length. We assume a perfect synchronization during the simulation. The receiver tries to estimate $L = 96$ taps. In this comparison, we have selected three Golay-ZCZ sequences of lengths 256, 384 and 512. The Golay sequence of length 256 and 512 can be generated by the methods proposed in [6] and [5]. The Golay-ZCZ sequence of length 384 is proposed this paper. Because in [5] cross-correlation has not been considered, then inter channel interference seriously affects the performance of channel estimation. The Golay-ZCZ sequences of length 256, constructed using the method proposed in [6], has ZCZ zone 64. However, since the maximum multipath delay is 80, its the channel estimation performance is not good. For a multipath delay of this kind, if we follow the construction proposed in [6], we have to use Golay-ZCZ sequence of length 512, to achieve the optimal channel estimation performance. However, as we can see in Fig.8, our proposed Golay-ZCZ sequences of length 384 can generate optimal channel estimation in this scenario. This significantly reduces the pilot overhead, as our proposed sequences have smaller length as compared to the previous constructions, which is required for optimal channel estimation. The performance of $\boldsymbol{h}_1$ is the same as that of $\boldsymbol{h}_0$ in Fig.8, so it is omitted.

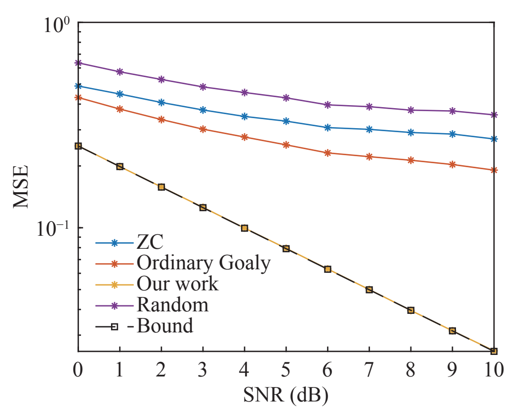

Fig. 6. MSE performances of $\boldsymbol{h}_0$ on MIMO.

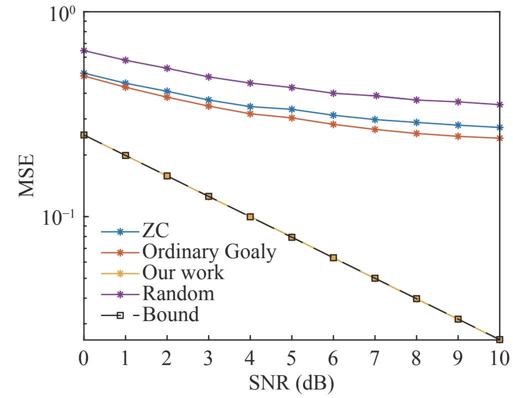

Fig. 7. MSE performances of $\boldsymbol{h}_1$ on MIMO.

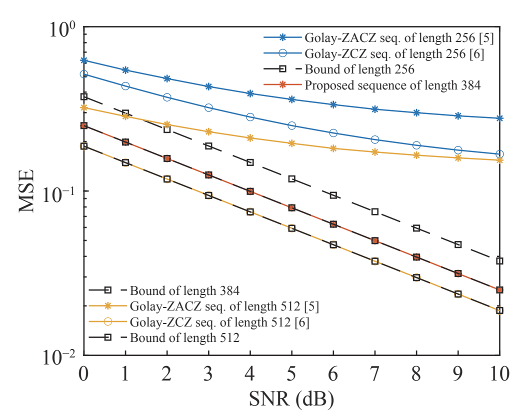

Fig. 8. MSE performances difference length Golay ZCZ sequences of $\boldsymbol{h}_0$ on MIMO.

Remark 5 In application scenarios, the statistical information of the channel is assumed to be known. Therefore, the maximum multipath delay can be used to determine the parameters of Golay-ZCZ sequence as pilot sequences. Specifically, to increase the efficiency of the system, the selection of parameters should be chosen, based on the following two criteria.

- The ZCZ width of Golay-ZCZ sequence set should be greater than the maximum multipath delay.
- The Golay-ZCZ sequence length should be as small as possible to reduce the pilot overhead.

VI. Discussion on Optimality of the Proposed Sequence Sets

For polyphase sequence sets consisting of M sequences each of length L, having ZCZ width Z, we have from [8]

$$Z \leq \frac{L}{M}$$

For binary sequence sets, the bound is conjectured [8] as

$$Z \leq \frac{L}{2M}$$

Assuming Z_{opti} to be the optimal value of Z, let us define the optimality factor (ρ) as

$$\rho = \frac{Z}{Z_{\text{opti}}}$$

Consider the $(2, 4N, N)$-Golay-ZCZ sequence sets described in Theorem 1. If we consider the binary cases, then we have $\rho = 1$. Hence the resultant Golay-ZCZ sequence pairs proposed in Theorem 1 are optimal.

Assuming that an (M, M, N)-CCC exists, from Theorem 2, we get $(M, M^2N, (M-1)N)$-Golay-ZCZ sequence set. To be an optimal Golay-ZCZ sequence set, the optimal ZCZ width $Z_{\text{opti}} = \frac{L}{M}$, i.e., in this case $Z_{\text{opti}} = MN$. However, the ZCZ we can achieve through Theorem 2 is $Z = (M-1)N$.

Therefore,

$$\rho = \frac{Z}{Z_{\text{opti}}} = \frac{(M-1)}{M}$$

Therefore, $\lim\limits_{M\to\infty} \rho = 1$, hence, the sequence sets proposed in Theorem 2 are asymptotically optimal, as the number of sequences increases.

VII. The Novelty of the Proposed Constructions as Compared to Previous Works

In [26], Han *et. al.* proposed a ZCZ sequence set based on CCC. However, the analysis of the complementary property of the proposed sequence sets was missing in [26]. In this section we compare the proposed constructions to the previous works, specifically with the works of Gong *et al.* [5] and Chen *et al.* [6] and discuss the novelty of the proposed constructions.

1) In [5], the authors only considered the complementary sequences of lengths of the form 2^m, and analysed their corresponding periodic zero auto-correlation zones. As compared to that, Theorem 1 and Theorem 2 considers complementary sequences of a more flexible form and analyses both periodic zero auto-correlation zone as well as zero crosscorrelation zone.

2) In [6], the authors proposed Golay-ZCZ sequence sets. However, the constituent sequences have lengths only of the form of 2^m. As compared to that, in Theorem 1 we have proposed Golay-ZCZ sequence pairs of length $4N$, where N is the length of a GCP. In Theorem 2, we have proposed Golay-ZCZ sequence sets, which consist of sequences with lengths M^2N, using an (M, M, N) CCC. As we see the lengths are more flexible as compared to the results in [6].

3) The results reported in [6] only achieves optimality for the case of binary $(2, 2^m, 2^{m-2})$-Golay-ZCZ sequence pairs. As compared to that, based on the discussions in Section V, we can see that Theorem 1 generates binary $(2, 4N, N)$-Golay-ZCZ sequence pair, which is optimal. Theorem 2 generates polyphase $(M, M^2N, (M-1)N)$ Golay-ZCZ sequence set, which is also asymptotically optimal, as the number of sequence increases.

4) Analysing the parameters of the Golay-ZCZ sequence sets of [6], we can observe that, for the same set size and sequence length, our proposed construction generates Golay sequences having larger ZCZ widths as compared to the Golay-ZCZ sequences, reported in [6].

5) To increase the availability of CCCs of various parameters, which in turn increases the flexibility of the parameters of the proposed Golay-ZCZ sequence sets, we have proposed a new iterative construction of CCCs. We have also provided computer search results for binary CCCs with parameters $(4, 4, N)$, for various values of N given in Table 3. Moreover, using those as seed CCCs in [23] and Theorem 3, we can construct several binary CCCs with new parameters, which are not reported before. As discussed in Remark 3, these CCCs are important in itself, since N is not a power-of-two.

6) In this paper, we have demonstrated the applicability of Golay-ZCZ sequences of various parameters and analysed their efficiency through numerical simulations.

VIII. Conclusions

In this paper, we have made two contributions.

Firstly, we have systematically constructed GCPs of lengths non-power-of-two where each of the constituent sequences have a periodic ZACZ and the pair have a periodic ZCCZ. We have also constructed complementary sets consisting of sequences having large periodic ZACZ and ZCCZ using CCCs and IDFT matrices. Notably, the second construction generates asymptotically optimal Golay-ZCZ sequences with respect to Tang-Fan-Matsufuji bound, as the number of sequence increases. To increase the availability of CCC for various parameters and consequently increase the flexibility of the proposed Golay-ZCZ sequence sets, we have also proposed a new iterative construction of CCCs. Moreover, we have found binary CCCs with parameters $(4,4,3)$, $(4,4,5)$, $(4,4,7)$, $(4,4,11)$, and $(4,4,13)$ based on computer search, which can be used as seed CCCs, and can hugely increase the flexibility of the proposed construction. Since the length of the resultant Golay-ZCZ sequences are more flexible, the proposed constructions partially fills the gap left by the previous remarkable works of Gong *et al.* and Chen *et al.* We have also demonstrated through simulation results that the proposed sequences perform better in ISI channel estimation, as compared to other traditional sequences. The proposed Golay-ZCZ sequences also have potential applications in uplink grant-free NOMA.

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