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Convolution Theorem Associated with the QWFRFT

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Abstract — The quaternion windowed fractional Fourier transform (QWFRFT) is a generalized form of the quaternion fractional Fourier transform (QFRFT), it plays a crucial role in signal processing for the analysis of multidimensional signals. In this paper, we first give the definition of the two-sided QWFRFT and some fundamental properties. Then, the quaternion convolution is proposed, and the relation between the quaternion convolution and the classical convolution is also given. Based on the quaternion convolution of the QWFRFT, relevant convolution theorems for the QWFRFT are studied. Moreover, the fast algorithm for QWFRFT is discussed. Finally, the complexity of QWFRFT and the quaternion windowed fractional convolution are given.

Key words — Windowed fractional Fourier transform, Convolution theorem, Quaternion algebra.

I. Introduction

The fractional Fourier transform (FRFT), which is very suitable for signal characterization and analysis, plays a vital role in quantum physics, applied mathematics, optics, communications, image and signal processing [1]–[4]. However, we all know that classical FRFT is ineffective in representing and calculating the local information of a signal. In order to overcome the above deficiencies, some scholars try to apply FRFT to research generalized windowed functions and propose windowed fractional Fourier transform (WFRFT). The WFRFT [5], also called short-time fractional Fourier transform (STFRFT), providing simultaneously information in time and frequency domains, has been widely used in applied mathematics, signal processing, radar system analysis, pattern recognition, and many other fields [6]–[9]. Some important properties of the WFRFT are discussed, including convolution, uncertainty principle (UP), reconstruction formula, etc [10]–[12].

Quaternions [13], [14] are one of the generalizations of complex numbers, which was introduced by Hamilton in 1843. It has been used for signal and color image processing [15], [16]. For the past few years, some authors have extended the integral transform into the quaternion algebra domain and established the theoretical system of quaternion fractional transform, such as quaternion Fourier transform (QFT) [17]–[20], QFRFT [21]–[23], and quaternion linear canonical transform (QLCT) [24], [25].

Recently, some studies have tried to generalize the windowed integral transform to quaternion algebra. Quaternion windowed Fourier transform (QWFT) was first proposed by Bahri and Hitzer [26], they extended the windowed Fourier transform (WFT) to the right-sided QWFT, and some important properties and applications to a linear time-varying system were analyzed. Then, some scholars have also paid attention to QWFT [27]–[31], they derived several important properties and a number of UP of QWFT. Roopkumar [32] extends the STFRFT to a suitable space of quaternion-valued functions on $\mathbb{R}$, proposed quaternionic short-time fractional Fourier transform (QSTFRFT), and some properties including Parseval's formula, inversion formulae and the UP are discussed. In [33], [34], Gao and Li presented quaternion windowed linear canonical transform (QWLCT) and obtained different kinds of UP for the QWLCT, such as the Lieb UP, the logarithmic UP, the entropy UP, Donoho-Stark's UP, and Heisenberg UP.

However, to the best of our knowledge, convolution and corresponding convolution theorem for the QWFRFT have not been presented in the literature. Since the convolution plays an essential part in math-

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ematics, the study of quaternion convolution in the QWFRFT domain is not only theoretically interesting but also practically useful. The main goal of this paper is to construct the quaternion convolution for the QWFRFT. Firstly, we propose the QWFRFT and some basic properties are obtained. Then, the quaternion convolution is given and the corresponding convolution theorem is derived. Thirdly, fast algorithm for QWFRFT are discussed, the complexity of QWFRFT and the quaternion windowed fractional convolution via the convolution and product theorem are discussed.

II. Preliminaries

In this section, we mainly review some essential facts of quaternion algebra and the QFRFT, which will be used in this article.

1. Quaternion algebra

Quaternion algebra [13], [14] was introduced by Hamilton in 1843, which is an extension of complex numbers to 4D algebra. A quaternion is denoted as $\mathbb{H}$ and each of its elements has form given by

$$\mathbb{H} = \{q | q = q_0 + iq_1 + jq_2 + kq_3,\ q_0,\ q_1,\ q_2,\ q_3 \in \mathbb{R}\} \tag{1}$$

which obey Hamilton's multiplication rules

$$ij = -ji = k,\ \ jk = -kj = i,\ \ ki = -ik = j,\\ i^2 = j^2 = k^2 = ijk = -1 \tag{2}$$

The conjugate of q is defined by $\overline{q} = q_0 - iq_1 - jq_2 - jq_3$. And the norm of $q \in \mathbb{H}$ defined as

$$|q| = \sqrt{q\overline{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} \tag{3}$$

So, we can get

$$\overline{pq} = \overline{q}\overline{p},\ \ |qp| = |q||p|,\ \ \forall q,\ p \in \mathbb{H} \tag{4}$$

Let $f(\boldsymbol{x}) \in L^2(\mathbb{R}^2, \mathbb{H}), \boldsymbol{x} = (x_1, x_2)$, then we have an expression.

$$f(\boldsymbol{x}) = f_0(\boldsymbol{x}) + f_1(\boldsymbol{x})i + f_2(\boldsymbol{x})j + f_3(\boldsymbol{x})k \tag{5}$$

where $f_0(\boldsymbol{x}), f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), f_3(\boldsymbol{x}) \in \mathbb{R}^2$.

The inner product of $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$ is given by

$$(f, g)_{L^2(\mathbb{R}^2, \mathbb{H})} = \int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{g(\boldsymbol{x})} d\boldsymbol{x} \tag{6}$$

The associated scalar norm of $f(\boldsymbol{x}) \in L^2(\mathbb{R}^2, \mathbb{H})$ is defined as:

$$\|f\|^2_{L^2(\mathbb{R}^2\mathbb{H})} = (f, f)_{L^2(\mathbb{R}^2, \mathbb{H})} = \int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 d\boldsymbol{x} < \infty \tag{7}$$

We have the quaternion Cauchy-Schwarz inequality:

$$\left| \int_{\mathbb{R}^2} f(\boldsymbol{x}) \overline{g}(\boldsymbol{x}) d\boldsymbol{x} \right| \le \|f\|_2 \|g\|_2 \tag{8}$$

2. The quaternion fractional Fourier transform

The QFRFT [22] is a generalization of the FRFT. Because quaternion multiplication does not satisfy the commutative law, there are three kinds the QFRFT: the left-sided QFRFT, the right-sided QFRFT, and the two-sided QFRFT. We emphatically discuss the two-sided QFRFT in this article.

Let $f(\boldsymbol{x}) \in L^2(\mathbb{R}^2, \mathbb{H})$, the two-sided QFRFT is defined as

$$F_{i,j}^{p_1,p_2}(\boldsymbol{v}) = \int_{\mathbb{R}^2} K_{p_1}^i(x_1, v_1) f(\boldsymbol{x}) K_{p_2}^j(x_2, v_2) d\boldsymbol{x} \tag{9}$$

where $K_{p_1}^i(x_1, v_1)$, $K_{p_2}^j(x_2, v_2)$ is given by

$$K_{p_1}^i(x_1, v_1) = A_\alpha e^{i((x_1^2 + v_1^2)C_\alpha - x_1 v_1 B_\alpha)} \tag{10}$$

$$K_{p_2}^j(x_2, v_2) = A_\beta e^{j((x_2^2 + v_2^2)C_\beta - x_2 v_2 B_\beta)} \tag{11}$$

and

$$A_\alpha = \sqrt{\frac{1 - i\cot\alpha}{2\pi}},\ \ A_\beta = \sqrt{\frac{1 - j\cot\beta}{2\pi}} \tag{12}$$

$$C_\alpha = \frac{\cot\alpha}{2},\ C_\beta = \frac{\cot\beta}{2},\ B_\alpha = \csc\alpha,\ B_\beta = \csc\beta,\\ \alpha = p_1\frac{\pi}{2},\ \ \beta = p_2\frac{\pi}{2} \tag{13}$$

A quaternion signal $f(\boldsymbol{x})$ can be reconstructed via QFRFT.

$$\begin{aligned} f(\boldsymbol{x}) &= F_{i,j}^{-p_1,-p_2}\{F_{i,j}^{p_1,p_2}(\boldsymbol{v})\} \\ &= \int_{\mathbb{R}^2} K_{-p_1}^i(x_1, v_1) F_{i,j}^{p_1,p_2}(\boldsymbol{v}) K_{-p_2}^j(x_2, v_2) d\boldsymbol{v} \end{aligned} \tag{14}$$

Let $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, the convolution operator $\star$ for QFRFT is defined as

$$\begin{aligned} &(f \star g)(\boldsymbol{x}) \\ &= (A_{\alpha,\beta} e^{i(x_1^2 C_\alpha + x_2^2 C_\beta)} e^{i(x_1^2 C_\alpha + x_2^2 C_\beta)} f(\boldsymbol{x})) \\ &\quad * (g(\boldsymbol{x}) e^{i(x_1^2 C_\alpha + x_2^2 C_\beta)}) \end{aligned} \tag{15}$$

where $A_{\alpha,\beta} = \frac{\sqrt{(1 - i\cot\alpha)(1 - i\cot\beta)}}{2\pi}$, $*$ is the classical convolution operator.

III. Quaternion Windowed Fractional Fourier Transform

In this section, the QWFRFT is proposed, the convolution operation for QWFRFT is defined, and the corresponding convolution theorem is derived. In detail, several basic properties for the QWFRFT are investigated.

1. 2D QWFRFT

In this subsection, two-sided QWFRFT is proposed, and the relationship between QWFRFT and QFRFT is also given.

Definition 1 Let $f(\boldsymbol{x}) \in L^2(\mathbb{R}^2, \mathbb{H})$, the two-sided QWFRFT of $f(\boldsymbol{x})$ is defined as

$$G_\phi^{p_1,p_2}(f)(\boldsymbol{v},\boldsymbol{u}) = \int_{\mathbb{R}^2} K_{p_1}^i(x_1,v_1) f(\boldsymbol{x}) \overline{\phi(\boldsymbol{x}-\boldsymbol{u})} K_{p_2}^j(x_2,v_2) \mathrm{d}\boldsymbol{x} \tag{16}$$

where $K_{p_1}^i(x_1,v_1)$, $K_{p_2}^j(x_2,v_2)$ are same in (10) and (11), $\phi \in L^2(\mathbb{R}^2, \mathbb{H})$ be a quaternion windowed function.

Note: 1) $\alpha = \beta = \frac{\pi}{2}$ ($p_1 = p_2 = 1$), the equation (16) is reduced to the QWFT [19]; 2) $\alpha = \frac{\pi}{2}$ and $\beta = 0$ ($p_1 = 1$ and $p_2 = 0$), or $\alpha = 0$ and $\beta = \frac{\pi}{2}$ ($p_1 = 0$ and $p_2 = 1$), the equation (16) is reduced to the QWFT [19] of $f(\boldsymbol{x})$ for variable x_1 or x_2; 3) $\alpha = 0$ and $\beta = 0$ ($p_1 = 0$ and $p_2 = 0$), the equation (16) is reduced to the $f(\boldsymbol{x})$.

Lemma 1 The relationship between the QWFRFT and QFRFT can be obtained as follows:

$$G_\phi^{p_1,p_2}(f)(\boldsymbol{v},\boldsymbol{u}) = F_{i,j}^{p_1,p_2}\{f(\boldsymbol{x})\overline{\phi(\boldsymbol{x}-\boldsymbol{u})}\}(\boldsymbol{v}) \tag{17}$$

2. Properties of QWFRFT

In this subsection, we will give some basic properties about the QWFRFT.

Property 1 (Linearity) Let $\phi \in L^2(\mathbb{R}^2, \mathbb{H})\backslash\{0\}$, $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, then we get

$$G_\phi^{p_1,p_2}(\lambda_1 f + \lambda_2 g)(\boldsymbol{v},\boldsymbol{u}) = \lambda_1 G_\phi^{p_1,p_2}(f)(\boldsymbol{v},\boldsymbol{u}) + \lambda_2 G_\phi^{p_1,p_2}(g)(\boldsymbol{v},\boldsymbol{u}) \tag{18}$$

where $\lambda_1, \lambda_2 \in \mathbb{H}$.

Proof From the equation (16), we can easily get it.

Property 2 (Boundedness) Let $\phi \in L^2(\mathbb{R}^2, \mathbb{H})\backslash\{0\}$ and $f(\boldsymbol{x}) \in L^2(\mathbb{R}^2, \mathbb{H})$, then we have

$$|G_\phi^{p_1,p_2}(f)(\boldsymbol{v},\boldsymbol{u})| \leq A_{\alpha,\beta} \|f\|_{L^2(\mathbb{R}^2)} \|\phi\|_{L^2(\mathbb{R}^2)} \tag{19}$$

Proof By using (8), we obtion

$$\begin{aligned} &|G_\phi^{p_1,p_2}(f)(\boldsymbol{v},\boldsymbol{u})|^2 \\ &= \left| \int_{\mathbb{R}^2} K_{p_1}^i(x_1,v_1) f(\boldsymbol{x}) \overline{\phi(\boldsymbol{x}-\boldsymbol{u})} K_{p_2}^j(x_2,v_2) \mathrm{d}\boldsymbol{x} \right|^2 \\ &= A_{\alpha,\beta} \left(\int_{\mathbb{R}^2} \left| f(\boldsymbol{x}) \overline{\phi(\boldsymbol{x}-\boldsymbol{u})} \right| \mathrm{d}\boldsymbol{x} \right)^2 \\ &\leq A_{\alpha,\beta} \left(\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} \right) \left(\int_{\mathbb{R}^2} \left| \overline{\phi(\boldsymbol{x}-\boldsymbol{u})} \right|^2 \mathrm{d}\boldsymbol{x} \right) \\ &= A_{\alpha,\beta} \|f\|_{L^2(\mathbb{R}^2)}^2 \|\phi\|_{L^2(\mathbb{R}^2)}^2 \end{aligned} \tag{20}$$

The proof is completed.

Property 3 (shift) Let $\phi \in L^2(\mathbb{R}^2, \mathbb{H})\backslash\{0\}$ and $f(\boldsymbol{x}) \in L^2(\mathbb{R}^2, \mathbb{H})$, then we obtian

$$\begin{aligned} &G_\phi^{p_1,p_2}(T_{\boldsymbol{h}} f)(\boldsymbol{v},\boldsymbol{u}) \\ &= \mathrm{e}^{i\frac{h_1^2}{2}\sin\alpha\cos\alpha} \mathrm{e}^{-ih_1 v_1 \sin\alpha} G_\phi^{p_1,p_2}(f)(\boldsymbol{z}, \boldsymbol{u}-\boldsymbol{h}) \\ &\cdot \mathrm{e}^{j\frac{h_2^2}{2}\sin\beta\cos\beta} \mathrm{e}^{-jh_2 v_2 \sin\beta} \end{aligned} \tag{21}$$

where $T_{\boldsymbol{h}} f = f(\boldsymbol{x}-\boldsymbol{h})$, $\boldsymbol{h} = (h_1, h_2)$, $\boldsymbol{z} = (z_1, z_2)$, $z_1 = v_1 - t_1 \cos\alpha$, $z_2 = v_2 - t_2 \cos\beta$.

Proof See Appendix A.

Property 4 (Modulation) Let $\phi \in L^2(\mathbb{R}^2, \mathbb{H})\backslash\{0\}$ and $f(\boldsymbol{x}) \in L^2(\mathbb{R}^2, \mathbb{H})$, $\mathbb{M}_{\boldsymbol{s}} f(\boldsymbol{x}) = \mathrm{e}^{ix_1 s_1} f(\boldsymbol{x}) \mathrm{e}^{jx_2 s_2}$ with $\boldsymbol{s} = (s_1, s_2)$, then we have

$$\begin{aligned} &G_\phi^{p_1,p_2}(\mathbb{M}_{\boldsymbol{s}} f)(\boldsymbol{v},\boldsymbol{u}) \\ &= \mathrm{e}^{iv_1 s_1 \cos\alpha} \mathrm{e}^{-i\frac{s_1^2}{2}\sin\alpha\cos\alpha} G_\phi^{p_1,p_2}(f)(\boldsymbol{z},\boldsymbol{u}) \\ &\cdot \mathrm{e}^{jv_2 s_2 \cos\beta} \mathrm{e}^{-j\frac{s_2^2}{2}\sin\beta\cos\beta} \end{aligned} \tag{22}$$

where $\boldsymbol{z} = (z_1, z_2)$, $z_1 = v_1 - s_1 \sin\alpha$, $z_2 = v_2 - s_2 \sin\beta$.

Proof See Appendix B.

Property 5 (Inversion formula) Let $\phi \in L^2(\mathbb{R}^2, \mathbb{H})\backslash\{0\}$ and $f \in L^2(\mathbb{R}^2, \mathbb{H})$. Then we give the inversion of the QWFRFT as

$$\begin{aligned} f(\boldsymbol{x}) = &\frac{1}{\| \phi \|^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \overline{K_{p_1}^i(x_1,v_1)} G_\phi^{p_1,p_2}(f)(\boldsymbol{v},\boldsymbol{u}) \\ &\cdot \overline{K_{p_2}^j(x_2,v_2)} \phi(\boldsymbol{x}-\boldsymbol{u}) \mathrm{d}\boldsymbol{v} \mathrm{d}\boldsymbol{u} \end{aligned} \tag{23}$$

Proof See Appendix C.

3. Convolution of QWFRFT

In this subsection, we introduce a new quaternion convolution operation for the QWFRFT. And the QWFRFT convolution theorem is derived.

Definition 2 Let $h(\boldsymbol{x}) = f(\boldsymbol{x}) \Theta g(\boldsymbol{x})$, $f(\boldsymbol{x}), g(\boldsymbol{x}) \in L^2(\mathbb{R}^2, \mathbb{H})$, the convolution operator Θ for the QWFRFT is defined as

$$\begin{aligned} h(\boldsymbol{x}) &= (f \Theta g)(\boldsymbol{x}) \\ &= \int_{\mathbb{R}^2} A_{\alpha,\beta} \mathrm{e}^{ix_1^2 C_\alpha} \mathrm{e}^{-jx_2^2 C_\beta} \mathrm{e}^{iz_1^2 C_\alpha} f(\boldsymbol{z}) \mathrm{e}^{jz_2^2 C_\beta} \\ &\cdot \mathrm{e}^{i(x_1-z_1)^2 C_\alpha} g(\boldsymbol{x}-\boldsymbol{z}) \mathrm{e}^{j(x_2-z_2)^2 C_\beta} \mathrm{d}\boldsymbol{z} \end{aligned} \tag{24}$$

Based on the classical convolution operator $*$, the QWFRFT convolution expression (24) can be rewritten as follows:

$$\begin{aligned} h(\boldsymbol{x}) = (f \Theta g)(\boldsymbol{x}) = &A_{\alpha,\beta} \mathrm{e}^{-ix_1^2 C_\alpha} \mathrm{e}^{-jx_2^2 C_\beta} (\mathrm{e}^{ix_1^2 C_\alpha} f(\boldsymbol{x}) \\ &\cdot \mathrm{e}^{jx_2^2 C_\beta}) * (\mathrm{e}^{ix_1^2 C_\alpha} g(\boldsymbol{x}) \mathrm{e}^{jx_2^2 C_\beta}) \end{aligned} \tag{25}$$

which means that the convolution operation in (25) can be expressed as classical form. From Fig.1, we can calculate convolution Θ in (25) by using classical $*$.

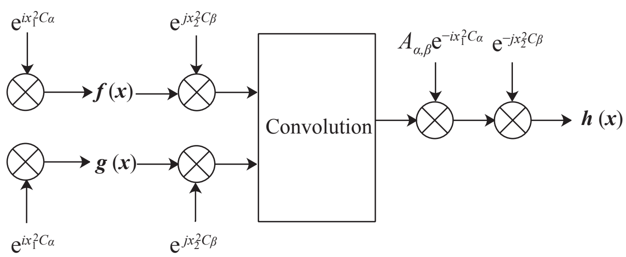

Fig. 1. Convolution operation for QWFRFT.

Theorem 1 Let $f,g\in L^2(\mathbb{R}^2,\mathbb{H})$, the convolution theorem for the WFRFT is obtained

$$\begin{aligned}&G_{\phi*\psi}^{p_1,p_2}(f\star g)(\boldsymbol{v},\boldsymbol{u})\\&=A_{\alpha,\beta}\mathrm{e}^{-i(v_1^2C_\alpha+v_2^2C_\beta)}\int_{\mathbb{R}^2}G_\phi^{p_1,p_2}(f)(\boldsymbol{v},\boldsymbol{y})\\&\quad\cdot G_\psi^{p_1,p_2}(g)(\boldsymbol{v},\boldsymbol{y})\mathrm{e}^{2i(y_1^2-(u_1-y_1)s_1-z_1y_1-z_1s_1)C_\alpha}\\&\quad\cdot\mathrm{e}^{2i(y_2^2-(u_2-y_2)s_2-z_2y_2-z_2s_2)C_\beta}\mathrm{d}\boldsymbol{y}\end{aligned}\tag{26}$$

Proof See Appendix D.

Theorem 2 Let $f,g\in L^2(\mathbb{R}^2,\mathbb{H})$, the convolution production theorem for WFRFT is achieved

$$\begin{aligned}&G_\phi^{p_1,p_2}\{\mathrm{e}^{i(x_1^2C\alpha+x_2^2C_\beta)}f(\boldsymbol{x})g(\boldsymbol{x})\overline{\phi(\boldsymbol{x}-\boldsymbol{u})}^2\}\\&\quad=A_{-\alpha,-\beta}(G_\phi^{p_1,p_2}f(\boldsymbol{v})\overline{\star}G_\phi^{p_1,p_2}g(\boldsymbol{v}))\end{aligned}\tag{27}$$

where

$$A_{-\alpha,-\beta}=-\frac{\sqrt{(1-i\cot\alpha)(1-i\cot\beta)}}{2\pi}$$

Proof See Appendix E.

Theorem 3 Let $f,g\in L^2(\mathbb{R}^2,\mathbb{H})$, $f(\boldsymbol{x})=f_0(\boldsymbol{x})+if_1(\boldsymbol{x})+jf_2(\boldsymbol{x})+kf_3(\boldsymbol{x})=f_a(\boldsymbol{x})+f_b(\boldsymbol{x})j$, $g(\boldsymbol{x})=g_0(\boldsymbol{x})+ig_1(\boldsymbol{x})+jg_2(\boldsymbol{x})+kg_3(\boldsymbol{x})=g_a(\boldsymbol{x})+g_b(\boldsymbol{x})j$, where $f_a(\boldsymbol{x})=f_0(\boldsymbol{x})+if_1(\boldsymbol{x})$, $f_b(\boldsymbol{x})=f_2(\boldsymbol{x})+if_3(\boldsymbol{x})$, $g_a(\boldsymbol{x})=g_0(\boldsymbol{x})+ig_1(\boldsymbol{x})$, $g_b(\boldsymbol{x})=g_2(\boldsymbol{x})+ig_3(\boldsymbol{x})$. Then the convolution theorem associated with the QWFRFT is given by

$$\begin{aligned}&G_{\phi*\psi}^{p_1,p_2}(f\Theta g)(\boldsymbol{v},\boldsymbol{u})\\&=\int_{\mathbb{R}^2}A_{\alpha,\beta}B_i[G_\phi^{p_1,p_2}(f_a)(\boldsymbol{v},\boldsymbol{y})+jG_\phi^{p_1,p_2}(f_b)(\boldsymbol{v},\boldsymbol{y})]\\&\quad\cdot[G_\psi^{p_1,p_2}(g_a)(\boldsymbol{v},\boldsymbol{u}-\boldsymbol{y})+jG_\psi^{p_1,p_2}(g_b)(\boldsymbol{v},\boldsymbol{u}-\boldsymbol{y})]B_j\mathrm{d}\boldsymbol{y}\end{aligned}\tag{28}$$

where

$$B_i=\mathrm{e}^{i(-v_1^2+y_1^2-2z_1(s_1+y_1)+(u_1-y_1)^2-2s_1(u_1-y_1))C_\alpha}$$

$$B_j=\mathrm{e}^{j(-v_2^2+y_2^2-2z_2(s_2+y_2)+(u_2-y_2)^2-2s_2(u_2-y_2))C_\beta}$$

Proof From the Definitions 1 and 2, we have

$$\begin{aligned}&G_{\phi*\psi}^{p_1,p_2}(f\Theta g)(\boldsymbol{v},\boldsymbol{u})\\&=\int_{\mathbb{R}^2}K_{p_1}^i(x_1,v_1)(f\Theta g)(\boldsymbol{x})\overline{\phi}*\overline{\psi}(\boldsymbol{x}-\boldsymbol{u})K_{p_2}^j(x_2,v_2)\mathrm{d}\boldsymbol{x}\\&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}A_\alpha^3\mathrm{e}^{i(v_1^2C_\alpha-x_1v_1B_\alpha)}\\&\quad\cdot\mathrm{e}^{iz_1^2C_\alpha}f(\boldsymbol{z})\mathrm{e}^{jz_2^2C_\beta}\mathrm{e}^{i(x_1-z_1)^2C_\alpha}g(\boldsymbol{x}-\boldsymbol{z})\mathrm{e}^{j(x_2-z_2)^2C_\beta}\\&\quad\cdot\mathrm{e}^{-ix_1^2C_\alpha}\mathrm{e}^{-jx_2^2C_\beta}\mathrm{e}^{ir_1^2C_\alpha}\overline{\phi}(\boldsymbol{r})\mathrm{e}^{jr_2^2C_\beta}\\&\quad\cdot\mathrm{e}^{i(x_1-u_1-r_1)^2C_\alpha}\overline{\psi}(\boldsymbol{x}-\boldsymbol{u}-\boldsymbol{r})\mathrm{e}^{j(x_2-u_2-r_2)^2C_\beta}\\&\quad\cdot A_\beta^3\mathrm{e}^{j(v_2^2C_\beta-x_2v_2B_\beta)}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{z}\mathrm{d}\boldsymbol{r}\end{aligned}\tag{29}$$

Let $\boldsymbol{r}=\boldsymbol{z}-\boldsymbol{y}$, $\boldsymbol{x}=\boldsymbol{z}+\boldsymbol{s}$, the equation (29) can be rewritten as

$$\begin{aligned}&G_{\phi*\psi}^{p_1,p_2}(f\Theta g)(\boldsymbol{v},\boldsymbol{u})\\&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}A_\alpha\mathrm{e}^{i(-v_1^2+y_1^2-2z_1(s_1+y_1))}\\&\quad\cdot\mathrm{e}^{i((u_1-y_1)^2-2s_1(u_1-y_1))C_\alpha}\cdot A_\alpha\mathrm{e}^{i((v_1^2+z_1^2)C_\alpha-z_1v_1B_\alpha)}\\&\quad\cdot[f_a(\boldsymbol{z})+f_b(\boldsymbol{z})j]\overline{\phi}(\boldsymbol{z}-\boldsymbol{y})\cdot A_\beta\mathrm{e}^{j(v_2^2+z_2^2)C_\beta-z_2v_2B_\beta)}\\&\quad\cdot\int_{\mathbb{R}^2}A_\alpha\mathrm{e}^{i(v_1^2+s_1^2)C_\alpha-s_1v_1B_\alpha)}\\&\quad\cdot[g_a(\boldsymbol{s})+g_b(\boldsymbol{s})j]\overline{\psi}(\boldsymbol{s}-(\boldsymbol{u}-\boldsymbol{y}))\\&\quad\cdot A_\beta\mathrm{e}^{j(v_2^2+s_2^2)C_\beta-s_2v_2B_\beta)}A_\beta\mathrm{e}^{j(-v_2^2+y_2^2-2z_2(s_2+y_2))}\\&\quad\cdot\mathrm{e}^{j((u_2-y_2)^2-2s_2(u_2-y_2))C_\beta}\mathrm{d}\boldsymbol{z}\mathrm{d}\boldsymbol{s}\mathrm{d}\boldsymbol{y}\\&=\int_{\mathbb{R}^2}A_{\alpha,\beta}B_i[G_\phi^{p_1,p_2}(f_a)(\boldsymbol{v},\boldsymbol{y})+jG_\phi^{p_1,p_2}(f_b)(\boldsymbol{v},\boldsymbol{y})]\\&\quad\cdot[G_\psi^{p_1,p_2}(g_a)(\boldsymbol{v},\boldsymbol{u}-\boldsymbol{y})+jG_\psi^{p_1,p_2}(g_b)(\boldsymbol{v},(\boldsymbol{u}-\boldsymbol{y})]B_j\mathrm{d}\boldsymbol{y}\end{aligned}\tag{30}$$

The proof is achieved.

Corollary 1 Let $f,g\in L^2(\mathbb{R}^2,\mathbb{H})$, $f(\boldsymbol{x})=f_a(\boldsymbol{x})+f_b(\boldsymbol{x})j$, $g(\boldsymbol{x})=g_a(\boldsymbol{x})+g_b(\boldsymbol{x})j$, where $f_a(\boldsymbol{x})=f_0(\boldsymbol{x})+if_1(\boldsymbol{x})$, $f_b(\boldsymbol{x})=f_2(\boldsymbol{x})+if_3(\boldsymbol{x})$, $g_a(\boldsymbol{x})=g_0(\boldsymbol{x})+ig_1(\boldsymbol{x})$, $g_b(\boldsymbol{x})=g_2(\boldsymbol{x})+ig_3(\boldsymbol{x})$. Then the convolution theorem associated with the QFRFT is given by

$$\begin{aligned}&F_{i,j}^{p_1,p_2}(f\Theta g)(\boldsymbol{v})\\&\quad=\mathrm{e}^{-i(v_1C_\alpha+v_2C_\beta)}[F_{i,j}^{p_1,p_2}(f_a)(\boldsymbol{v})+jF_{i,j}^{p_1,p_2}(f_b)(\boldsymbol{v})]\\&\qquad\cdot[F_{i,j}^{p_1,p_2}(g_a)(\boldsymbol{v})+jF_{i,j}^{p_1,p_2}(g_b)(\boldsymbol{v})]\end{aligned}\tag{31}$$

IV. Fast Algorithm of QWFRFT

In this section, we will discuss the fast algorithm of QWFRFT and estimate the complexity of quaternion windowed fractional convolution for $f,g\in L^2(\mathbb{R}^2,\mathbb{H})$ in the transform domain.

For $f(\boldsymbol{x})\in L^2(\mathbb{R}^2,\ \mathbb{H})$, $p_1,p_2\in[-1,1]$, from (16), we have

$$\begin{aligned} & G_\phi^{p_1,p_2}\{f(\boldsymbol{x})\} \\ & = \int_{\mathbb{R}^2} K_{p_1}^i(x_1,v_1) f(\boldsymbol{x})\overline{\phi(\boldsymbol{x}-\boldsymbol{u})} K_{p_2}^j(x_2,v_2)\mathrm{d}\boldsymbol{x} \\ & = A_i(v_1)\int_{\mathbb{R}^2} \mathrm{e}^{-i(v_1x_1\csc\alpha)} q(\boldsymbol{x})\mathrm{e}^{-j(v_2x_2\csc\beta)}\mathrm{d}\boldsymbol{x} A_j(v_2) \end{aligned} \tag{32}$$

where

$$q(\boldsymbol{x}) = \mathrm{e}^{ix_1^2C_\alpha} f(\boldsymbol{x})\overline{\phi(\boldsymbol{x}-\boldsymbol{u})}\mathrm{e}^{jx_2^2}C_\beta \tag{33}$$

$$A_i(v_1) = A_\alpha \mathrm{e}^{i\frac{v_1^2}{2}\cot\alpha}, A_j(v_2) = A_\beta \mathrm{e}^{j\frac{v_2^2}{2}\cot\beta} \tag{34}$$

Let

$$M(\boldsymbol{v}) = \int_{\mathbb{R}^2} \mathrm{e}^{-i(v_1x_1\csc\alpha)} q(\boldsymbol{x})\mathrm{e}^{-i(v_2x_2\csc\beta)}\mathrm{d}\boldsymbol{x} \tag{35}$$

then

$$\begin{aligned} & \frac{M(v_1,v_2)+M(v_1,-v_2)}{2} \\ & = \int_{\mathbb{R}^2} \mathrm{e}^{-i(v_1x_1\csc\alpha)} q(\boldsymbol{x})\cos(v_2x_2\csc\beta)\mathrm{d}\boldsymbol{x} \end{aligned} \tag{36}$$

$$\begin{aligned} & \frac{M(v_1,v_2)-M(v_1,-v_2)}{2} \\ & = \int_{\mathbb{R}^2} \mathrm{e}^{-i(v_1x_1\csc\alpha)} q(\boldsymbol{x})(-i)\sin(v_2x_2\csc\beta)\mathrm{d}\boldsymbol{x} \end{aligned} \tag{37}$$

therefore,

$$\begin{aligned} & \frac{M(v_1,v_2)+M(v_1,-v_2)}{2} + \frac{M(v_1,v_2)-M(v_1,-v_2)}{2}(-k) \\ & = \int_{\mathbb{R}^2} \mathrm{e}^{-i(v_1x_1\csc\alpha)} q(\boldsymbol{x})\mathrm{e}^{-j(v_2x_2\csc\beta)}\mathrm{d}\boldsymbol{x} \end{aligned} \tag{38}$$

Then we obtain

$$\begin{aligned} & G_\phi^{p_1,p_2} f(\boldsymbol{v},\boldsymbol{u}) \\ & = A_i(v_1)\frac{M(v_1,v_2)(1-k)+M(v_1,-v_2)(1+k)}{2}A_j(v_2) \end{aligned} \tag{39}$$

From the quaternion algebra, $q(\boldsymbol{x})$ in (33) can be expressed as

$$\begin{aligned} q(\boldsymbol{x}) &= q_0(\boldsymbol{x}) + iq_1(\boldsymbol{x}) + jq_2(\boldsymbol{x}) + kq_3(\boldsymbol{x}) \\ &= q_a(\boldsymbol{x}) + q_b(\boldsymbol{x})j \end{aligned} \tag{40}$$

where $q_a(\boldsymbol{x}) = q_0(\boldsymbol{x}) + iq_1(\boldsymbol{x})$, $q_b(\boldsymbol{x}) = q_2(\boldsymbol{x}) + iq_3(\boldsymbol{x})$, we have

$$\begin{aligned} M(\boldsymbol{v}) &= \int_{\mathbb{R}^2} \mathrm{e}^{-i(v_1x_1\csc\alpha)} q_a(\boldsymbol{x})\mathrm{e}^{-j(v_2x_2\csc\beta)}\mathrm{d}\boldsymbol{x} \\ &\quad + (\int_{\mathbb{R}^2} \mathrm{e}^{-i(v_1x_1\csc\alpha)} q_b(\boldsymbol{x})\mathrm{e}^{-j(v_2x_2\csc\beta)}\mathrm{d}\boldsymbol{x})j \\ &= F\{q_a(\boldsymbol{x})\}(v_1\csc\alpha, v_2\csc\beta) \\ &\quad + F\{q_b(\boldsymbol{x})\}(v_1\csc\alpha, v_2\csc\beta)j \end{aligned} \tag{41}$$

The QWFRFT can be calculated by 2D FFT. The calculation steps of QWFRFT are as follows:

Step 1: Calculate $q(\boldsymbol{x})$ from $f(\boldsymbol{x})$ using equations (33) and (40).

Step 2: Calculate $M(\boldsymbol{v})$ from $q(\boldsymbol{x})$ using (41).

Step 3: Calculate $A_i(v_1)$ and $A_j(v_2)$ using (34).

Step 4: Calculate $G_\phi^{p_1,p_2} f(\boldsymbol{x})$ using (39).

Now, we give the computational complexity of QWFRFT. For a 2D discrete signal of size $M\times N$, a 2D discrete Fourier transform requires a $MN\log_2(MN)$ real-number multiplications [20], [35], [36]. By (39) and (41), we can calculate the complexity of QWFRFT for quaternion signals $f(\boldsymbol{x})$ is $O(2MN\log_2(MN))$. Obviously, the computational complexity of QWFRFT is the same as that of QFRFT. Therefore, we can seen that the major computational for QWFRFT load of calculation of $q(\boldsymbol{x})$ and $M(\boldsymbol{v})$ due to the windowed function, which leads to an increase in the amount of calculation. But we can calculate QWFRFT by using classical 2D FFT from Fig.2 which is very important to engineering usage.

Next, we estimate the complexity of the QW-FRFT convolution by taking $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ ($x_1 \in [1,M]$, $x_2 \in [1,N]$) over the transform domain. Given $f(\boldsymbol{x})$, $g(\boldsymbol{x}) \in L^2(\mathbb{R}^2,\mathbb{H})$, we can rewrite them as follows:

$$\begin{aligned} f(\boldsymbol{x}) &= f_0(\boldsymbol{x}) + if_1(\boldsymbol{x}) + jf_2(\boldsymbol{x}) + kf_3(\boldsymbol{x}) \\ &= f_a(\boldsymbol{x}) + f_b(\boldsymbol{x})j \end{aligned} \tag{42}$$

$$\begin{aligned} g(\boldsymbol{x}) &= g_0(\boldsymbol{x}) + ig_1(x) + jg_2(\boldsymbol{x}) + kg_3(\boldsymbol{x}) \\ &= g_a(\boldsymbol{x}) + g_b(\boldsymbol{x})j \end{aligned} \tag{43}$$

Substitute $f(\boldsymbol{x}) = f_a(\boldsymbol{x}) + f_b(\boldsymbol{x})j$ and $g(\boldsymbol{x}) = g_a(\boldsymbol{x}) + g_b(\boldsymbol{x})j$ into (25), we have

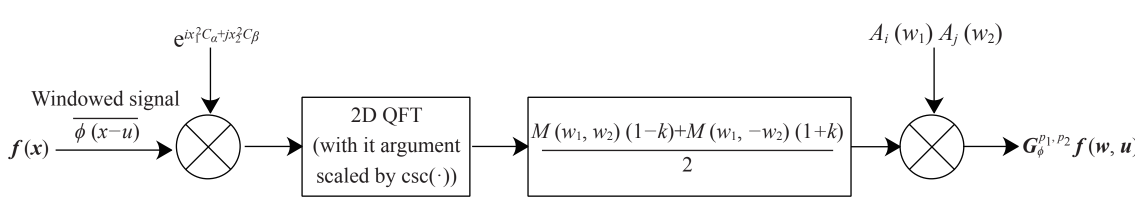

Fig. 2. The calculation process of QWFRFT.

$$
\begin{aligned}
&(f\Theta g)(\boldsymbol{x})\\
&= A_{\alpha,\beta}\mathrm{e}^{-ix_1^2C_\alpha}\mathrm{e}^{-jx_2^2C_\beta}\\
&\quad\cdot\Big\{(\mathrm{e}^{ix_1^2C_\alpha}f_a(\boldsymbol{x})\mathrm{e}^{jx_2^2C_\beta})*(\mathrm{e}^{ix_1^2C_\alpha}g_a(\boldsymbol{x})\mathrm{e}^{jx_2^2C_\beta})\\
&\quad+(\mathrm{e}^{ix_1^2C_\alpha}f_a(\boldsymbol{x})\mathrm{e}^{jx_2^2C_\beta})*(\mathrm{e}^{ix_1^2C_\alpha}g_b(\boldsymbol{x})\mathrm{e}^{jx_2^2C_\beta})j\\
&\quad+(\mathrm{e}^{ix_1^2C_\alpha}f_b(\boldsymbol{x})\mathrm{e}^{jx_2^2C_\beta})*(\mathrm{e}^{ix_1^2C_\alpha}g_a(\boldsymbol{x})\mathrm{e}^{jx_2^2C_\beta})j\\
&\quad-(\mathrm{e}^{ix_1^2C_\alpha}f_b(\boldsymbol{x})\mathrm{e}^{jx_2^2C_\beta})*(\mathrm{e}^{ix_1^2C_\alpha}g_b(\boldsymbol{x})\mathrm{e}^{jx_2^2C_\beta})\Big\}
\end{aligned}\tag{44}
$$

From the Theorem 1, Theorem 3, Corollary 1 and the calculation steps of QWFRFT, we obtain the complexity of the quaternion windowed fractional convolution via 2D FFT is $O(16MN\log_2(MN))$.

V. Conclusions

In this research, we proposed the QWFRFT based on quaternion algebra and FRFT, which can be seen as the generalization of the QWFT. Some properties for the QWFRFT are given. A novel QWFRFT convolution operation is defined, and the convolution and product theorem associated with the QWFRFT is derived. Finally, the fast algorithm for QWFRFT is presented, the complexity of QWFRFT, quaternion windowed fractional convolution via the convolution and product theorem are discussed in this paper. Although QWFRFT requires a large amount of computation, its complexity is the same as 2D FFT, hence, we can calculate QWFRFT by using classical 2D FFT.

Since quaternion plays an important role in color images processing for the multidimensional signal analysis, in practical application, the most important part is to seek the appropriate convolution theorem and the corresponding filter design. Our future work will be further focused on the convolution theorem in the quaternion fractional domain and its filter design in color image processing.

Appendix A. Proof of the Property 3

By Definition 1, we have

$$
\begin{aligned}
&G_\phi^{p_1,p_2}(T_{\boldsymbol{h}}f)(\boldsymbol{v},\boldsymbol{u})\\
&=\int_{\mathbb{R}^2}K_{p_1}^i(x_1,v_1)(f)(\boldsymbol{x}-\boldsymbol{h})\overline{\phi(\boldsymbol{x}-\boldsymbol{u})}K_{p_2}^j(x_2,v_2)\mathrm{d}\boldsymbol{x}
\end{aligned}\tag{A-1}
$$

Let $\boldsymbol{t}=\boldsymbol{x}-\boldsymbol{h}$ in (A-1) we obtain

$$
\begin{aligned}
&G_\phi^{p_1,p_2}(T_{\boldsymbol{h}}f)(\boldsymbol{v},\boldsymbol{u})\\
&=\int_{\mathbb{R}^2}A_\alpha\mathrm{e}^{i\left(\frac{(h_1+t_1)^2+v_1^2}{2}\cot\alpha-(h_1+t_1)v_1\csc\alpha\right)}\\
&\quad\cdot f(\boldsymbol{t})\overline{\phi(\boldsymbol{t}-(\boldsymbol{u}-\boldsymbol{h}))}\\
&\quad\cdot A_\beta\mathrm{e}^{j\left(\frac{(h_2+t_2)^2+v_2^2}{2}\cot\beta-(h_2+t_2)v_2\csc\beta\right)}\mathrm{d}\boldsymbol{t}\\
&=\mathrm{e}^{i\frac{h_1^2}{2}\sin\alpha\cos\alpha}\mathrm{e}^{-ih_1v_1\sin\alpha}G_\phi^{p_1,p_2}(f)(\boldsymbol{z},\boldsymbol{u}-\boldsymbol{h})\\
&\quad\cdot\mathrm{e}^{j\frac{h_2^2}{2}\sin\beta\cos\beta}\mathrm{e}^{-jh_2v_2\sin\beta}
\end{aligned}\tag{A-2}
$$

The proof of Property 3 completed.

Appendix B. Proof of the Property 4

From Definition 1, it follows that

$$
\begin{aligned}
&G_\phi^{p_1,p_2}(\mathbb{M}_{\boldsymbol{s}}f)(\boldsymbol{v},\boldsymbol{u})\\
&=\int_{\mathbb{R}^2}A_\alpha\mathrm{e}^{i\left(\frac{x_1^2+v_1^2}{2}\cot\alpha-x_1v_1\csc\alpha\right)}\mathrm{e}^{ix_1s_1}f(\boldsymbol{x})\\
&\quad\cdot\overline{\phi(\boldsymbol{x}-\boldsymbol{u})}\mathrm{e}^{jx_2s_2}A_\beta\mathrm{e}^{j\left(\frac{x_2^2+v_2^2}{2}\cot\beta-x_2v_2\csc\beta\right)}\mathrm{d}\boldsymbol{x}\\
&=\int_{\mathbb{R}^2}A_\alpha\mathrm{e}^{iv_1s_1\cos\alpha}\mathrm{e}^{-i\frac{s_1^2}{2}\sin\alpha\cos\alpha}\\
&\quad\cdot\mathrm{e}^{i\left(\frac{x_1^2+z_1^2}{2}\cot\alpha-x_1z_1\csc\alpha\right)}f(\boldsymbol{x})\overline{\phi(\boldsymbol{x}-\boldsymbol{u})}A_\beta\\
&\quad\cdot\mathrm{e}^{j\left(\frac{x_2^2+z_2^2}{2}\cot\beta-x_2z_2\csc\beta\right)}\mathrm{e}^{jv_2s_2\cos\beta}\mathrm{e}^{-j\frac{s_2^2}{2}\sin\beta\cos\beta}\mathrm{d}\boldsymbol{x}\\
&=\mathrm{e}^{iv_1s_1\cos\alpha}\mathrm{e}^{-i\frac{s_1^2}{2}\sin\alpha\cos\alpha}G_\phi^{p_1,p_2}(f)(\boldsymbol{z},\boldsymbol{u})\\
&\quad\cdot\mathrm{e}^{jv_2s_2\cos\beta}\mathrm{e}^{-j\frac{s_2^2}{2}\sin\beta\cos\beta}
\end{aligned}\tag{B-1}
$$

The proof is completed.

Appendix C. Proof of the Property 5

Applying the inversion QFRFT of $f(\boldsymbol{x})\in L^2(\mathbb{R}^2,\mathbb{H})$, we get

$$
\begin{aligned}
&f(\boldsymbol{x})\overline{\phi(\boldsymbol{x}-\boldsymbol{u})}\\
&=F_{i,j}^{-p_1,-p_2}\{G_\phi^{p_1,p_2}(f)(\boldsymbol{v},\boldsymbol{u})\}\\
&=\int_{\mathbb{R}^2}\overline{K_{p_1}^i(x_1,v_1)}G_\phi^{p_1,p_2}(f)(\boldsymbol{v},\boldsymbol{u})\overline{K_{p_2}^j(x_2,v_2)}\mathrm{d}\boldsymbol{v}
\end{aligned}\tag{C-1}
$$

multiplying by $\phi(\boldsymbol{x}-\boldsymbol{u})$ on the both side of (C-1), we have

$$
\begin{aligned}
\int_{\mathbb{R}^2}f(\boldsymbol{x})\parallel\phi\parallel^2\mathrm{d}\boldsymbol{u}=&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{K_{p_1}^i(x_1,v_1)}G_\phi^{p_1,p_2}(f)(\boldsymbol{v},\boldsymbol{u})\\
&\cdot\overline{K_{p_2}^j(x_2,v_2)}\phi(\boldsymbol{x}-\boldsymbol{u})\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{u}
\end{aligned}\tag{C-2}
$$

hence, we get

$$
\begin{aligned}
f(\boldsymbol{x})=&\frac{1}{\parallel\phi\parallel^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{K_{p_1}^i(x_1,v_1)}G_\phi^{p_1,p_2}(f)(\boldsymbol{v},\boldsymbol{u})\\
&\cdot\overline{K_{p_2}^j(x_2,v_2)}\phi(\boldsymbol{x}-\boldsymbol{u})\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{u}
\end{aligned}\tag{C-3}
$$

The proof is achieved.

Appendix D. Proof of the Theorem 1

By the equation (15), we have

$$
\begin{aligned}
&G_{\phi*\psi}^{p_1,p_2}(f\star g)(\boldsymbol{v},\boldsymbol{u})\\
&=\int_{\mathbb{R}^2}K_{p_1}(x_1,v_1)\{f\star g\}(\boldsymbol{x})\overline{\phi}*\overline{\psi}(\boldsymbol{x}-\boldsymbol{u})K_{p_2}(x_2,v_2)\mathrm{d}\boldsymbol{x}\\
&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}A_{\alpha,\beta}^2\mathrm{e}^{i((x_1^2+v_1^2)C_\alpha-x_1v_1B_\alpha)}\mathrm{e}^{-i(x_1^2C_\alpha+x_2^2C_\beta)}\\
&\quad\cdot\mathrm{e}^{i(z_1^2C_\alpha+z_2^2C_\beta)}f(\boldsymbol{z})g(\boldsymbol{x}-\boldsymbol{z})\\
&\quad\cdot\mathrm{e}^{i((x_1-z_1)^2C_\alpha+(x_2-z_1)^2C_\beta)}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{z}\\
&\quad\cdot\int_{\mathbb{R}^2}A_{\alpha,\beta}\mathrm{e}^{-i(x_1^2C_\alpha+x_2^2C_\beta)}\mathrm{e}^{i(r_1^2C_\alpha+r_2^2C_\beta)}\overline{\phi}(\boldsymbol{r})\\
&\quad\cdot\overline{\psi}(\boldsymbol{x}-\boldsymbol{u}-\boldsymbol{r})\mathrm{e}^{i((x_1-u_1-r_1)^2C_\alpha+(x_2-u_2-r_2)^2C_\beta)}\\
&\quad\cdot\mathrm{e}^{i((x_2^2+v_2^2)C_\beta-x_2v_2B_\beta)}\mathrm{d}\boldsymbol{r}
\end{aligned}\tag{D-1}
$$

Let $\boldsymbol{r}=\boldsymbol{z}-\boldsymbol{y}$, $\boldsymbol{x}=\boldsymbol{z}+\boldsymbol{s}$, we get

$$\begin{aligned}
&G_{\phi*\psi}^{p_1,p_2}(f\star g)(\boldsymbol{v},\boldsymbol{u})\\
&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}A_{\alpha,\beta}^3\mathrm{e}^{i(v_1^2C_\alpha+v_2^2C_\beta)}\mathrm{e}^{-i(z_1+s_1)^2C_\alpha}\\
&\quad\cdot\mathrm{e}^{-i(z_2+s_2)^2C_\beta}\mathrm{e}^{-i((z_1+s_1)v_1B_\alpha+(z_2+s_2)v_2B_\beta)}\\
&\quad\cdot\mathrm{e}^{i(z_1^2C_\alpha+z_2^2C_\beta)}f(\boldsymbol{z})g(\boldsymbol{s})\mathrm{e}^{i(s_1^2C_\alpha+s_2^2C_\beta)}\\
&\quad\cdot\mathrm{e}^{i((z_1-y_1)^2C_\alpha+(z_2-y_2)^2C_\beta)}\overline{\phi}(\boldsymbol{z}-\boldsymbol{y})\overline{\psi}(\boldsymbol{s}-(\boldsymbol{u}-\boldsymbol{y}))\\
&\quad\cdot\mathrm{e}^{i(s_1-(u_1-y_1))^2C_\alpha+(s_2-(u_2-y_2))^2C_\beta}\mathrm{d}\boldsymbol{z}\mathrm{d}\boldsymbol{s}\mathrm{d}\boldsymbol{y}
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&G_{\phi*\psi}^{p_1,p_2}(f\star g)(\boldsymbol{v},\boldsymbol{u})\\
&=A_{\alpha,\beta}\mathrm{e}^{-\mathrm{i}(v_1^2C_\alpha+v_2^2C_\beta)}\\
&\quad\cdot\int_{\mathbb{R}^2}G_\phi^{p_1,p_2}(f)(\boldsymbol{v},\boldsymbol{y})G_\psi^{p_1,p_2}(g)(\boldsymbol{v},\boldsymbol{y})\\
&\quad\cdot\mathrm{e}^{2\mathrm{i}(y_1^2-(u_1-y_1)s_1-z_1y_1-z_1s_1)C_\alpha}\\
&\quad\cdot\mathrm{e}^{2\mathrm{i}(y_2^2-(u_2-y_2)s_2-z_2y_2-z_2s_2)C_\beta}\mathrm{d}\boldsymbol{y}
\end{aligned}\tag{D-2}$$

which completes the proof of Theorem 1.

Appendix E. Proof of the Theorem 2

With Definition 1, we have

$$\begin{aligned}
&F_\phi^{-p_1,-p_2}A_{-\alpha,-\beta}(G_\phi^{p_1,p_2}(f)(\boldsymbol{v})\overline{\star}G_\phi^{p_1,p_2}(g)(\boldsymbol{v}))\\
&=A_{-\alpha,-\beta}\int_{\mathbb{R}^2}\mathrm{e}^{-i((x_1^2+v_1^2)C\alpha+(x_2^2+v_2^2)C_\beta)}\\
&\quad\cdot\{A_{-\alpha,-\beta}(G_\phi^{p_1,p_2}f(\boldsymbol{v})\overline{\star}G_\phi^{p_1,p_2}g(\boldsymbol{v}))\}\\
&\quad\cdot\mathrm{e}^{i(x_1v_1B\alpha+x_2v_2B_\beta)}\mathrm{d}\boldsymbol{v}\\
&=(A_{-\alpha,-\beta})^2\mathrm{e}^{-i(x_1^2C\alpha+x_2^2C_\beta)}\\
&\quad\cdot\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(G_\phi^{p_1,p_2}g(\boldsymbol{v}-\boldsymbol{z})\\
&\quad\cdot\mathrm{e}^{-i((v_1-z_1)^2)C\alpha+(v_2-z_2)^2C_\beta)})\\
&\quad\cdot(G_\phi^{p_1,p_2}f(\boldsymbol{z})\mathrm{e}^{-i(z_1^2C_\alpha+z_1^2C_\beta)})\\
&\quad\cdot\mathrm{e}^{i(x_1v_1B\alpha+x_2v_2B_\beta)}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{z}
\end{aligned}\tag{E-1}$$

Let $\boldsymbol{s}=\boldsymbol{v}-\boldsymbol{z}$, we have

$$\begin{aligned}
&F_\phi^{-p_1,-p_2}A_{-\alpha,-\beta}(G_\phi^{p_1,p_2}f(\boldsymbol{v})\overline{\star}G_\phi^{p_1,p_2}g(\boldsymbol{v}))\\
&=(A_{-\alpha,-\beta})^2\mathrm{e}^{-i(x_1^2C\alpha+x_2^2C_\beta)}\\
&\quad\cdot\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(G_\phi^{p_1,p_2}g(\boldsymbol{s})\mathrm{e}^{-i(s_1^2C\alpha+s_2^2C_\beta)})\\
&\quad\cdot(G_\phi^{p_1,p_2}f(\boldsymbol{z})\mathrm{e}^{-i(z_1^2C_\alpha+z_2^2C_\beta)})\\
&\quad\cdot\mathrm{e}^{i((s_1+z_1)x_1B\alpha+(s_2+z_2)x_2B_\beta)}\mathrm{d}\boldsymbol{s}\mathrm{d}\boldsymbol{z}\\
&=\mathrm{e}^{-i(x_1^2C\alpha+x_2^2C_\beta)}\\
&\quad\cdot\int_{\mathbb{R}^2}A_{-\alpha,-\beta}(G_\phi^{p_1,p_2}g(\boldsymbol{s})\\
&\quad\cdot\mathrm{e}^{-i(s_1^2C\alpha+s_2^2C_\beta)})\mathrm{e}^{i(s_1v_1B_\alpha+s_2v_2B_\beta)})\\
&\quad\cdot\int_{\mathbb{R}^2}A_{-\alpha,-\beta}(G_\phi^{p_1,p_2}f(\boldsymbol{z})\\
&\quad\cdot\mathrm{e}^{-i(z_1^2C\alpha+z_2^2C_\beta)}\mathrm{e}^{i(z_1v_1B_\alpha+z_2v_2B_\beta)})\mathrm{d}\boldsymbol{s}\mathrm{d}\boldsymbol{z}\\
&=f(\boldsymbol{x})g(\boldsymbol{x})\overline{\phi(\boldsymbol{x}-\boldsymbol{u})}^2\mathrm{e}^{i(x_1^2C\alpha+x_2^2C_\beta)}
\end{aligned}\tag{E-2}$$

Hence, we achieve

$$\begin{aligned}
&(G_\phi^{p_1,p_2}f(\boldsymbol{v})\overline{\star}G_\phi^{p_1,p_2}g(\boldsymbol{v}))\\
&=(A_{-\alpha}\ \ {}_{-\beta})^2\mathrm{e}^{-i(v_1^2C\alpha+v_2^2C_\beta)}(G_\phi^{p_1,p_2}f(\boldsymbol{v})\\
&\quad\cdot\mathrm{e}^{-i(v_1^2C\alpha+v_2^2C_\beta}\ \ ))*(G_\phi^{p_1,p_2}g(\boldsymbol{v})\mathrm{e}^{-i(v_1^2C\alpha+v_2^2C_\beta)})
\end{aligned}$$

The proof of Theorem 2 is achieved.

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