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A review on applications of holomorphic embedding methods

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ABSTRACT

The holomorphic embedding method (HEM) stands as a mathematical technique renowned for its favorable convergence properties when resolving algebraic systems involving complex variables. The key idea behind the HEM is to convert the task of solving complex algebraic equations into a series expansion involving one or multiple embedded complex variables. This transformation empowers the utilization of complex analysis tools to tackle the original problem effectively. Since the 2010s, the HEM has been applied to steady-state and dynamic problems in power systems and has shown superior convergence and robustness compared to traditional numerical methods. This paper provides a comprehensive review on the diverse applications of the HEM and its variants reported by the literature in the past decade. The paper discusses both the strengths and limitations of these HEMs and provides guidelines for practical applications. It also outlines the challenges and potential directions for future research in this field.

KEYWORDS

Holomorphic embedding method, power flow, polynomial solutions, nonlinear algebraic equations, differential equations.

Electric power systems are undergoing an architectural transformation towards an efficient, sustainable grid. A high-penetration renewable and inverter-based-resources (IBRs) dominated system will induce severe operational challenges for both steady-state and dynamics analysis. However, the deployment of information, communication, and computing advanced technologies evolves traditional power systems towards smart grids and could offer an opportunity to overcome these challenges through real-time reactions and online assessment. Power flow analysis, commonly referred to as load flow analysis, constitutes a fundamental cornerstone in power system studies. It plays a crucial role in the domains of power system planning, operation, and design, which is used to determine the steady-state operating conditions of a power system, providing information about the voltage magnitudes, phase angles, and power throughout the network. The predominant power flow computational methods primarily rely on iterative approaches, including Newton–Raphson (NR)[1], fast–decoupled (FD)[2] Newton, Gauss–Seidel (GS)[3], and various other NR-based variants. Similar to other iterative techniques, those approaches also encounter challenges like slow convergence rates or divergence, which continue to be important issues, moreover, the emergence of new control devices, IBRs, and heavy-load conditions could further exacerbate these issues, creating urgent needs to develop fast and robust power flow algorithms. Also, the increase in solar energy, wind power, energy storage resources, and upgrading infrastructures has driven the system toward the voltage stability boundary, and how to assess voltage stability online becomes a common concern. Traditionally continuation power flow (CPF)[4] is utilized to estimate the margin of voltage stability. By employing an iterative predictor-corrector scheme, the CPF method effectively retains the nonlinear characteristics of power flow analysis and addresses the issue of singularities by incrementally increasing the system load, gradually approaching the maximum loading point (nose point). Despite its success, CPF's primary drawback lies in its substantial computational complexity, making it less suitable for real-time online applications[5]. This requires a urgent powerful new algorithm to assess voltage stability online with enough robustness and enjoyable speed. Furthermore, dynamic performance is a critical aspect of power systems since it refers to the system's behavior and response during transient events —such as faults, switching operations, sudden load, or generation changes. These dynamics hold critical implications for evaluating system stability, voltage and frequency regulation, the integration of renewable energy sources, power quality, and the design of protection systems. For a complex power system, which is often modeled by differential algebraic equations (DAEs) or ordinary differential equations (ODEs), an integration scheme such as the Runge–Kutta method cooperating with NR or GS is utilized to solve the dynamics of the system. Compared with the steady-state problem, time-domain simulation is much more time-consuming and also suffers divergence issues especially in DAEs. To overcome those challenges induced by traditional numerical computational methods, the holomorphic embedding method (HEM) is proposed initially to solve steady-state problems such as power flow computations[6–9], voltage security assessments[10–12], transfer capacity analysis[13], contingency analysis[14,15], network reductions[16,17], control strategy design[18–20] and optimal power flow[21,22] due to satisfying convergence rate, efficient and robust properties, and non-iterative feature, and later is extended to accelerate dynamic simulations[23,24]. To further investigate the value of the HEM in power system analysis, it is very important to summarize present related research and then give a direction for future research. This paper will introduce various HEM algorithms, provide examples of applying the HEMs to power systems, and discuss critical issues in their application. In conducting the literature review for this study, we have considered publications in the past decade. This period is chosen to capture the most recent developments of HEM in the field. The databases utilized for the literature search include IEEE Xplore, ScienceDirect, Google Scholar, and the arXiv preprint server. These platforms are selected for their comprehensive coverage of scholarly articles and conference papers related to the specific topic under investigation. The search terms used included holomorphic embedding method, power flow, polynomial solutions, nonlinear algebraic equations, and differential equations. This literature review aims

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to present a comprehensive overview of relevant studies and developments within the defined timeframe and from reputable academic sources. The main contributions of this paper are listed below.

(1) This paper offers a comprehensive and structured overview of the HEM, covering fundamental concepts, theoretical foundations, convergence properties, and cutting-edge HEM techniques.

(2) Several key applications such as power flow analysis in various systems, voltage stability analysis, and dynamic simulations are selected to illustrate the overall procedure of applying the HEMs to the control and assessment of power systems.

(3) In-depth, this paper discusses the critical challenges, benefits, limitations, and future directions for applying the HEMs to power system problems.

The rest of the paper is organized as follows. Section 1 reviews the necessary algorithms and theoretical properties of the original HEM, and some basic implementations of the HEM for solving power flow, besides, two variants of the HEM are also introduced. Section 2 reviews the state-of-the-art applications utilizing the HEMs for steady-state problems such as power flow analysis in AC/DC systems and distribution systems. Furthermore, applications of the HEMs are also reviewed for voltage stability assessment and transient stability problems in Section 3. Some necessary clarifications and discussions of the HEMs are summarized in Section 4. Finally, the conclusion is drawn in Section 5.

1 Introduction of the HEM

1.1 Basics of the HEM using a non-physical germ

In this section, the essential steps for employing the HEM to address the power flow problem are comprehensively outlined. Let's consider an N-bus system comprising PQ, PV buses, and the swing bus, identified respectively as N_{p}, N_{v}, and N_{s}. The classical power flow equations could be expressed as

$$\begin{aligned}&\sum_{k=1}^{N} Y_{ik}V_k = S_i^*/V_i^*, \forall i\in N_{\mathrm{p}}\\&\begin{cases} P_i = \mathrm{Re}\left(V_i \sum_{k=1}^{N} Y_{ik}^* V_k^*\right), \forall i \in N_{\mathrm{v}}\\ |V|_i = |V_i^{\mathrm{sp}}|\end{cases}\\&V_i = V_i^{\mathrm{SL}}, \forall i\in N_{\mathrm{s}}\end{aligned} \tag{1}$$

where $Y=(Y_{ik})_{N\times N}$ denotes the admittance matrix, the injection power is $S_i = P_i + jQ_i$, $\forall i$, $S_i^* = P_i - jQ_i$ represents the conjugate of S_i, $V_i \in C$ is the voltage at bus i, $|V_i^{\mathrm{sp}}|$ denotes the specified voltage magnitude at a PV bus i and V_i^{SL} denotes the voltage magnitude at the slack bus i. In general, Eq. (1) is highly nonlinear, typically solved by iterative solvers like the NR method and its variants. The foundational concept of the HEM revolves around the idea of not directly solving the original system Eq. (1). Instead, a new embedded system is created by introducing an extra embedding variable $s\in C$. This construction builds an easy solution under specific circumstances (e.g., when $s=0$). The embedding system associated with Eq. (1) is represented in Ref. [7] as follows:

$$\begin{aligned}&\sum_{k=1}^{N} Y_{ik,\mathrm{trans}} V_k(s) = \frac{sS_i^*}{V_i^*(s^*)} - sY_{i,\mathrm{shunt}}V_i(s), \forall i\in N_{\mathrm{p}}\\&V_i(s)\times V_i^*(s) = 1 + s\left(|V_{i,\mathrm{sp}}|^2 - 1\right), \forall i \in N_{\mathrm{v}}\\&\sum_{k=1}^{N} Y_{ik,\mathrm{trans}} V_k(s) = \frac{sP_i - jQ_i(s)}{V_i^*(s^*)} - sY_{i,\mathrm{shunt}}V_i(s), \forall i\in N_{\mathrm{v}}\\&V_i(s) = 1 + s(V_i^{\mathrm{SL}} - 1), \forall i \in N_{\mathrm{s}}\end{aligned} \tag{2}$$

where s denotes the embedded complex variable, $Y_{ik,\mathrm{trans}}$ and $Y_{i,\mathrm{shunt}}$ represent the series-branch and shunt components of the admittance matrix Y, respectively. Additionally, Q_i stands for reactive power. It is important to highlight that as an effective embedded system, Eq. (2) must satisfy two essential conditions. First, at the reference state $s=0$, the embedded system Eq. (2) should have at least one straightforward trivial solution(e.g., zero-current injection solution: $V_i = 0, \forall i \in N$ and $Q_i = 0, \forall i\in N_{\mathrm{p}}$, which is also called germ[6]). Second, at the target state $s=1$, the solution of Eq. (2) should correspond to the desired solution—namely, the solution of the original system Eq. (1). It is noteworthy that, to uphold the $V(s)$ holomorphic, its conjugate V^* is distinctly defined as $V^*(s^*)$, rather than $V^*(s)$. This distinction arises from the fact that the former adheres to the Cauchy–Riemann equations, while the latter does not, as explained comprehensively in Ref. [25] within the context of a conventional HEM-based load flow calculation.

The next step will be representing solutions $V_i(s)$ in a power series form. The definition of holomorphic property is given below.

Theorem (holomorphic property[9]) *A complex function V is holomorphic at a point $s\in\Omega\subseteq C$ if the following limit exists,*

$$V'(s) = \lim_{h\in C, h\to 0}[V(s+h) - V(s)]/h,$$

where $h\in C$ with $h\neq 0, s+h\in\Omega$.

In particular, a complex function f is said to be holomorphic in Ω if it is holomorphic at every point in Ω. An embedded system is called holomorphic if the solution functions to the embedded system are holomorphic in s. Benefiting from the holomorphic complex analytic property, the actual solution (at $s=1$) can be effectively attained as a power series centered around the reference easy solution. Suppose $V_i(s)$ is holomorphic for all i in a neighborhood of $s=0$, then there is a converged power series expansion centered at $s=0$ as well as $V^*(s^*)$[26], i.e.,

$$\begin{aligned}V_i(s) &= \sum_{q=0}^{\infty} V_i[q]s^q,\\ V_i^*(s^*) &= \sum_{q=0}^{\infty} V_i^*[q]s^q,\end{aligned} \tag{3}$$

where $V_i[q]$ denotes the q-th order coefficient of V_i, practically, the computation of $V_i[q]$ up to a specified order K requires a rearrangement of the solution of Eq. (1) from $\boldsymbol{V}$ to $[\boldsymbol{V},\boldsymbol{W}]^\top$, where $W_i(s) = 1/V_i(s)$. Then, both $V_i[q]$ and $W_i[q]$ can be solved in a recursive manner, from 0 to the fixed order K. The procedure can be expressed as

$$\boldsymbol{A}\boldsymbol{x}[q+1] = \boldsymbol{b}[q], q = 0, \ldots, K, \tag{4}$$

where $\boldsymbol{x} = [\boldsymbol{V},\boldsymbol{W}]^\top$ is the vector composed of coefficients of solutions with order q and $\boldsymbol{A}$ is the transfer matrix, and $\boldsymbol{b}[q]$ is related with order q. For the sake of simplicity, the detailed construction process of $\boldsymbol{A}$ and $\boldsymbol{b}[q]$ is not presented here. Comprehensive and systematic explanations of this construction can be found in references such as [7, 9, 27]. Note that for a model different from Eq. (1), $\boldsymbol{A}$ and $\boldsymbol{b}[q]$ are different. Upon calculating $V_i[q]$ up to a specific order K, the solution is approximated as $V_i(s)\approx\sum_{q=0}^{K} V_i[q]s^q$. Considering K is always finite in practice, the truncation error is inevitable, and the target variable $s=1$ might not always fall within the convergence domain. Consequently, the next step involves employing rational approximants, such as the Padé approximant[7,27], to achieve enhanced approximations within, and sometimes even beyond, the original convergence radius. For

$V_i(s)$ in the form of Eq. (3), a corresponding Padé approximant is an algebraic fraction where both the denominator and the numerator are polynomials, i.e.,

$$[m/n]_i(s) = \frac{a_{k0} + a_{k1}s + \cdots + a_{im}s^m}{b_{i0} + b_{i1}s + \cdots + b_{in}(s)^n}, \tag{5}$$

where the integers m and n represent the degrees of the numerator and denominator polynomials, respectively. Coefficients a_{kq} and b_{kq} are candidate coefficients, which can be computed using Viskovatov's method[28] or by solving a linear equation formulated based on Eq. (3)[9, 29]. Through a thoughtful selection of m and n, it becomes possible to enhance the truncation error's order compared to the original series Eq. (3):

$$\begin{aligned} &\sum_{q=0}^{\infty} V_i[q]s^q - \sum_{q=0}^{K} V_i[q]s^q = O(s^{K+1}), \\ &\sum_{q=0}^{\infty} V_i[q]s^q - [m/n]_k(s) = O(s^{m+n+1}). \end{aligned} \tag{6}$$

It is worthwhile to mention that the sequence of near-to-diagonal Padé approximants converges to the maximal analytic continuation of the corresponding power series Eq. (3), its value is guaranteed to be the high-voltage solution for the aforementioned embedding system Eq. (2), for $s=1$ within the function's extremal domain, under the condition that Stahl's theorem[30, 31] are satisfied. This statement is mostly used in the literature. It will be explained later this statement is correct theoretically, but it does not mean divergence can not happen theoretically in the whole function domain. This requires a better understanding of the original statement. The final step will be the evaluation of the result. With $V_i(s) \approx [m/n]_i(s)$ for all i in mind, the mismatch of Eq. (1) can be calculated using $V_i(s)$ and a termination criterion can be set as ε, i.e., when the norm of mismatch is less than ε, the power flow solution is obtained, otherwise, recalculate $V_i(s)$ by a higher order approximation with more power series and recalculate Padé approximation until the criterion is satisfied.

1.2 Fast and flexible holomorphic embedding method

Most of the previously mentioned steps build upon the foundational HEM for the classical power flow model from Refs. [6, 32]. Some papers focus on the exploration and proposition of new fundamental HEMs, distinct from Refs. [6, 32]. In this section and the next section, two comprehensive variants of the original HEM: the fast and flexible holomorphic embedding method (FFHEM)[9], and the multi-dimensional holomorphic embedding method (MDHEM)[8] would be introduced. A notable characteristic of FFHEM is that the proposed new embedded systems don't necessarily require a flat start or a zero-current injection point but any point except 0 as the reference germ solution at $s=0$. With original embedded systems (2) in mind, new systems proposed by Ref. [9] are

$$\begin{aligned} &V_i^*(s^*)\textstyle\sum_k Y_{ik}V_k(s) = c_i^*\textstyle\sum_k Y_{ik}c_k + s\left(S_i^* - c_i^*\textstyle\sum_k Y_{ik}c_k\right), \\ &V_i(s)V_i^*(s^*) = c_ic_i^* + s\left(|V_{i,\mathrm{sp}}|^2 - c_i\cdot c_i^*\right), \\ &V_i^*(s^*)\textstyle\sum_k Y_{ik}V_k(s) + V_i(s)\textstyle\sum_k Y_{ik}^*V_k^*(s^*) = c_i^*\textstyle\sum_k Y_{ik}c_k \\ &+c_i\Sigma_k Y_{ik}^*c_k^* + s\left[2P_i - \left(c_i^*\textstyle\sum_k Y_{ik}c_k + c_i\textstyle\sum_k Y_{ik}^*c_k^*\right)\right], \end{aligned} \tag{7}$$

where Σ_k denotes $\sum_{k=1}^{N}$, $c_i \in C\backslash\{0\}$ is a constant. Notice that the first equation of Eq. (7) corresponds to PQ buses, and the last two correspond to PV buses, with the reference state at $s=0$ and the target state at $s=1$. One can check two conditions mentioned before in Section 1 are satisfied, i.e., $V_i(0)=c_i$ is a solution for Eq. (7) at $s=0$, and specifically, the embedded equation of slack bus is $V_i(s) = V_i^{\mathrm{SL}}, \forall i \in N_s$, for all $s \in C$. The germ solution of Eq. (7), $V_i(0) = c_i$ (note $c_i = V_i^{\mathrm{SL}}, \forall i \in N_s$), is compatible to any solution except 0. This aspect offers the advantage of designing a specific initial starting point, based on the solution of Eq. (1). Examples include warm starts, cold starts, solutions derived from a DC power flow, and an outcome from traditional power flow solvers like NR, FD, or any intelligent start at $s=0$. Results in Refs. [7, 33] have demonstrated that, while constrained by the fixed initial point in Eq. (2), more than 100 terms in the expansion Eq. (3) are often necessary to calculate a desired solution using rational approximants for certain cases, even with $\varepsilon = 1.0\times10^{-10}$. While with the incorporation of iterative methods, an approximately 20-fold acceleration can be achieved when compared to the original HEM, without compromising accuracy[9]. This enhancement places the overall performance of FFHEM on par with NR and FD methods. Moreover, Ref. [9] introduces the GS-assisted FFHEM and the FD-assisted FFHEM.

Another innovation of FFHEM is its algorithm for Padé approximants calculation. The rational approximant is defined as

$$[m/n](s) = \det\left(P^{[m/n]}(s)\right)/\det\left(Q^{[m/n]}(s)\right),$$

where "det" denotes the determinant of a matrix, and matrices $P^{[m/n]}(s)$ and $Q^{[m/n]}(s)$ are decided by the power series terms $V_i[q]$ in Eq. (3) This approximation can be further evaluated at $s=1$ as shown in Ref. [9]:

$$[m/n](1) = \det\left(\Psi(P^{[m/n]})(1)\right)/\det\left(\Psi(Q^{[m/n]})(1)\right), \tag{8}$$

where Ψ represents a full-rank pre-conditioner linear map. Mostly detailed closed analytical form of Eq. (5) $[m/n](s)$ is not interested, only $[m/n](1)$ is required, Ref. [9] directly computes a ratio of matrix determinants without calculating all coefficients a_{kq} and b_{kq} of the Padé approximant. Results from Ref. [9] demonstrated that this proposed approximation algorithm offers faster performance compared to Eq. (5).

There are several following papers under this FFHEM framework, specifically, theoretical properties such as the holomorphic and conjugate properties[5], the algebraic property[33] are investigated based on the frame of FFHEM, also, some applications research such as modeling of static synchronous series compensator (SSSC) and interline power flow controller (IPFC) in power flow[34], FFHHEM-based energy flow in integrated system[35] and large-scale power system voltage stability[12] also utilize FFHEM instead of the original HEM. The numerical convergence of FFHEM is also compared with NR and GS methods in Ref. [36], and a practical domain of reference state solution c_k is proposed to achieve a better convergence.

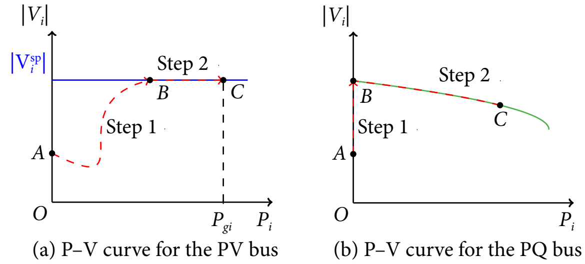
1.3 Multi-dimensional holomorphic embedding method using a physical germ

In Ref. [8], a physical germ solution is first defined. The procedure for finding the physical germ solution is shown in Figure 1. The concept of a physical germ solution introduces an original reference point within the solution space, representing an operational state with a physical meaning (different from the virtual germ used in the HEM). In essence, any power flow solution can be conceptualized as a physical germ solution. Specifically, in MDHEM, the physical germ solution corresponds to an operating state where each load bus (PQ bus) carries neither load nor generation, and each generator bus (PV bus) maintains a specified active power output, with the reactive power output adjusted to regulate the voltage magnitude to a predetermined value, i.e., point B in Figures

1(a) and 1(b). While alternative germ solutions are acceptable, this particular physical germ solution offers two advantages. First, it characterizes a scenario devoid of any load at PQ buses. Second, the embedded variables s can depict physical loading scales, ranging from zero and expansible throughout the entire solution space. To systematically solve the physical germ solution, i.e., point B in Figure 1, a two-stage algorithm is proposed in Ref. [8]. The first stage is to calculate point A for PV and PQ buses, assuming zero power injects from all PQ and PV buses and initial values of all bus voltages can be determined, using Eqs. (15)–(16) in Ref. [8]. The second stage assumes the injected reactive power as a function of s, denoted as $Q_{gi}(s)$, for each PV bus $i \in N_v$. Then this stage enables the user to manipulate the voltage magnitude, directing it from the initial voltage towards the desired voltage, which involves solving Eqs. (12)–(14) in Ref. [8]. The process is illustrated by finding point B in Figure 1. Simultaneously, the active power of each PV bus is regulated to a specified value, which is denoted as P_{gi} in Figure 1(a). The newly proposed HEM initiates by identifying the proposed physical germ solution (point B), which can serve as the initial guess in the solution space, in contrast to a non-physical solution as in Eq. (2). It allows the HEM to extend from the physical germ solution while attributing physical meanings to the embedded variables s, such as controlling the loading levels of load buses through loading scales. This special feature enhances the robustness of the HEM as any solution at $s \neq 1$ is meaningful and could cooperate with other methods such as multi-stage strategy[37]. Also, the original HEM embeds one dimension s into the power flow equations, making the solution a single-dimensional power series. One drawback is that all loads in the system are uniformly scaled at the same rate, without different ratios. As a result, it cannot cover all operating conditions. To address this limitation, Ref. [38] considers a bi-variate HEM, and Ref. [8] finally generalizes the HEM as a multi-dimension approach, expanding V_i as a multi-variable function as

$$\begin{aligned} &V_i(s_1, s_2, \cdots, s_j, \cdots, s_D) \\ &= \sum_{n_D=0}^{\infty} \cdots \sum_{n_j=0}^{\infty} \cdots \sum_{n_1=0}^{\infty} V[n_1, \cdots, n_j, \cdots, n_D] \times s_1^{n_1} \cdots s_j^{n_j} \cdots s_D^{n_D}, \\ &= V_i[\underbrace{0, 0, \cdots, 0}_{D}] + V_i[1, 0, \cdots, 0]s_1 + V_i[0, 1, \cdots, 0] \\ &\quad \times s_2 + \cdots + V_i[2, 0, \cdots, 0]s_1^2 + V_i[1, 1, \cdots, 0] \\ &\quad \times s_1 s_2 + V_i[0, 2 \cdots, 0]s_2^2 + \cdots \end{aligned}$$

where $s_1, ..., s_D$ are embedded variables similar to s in the original HEM, D is the dimension, and each s_j governs the scale level of the injected active or reactive power, independently controlling either individual loads or groups of loads. MDHEM tries to track solutions of embedded systems along a path that has physical meanings, starting from a point exactly at the PV curve instead of the germ solution of the original HEM, this is similar to Ref. [10]. The corresponding properties such as scale invariance, holomorphy, and reflection condition are verified, and the recursive equations involving multi-dimensional discrete convolutions are derived, followed by the application of a multi-Padé approximation[8]. The result has shown that MDHEM can screen power flow solutions faster than the NR method(1.04 s vs 24.1 s, Case 1, 4-bus system). Several papers have adopted the MDHEM instead of the original HEM. For instance, Liu et al. consider a novel analytical probabilistic power flow approach employing MDHEM to construct an explicit, analytical power flow solution in the form of a multivariate power series, allowing specific power injections to be modeled as random inputs[39]. Ref. [40] introduces a multi-variable V–Q sensitivity method for voltage stability based on MDHEM, effectively avoiding repetitive power flow calculations. Also, a new interval power flow model is proposed based on MDHEM, in which embedded variables represent {renewable energy uncertain parameters} in different zones[41]. Similarly, they derive a multi-variable analytical calculation method for energy flow in natural gas systems using MDHEM[42]. An MDHEM-based power flow approach for active distribution networks is also proposed, which is well-suited for multi-scenario analysis[43]. Moreover, MDHEM is applied to the power flow analysis of HVDC systems, dividing the system into typical regions, where each embedding variable characterizes the uncertain generation/load of the respective region[13]. Lastly, an analytical expression of the static voltage stability boundary using MDHEM is derived in Ref. [44]. In summary, the Table 1 compares different HEMs from different perspectives. Note that here multi-stage HEM method (MSHEM) will be explained later.



(a) P–V curve for the PV bus (b) P–V curve for the PQ bus

Fig. 1 The procedure for finding the physical germ solution.

2 Applications for power flow analysis

2.1 Motivation

The HEM was first proposed to solve power flow by Trias in 2012[6], in which only pure PQ buses were considered. The motivation is that traditional iterative methods such as GS, NR, and FD struggle to ensure convergence, particularly when confronting heavy load conditions. Even convergence is achieved, for a complex-variable problem, numerous potential solutions exist, and it is impossible to control to which the method will converge. The presence of Newton fractal has already been found in power flow computations, which means that iterative methods that

Table 1 Summary of holomorphic embedding methods

Method	Germ solution	Number of dimensions	Special features	Test systems	References
HEM	Virtual germ	1 (s)	Free of iterations, strong convergence	3120-bus system	Refs. [7, 25]
FFHEM	Any non-zero point as reference germ	1 (s)	No flat start required, numerical methods-assisted, novel PAs computations.	70k-bus system	Refs. [5, 9, 12, 33, 35, 36]
MDHEM	Physical germ (point B)	D (s_D)	A general form of the HEM, multi-operation conditions	2383-bus system	Refs. [8, 10, 13, 39–41, 43, 44]
MSHEM	Physical germ (point B)	1 (s)	Multi-stage calculations, high precision	2383-bus system	Refs. [24, 37, 45]

attempt to locate a particular solution by initializing guesses, might ultimately converge towards entirely different solutions[6, 46]. Those issues can be partially fixed by selecting a good initial condition, however, designing a systematic strategy for selecting suitable initial guesses for arbitrary load scenarios remains challenging. Some efforts have been made to improve the quality of initial guess[46, 47], but success is not guaranteed. As mentioned by Ref. [6], different from iterative methods, the HEM possesses the ability to produce the correct solution when it exists, establishing a deterministic and robust way to solve physical power flows. This distinctive characteristic not only facilitates reliable, real-time, intelligent applications but also offers fresh perspectives for power flow computations.

2.2 Considering voltage magnitude constraints

Different from the method proposed in Ref. [6] which only deals with PQ buses, several papers have explored the derivation of embedding equations of voltage magnitude constraints, i.e., PV buses. It is important to acknowledge that these embedding equations are not unique. Among the various formulations, one such representation is Eq. (2). In Ref. [48], a novel approach is introduced to embed the AC power flow problem with voltage magnitude constraints, i.e., PV buses. This is achieved through the introduction of additional analytic functions $\bar{V}_i(s)$ and $\bar{S}_i(s)$ that are independent of $V_i(s)$ and $S_i(s)$, respectively. Additional embedded equations are incorporated from Eqs. 18(a)–18(d) in Ref. [48] to ensure that $V_i(s)$ possesses the reference solution V_r at $s=0$ and enforce $\bar{V}_i(s)=(V_i(s^*))^*$ at $s=1$. Similarly, $\bar{S}_i(s)\neq(S_i(s^*))^*$ for all $i\in N_v$ except at $s=1$. Note that another type embedded forms demonstrated in Eq. (2), which is frequently employed embedded systems for PV buses in the literature, are derived in Refs. [7, 32]. Other alternative embedded forms for PV buses are also proposed, such as those presented in Refs. [49–51]. In Ref. [51], a new model of PV buses is proposes to address accuracy issues arising from Ref. [49]. In Ref. [50], authors propose two distinct approaches for modeling the voltage control of PV buses and also mention that the method proposed in Ref. [49] does not satisfactorily constrain voltage magnitude, implying that the obtained results do not conform to either voltage magnitude or active power constraints.

2.3 Modeling of FACTS devices

Except the focus on PV bus modeling, recognizing the importance of integrating flexible ac transmission system (FACTS) devices into power system planning, control, and protection, the formulation of embedded equations for these devices becomes crucial. Recently several Refs. [34, 52–54] have contributed to novel forms of embedded systems considering FACTS. In Ref. [52], several thyristor-based FACTS controllers, e.g., static var compensator (SVC), thyristor controlled switched capacitor (TCSC), thyristor controlled voltage regulator (TCVR), and thyristor controlled phase angle regulator (TCPAR) are integrated into the AC power flow problem. The corresponding embedded systems are formulated in a way that the overall solution functions are holomorphic. Furthermore, the embedded formulations of VSC-based FACTS controller, i.e., static synchronous compensator (STATCOM) have been developed in Ref. [34]. Notably, an embedded system including the SSSC and IPFC, guaranteeing convergence, is introduced in Ref. [53] follow fundamental HEM principles. Later, the embedded systems of STATCOM and unified power flow controller (UPFC) are modeled based on the frame of FFHEM in Ref. [54], the case study shows that the proposed embedded system requires less execution time and error is reduced compared with the original HEM[32].

2.4 HEMs for hybrid AC/DC systems

Various HEM-based power flow models introduced before mainly focusing on AC transmission networks. There are many papers extending the HEM to other types of power systems such as DC systems[33], AC/DC systems[55–59], three-phase distribution networks[60–64], integrated systems[35, 42, 65, 66] and isolated systems[58, 67]. It is important to note that unlike the AC power flow, in which one has to take particular attention to deal with the complex conjugation operation by considering additional independent variables $V_i^*(s^*)$, all voltages variables involved in DC systems[33] are real instead of complex, so $V_i(s)=V_i^*(s^*)$ is required and satisfied, ensuring that physical solutions are real when s is real. Nonetheless, it is crucial that the embedding variable s remains complex. This complexity is essential because it is only within the complex plane that the complete structure of the solution branches can be maintained and Stahl's theorem[30] is guaranteed to extend the convergence radius utilized in the numerical partial sum of a power series[33].

Different from pure AC and DC systems, a typical hybrid AC/DC system often consists of one or more interconnected AC systems along with other DC systems, connected by interlinking converters. These converters can be in the form of back-to-back converters or voltage source converters (VSCs). Furthermore, these hybrid AC/DC systems can be operated in both grid-connected mode[55–57], and islanded mode[58,59]. All those features increase complexity to design embedded systems for AC/DC systems. Ref. [55] introduces a HEM-based power flow model for AC/DC systems, employing separate AC-HEM and DC-HEM algorithms. In Ref. [56], a systematic HEM-based power flow model is developed for the line-commutated converter based high voltage direct current (LCC-HVDC) system, including various control modes. This study also proposes a computational strategy for efficient Padé approximants using Bauer's Eta algorithm. Additionally, Ref. [57] concentrates on the VSC-HVDC system, leveraging pre-stored sparse recursive matrices to optimize computation time and memory usage. The incorporation of iterative inverse matrix calculation into Padé approximants is proposed to further enhance computation speed.

2.5 HEMs for distribution systems

For distribution systems, there are several key features, first the distribution system is characterized by radial or weakly meshed structures[62], second there are many unbalanced three-phase operations[61]. Also, the accurate modeling of ZIP loads is necessary which will introduce more nonlinearity to the system[63]. Furthermore, the high R/X ratio in distribution systems can lead to slow or non-convergence of the NR method because it may encounter difficulties in finding the appropriate initial guesses for voltage magnitudes and angles. So it is necessary to better utilize and develop HEM-based algorithms in distribution systems to achieve a better convergence rate. Numerous contributions exist in this area, Ref. [61] address power flow with the HEM in radial distribution networks under unbalanced conditions, assuming constant-power and voltage-independent constant-current wye-connected loads, a similar load model is also employed in Ref. [64]. And Ref. [60] examines HEM's performance in distribution networks with voltage-dependent ZIP loads. However, it is highlighted in Ref. [63] that the holomorphy assumption for solution functions might not be met for that derived embedded system in Ref. [60]. To address this issue, Ref. [63] considers an embedded system satisfying the holomorphy assumption for three-phase distribution networks, considering comprehensive voltage-dependent ZIP load models for both wye and delta connections. Case studies find significant errors on previous HEM-based power flow models if voltage-independent constant-current loads are assumed[61, 64].

2.6 HEMs for microgrids and integrated systems

Besides, HEM-based power flow formulations of isolated AC microgrids with hierarchical control and isolated AC/DC systems with droop controls are investigated in Ref. [67] and Refs. [58, 59], respectively. It should be noticed that generally the frequency of the AC network cannot be assumed constant in islanded microgrids, so the assumption that Y matrix is constant becomes invalid. In Refs. [58, 59], Y_{bus} matrix is expressed as an embedded function of the complex embedding variable s to handle this issue, and the secondary control scheme is considered in Ref. [67], thus the actual angular frequency matches the nominal value, Y_{bus} matrix can be assumed constant.

As a natural extension, NR and GS can be replaced by the HEM in the calculation of the power flow of the integrated system, which is first applied in electricity–gas–heat energy systems[65, 66], and later the HEM formulations of gas flow of the gas system are also investigated in Refs. [35, 42], the heat flow is considered in Ref. [68].

3 Applications for stability and dynamic problems

3.1 Voltage stability assessment

Capitalizing on the HEM's desirable convergence properties, its applications naturally extend to static voltage stability analysis, which could be performed by computing the power flow solution (i.e., the PV curve). A traditional approach CPF[4, 69] has been proposed to estimate stability limits and load margins, which is an NR-based predictor-corrector method. However, it is important to note that the computational complexity of CPF is significantly greater than a simple power flow solved by NR. This is due to the necessity of solving a new power flow solution at each step while traversing the PV curve towards the voltage collapse point[4]. It is worth noting that traditional HEM (2) cannot benefit the performance of CPF if it is applied iteratively, similar to the NR method[10]. To see this point, one can observe that the above formulation of traditional HEM (2) represents the power system solely at $s = 1$ without direct physical interpretation at other s values, because $Y_{i,\text{shunt}}$ is scaled by s along the path, from $s = 0$ to $s = 1$, the resultant solution represents a network with shunt components differing from the original network. To enhance flexibility and robustness, it would be advantageous if the path from $s = 0$ to $s = 1$, as well as the variable s itself, could hold physical meanings, such as representing load scaling levels or variations. By incorporating these meaningful interpretations, the PV curve could be efficiently characterized using a single-stage HEM, significantly reducing complexity compared to CPF.

The modified version of the HEM aimed at achieving this objective is initially introduced in Ref. [10], four methods for evaluating saddle-node bifurcation point (SNBP) (Padé approximant search, power flow search, roots method, and extrapolating sigma method) are proposed and compared in terms of both accuracy and efficiency. Conditions such as uniformly scaled loads at different buses, loads scaled by varying amounts at different buses, the inclusion of var limits, and the application of polynomial ZIP loads are all considered and discussed. One of the biggest advantages of the Padé approximant search method and the roots method is that the power flow problem needs to be solved just once, and then only the Padé approximants have to be evaluated at different values of s. Note that the proximity of the system to its nose point is determined by assessing the smallest real zero or the smallest real pole of the Padé approximant in the roots method which has a tighter estimation of the nose point compared with other methods.

Similarly, Ref. [11] considers a two-step physical germ solution for online voltage stability, after locating point B shown in Figure 1, the variable s is embedded to represent a meaningful loading scale in the power flow, enabling the analytic trajectories to represent varying operating conditions, such as the trajectories from point B to point C. Notably, the second-stage HEM formulations proposed in Refs. [10, 11] share similarities, differing mainly in the use of the Padé approximant. An adaptive two-stage Padé approximant is proposed in Ref. [11], capable of providing optimal Padé approximant orders for individual PQ buses. Case studies demonstrate the algorithm's effectiveness and accuracy in comparison to the normal Padé approximant algorithm[10]. However, as noted by Ref. [37], the use of the Padé matrix in the HEM increases the condition number when the system approaches the nose point, consequently slowing down the HEM's convergence rate and introducing significant errors due to limited computational precision. Addressing this, a multi-stage HEM method MSHEM is proposed in Ref. [37] based on Ref. [11]. MSHEM employs a predictor-corrector scheme to eliminate errors without resorting to the Padé approximants. The results highlight that MSHEM maintains the non-iterative advantage compared to CPF[69] and, notably, enhances precision compared to the conventional HEM[11]. It is important to note that MSHEM builds upon the physical germ-inspired HEM approach, which allows it to initiate from a physical germ and extend solutions along the PV curve.

There are some other methods to detect SNBP and weak buses such as V–Q curve[70] and HEM-based sigma method[71–75]. In patent[71], the sigma method is introduced to identify weak buses and estimate the distance from the operational state to the SNBP based on the HEM but later in Ref. [70], authors demonstrate that the voltage criticality quantification in Ref. [71] lacked reliability. A slight modification is applied to the sigma method to estimate SNBP, and V–Q sensitivity functions for voltage stability analysis are proposed based on the formulation of the HEM in Ref. [10]. Additionally, in Ref. [76]. A HEM-based sensitivity analysis approach is introduced to mitigate ill-conditioned Jacobian matrices in the traditional sensitivity analysis.

However, current HEM-based methods for voltage stability analysis have limitations in their scope. Most of them embed only a single variable that uniformly controls generations or loads such as Refs. [11, 37, 71, 72]. This approach proves inadequate when loads or generations change in varying proportions. Repetitive power flow calculations become essential but time-consuming for voltage stability analysis involving multiple operating conditions with diverse load and generation scales. To address those issues, Ref. [40] proposes a multi-variable V–Q sensitivity method based on MDHEM to quickly analyze voltage stability across multiple operating conditions which aims to identify weak buses and the SNBP. Another limitation is that there is no HEM except[12] that could trace the whole PV curve containing low voltage solutions as the rational approximation will begin oscillations when the solution is close to the nose point, i.e., SNBP. In Ref. [12], a new method called HEAP is introduced, which utilizes an arc-length parametrization to represent a nonlinear system for tracking the PV curve. This approach combines the advantages of CPFLOW[69] (passing through the nose without numerical difficulties) with the robustness enhancement provided by piecewise approximants, inspired by the flexibility seen in FFHEM. Results have shown a speedup of at least 19 times for systems with 10, 000 or more buses compared to CPF.

3.2 Time-domain simulations

In previous sections, the embedded variable s is commonly complex when solving AC power flow problems or represents the load scaling level when dealing with static voltage stability. In dynamic problems, where the system is represented by ODEs or DAEs, the embedded variable s naturally means the time variable. Replacing s with variable t as the embedding variable inherently produces a solution that illustrates temporal state evolution.

With the above consideration in mind, the HEM is first applied to voltage stability assessment involving dynamics in Ref. [23] where load factor increases linearly over time, s inherently means time evolution. Different from previous static voltage stability problems, embedded formulations of induction motor models are first proposed, which enrich the capability of the HEM in power system dynamic analysis. The authors introduce three distinct modules of the HEM, to address specific time scales of power systems: steady-state power flow, quasi-steady-state (QSS) under load escalation, and dynamic state under increased loading conditions. Participation factors-based algorithm is investigated allowing for the realization of partial-QSS simulation and full-dynamic simulation methods based on the HEM. Note that in the voltage stability assessment, the system is free from any angle stability issues. As a result, the dynamic models of synchronous generators are excluded from consideration and are represented as constant-voltage sources connected to PV or slack buses in Ref. [23].

In a subsequent Ref. [24], the HEM is modified and applied to time-domain simulations. The study outlines guidelines for developing and solving HE formulations of DAEs, including synchronous generators and controllers in power systems. The work considers coordinate transformations and proposes embedded formulations for resolving system states post-fault. The multi-stage HEM scheme is introduced to link solutions, ensuring computational accuracy and error control. Results verify HEM's potential for time-domain simulations, showcasing larger effective time steps than traditional numerical integration methods. With its non-iterative semi-analytical property for solving algebraic equations, the HEM exhibits advantages in computational efficiency and robustness. This property is quite helpful in solving DAEs since numerical divergence or wrong convergence could happen during grounding faults, especially implementing NR methods for large-scale systems. Furthermore, in Ref. [45], the HEM is applied to develop an extended-term simulation framework. The paper presents embedded formulations for solving common atomic events and introduces a hybrid simulation approach that combines dynamic and QSS simulations under the HEM frame. This approach enhances simulation across multiple timescales and ensures accuracy. The efficient and practical model switching between dynamic and QSS simulations, guided by HEM coefficients, is highlighted. And the continuity of HEM solutions in the time domain allows it to effectively handle various events in extended-term simulations. Test cases show the proposed hybrid event-driven extended-term simulation accurate and efficient, proving its suitability for simulating complex power systems events like restoration and cascading outages.

Efficiency in dynamic simulations is crucial, however, the computation of Padé approximants for large-scale power systems can be time-consuming. In Ref. [77], authors present a rapid, vectorized algorithm for calculating Padé approximants in HEM-based dynamic simulations. They utilize the Levinson algorithm to reduce temporal and spatial complexities. Considering computational consistency, they develop a vectorized version of the Levinson algorithm to exploit instruction-level parallelism. Evaluation of the Polish 2383-bus system confirms accelerated computation speed due to the advantageous combination of the Levinson algorithm and vectorization.

It is noteworthy that the primary focus of the HEM-based power flow model is to identify stable equilibrium points (SEPs) in steady-state power flow analysis. In contrast, traditional numerical methods struggle to reliably locate the closest, or controlling, unstable equilibrium points (UEPs). HEM's applications in finding UEPs begin with a two-bus system, offering theoretical guarantees for reaching the UEP if it exists[78]. Subsequently, the HEM is extended to tackle multiple UEPs in a multi-generator power system, demonstrating its effectiveness in rapidly identifying type-1 UEPs in a 3-generator setup[79] by adjusting the initial germs.

4 Discussions

While the applications discussed in this paper showcase promising outcomes, it is important to acknowledge that the HEMs also have limitations. Several concerns and misunderstandings regarding solution existence, uniqueness, holomorphism of solution functions, and convergence properties have been raised. This section suggests some guidelines for understanding the HEMs and Padé approximants, and addresses some misconceptions in the literature.

4.1 Existence, uniqueness, and holomorphism of solutions

The holomorphic property of solutions in Eq. (2) significantly influences the effectiveness of the power series expansion Eq. (3) and the convergence of derived rational approximants Eq. (5) within the HEM framework. It is crucial to note that the existence, uniqueness, and holomorphism of solutions in Eq. (2) are not straightforward, particularly when dealing with non-polynomial systems involving elements like ZIP loads in distribution networks[63], where the unknown complex functions $(V(s), V^*(s^*))$ in the embedded system are not algebraic in general. One must carefully design the embedded system among infinite choices. To ensure existence and uniqueness of holomorphic solutions $(V(s), V^*(s^*))$, validation of sufficient conditions presented in Ref. [5, *Theorem 2*] is advised. In particular, the existence of function solutions is guaranteed by the implicit function theorem. So the initial point is very crucial for both polynomials and non-polynomial systems when the flat initial state is not required, e.g., FFHEM, the algebraic vector system should be non-singular and holomorphic around the initial state. Also, the reason for selecting real values for s at the reference (germ) $s_0 = 0$ and target state $s_1 = 1$ holds significance. This choice emerges from the fact that $s_0 = s_0^*$ if $s_0 \in R$ and $V^*(s^*)$ is holomorphic at $s = s_0$ if and only if $V(s)$ is holomorphic at $s = s_0$. Similar property relations are valid for $s = s_1$, which is a critical consideration when designing embedded systems to ensure the holomorphic and conjugate properties of solution functions.

The HEM approach benefits from the remarkable convergence properties of algebraic functions in a polynomial system, notably the analytic continuation of solution functions across a trajectory extending from the reference state to almost any point within the complex plane. However, several influencing papers[6, 32, 33] have previously assumed the algebraic property of solution functions, often taking for granted the existence of a bivariate annihilating polynomial for all solution functions within a polynomial embedded system. Nevertheless, recent work[33] has underscored that these conclusions are not rigorously proven, but rather empirical observations. A counterexample is provided, while a set of sufficient conditions are established to ensure the algebraic property of solution functions through the existence of bivariate annihilating polynomials when employing a general polynomial embedded system within the HEM framework.

It is worth noting that within a general embedded system, certain solutions may not hold to the reflection condition[6, 32, 48], so they are not physical solutions to the power flow problem, named ghost solutions by Ref. [6]. In the literature, ensuring the physical validity of solutions $V(s)$ is typically achieved by directly incorporating the reflection condition into the general embedded system, i.e., directly employing $V^*(s^*)$ in the general embedded system, resulting in the well-known formulation (2). To complete the discussion of the conjugate property, a theorem and a set of sufficient conditions in terms of the structure of the general embedded systems are developed in Ref. [5] to guarantee the satisfaction of the reflection condition throughout the trajectory from $s=0$ to $s=1$, rather than requiring its forcing in advance.

4.2 Theoretical convergence v.s. numerical convergence

The HEM is highlighted by a claim of universal convergence guarantee, as stated in Ref. [6] and supported by Stahl's theorems[30]. Notice that a holomorphic function solution can be locally represented by a convergent power series around a reference state, such as $s_0=0$. So the Pade approximants are utilized to extend the convergence domain[6]. Theoretically, the statement in Ref. [6]: "*any close-to-diagonal sequence of Pade approximants converges in capacity to said function in the extremal domain*" holds theoretically and has been cited as the evidence for the universal convergence of the HEMs across various studies. However, it is crucial to recognize that the extremal domain is distinct from the function's domain. In Ref. [80], Stahl's theorems are reinterpreted and adapted for the power system community, highlighting the distinctions among the function's domain, the extremal domain, and the convergence domain. The latter one is the domain in which a sequence of near-diagonal Padé approximants converges to the function in capacity. The confusion surrounding the interpretation of Stahl's theorem's theoretical convergence guarantee emerges from a conflation of those three different domains.

While Stahl's theorem guarantees theoretical convergence, numerical convergence is not established. It is claimed and admitted that universal convergence cannot be guaranteed due to precision issues[32, 37, 81, 82]. Specifically, finite computing precision restricts the meaningful number of power series terms when generating Padé approximants. In power flow problems, a suggested maximum range of 40 to 50 terms is considered acceptable[82]. Also, the convergence rate of Padé approximants is related to the number and placement of singular points, influenced by the original problem model and chosen embedded systems. Furthermore, eight different approaches to enhance the convergence of a power series are compared in Ref. [82]. Notably, methods like the Van Wijngaarden method and the Theta method, distinct from Padé approximants, are also evaluated. The matrix method of obtaining near-diagonal Padé approximants and the eta method emerge as the most numerically robust and efficient options, verified through case studies.

4.3 Time performance and scalability

The original HEM (2) does not come with a theoretical guarantee of better time performances compared to traditional methods like the FD, NR, and GS methods for solving individual power flow solutions. Observations indicate that employing the HEM (2) typically results in a performance slowdown of approximately 30 times or even more when compared to the NR and FD methods, particularly evident in test cases with 1000 buses or more[7, 33, 83]. Comprehensive comparisons reported in Ref. [83] acknowledge the robustness and reliability of the HEM, yet it is highlighted that the average speed of the HEM (2) is notably slower compared with NR and GS. It is important to note that the embedded system (2) supports a specific flat start as the initial guess, implying that if the target state solution is significantly distant from the flat start, the number of required terms must be increased. Therefore, to attain comparable speed using the HEMs to solve an individual power flow solution, it is recommended to explore variants of the original HEM that can accommodate any state vector as the reference solution. Variants such as FFHEM[9], or other numerical-assisted adaptations like GS-assisted FFHEM or NR-assisted FFHEM[9], are suggested alternatives, the scalability is verified up to the 13659 buses system.

However, in applications for voltage stability assessment, the HEMs do indeed present a significant speed advantage. The utilization of the HEMs permits the tracing of the complete PV curve up to the nose point through nearly identical complexity used for solving a single power flow. This stands in contrast to the CPF method, which necessitates repetitive power flow calculations. The remarkable speed exhibited by the HEMs for voltage stability problems has been confirmed in studies such as Refs. [11, 12] up to the 70, 000 bus system. Additionally, MDHEM showcases impressive speed enhancements when dealing with multi-scenario problems[40] up to the 2383 system.

Moreover, the application of the HEMs in dynamic simulations also presents significant advantages over traditional methods. Dynamic simulations utilizing DAEs numerical solvers typically involve multi-stage processes, requiring iterations for algebraic equations at each stage which is avoided in the HEMs, leading to more favorable speeds, as indicated in Refs. [24, 84] up to the 2383 system.

5 Conclusions

In summary, this review comprehensively explores the applications of the HEMs in power systems, certain classes of the HEMs offer remarkable advantages like expansive convergence regions, rapid convergence rate, and precise detection of solution nonexistence, setting them apart from traditional numerical iterative approaches.

Furthermore, the HEMs do provide new insights and views for solving a wide range of computational algebraic and dynamic problems in the power system community. Its ability to handle both reliable steady-state computations and efficient time-domain simulations make it a valuable tool for researchers and engineers. Indeed, the HEMs have proven invaluable in addressing large-scale nonlinear problems and several toolboxes are built based on the HEMs. Also, the development of other variants, like FFHEM, MDHEM, and MSHEM proposed in recent research, improve the adaptability and robustness of the HEM in handling different power system problems.

Despite the promising results presented in the applications considered in this paper, the HEMs certainly come with their share of drawbacks. There are even some misconceptions existing in historical literature. Statements related to solution existence, uniqueness, holomorphism, and convergence properties have been revisited and claimed in this paper. These issues have prompted the need for guidelines and clarifications to ensure the correct application of the HEMs.

First, the derivation of an effective embedded system requires many considerations since it will influence the holomorphic property of solution functions and further the convergence property of HEM. One should check that the initial state is non-singular and the embedded system is holomorphic in a certain domain.

Second, the HEMs generally don't have a better time performance compared with NR and GS when solving a single power flow. Traditional HEM couldn't select the initial state (fixed), power coefficients are necessary to be calculated up to a high enough order when the target state is far from the initial state. And Padé approximants are very time-consuming at that time. Due to the precision issues, the maximum number of power series terms meaningful in generating Pade approximants is suggested around 40 to 50. If a high computational speed is preferred under heavy load conditions, FFHEM should be utilized together with the assistance of NR or GS.

Finally, for voltage stability assessments, variants such as FFHEM, and MSHEM do achieve a much better time performance, since there is no need to calculate power flow repeatedly at each load level. MSHEM and MDHEM are suggested to handle these kinds of problems as they are based on the physical germ solution, one can still get useful insights even if the calculation fails at $s = 1$. For dynamic simulations, the speedup comes from the large step size and non-iterative properties when dealing with DAEs. On the contrary, for a multi-stage numerical scheme, NR is necessary at every stage which slows down speed compared with MSHEM.

The HEMs will provide some interesting opportunities for extension in the future due to their enjoyable convergence property and special semi-analytical form for solutions. Future research directions mainly fall into two aspects: comprehensive applications and theoretical studies. For instance, multi-scenario stability assessments are still time-consuming, so it is valuable to decide suitable dimensions of MDHEM to efficiently handle large-scale problems. The HEMs can provide power flow solutions in power series form, how to obtain insight to handle optimal power flow is an interesting topic. For theoretical studies, the algorithm for determining the optimal order of the HEMs for a special problem should be designed and the estimation of convergence domains, is valuable to further enhance the capabilities of the HEMs. The success of HEM lies not only in its theoretical foundations but also in its practical implementation, making it an indispensable tool for ensuring the stability and reliability of modern power systems.

Nomenclature

Abbreviations	
CPF	Continuation power flow
DAEs	Differential algebraic equations
FD	Fast decoupled
FFHEM	Fast and flexible holomorphic embedding method
GS	Gauss–Seidel
HEM	Holomorphic embedding method
HVDC	High-voltage direct current
IBR	Inverter-based resources
IPFC	Interline power flow controller
LCC-HVDC	Line-commutated converter based high voltage direct current
MDHEM	Multi-dimensional holomorphic embedding method
MSHEM	Multi-stage holomorphic embedding method
NR	Newton–Raphson
ODEs	Ordinary differential equations
PA	Padé approximant
QSS	Quasi-steady-state
SEPs	Stable equilibrium points
SNBP	Saddle-node bifurcation point
SSSC	Static synchronous series compensator
STATCOM	Static synchronous compensator
SVC	Static var compensator
UEPs	Unstable equilibrium points
Key notations	
$\boldsymbol{A}$, $\boldsymbol{x}[q]$, $\boldsymbol{b}[q]$	Transfer matrix, coefficients vector, vector related to order q in the solution process
$[m/n]_i(s)$	Padé approximant for $V_i(s)$
N	Number of buses in a power system
N_p, N_v, N_s	Number of PQ, PV, and Slack buses
s	Embedded complex variable
S_i, P_i, Q_i	Injection power, active power, reactive power at bus i
V_i	Voltage at bus i
$V_i[q]$, $V_i^*[q]$	Coefficients in the power series expansion of $V_i(s)$ and $V_i^*(s^*)$
$V_i(s)$, $V_i^*(s^*)$	Solution function of voltage and its conjugate at bus i in the embedded system
Y_{ik}	Admittance matrix element between buses i and k
ε	Termination criterion for the mismatch

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References

[1] Tinney, W., Hart, C. (1967). Power flow solution by Newton's method. *IEEE Transactions on Power Apparatus and Systems*, PAS-86: 1449–1460.

[2] Stott, B., Alsac, O. (1974). Fast decoupled load flow. *IEEE Transactions on Power Apparatus and Systems*, PAS-93: 859–869.

[3] Ward, J. B., Hale, H. W. (1956). Digital computer solution of power-flow problems [includes discussion]. *Transactions of the American Institute of Electrical Engineers Part III: Power Apparatus and Systems*, 75: 398–404.

[4] Ajjarapu, V., Christy, C. (1992). The continuation power flow: A tool for steady state voltage stability analysis. *IEEE Transactions on Power Systems*, 7: 416–423.

[5] Wang, T., Chiang, H. D. (2020). On the holomorphic and conjugate properties for holomorphic embedding methods for solving power flow equations. *IEEE Transactions on Power Systems*, 35: 2506–2515.

[6] Trias, A. (2012). The holomorphic embedding load flow method. In: Proceedings of the 2012 IEEE Power and Energy Society General Meeting, San Diego, CA, USA.

[7] Rao, S., Feng, Y., Tylavsky, D. J., Subramanian, M. K. (2016). The holomorphic embedding method applied to the power-flow problem. *IEEE Transactions on Power Systems*, 31: 3816–3828.
[8] Liu, C., Wang, B., Xu, X., Sun, K., Shi, D., Bak, C. L. (2017). A multi-dimensional holomorphic embedding method to solve AC power flows. *IEEE Access*, 5: 25270–25285.
[9] Chiang, H. D., Wang, T., Sheng, H. (2018). A novel fast and flexible holomorphic embedding power flow method. *IEEE Transactions on Power Systems*, 33: 2551–2562.
[10] Rao, S. D., Tylavsky, D. J., Feng, Y. (2017). Estimating the saddle-node bifurcation point of static power systems using the holomorphic embedding method. *International Journal of Electrical Power & Energy Systems*, 84: 1–12.
[11] Liu, C., Wang, B., Hu, F., Sun, K., Bak, C. L. (2018). Online voltage stability assessment for load areas based on the holomorphic embedding method. *IEEE Transactions on Power Systems*, 33: 3720–3734.
[12] Zhang, W., Wang, T., Chiang, H. D. (2022). A novel FFHE-inspired method for large power system static stability computation. *IEEE Transactions on Power Systems*, 37: 726–737.
[13] Sun, Y., Ding, T., Qu, M., Wang, F., Shahidehpour, M. (2022). Interval total transfer capability for mesh HVDC systems based on sum of squares and multi-dimensional holomorphic embedding method. *IEEE Transactions on Power Systems*, 37: 4157–4167.
[14] Yao, R., Qiu, F. (2020). Novel AC distribution factor for efficient outage analysis. *IEEE Transactions on Power Systems*, 35: 4960–4963.
[15] Yao, R., Qiu, F., Sun, K. (2022). Contingency analysis based on partitioned and parallel holomorphic embedding. *IEEE Transactions on Power Systems*, 37: 565–575.
[16] Rao, S., Tylavsky, D. (2016). Nonlinear network reduction for distribution networks using the holomorphic embedding method. In: Proceedings of the 2016 North American Power Symposium (NAPS), Denver, CO, USA.
[17] Zhu, Y., Tylavsky, D., Rao, S. (2018). Nonlinear structure-preserving network reduction using holomorphic embedding. *IEEE Transactions on Power Systems*, 33: 1926–1935.
[18] Liu, C., Qin, N., Sun, K., Bak, C. L. (2019). Remote voltage control using the holomorphic embedding load flow method. *IEEE Transactions on Smart Grid*, 10: 6308–6319.
[19] Rao, B. V., Stefan, M., Schwalbe, R., Karl, R., Kupzog, F., Kozek, M. (2021). Stratified control applied to a three-phase unbalanced low voltage distribution grid in a local peer-to-peer energy community. *Energies*, 14: 3290.
[20] Luo, Y., Liu, C. (2023). Area interchange control using the holomorphic embedding load flow method considering automatic generation control. *Electric Power Systems Research*, 225: 109721.
[21] Rao, B., Kupzog, F., Kozek, M. (2019). Three-phase unbalanced optimal power flow using holomorphic embedding load flow method. *Sustainability*, 11: 1774.
[22] Sayed, A. R., Zhang, X., Wang, G., Wang, C., Qiu, J. (2023). Optimal operable power flow: Sample-efficient holomorphic embedding-based reinforcement learning. *IEEE Transactions on Power Systems*, https://doi.org/10.1109/TPWRS.2023.3266773.
[23] Yao, R., Sun, K., Shi, D., Zhang, X. (2019). Voltage stability analysis of power systems with induction motors based on holomorphic embedding. *IEEE Transactions on Power Systems*, 34: 1278–1288.
[24] Yao, R., Liu, Y., Sun, K., Qiu, F., Wang, J. (2020). Efficient and robust dynamic simulation of power systems with holomorphic embedding. *IEEE Transactions on Power Systems*, 35: 938–949.
[25] Subramanian, M. K. (2014). Application of holomorphic embedding to the power-flow problem. Master's Thesis, Arizona State University, USA.
[26] Remmert, R. (1991). Complex integral calculus. In: *Theory of Complex Functions*. New York: Springer.
[27] Trias, A., Marin, J. L. (2016). The holomorphic embedding loadflow method for DC power systems and nonlinear DC circuits. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 63: 322–333.
[28] Cuyt, A., Petersen, V. B., Verdonk, B., Waadeland, H., Jones, W. B. (2008). Exponential integrals and related functions. In: *Handbook of Continued Fractions for Special Functions*. Dordrecht: Springer.
[29] Baker, G., Graves-Morris, P. (1996). P*adé Approximants*. Cambridge: Cambridge University Press.
[30] Stahl, H. (1989). On the convergence of generalized Padé approximants. *Constructive Approximation*, 5: 221–240.
[31] Stahl, H. (1997). The convergence of padé approximants to functions with branch points. *Journal of Approximation Theory*, 91: 139–204.
[32] Trias, A. (2015). Fundamentals of the holomorphic embedding load-flow method. *arXiv preprint*, arXiv: 1509.02421.
[33] Wang, T., Chiang, H. D. (2021). Theoretical study of non-iterative holomorphic embedding methods for solving nonlinear power flow equations: Algebraic property. *IEEE Transactions on Power Systems*, 36: 2934–2945.
[34] Singh, P., Tiwari, R. (2019). STATCOM model using holomorphic embedding. *IEEE Access*, 7: 33075–33086.
[35] Zhang, T., Li, Z., Zheng, J. H., Wu, Q. H., Zhou, X. (2020). Power flow analysis of integrated gas and electricity systems using the fast and flexible holomorphic embedding method. In: Proceedings of the 2020 IEEE Power & Energy Society General Meeting (PESGM), Montreal, QC, Canada.
[36] Pan, S., Li, Z., Zheng, J. H., Wu, Q. H. (2020). On convergence performance and its common domain of the fast and flexible holomorphic embedding method for power flow analysis. In: Proceedings of the 2020 IEEE Power & Energy Society General Meeting (PESGM), Montreal, QC, Canada.
[37] Wang, B., Liu, C., Sun, K. (2018). Multi-stage holomorphic embedding method for calculating the power–voltage curve. *IEEE Transactions on Power Systems*, 33: 1127–1129.
[38] Zhu, Y., Tylavsky, D. (2016). Bivariate holomorphic embedding applied to the power flow problem. In: Proceedings of the 2016 North American Power Symposium (NAPS), Denver, CO, USA.
[39] Liu, C., Sun, K., Wang, B., Ju, W. (2018). Probabilistic power flow analysis using multidimensional holomorphic embedding and generalized cumulants. *IEEE Transactions on Power Systems*, 33: 7132–7142.
[40] Sun, Y., Ding, T., Han, O., Liu, C., Li, F. (2023). Static voltage stability analysis based on multi-dimensional holomorphic embedding method. *IEEE Transactions on Power Systems*, 38: 3748–3759.
[41] Sun, Y., Ding, T., Qu, M., Li, F., Shahidehpour, M. (2021). Tight semidefinite relaxation for interval power flow model based on multi-dimensional holomorphic embedding method. *IEEE Transactions on Power Systems*, 36: 2138–2148.
[42] Sun, Y., Ding, T., Shahidehpour, M. (2023). A multi-variable analytical method for energy flow calculation in natural gas systems. *IEEE Transactions on Power Systems*, 38: 1767–1770.
[43] Sun, Y., Ding, T., Xu, T., Mu, C., Siano, P., Catalao, J. P. S. (2022). Power flow analytical method for three-phase active distribution networks based on multi-dimensional holomorphic embedding method. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 69: 5069–5073.
[44] Lai, Q., Liu, C., Sun, K. (2022). Analytical static voltage stability boundary based on holomorphic embedding. *International Journal of Electrical Power & Energy Systems*, 134: 107386.
[45] Yao, R., Zhao, D., Sakis Meliopoulos, A. P., Singh, C., Mitra, J., Qiu, F. (2022). Advanced extended-term simulation approach with flexible quasisteady-state and dynamic semi-analytical simulation engines. *iEnergy*, 1: 124–132.
[46] Klump, R. P., Overbye, T. J. (2000). A new method for finding low-voltage power flow solutions. In: Proceedings of the 2000 Power Engineering Society Summer Meeting (Cat. No.00CH37134), Seattle, WA, USA.
[47] Leonidopoulos, G. (1995). Approximate linear decoupled solution as the initial value of power system load flow. *Electric Power Systems Research*, 32: 161–163.
[48] Baghsorkhi, S. S., Suetin, S. P. (2015). Embedding AC power flow with voltage control in the complex plane: The case of analytic continuation via Padé approximants. *arXiv preprint*, arXiv: 1504.03249.
[49] Subramanian, M. K., Feng, Y., Tylavsky, D. (2013). PV bus modeling in a holomorphically embedded power-flow formulation. In: Pro-

ceedings of the 2013 North American Power Symposium (NAPS), Manhattan, KS, USA.

[50] Baghsorkhi, S. S., Suetin, S. P. (2016). Embedding AC power flow in the complex plane Part I: Modelling and mathematical foundation. *arXiv preprint*, arXiv: 1609.01211.

[51] Wallace, I., Roberts, D., Grothey, A., McKinnon, K. (2016). Alternative PV bus modelling with the holomorphic embedding load flow method. *arXiv preprint*, arXiv: 1607.00163.

[52] Basiri-Kejani, M., Gholipour, E. (2017). Holomorphic embedding load-flow modeling of thyristor-based FACTS controllers. *IEEE Transactions on Power Systems*, 32: 4871–4879.

[53] Singh, P., Senroy, N., Tiwari, R. (2021). Guaranteed convergence embedded system for SSSC and IPFC. *IEEE Transactions on Power Systems*, 36: 2725–2728.

[54] Singh, P., Senroy, N. (2021). Steady-state models of STATCOM and UPFC using flexible holomorphic embedding. *Electric Power Systems Research*, 199: 107390.

[55] Sur, U., Biswas, A., Bera, J. N., Sarkar, G. (2020). A modified holomorphic embedding method based hybrid AC-DC microgrid load flow. *Electric Power Systems Research*, 182: 106267.

[56] Zhao, Y., Li, C., Ding, T., Hao, Z., Li, F. (2021). Holomorphic embedding power flow for AC/DC hybrid power systems using bauer's ETA algorithm. *IEEE Transactions on Power Systems*, 36: 3595–3606.

[57] Huang, Y., Ai, X., Fang, J., Yao, W., Wen, J. (2020). Holomorphic embedding approach for VSC-based AC/DC power flow. *IET Generation, Transmission & Distribution*, 14: 6239–6249.

[58] Morgan, M. Y., Shaaban, M. F., Sindi, H. F., Zeineldin, H. H. (2022). A holomorphic embedding power flow algorithm for islanded hybrid AC/DC microgrids. *IEEE Transactions on Smart Grid*, 13: 1813–1825.

[59] Huang, Y., Ai, X., Fang, J., Cui, S., Zhong, R., Yao, W., Wen, J. (2023). Holomorphic embedding power flow modeling of autonomous AC/DC hybrid microgrids. *International Journal of Electrical Power & Energy Systems*, 145: 108549.

[60] Sun, L., Ju, Y., Yang, L., Ge, S., Fang, Q., Wang, J. (2018). Holomorphic embedding load flow modeling of the three-phase active distribution network. In: Proceedings of the 2018 International Conference on Power System Technology (POWERCON), Guangzhou, China.

[61] Keihan Asl, D., Mohammadi, M., Reza Seifi, A. (2019). Holomorphic embedding load flow for unbalanced radial distribution networks with DFIG and tap-changer modelling. *IET Generation, Transmission & Distribution*, 13: 4263–4273.

[62] Heidarifar, M., Andrianesis, P., Caramanis, M. (2019). Efficient load flow techniques based on holomorphic embedding for distribution networks. In: Proceedings of the 2019 IEEE Power & Energy Society General Meeting (PESGM), Atlanta, GA, USA.

[63] Heidarifar, M., Andrianesis, P., Caramanis, M. (2023). Holomorphic embedding load flow method in three-phase distribution networks with ZIP loads. *IEEE Transactions on Power Systems*, 38: 4605–4616.

[64] Shamseldein, M. (2022). A fast holomorphic embedding power flow approach for meshed distribution networks. *International Transactions on Electrical Energy Systems*, 2022: 9561385.

[65] Massrur, H. R., Niknam, T., Aghaei, J., Shafie-khah, M., Catalao, J. P. S. (2018). Fast decomposed energy flow in large-scale integrated electricity–gas–heat energy systems. *IEEE Transactions on Sustainable Energy*, 9: 1565–1577.

[66] Massrur, H. R., Niknam, T., Fotuhi-Firuzabad, M. (2018). Investigation of carrier demand response uncertainty on energy flow of renewable-based integrated electricity–gas–heat systems. *IEEE Transactions on Industrial Informatics*, 14: 5133–5142.

[67] Huang, Y., Ai, X., Fang, J., Cui, S., Zhong, R., Yao, W., Wen, J. (2022). Holomorphic embedding power flow algorithm for isolated AC microgrids with hierarchical control. *IEEE Transactions on Smart Grid*, 13: 1679–1690.

[68] Sun, Y., Ding, T., Zhang, J., Shahidehpour, M., Jia, W., Xue, Y. (2023). An analytical steady heat flow calculation method for district heating networks. *IEEE Transactions on Sustainable Energy*: https://doi.org/10.1109/TSTE.2023.3328155.

[69] Chiang, H. D., Flueck, A. J., Shah, K. S., Balu, N. (1995). CPFLOW: A practical tool for tracing power system steady-state stationary behavior due to load and generation variations. *IEEE Transactions on Power Systems*, 10: 623–634.

[70] Rao, S., Tylavsky, D., Vittal, V., Yi, W., Shi, D., Wang, Z. (2018). Fast weak-bus and bifurcation point determination using holomorphic embedding method. In: Proceedings of the 2018 IEEE Power & Energy Society General Meeting (PESGM), Portland, OR, USA.

[71] Trias, A. (2018). Sigma algebraic approximants as a diagnostic tool in power networks. US Patent, 9,563,722, 2017-2-7.

[72] Lai, Q., Liu, C., Sun, K. (2020). A network decoupling method for voltage stability analysis based on holomorphic embedding. *arXiv preprint*, arXiv: 2003.12287.

[73] Singh, P., Tiwari, R. (2020). Extended holomorphic embedded load-flow method and voltage stability assessment of power systems. *Electric Power Systems Research*, 185: 106381.

[74] Lai, Q., Liu, C., Sun, K. (2021). Vulnerability assessment for voltage stability based on solvability regions of decoupled power flow equations. *Applied Energy*, 304: 117738.

[75] Lai, Q., Liu, C., Sun, K. (2022). Formulation and visualization of bus voltage-var safety regions for a power system. *IEEE Transactions on Power Systems*, 37: 3153–3156.

[76] Gao, H., Chen, J., Diao, R., Zhang, J. (2021). A HEM-based sensitivity analysis method for fast voltage stability assessment in distribution power network. *IEEE Access*, 9: 13344–13353.

[77] Yao, R., Sun, K., Qiu, F. (2019). Vectorized efficient computation of padé approximation for semi-analytical simulation of large-scale power systems. *IEEE Transactions on Power Systems*, 34: 3957–3959.

[78] Feng, Y., Tylavsky, D. (2013). A novel method to converge to the unstable equilibrium point for a two-bus system. In: Proceedings of the 2013 North American Power Symposium (NAPS), Manhattan, KS, USA.

[79] Xu, X., Liu, C., Sun, K. (2018). A holomorphic embedding method to solve unstable equilibrium points of power systems. In: Proceedings of the 2018 IEEE Conference on Decision and Control (CDC), Miami Beach, FL, USA.

[80] Li, S., Tylavsky, D., Shi, D., Wang, Z. (2021). Implications of Stahl's theorems to holomorphic embedding Part I: Theoretical convergence. *CSEE Journal of Power and Energy Systems*, 7: 761–772.

[81] Rao, S. D., Tylavsky, D. J. (2018). Theoretical convergence guarantees versus numerical convergence behavior of the holomorphically embedded power flow method. *International Journal of Electrical Power & Energy Systems*, 95: 166–176.

[82] Dronamraju, A., Li, S., Li, Q., Li, Y., Tylavsky, D., Shi, D., Wang, D. (2021). Implications of stahl's theorems to holomorphic embedding Part II: Numerical convergence. *CSEE Journal of Power and Energy Systems*, 7: 773–784.

[83] Sauter, P. S., Braun, C. A., Kluwe, M., Hohmann, S. (2017). Comparison of the holomorphic embedding load flow method with established power flow algorithms and a new hybrid approach. In: Proceedings of the 2017 Ninth Annual IEEE Green Technologies Conference (GreenTech), Denver, CO, USA.

[84] Liu, J., Yao, R., Qiu, F., Liu, Y., Sun, K. (2023). PowerSAS.m—An open-source power system simulation toolbox based on semi-analytical solution technologies. *IEEE Open Access Journal of Power and Energy*, 10: 222–232.