

# Navigation Systems May Deteriorate Stability in Traffic Networks

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**ABSTRACT** Advanced traffic navigation systems, which provide routing recommendations to drivers based on real-time congestion information, are nowadays widely adopted by roadway transportation users. Yet, the emerging effects on the traffic dynamics originating from the widespread adoption of these technologies have remained largely unexplored until now. In this paper, we propose a dynamic model where drivers imitate the path preferences of previous drivers, and we study the properties of its equilibrium points. Our model is a dynamic generalization of the classical *traffic assignment framework*, and extends it by accounting for dynamics both in the path decision process and in the network's traffic flows. We show that, when travelers learn shortest paths by imitating other travelers, the overall traffic system benefits from this mechanism and transfers the maximum admissible amount of traffic demand. On the other hand, we demonstrate that, when the travel delay functions are not sufficiently steep or the rates at which drivers imitate previous travelers are not adequately chosen, the trajectories of the traffic system may fail to converge to an equilibrium point, thus compromising asymptotic stability. Illustrative numerical simulations combined with empirical data from highway sensors illustrate our findings.

**INDEX TERMS** Network control, dynamical flow networks, traffic networks

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## I. INTRODUCTION

Roadway traffic networks are fundamental components of modern societies, making economic activity possible by enabling the transfer of passengers, goods, and services in a timely and reliable fashion. Despite their critical role, these transportation systems are impaired by the long-standing problem of traffic congestion, which wastes billions of gallons of fuel each year [1], [2]. Advanced navigation systems are nowadays widely adopted by travelers, largely thanks to the widespread use of smartphone-based navigation apps (such as Google Maps, Inrix, Waze, Apple Maps, etc.) [3]. Advanced navigation systems provide shortest-path routing recommendations based on real-time global travel time information. On the one hand, these technologies have enabled travelers to save time and fuel but, on the other hand, they have transformed the transportation infrastructure originating unanticipated effects and disrupting existing traffic flow patterns [4]. While the implications of the widespread adoption of advanced navigation systems have been analyzed game-theoretically [5], a characterization of the impact of

these technologies on dynamic models of traffic for general, dynamic, traffic networks has remained elusive until now.

In this work, we study the stability properties of a traffic system composed of the interconnection between a dynamic model of traffic flows and a dynamic model of route selection (derived from the Replicator Dynamics [6]). Our choice of using the replicator equation is motivated by recent studies that showed that this model emerges as an aggregate description of learning processes in large populations and as the limiting case of the Best Response dynamics [7]. We show that, at equilibrium, our model shares the same properties as the well-studied routing game [5], and thus it is consistent with existing studies that focus on systems operating at equilibrium. It is worth noting that, with respect to the classical routing-game framework, our model accounts for dynamics both in the *route selection process* as well as in the *traffic flows*. Our dynamical model suggests that systems where travelers continuously prefer highways with minimal latency to destination – and select these highways by imitating other travelers already in the network – admit an equilibrium point, provided that the external inflow is

bounded above by the min-cut capacity of the network. This implies that traffic systems where the users learn through imitation transfer the maximum amount of flow that is transferable by that network. This connects our work with classical static flow models used in the transportation literature. Moreover, our results show that when the rate of imitation (namely, the frequency at which new users imitate the path preferences of other users) is either too small or too large, the equilibrium points may fail asymptotic stability, thus implying that in unregulated networks the congestion state may oscillate around or diverge from the equilibria.

*Related Work.* The traffic model proposed here finds its roots in the well-established routing game [8] and corresponding traffic assignment problem [5], which have been used in the transportation literature to model how travelers make decisions in congested traffic. Recently, this framework has received increased attention with several studies investigating the impact of different sources of information on the traffic system; e.g., see [9]–[12] and the references therein. One of the main limitations of this classical approach is that it models systems operating at equilibrium, thus neglecting dynamics near these points. For this reason, evolutionary dynamics [6] have been proposed to study the dynamic properties of equilibria [13], [14]. Although these works represent a step forward toward understanding the impact of advanced navigation systems on traffic patterns, the used models still rely on *static descriptions*, where traffic flows propagate instantaneously across the network. It is immediate to realize that such models are accurate only when the routing preferences update at a *slower timescale* than that of the traffic dynamics (e.g., when drivers update their path preferences from day-to-day as a result of a personal observation). On the other hand, in modern traffic networks, advanced navigation systems allow drivers to update their routing preferences *at the same timescale* as the traffic flows, thanks to real-time traffic state measurements. This connects our work with the body of literature on dynamic traffic flow models. Our model is a simplified and continuous-time version of the Cell-Transmission Model [15] and related to the model studied in [16]. Dynamic traffic models with static routing preferences have been studied in [17] using monotonicity, in [18] using mixed monotonicity, in [19] using passivity. Of particular relevance to the framework studied here are [17], [20]. With respect to these works, here we study path selection mechanisms governed by the replicator equation and we focus on the game-theoretic properties of this model and its stability analysis. This work extends the preliminary work of the authors [19] in several directions, including a formal proof of uniqueness and evolutionary stability of the Nash equilibrium, and a sufficient condition to ensure asymptotic stability of the equilibrium point. Finally, the recent works [21], [22] also highlighted detrimental effects of navigation systems in a small-scale (two-link) network.

*Contribution.* The contribution of this work is threefold. First, we propose a dynamic model derived from the replicator dynamics to describe the path selection mechanism underlying drivers' decisions in congested traffic. We then couple this routing model with a dynamic model of traffic, which describes the evolution of traffic flows in the network in relation to the instantaneous routing choices. Relative to the classical traffic assignment framework, the use of a dynamic traffic model describes modern networks where *routing decisions and traffic flows update at the same timescale*. As illustrated in Section V, this model allows us to capture dynamic behaviors observed in practice, which could not be explained using static models [23], [24]. Second, we study the game-theoretic properties of the equilibria of the interconnected model. We show that, under suitable assumptions, an equilibrium point exists, is unique, and coincides with an evolutionary stable Nash (or Wardrop) equilibrium [25]. This relates our work with the well-studied routing game [8]. Third, we study the stability properties of the equilibrium. By using a Lyapunov-based reasoning, we derive sufficient conditions under which the equilibrium is asymptotically stable. In simulation and through an example, we show that the conditions are tight and that oscillating trajectories can emerge when our conditions do not hold. Intuitively, oscillations originate because the population is overreacting to small changes in congestion. In practice, this occurs because individual users update their routing preferences without anticipating the preferences of the rest of the population. This behavior is consistent with field data (see, e.g., [23], [24]).

*Organization.* This paper is organized as follows. Section II presents the proposed model. Section III derives conditions for existence and uniqueness of an equilibrium point and in Section IV we study the stability properties of the equilibria. Section V illustrates our findings via numerical simulations and Section VI concludes the paper.

*Notation.* Given  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ , we let  $(x, u) \in \mathbb{R}^{n+m}$  denote their concatenation; if  $n = m$ ,  $\langle x, u \rangle$  denotes the inner product. For symmetric matrix  $M$ ,  $\lambda_{\max}(M)$  and  $\lambda_{\min}(M)$  denote its largest and smallest eigenvalue, respectively.

## II. Model of traffic network

In this section, we present our models of traffic flows and routing decisions, and we formalize the problem we study.

### A. Traffic flow model

We model a transportation network using a digraph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ , where  $\mathcal{V}$  is the set of nodes and  $\mathcal{L}$  is the set of links. In what follows, we let  $\mathcal{L} = \{1, \dots, n\}$ ,  $n \in \mathbb{N}_{>0}$ . For a link  $i \in \mathcal{L}$ , we denote by  $o_i \in \mathcal{V}$  its origin node and by  $d_i \in \mathcal{V}$  its destination node. Motivated by real-world transportation networks with parallel highways, we will allow for parallel links, namely, we admit  $i, j \in \mathcal{L}$  such that  $i \neq j$  and have the same origin and destination:  $o_i = o_j$  and  $d_i = d_j$ . A *path* in  $\mathcal{G}$  is a sequence of links  $\{i_1, i_2, \dots\}$  such that the origin node of each link is the destination node of the

preceding one. Notice that a path may contain repeated links and, going along the path, one may reach repeated nodes. A path is *simple* if it does not contain the same link more than once. The *length* of a path is the number of edges contained in  $\{i_1, i_2, \dots\}$ . Following the Cell Transmission Model [15], we describe the macroscopic behavior of traffic on each link  $i \in \mathcal{L}$  over time  $t \geq 0$  using the conservation law:

$$\dot{x}_i(t) = f_i^{\text{in}}(x(t)) - f_i^{\text{out}}(x(t)), \quad (1)$$

where  $x_i(t) \in \mathbb{R}$  is the traffic density in link  $i$ ,  $f_i^{\text{in}}(x(t))$  is the traffic inflow entering at upstream, and  $f_i^{\text{out}}(x_i(t))$  is the traffic outflow exiting at downstream. We make the following assumptions on the outflow functions.

**Assumption 1:**

For all  $i \in \mathcal{L}$ , the outflow function  $f_i^{\text{out}}(x)$  depends only on the density  $x_i$ , namely,  $f_i^{\text{out}}(x) = f_i(x_i)$ . Moreover,  $f_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  satisfies  $f_i(x_i) = 0$  if and only if  $x_i = 0$ , it is continuous, and strongly monotone; namely,

$$(x_i - \bar{x}_i)(f_i(x_i) - f_i(\bar{x}_i)) \geq \mu |x_i - \bar{x}_i|^2, \quad (2)$$

for some  $\mu > 0$  and for all  $x_i, \bar{x}_i \in \mathbb{R}_{\geq 0}$ .  $\square$

We discuss this assumption in Remark 1 and we illustrate some choices of outflow functions in Example 2.

Assumption 1 guarantees that (1) is a positive system [26], namely, for every non-negative initial state and every non-negative input at all times, its state trajectory is non-negative. In what follows, for all  $i \in \mathcal{L}$ , we let

$$C_i := \sup_{z \in \mathbb{R}} f_i(z),$$

and  $C = (C_1, \dots, C_n)$ . If  $f_i(\cdot)$  is unbounded,  $C_i = +\infty$ .

**Remark 1 (Validity of Assumption 1):**

It is known (see, e.g., [16]) that the assumption that  $f_i(x_i)$  only depends on  $x_i$  and is strictly increasing is valid provided that we restrict our focus to free-flow regimes [15]. More precisely, it has been shown in [16] that the free-flow equilibrium points of a more complete traffic model (that accounts for congestion regimes and backpropagation through the junctions) inherit the same stability properties of the model considered here. Hence, the conclusions drawn here will be applicable also to more complete models, provided that their operation is restricted to the free-flow regimes [16]. While we acknowledge that accounting for congested regimes is an important problem, due to the technical challenges in dealing with a more complete model, we leave a generalization of our framework as the focus of future works. Regarding the condition  $f_i(x_i) = 0$  if and only if  $x_i = 0$ , the “if” part ensures that no vehicle density can flow out of a link when there is no density on it, and the “only if” part guarantees that any density is allowed to exit.  $\square$

**Example 2 (Flow functions that satisfy Assumption 1):**

A class of functions satisfying Assumption 1 (and used in, e.g., [27]) is that of linear outflow functions, given by

$$f_i^{\text{out}}(x_i) = \alpha_i x_i, \quad \alpha_i > 0.$$

In this case,  $C_i = +\infty$  and  $\mu = \min\{\alpha_i\}_{i \in \mathcal{L}}$ . A second class of functions satisfying Assumption 1 and used in [28] is

$$f_i^{\text{out}}(x_i) = C_i(1 - e^{-\beta_i x_i}), \quad \beta_i > 0,$$

which is strongly monotone on any bounded set.  $\square$

Throughout this paper, we will focus on single-commodity networks, namely, networks for which there is a single origin node  $o$  where exogenous traffic flows enter the network, and a single destination node  $d$ , where flows exit the network. We assume that  $\mathcal{G}$  is outflow-connected, namely, there is a path in  $\mathcal{G}$  from every  $i \in \mathcal{L}$  to  $d$ . To avoid trivial cases, we will also assume that there exists at least one path from  $o$  to  $d$ . We denote by  $\lambda \in \mathbb{R}_{>0}$  the commodity inflow rate at  $o$ .

To model mass propagation through the nodes, we introduce the scalar routing ratios (or routing splits)

$$\{r_{ij}(t)\}_{i,j \in \mathcal{L}}, \quad t \geq 0,$$

where  $r_{ij}(t) \in [0, 1]$  models the fraction of flow exiting link  $i$  that proceeds toward  $j$ . Because exchange of flow is allowed only between consecutive links in the network, we have  $r_{ij}(t) > 0$  only if  $d_i = o_j$ . Finally, mass is conserved through the nodes when  $\sum_j r_{ij}(t) = 1$ . Similarly, we let  $r_{oi}(t) \in [0, 1]$  be the fraction of exogenous inflow  $\lambda$  that is routed from the origin node  $o$  to link  $i$ ; analogously, we have  $r_{oi}(t) = 0$  if  $o_i \neq o$ , and  $\sum_{i \in \mathcal{L}} r_{oi} = 1$ . In what follows, it will be useful to combine the network routing ratios into a matrix  $R(t) = [r_{ij}(t)] \in \mathbb{R}^{n \times n}$  and the routing ratios at the origin into a vector  $R_o = (r_{o1}, \dots, r_{on}) \in \mathbb{R}^n$ . See Example 4 for an illustration of the model and notation.

**Remark 3 (Temporal dependence in the routing ratios):**

In this discussion, we treated  $\{r_{ij}(t)\}_{i,j \in \mathcal{L}}$  as time-varying quantities; we will see shortly below (cf. Section II.B) that the time-dependency in  $r_{ij}(t)$  implicitly originates as a result of the dependence between the routing ratios and the traffic state  $x(t)$ .  $\square$

At every node of  $\mathcal{G}$ , traffic flows are conserved, and thus the inflow to each link  $i \in \mathcal{L}$  is given by

$$f_i^{\text{in}}(x) = r_{oi}(t)\lambda + \sum_{j \in \mathcal{L}} r_{ji}(t)f_j(x_j(t)).$$

By substituting into (1), the density on each link evolves as:

$$\dot{x}_i(t) = r_{oi}(t)\lambda + \sum_{j \in \mathcal{L}} r_{ji}(t)f_j(x_j(t)) - f_i(x_i(t)). \quad (3)$$

By letting

$$x := (x_1, \dots, x_n), \quad f(x) := (f_1(x_1), \dots, f_n(x_n)),$$

be the joint vectors of densities and flows, respectively, the network state evolves according to:

$$\dot{x}(t) = (R(t)^\top - I)f(x(t)) + R_o(t)\lambda. \quad (4)$$

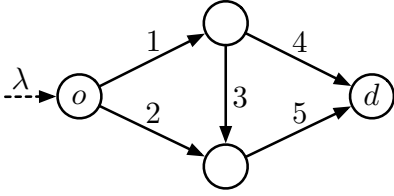


Fig. 1: Graph topology used to illustrate our model. See examples 4 and 7. Nodes labeled by ‘o’ and ‘d’ describe the origin and destination, where exogenous inflows enter and exit the network, respectively. The dashed arrow illustrates the exogenous inflow.

We illustrate this traffic model in Example 4.

**Example 4 (Illustration of traffic flow model):**

Consider the network topology in Fig. 1. The model (4) reads as:

$$\begin{aligned} \dot{x}_1 &= -f_1(x_1) + r_{o1}\lambda, & \dot{x}_3 &= -f_3(x_3) + r_{13}f_1(x_1), \\ \dot{x}_2 &= -f_2(x_2) + r_{o2}\lambda, & \dot{x}_4 &= -f_4(x_4) + r_{14}f_1(x_1), \\ \dot{x}_5 &= -f_5(x_5) + f_2(x_2) + f_3(x_3), \end{aligned}$$

where time dependencies have been dropped for compactness. Notice that the routing ratios satisfy  $r_{o1} + r_{o2} = 1, r_{13} + r_{14} = 1$ .  $\square$

**B. Congestion-responsive path selection model**

We next propose a model to describe the path selection process followed by drivers that seek to minimize their travel time to destination. Let  $\mathcal{P}$  denote the set of simple paths from  $o$  to  $d$ . To model this process, we introduce the variables  $\{y_p(t)\}_{p \in \mathcal{P}}$ , where  $y_p(t)$  denotes the fraction of exogenous inflow  $\lambda$  that is routed through path  $p$  at time  $t$ . We stress that  $y_p(t)$  models a virtual amount of flow that may never be observed in the network: indeed,  $y_p(t)$  describes the fraction of  $\lambda$  that is routed through  $p$  at time  $t$ , but the actual traffic flows on the links of  $p$  will be determined by the traffic flow model, as described shortly below. Hence, in what follows, we call  $y_p(t)$  *demanded path flow* for path  $p$  (for a discussion on this wording choice, see Remark 10 shortly below). See Fig. 2 for an illustration. Then, the set of admissible path flow demands is the scaled simplex:

$$\Delta := \{y \in \mathbb{R}_{\geq 0}^{|\mathcal{P}|} : \sum_{p \in \mathcal{P}} y_p = \lambda\}. \quad (5)$$

For link  $i \in \mathcal{L}$ , we let

$$y_i^l(t) := \sum_{p \in \mathcal{P}: i \in p} y_p(t), \quad (6)$$

be the *demanded link flows*. Similarly to the demanded path flows, the demanded link flow  $y_i^l(t)$  describes the fraction of  $\lambda$  that is routed through link  $i$  at time  $t$ . In vector form,  $y(t) = (y_1(t), \dots, y_{|\mathcal{P}|}(t))$  and  $y^l(t) := (y_1^l(t), \dots, y_n^l(t))$ . Notice that [5, Thm 2.2] guarantees that for any  $y(t) \in \Delta$ ,  $y^l(t)$  is uniquely determined.

To every link  $i \in \mathcal{L}$ , we associate a latency function  $\ell_i^l(x_i)$  mapping traffic density into latency, and describing

the travel time or latency required to traverse that link. With this notation, the total *demanded traffic latency* for path  $p$  is given by the sum of latencies of the links in that path:

$$\ell_p(x) := \sum_{i \in p} \ell_i^l(x_i). \quad (7)$$

In vector form,  $\ell(x) := (\ell_1(x), \dots, \ell_{|\mathcal{P}|}(x))$  and  $\ell^l(x) := (\ell_1^l(x_1), \dots, \ell_n^l(x_n))$ .

**Remark 5 (Choice of relating latencies to densities):**

In this work, we make the assumption that the link latencies are functions of the densities, rather than of the flows (as in, e.g., [17]). The reason for this choice stems from the transportation literature, where it is well-established that one can model the relationship velocity-density using a bijective map, while the map velocity-flow is not bijective (see, e.g. [29, Fig. 4.12]). Although in our setting the two frameworks are equivalent (by virtue of Assumption 1), we believe that our framework is more amenable to generalizations to non-free-flow conditions (where Assumption 1 does not hold) as compared to models that relate latencies to flows.  $\square$

Motivated by [30], we make the following assumption.

**Assumption 2:**

For all  $i \in \mathcal{L}$ ,  $\ell_i^l : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is non-negative, continuous, and such that

$$\lim_{x_i \rightarrow f_i^{-1}(C_i)} \ell_i^l(x_i) = +\infty. \quad (8)$$

Assumption 2 is very mild, as it requires that every link has a non-negative travel time that varies smoothly as a function of the traffic densities and that tends to infinity as the link flow approaches the flow capacity; we refer to [30] for a detailed discussion on the validity of this assumption.

We consider a model where the vector of path preferences  $y(t)$  is continuously updated over time based on the traffic state of the network. We adopt a model of path selection where the preference for a certain path  $p \in \mathcal{P}$  will increase or decrease depending on whether that path has a better or worse travel time compared to the network average. To this end, we model the time-evolution of the flow demands using the replicator dynamics [31]:

$$\dot{y}_p(t) = y_p(\bar{\ell}(x(t), y(t)) - \ell_p(x(t))), \quad p \in \mathcal{P}, \quad (9)$$

where

$$\bar{\ell}(x(t), y(t)) = \lambda^{-1} \sum_{p \in \mathcal{P}} y_p(t) \ell_p(x(t)), \quad (10)$$

is the average latency of traversing the network from  $o$  to  $d$ . Equation (9) states that the growth rate of flow demand for path  $p$  is proportional to the difference between the average latency of traversing the network  $\bar{\ell}(x(t), y(t))$  and the latency of that path  $\ell_p(x(t))$ . We motivate our choice of adopting the replicator dynamics in Remark 6; we also note

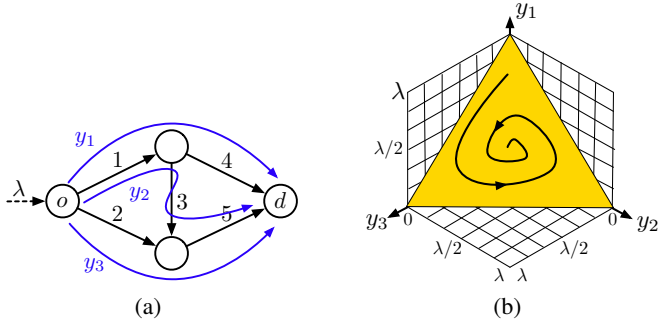


Fig. 2: (a) Demanded path flows  $y_1(t), y_2(t), y_3(t)$  describe the fraction of  $\lambda$  that is routed through the paths, respectively,  $p_1, p_2, p_3$ . (b) The demanded path flows follow a model of path selection (cf. (9)) where the preference for a certain path will increase or decrease depending on whether that path has a better or worse travel time compared to the network average. This model ensures that the trajectories  $y_1(t), y_2(t), y_3(t)$  are trapped inside the simplex (5).

that this model has been widely adopted in the transportation literature to study dynamics in the routing game [13], [32].

**Remark 6 (Choice of the replicator dynamics):**

The replicator equation is a deterministic model of imitation, where future path preferences are selected by imitating successful path preferences of previous users. Replicator dynamics have originated in biology, arising in the study of animal behavior and evolution, and researchers later proved that this model is also an accurate description of learning processes in large populations [7]. Interestingly, this is a good model to describe the outcome of machine learning processes as its dynamics hinge on historical data and the paradigm of imitation (i.e., users observe others' travel times and change their own strategies based on these observations). Although our analysis is tailored to the replicator model, other selection models could be also considered – such as the Best Response dynamics [33]. It is worth noting that the asymptotic properties of the trajectories are common across several different models of selection: for instance, [33] showed that noisy versions of the best-response dynamics have the same qualitative properties as the replicator model.  $\square$

**Example 7 (Illustration of the path selection model):**

Consider the network illustrated in Fig. 1 and discussed in Example 4. This graph includes three simple paths  $\mathcal{P} = \{p_1, p_2, p_3\}$  (see Fig. 2(a)) given by

$$p_1 = (1, 4), \quad p_2 = (1, 3, 5), \quad p_3 = (2, 5).$$

The demanded path flows  $y_1, y_2, y_3$  are scalar quantities that model the fraction of exogenous inflow  $\lambda$  that is routed through, respectively, paths  $p_1, p_2, p_3$ . According to (6), the demanded flows on the links  $y_1^l, y_2^l, y_3^l, y_4^l, y_5^l$  can be

computed from the demanded flows on the path as follows:  
 $y_1^l = y_1 + y_2, \quad y_2^l = y_3, \quad y_3^l = y_2, \quad y_4^l = y_1, \quad y_5^l = y_2 + y_3.$

In other words, the above relationships state that the flow on each link is the sum of the flows on paths passing through that link. The demanded traffic latencies of the paths (7) are:

$$\ell_1(x) = \ell_1^l(x_1) + \ell_4^l(x_4), \quad \ell_3(x) = \ell_2^l(x_2) + \ell_5^l(x_5), \\ \ell_2(x) = \ell_1^l(x_1) + \ell_3^l(x_3) + \ell_5^l(x_5).$$

Namely, the latency of each path is the sum of latencies of all links that compose that path. The average latency of traversing the network (10) is:

$$\bar{\ell}(x, y) = \lambda^{-1}(y_1 \ell_1(x) + y_2 \ell_2(x) + y_3 \ell_3(x)),$$

and models the latency required to traverse the network, averaged over all paths. The replicator model (9) proposed to describe the path selection process in this case reads as:

$$\dot{y}_1 = y_1(\bar{\ell}(x, y) - \ell_1(x)), \quad \dot{y}_2 = y_2(\bar{\ell}(x, y) - \ell_2(x)), \\ \dot{y}_3 = y_3(\bar{\ell}(x, y) - \ell_3(x)).$$

In words, the preference of users for a certain path grows proportionally to the difference between the average delay in the network and the delay of that path. See Fig. 2(b).  $\square$

**Remark 8 (Forward invariance of replicator dynamics):**

We state three important properties of (9) that will be used throughout this paper.

*P1 - Forward invariance of  $\Delta$ .* If  $y(0) \in \Delta$ , then  $y(t) \in \Delta$  for all  $t \in \mathbb{R}_{>0}$ . This follows by noting that, when  $y(0) \in \Delta$ , from (9)-(10), we have  $\sum_{p \in \mathcal{P}} \dot{y}_p(t) = 0 \quad \forall t \in \mathbb{R}_{>0}$ .

*P2 - Forward invariance of boundary faces.* Define the boundary faces of  $\Delta$  as:

$$\text{bf}_i \Delta := \{y : \sum_{p \in \mathcal{P}} y_p = \lambda, y_i = 0\}. \quad i \in \mathcal{P}.$$

By inspection of (9), we have that all boundary faces are forward invariant. In addition, the boundary  $\text{bd}\Delta$  (i.e., the union of all the boundary faces) is also forward invariant.

*P3 - Forward invariance of  $\text{int}\Delta$ .* Let  $\text{int}\Delta$  denote the interior of  $\Delta$ , namely, the subset satisfying  $y_i > 0 \quad \forall i$ . If  $y(0) \in \text{int}\Delta$ , the trajectories of (9) satisfy  $y(t) \in \text{int}\Delta$  for all  $t \in \mathbb{R}_{>0}$ . To see this, notice that

$$\lim_{y_p \rightarrow 0} y_p(\bar{\ell}(x(t), y(t)) - \ell_p(x(t))) = 0,$$

and thus the trajectories of (9) may converge to the boundary of  $\Delta'$  only for  $t \rightarrow +\infty$ , and are thus confined to  $\text{int}\Delta$  for all finite  $t$ .  $\square$

By property P2 in Remark 8, if  $y_p(0) \in \text{bf}_p$  for some  $p \in \mathcal{P}$ , the replicator equation will satisfy

$$y_p(t) = 0 \quad \forall t \geq 0,$$

namely, the dynamics will ‘ignore’  $y_p$ . In this case, by letting  $\mathcal{P}'$  denote the set of simple paths from  $o$  to  $d$  such that

$y_p(0) > 0$ , one can replace (9) with a new set of  $|\mathcal{P}'|$ -dimensional dynamics where the variables such that  $y_p(0) = 0$  are removed:

$$\dot{y}_p(t) = y_p(\bar{\ell}(x(t), y(t)) - \ell_p(x(t))), \quad p \in \mathcal{P}'. \quad (11)$$

In this case, the trajectories of (9) and (11) coincide at all times, with the additional condition

$$y_p(t) = 0, \quad \forall p \in \mathcal{P} \setminus \mathcal{P}'.$$

Motivated by this observation, in what follows it will be convenient to consider (11) in place of (9), as well as a restricted state space:

$$\Delta' := \{y \in \mathbb{R}_{>0}^{|\mathcal{P}'|} : \sum_{p \in \mathcal{P}'} y_p = \lambda\}, \quad (12)$$

and initial conditions  $y(0) \in \text{int}\Delta'$ . Notice that, by properties P1 and P3 in Remark 8, if  $y(0) \in \text{int}\Delta'$ , then  $y(t) \in \text{int}\Delta'$  for all  $t \in \mathbb{R}_{>0}$ .

### C. Combined model of traffic with congestion-responsive routing

In this section, we connect the traffic flow model (4) with the path selection model (9) to derive a model of traffic network with congestion-responsive routing. The key observation to relate the two models is that the set of demanded link flows  $y(t)$  implicitly determines the routing ratios  $r_{ij}(t)$ , as described next. For a link  $j \in \mathcal{L}$ , let  $\theta_j := \sum_{i \in \mathcal{L}: o_i = o_j} y_i^l$  denote the total demanded flow transferred by its origin node  $o_j$ . Then, we let the routing ratios depend on the demanded flows as follows: for all  $j$  such that  $o_j = d_i$ ,

$$r_{ij}(y) = \begin{cases} y_j^l / \theta_j & \text{if } \theta_j > 0, \\ \frac{1}{|\{k \in \mathcal{L}: o_k = o_j\}|} & \text{otherwise,} \end{cases} \quad (13)$$

and  $r_{ij}(y) = 0$  for all  $j$  such that  $o_j \neq d_i$ . Note that  $y_j^l$  is implicitly defined by  $y$  through (6). The model (13) states that the outflow exiting link  $i$  splits among the available downstream links proportionally to the flow demand of each downstream link, provided that the intermediate node carries a nontrivial amount of demanded flow; the outflow is split uniformly if the intermediate node transfers zero demanded flow. Notice that other allocation rules may be considered (e.g., where splits are non-uniform when  $\theta_j = 0$ ).

#### Remark 9 (An online path selection mechanism):

According to (13), the routing ratios in the entire network are instantaneously imposed by the demanded path flows (which are governed by the path selection mechanism (11) occurring at the network origin). In turn, this implies that our model describes traffic systems where travelers update their path while they are traversing the network (and do not necessarily follow the path chosen upon entering the network) in the interest of minimizing their travel time to destination.  $\square$

By combining (4) and (11), we obtain the following joint traffic flow model with congestion-responsive routing:

$$\dot{x}(t) = T(x(t), y(t)), \quad (14a)$$

$$\dot{y}(t) = F(x(t), y(t)), \quad (14b)$$

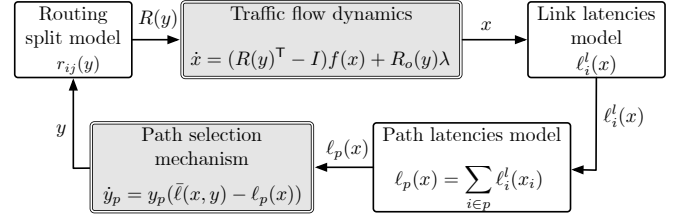


Fig. 3: The proposed model couples a compartmental-like model of traffic flows (see Section II.A) with an economic model of route selection (see Section II.B). Grey-shaded blocks illustrate dynamic models while white blocks illustrate algebraic relationships.

where  $T : \mathbb{R}_{\geq 0}^n \times \text{int}\Delta' \rightarrow \mathbb{R}^n$ ,  $F : \mathbb{R}_{\geq 0}^n \times \text{int}\Delta' \rightarrow \mathbb{R}^p$ , are defined entry-wise, for all  $i \in \{1, \dots, n\}$  and  $p \in \mathcal{P}'$ , as:

$$T_i(x, y) = r_{oi}(y)\lambda + \sum_{j \in \mathcal{L}} r_{ji}(y)f_j(x_j) - f_i(x_i),$$

$$F_p(x, y) = \eta y_p(\bar{\ell}(x, y) - \ell_p(x)).$$

Here, the scalar  $\eta > 0$  is a design parameter that we have introduced to modify the rate at which path preferences are updated. When the equation (14b) describes the behavior of users following routing recommendations provided by a navigation system,  $\eta$  can be modified by deciding the frequency at which travel recommendations are updated. For this reason, in what follows, we refer to  $\eta$  to as *imitation rate*. We illustrate the interconnection (14) and the quantities that establish the coupling between the two models in Fig. 3.

We next introduce some basic notation that will be used in the remainder. Since (13) ensures conservation of flows at the nodes, it guarantees that the vector of link flows  $y^l$  is an equilibrium for (4), namely, at all times:

$$0 = (R(y)^T - I)y^l + R_o(y)\lambda. \quad (15)$$

From (15), we deduce that a set of demanded path flows  $y$  implicitly defines a set of *demanded densities* corresponding to these flows. These are defined as:

$$\varphi(y) := f^{-1}(y^l), \quad \text{where } y_i^l = \sum_{p \in \mathcal{P}': i \in p} y_p, \quad \forall i \in \mathcal{L},$$

and  $f^{-1} : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}_{\geq 0}^n$  denotes the entrywise inverse function of  $f(\cdot)$ . In words, the function  $\varphi(y)$  maps a vector of demanded flows into the corresponding (demanded) densities. Similarly to demanded flows, demanded densities are virtual densities, which may never be observed in the network, describing the amount of traffic density needed to support the instantaneous demanded flows  $y(t)$ .

We conclude this section by stressing that the flows on the links  $f(x(t))$  imposed by the traffic dynamics differ from the demanded flows  $y^l(t)$ , which are imposed by the path selection model. We discuss in Remark 10 the important differences between these two quantities.

#### Remark 10 (Demanded flows vs actual flows):

It is important to stress a conceptual difference between “demanded” traffic variables and traffic variables imposed by

the traffic dynamics. Regarding traffic flows, the vector of *traffic flows*  $f(x(t))$  describes the flows on the links imposed by the traffic dynamics; on the other hand, the vector of *demanded traffic flows*  $y^l(t)$  describes the fraction of flow demand  $\lambda$  entering at  $o$  and that is routed to the links based on an economic process of path selection. Analogously, the *traffic densities*  $x(t)$  are quantities that are imposed by the physics, while the *demanded traffic densities*  $\varphi(y)$  are virtual quantities describing the densities associated with the traffic demand. Importantly, the *traffic latencies*  $\ell(x)$  describe the actual travel latencies imposed by the physics, which in general differ from the *demanded traffic latencies*  $\ell(\varphi(y))$ . This discrepancy differentiates our framework from the classical routing game [5], where the dynamics of traffic are infinitely fast.  $\square$

#### D. Connections with game-theoretic framework

In this section, we show that our framework can be related to a population game [34]. This will allow us to connect our settings to the routing game [8] and to relate the equilibrium points of the model (14) to Wardrop equilibria [25].

The replicator equation (9) naturally defines an associated population game [34], as described next. A (cost-minimization) population game is defined by the triple  $(\mathcal{S}, \mathcal{X}, \kappa)$ , where  $\mathcal{S}$  is a set of pure strategies,  $\mathcal{X}$  is a (generalized) simplex, and  $\kappa : \mathcal{X} \rightarrow \mathbb{R}^{|\mathcal{S}|}$  is a vector-valued cost function describing the cost associated with each strategy, see [34, Sec. 13.2]. The replicator equation (9) implicitly defines a population game defined by

$$\mathcal{S} = \mathcal{P}', \quad \mathcal{X} = \Delta', \quad \kappa(y) = \ell \circ \varphi(y), \quad (16)$$

which in what follows we denote by  $\mathcal{R}_{\Delta'} := (\mathcal{P}', \Delta', \ell \circ \varphi)$ . In line with the existing literature [34], we will call a vector of the simplex  $y \in \Delta'$  a (mixed) *strategy*. To this end, we will say that a strategy  $y_{br} \in \Delta'$  is a *best reply* to  $y$  if:

$$y_{br}^T \ell(\varphi(y)) \leq w^T \ell(\varphi(y)), \quad \forall w \in \Delta'.$$

Associated with  $\mathcal{R}$ , we have the following classical notion.

#### Definition 11 (Nash Equilibrium):

A vector  $y^* \in \Delta'$  is said to be a *Nash equilibrium* of  $\mathcal{R}_{\Delta'}$  if

$$\langle y^*, \ell(\varphi(y^*)) \rangle \leq \langle y, \ell(\varphi(y^*)) \rangle, \quad \forall y \in \Delta'. \quad (17)$$

$\square$

In other words, a Nash equilibrium is a best reply to itself. By noting that  $y^T \ell(\varphi(y))$  is the average population latency or cost (cf. (10)), a Nash equilibrium describes a situation where the population has no incentive to deviate away from strategy  $y$  as any other strategy will yield a non-smaller latency. Nash equilibria are used to describe routing games governed by selfish individuals, where each individual chooses their path to minimize their travel cost.

A very useful reformulation of the notion of Nash equilibrium is that of Wardrop equilibrium [35]:  $y$  is a Wardrop

equilibrium if, for all  $p \in \mathcal{P}'$ ,

$$y_p > 0 \text{ implies } \ell_p(\varphi(y)) \leq \ell_{p'}(\varphi(y)), \quad \forall p' \in \mathcal{P}'. \quad (18)$$

In line with the findings of [35], in what follows we will use the wording Nash equilibrium and Wardrop equilibrium interchangeably.

A desirable property for Nash equilibria is that of *evolutionary stability*. Intuitively, a strategy  $y \in \Delta'$  is evolutionary stable if it is a Nash equilibrium and small perturbations from this strategy have a strictly larger average latency.

#### Definition 12 (Evolutionary stable point):

A vector  $y \in \Delta'$  is said to be an *evolutionary stable point* of  $\mathcal{R}_{\Delta'}$  if it is a Nash equilibrium and, for all  $w \in \Delta', w \neq y$ ,

$$w^T \ell(\varphi(y)) = y^T \ell(\varphi(y)) \text{ implies } w^T \ell(w) > y^T \ell(\varphi(w)). \quad (19)$$

$\square$

In words,  $y$  is evolutionary stable if any other best response  $w$  to  $y$  is not a Nash equilibrium. It is worth stressing that evolutionary stability is a property of the game  $\mathcal{R}_{\Delta'}$  as it is defined independently of the choice of the vector field in (9).

We conclude this section with an important observation, which highlights the novelty of the model in Section II with respect to the classical routing game framework [8]. We remark that, in the routing game, both the traffic and path selection mechanisms operate at the Nash equilibrium [8] at all times. This requirement implicitly makes two highly limiting assumptions: (i) the highways have trivial (infinitely fast) dynamics so that the traffic flows can be modeled as an algebraic map  $\varphi(y)$  of the flow demands; (ii) there are no transients in the path selection process, so that the path preferences can be described by a Nash equilibrium (17) at all times. Remarkably, when the routing game framework was proposed in the 1950s [8], travelers could update their routing preferences only from day to day and networks would often operate near equilibrium as traffic demands would change slowly. In contrast, in modern networks, travelers can update their routing preferences at a faster timescale, as they have access to instantaneous real-time traffic information, and traffic demands are highly dynamic. Hence, we conjecture that the model proposed here is a more accurate description of modern traffic systems.

### III. Properties of the equilibrium points

In this section, we study the properties of the equilibrium points of (14). We begin by showing that solutions to (14) are well-defined.

#### Proposition 13 (Well-posedness of solutions):

Let Assumptions 1-2 hold and  $(x_0, y_0) \in \mathbb{R}_{\geq 0}^n \times \text{int}\Delta'$ . There exists a unique solution  $(x(t), y(t)) \in \mathbb{R}_{\geq 0}^n \times \text{int}\Delta' \forall t \in \mathbb{R}_{\geq 0}$  to (14) with  $x(0) = x_0$  and  $y(0) = y_0$ .

*Proof:*

Notice that, in the  $\text{int}\Delta'$ , the function  $r_{ij}(y)$  in (13) is continuously differentiable since  $\theta_j > 0 \forall j$ . This implies that, under Assumptions 1-2, the vector field in (14) is Lipschitz continuous everywhere in its domain (namely,  $\mathbb{R}_{\geq 0}^n \times \text{int}\Delta'$ ). By [36, Thm 3.1] existence and uniqueness of solutions to (14) follows. Finally,  $(x(t), y(t)) \in \mathbb{R}_{\geq 0}^n \times \text{int}\Delta'$  follows from Remark 8 and (12). ■

**Remark 14 (Lipschitz continuity of the vector fields (14)):**

Note that, from Remark 8, the set  $\mathbb{R}_{\geq 0}^n \times \text{int}\Delta'$  is forward-invariant for (14). Note also (cf. proof of Proposition 13) that the vector fields  $T(\cdot, \cdot)$ , and  $F(\cdot, \cdot)$  of (14) are Lipschitz continuous everywhere in their domain of definition. Altogether, these two properties guarantee that the state variables  $(x, y)$  do not leave the set  $\mathbb{R}_{\geq 0}^n \times \text{int}\Delta'$  and that the vector fields are Lipschitz continuous everywhere in this set. □

**A. Existence of fixed points**

We begin by investigating under what conditions the interconnected model (14) admits equilibrium points. Interestingly, we will show that their existence depends solely on the magnitude of external inflows entering the network. To this end, the min-cut capacity of the traffic flow model is:

$$C_{\text{cut}} = \min_{\substack{S \subset \mathcal{V}: \\ o_i \in S, d_i \notin S}} \sum_{\substack{i \in \mathcal{L}: \\ o_i \in S, d_i \notin S}} C_i.$$

Notice that  $C_{\text{cut}}$  may or may not be finite, precisely,  $C_{\text{cut}} \in [0, +\infty]$ .

**Proposition 15 (Existence of equilibria):**

Let Assumptions 1 and 2 be satisfied. If  $\lambda < C_{\text{cut}}$ , then the interconnected system (14) admits an equilibrium point that is a Nash equilibrium. Conversely, if  $\lambda > C_{\text{cut}}$ , then, no equilibrium point exists for (14). □

*Proof:*

(Case  $\lambda < C_{\text{cut}}$ ) To prove this implication, we show the existence of a point that satisfies the Wardrop conditions and that is an equilibrium of (14). Following [5, Thm. 2.1], a vector of path flows  $\bar{y} \in \mathbb{R}^{|\mathcal{P}'|}$  is a Wardrop equilibrium if and only if it satisfies the first-order optimality conditions of the following optimization problem:

$$\min_{y_1, \dots, y_{|\mathcal{P}'|} \in \mathbb{R}} \sum_{i \in \mathcal{L}} \int_0^{y_i^l} \ell_i^l(s) ds, \quad (20a)$$

$$\text{s. to} \quad \sum_{p \in \mathcal{P}'} y_p = \lambda, \quad (20b)$$

$$y_p \geq 0, \quad \forall p \in \mathcal{P}', \quad (20c)$$

$$\sum_{p \in \mathcal{P}': i \in \mathcal{P}'} y_p = y_i^l, \quad \forall i \in \mathcal{L}, \quad (20d)$$

In (20),  $y_1^l, \dots, y_n^l$  are dependent variables (describing link flows) that are uniquely determined by (20d) (see [5, Thm

2.2]). Since the objective function is continuous (cf. Assumption 2), according to Weierstrass' Theorem, it admits a minimum provided that the feasible set is closed, bounded, and nonempty. To see that the feasible set of (20) is bounded, notice that from (20b):

$$y_i = \lambda - \sum_{p \in \mathcal{P}': p \neq i} y_p \leq \lambda,$$

from the positiveness of the path flow variables. Hence, the feasible set can be made closed and bounded by adding the constraints  $y_p \leq \lambda \forall p \in \mathcal{P}'$  without affecting the solution. Since  $\lambda < C_{\text{cut}}$ , by the max-flow min-cut theorem [37, Thm 4.1], the feasible set is nonempty. Hence, by Weierstrass' Theorem, the game  $\mathcal{R}_{\Delta'}$  admits a Nash equilibrium.

Let  $y^*$  denote a Nash equilibrium of  $\mathcal{R}_{\Delta'}$ ; we next show that  $y^*$  is an equilibrium flow for (14a). Let  $R(y^*)$  be the routing matrix obtained from  $y^*$  via (13); by using (15), we have

$$(R(y^*)^T - I)\varphi(y^*) + \lambda = 0,$$

and thus we conclude that the pair  $(x^*, y^*)$ ,  $x^* := \varphi(y^*)$ , is an equilibrium of (14a). We are left to show that  $(x^*, y^*)$  is also an equilibrium of (14b). Since  $y^*$  is a Nash equilibrium, it satisfies:

$$\ell_p(\varphi(y^*)) = c, \quad \forall p \in \mathcal{P}' : \bar{y}_p > 0,$$

and thus we have  $\bar{\ell}(\varphi(y^*), y^*) = c$ . This proves that  $(x^*, y^*)$  is an equilibrium of (14b).

(Case  $\lambda > C_{\text{cut}}$ ) By contradiction, assume that an equilibrium point  $(x^*, y^*)$  exists. Because the replicator equation guarantees  $y(t) \in \Delta' \forall t \geq 0$ , we must have

$$\sum_{p \in \mathcal{P}'} y_p^* = \lambda \text{ and } y_p^* \geq 0, \quad \forall p \in \mathcal{P}'.$$

Under these two conditions, the max-flow min-cut theorem is applicable, which guarantees that, for some  $i \in \mathcal{L}$ ,

$$\sum_{p \in \mathcal{P}': i \in \mathcal{P}'} y_p^* > C_i, \quad (21)$$

but this contradicts the equilibrium condition  $(R(y^*)^T - I)\varphi(y^*) + \lambda = 0$ , thus proving the claim. ■

In words, fixed points exist when the external flow demand is bounded above by the min-cut capacity; moreover, at least one equilibrium point is a Wardrop equilibrium. This has two important implications. First, it ensures that our model is consistent with the classical literature, in particular, with the established notion of Wardrop equilibrium. Importantly, while Wardrop equilibria were developed for static models operating at equilibrium, our model instead is a dynamic generalization of this classical framework [5]. Second, the result relates our work with the fundamental bound concerning the maximum amount of flow transferable by a static graph (as given by the max-flow min-cut theorem [37]): it shows that traffic systems where users learn through imitation can transfer, asymptotically, the same amount of flow as static graphs with arbitrary routing. This implies that imitation-based routing benefits the overall traffic system, enabling it to



transfer the maximum admissible amount of flow. We remark that this property is in contrast with dynamic traffic flow models with static routing, which may not admit equilibrium points even when  $\lambda < C_{\text{cut}}$  (see, e.g., [15], [16, Prop. 2]).

### B. Conditions for uniqueness of the Nash equilibrium

While Proposition 15 guarantees existence of a Nash equilibrium, it remains unclear whether such an equilibrium is unique or evolutionary stable. We address this aspect next.

#### Proposition 16 (Uniqueness and evolutionary stability):

Let Assumptions 1–2 be satisfied and  $\mathcal{R}_{\Delta'}$  be the game induced by (9) and defined by (16). Further, assume that the latency functions are strictly monotone, namely, for all  $i \in \mathcal{L}$ ,

$$(x_i - \bar{x}_i)(\ell_i^l(x_i) - \ell_i^l(\bar{x}_i)) > 0, \quad (22)$$

for all  $x_i, \bar{x}_i \in [0, C_i], x_i \neq \bar{x}_i$ . Then, the game  $\mathcal{R}_{\Delta'}$  admits a unique Nash equilibrium. Moreover, such equilibrium is evolutionary stable.

The following lemma is instrumental for the proof.

#### Lemma 1 (Strict monotonicity of the flow latencies):

Under the assumptions of Proposition 16, the demanded path flow latency functions are strictly monotone, namely,

$$\langle y - \bar{y}, \ell(\varphi(y)) - \ell(\varphi(\bar{y})) \rangle > 0, \quad \forall y, \bar{y} \in \Delta', y \neq \bar{y}. \quad (23)$$

□

*Proof:*

We have:

$$\begin{aligned} & \langle y - \bar{y}, \ell(\varphi(y)) - \ell(\varphi(\bar{y})) \rangle \\ &= \sum_{p \in \mathcal{P}'} (y_p - \bar{y}_p) (\ell_p(\varphi(y)) - \ell_p(\varphi(\bar{y}))) \\ &= \sum_{p \in \mathcal{P}'} (y_p - \bar{y}_p) \left( \sum_{i \in p} \ell_i^l(\varphi_i(y)) - \sum_{i \in p} \ell_i^l(\varphi_i(\bar{y})) \right) \\ &= \sum_{i \in \mathcal{L}} \sum_{p \in \mathcal{P}': i \in p} (y_p - \bar{y}_p) (\ell_i^l(\varphi_i(y)) - \ell_i^l(\varphi_i(\bar{y}))) \\ &= \sum_{i \in \mathcal{L}} (y_i^l - \bar{y}_i^l) (\ell_i^l(\varphi_i(y)) - \ell_i^l(\varphi_i(\bar{y}))), \end{aligned}$$

where the second identity follows from (7) and the fourth identity from (6). Next, let  $i \in \mathcal{L}$  be fixed, and distinguish among three cases. (Case 1) Assume  $y_i^l > \bar{y}_i^l$ , we have:

$$(y_i^l - \bar{y}_i^l) (\ell_i^l(\varphi_i(y)) - \ell_i^l(\varphi_i(\bar{y}))) > 0, \quad (24)$$

since  $\varphi_i$  and  $\ell_i^l$  are strictly increasing. (Case 2) Assume  $y_i^l < \bar{y}_i^l$ . In this case, (24) also holds since  $\varphi_i$  and  $\ell_i^l$  are strictly increasing. (Case 3) Assume  $y_i^l = \bar{y}_i^l$ . We have:

$$(y_i^l - \bar{y}_i^l) (\ell_i^l(\varphi_i(y)) - \ell_i^l(\varphi_i(\bar{y}))) = 0.$$

Since  $y \neq \bar{y}$ , there exists at least one link  $i \in \mathcal{L}$  for which (24) is satisfied, from which we conclude that (23) holds. ■

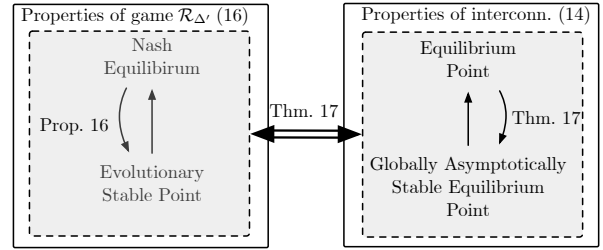


Fig. 4: Summary of the properties established in the main results of this paper. Proposition 16 shows existence of a unique Nash equilibrium and that such equilibrium is evolutionary stable; Theorem 17 shows that the unique Nash equilibrium of  $\mathcal{R}_{\Delta'}$  is also an equilibrium point of (14) and that such a point is globally asymptotically stable.

We are now ready to prove the proposition

*Proof of Proposition 16:* Let  $y^* \in \Delta'$  denote a Nash equilibrium of  $\mathcal{R}_{\Delta'}$  and  $y \in \Delta'$ . Using (17), we have

$$\langle y^*, \ell(\varphi(y^*)) \rangle + \langle y, \ell(\varphi(y)) \rangle \leq \langle y, \ell(\varphi(y^*)) \rangle + \langle y^*, \ell(\varphi(y)) \rangle,$$

by re-arranging:

$$\begin{aligned} \langle y, \ell(\varphi(y)) \rangle &\geq \langle y^*, \ell(\varphi(y^*)) \rangle + \langle y, \ell(\varphi(y)) - \ell(\varphi(y^*)) \rangle \\ &> \langle y^*, \ell(\varphi(y^*)) \rangle + \langle y^*, \ell(\varphi(y)) - \ell(\varphi(y^*)) \rangle \\ &= \langle y^*, \ell(\varphi(y)) \rangle, \end{aligned} \quad (25)$$

where the second row follows from (23) and the third row follows from re-arranging the terms. Inequality (25) proves (19), thus showing that  $y^*$  is evolutionary stable. Finally, since the above condition holds for all  $y \in \Delta'$ , no other point in  $\Delta'$  other than  $y^*$  can satisfy (17), thus proving uniqueness. ■

Proposition 16 shows that, under an additional monotonicity requirements on the latency functions, the game  $\mathcal{R}_{\Delta'}$  admits a unique Nash equilibrium that is evolutionary stable. See Fig. 4 (left) for a summarizing illustration of the properties proven for  $\mathcal{R}_{\Delta'}$ . We stress that uniqueness and evolutionary stability are properties of the Nash equilibrium of the game  $\mathcal{R}_{\Delta'}$ , (and not of the joint dynamics (14)). However, we will show in the next section that these properties can be harnessed to study the asymptotic properties of the trajectories of (14).

## IV. Asymptotic stability of the Nash equilibrium

In this section, we study the stability properties of the equilibrium points of the interconnection (14).

### A. Motivating example: existence of period orbits

It is a known result [38] that the solution trajectories of compartmental models (as in (4)) with static routing are not oscillatory orbits [38, Thm. 1]. Similarly, (under suitable monotonicity-type assumptions) the trajectories of the replicator equation with algebraic latency functions are also known to converge asymptotically to a Nash equilibrium [31, Sec. 3]. Interestingly, when the two models are intercon-

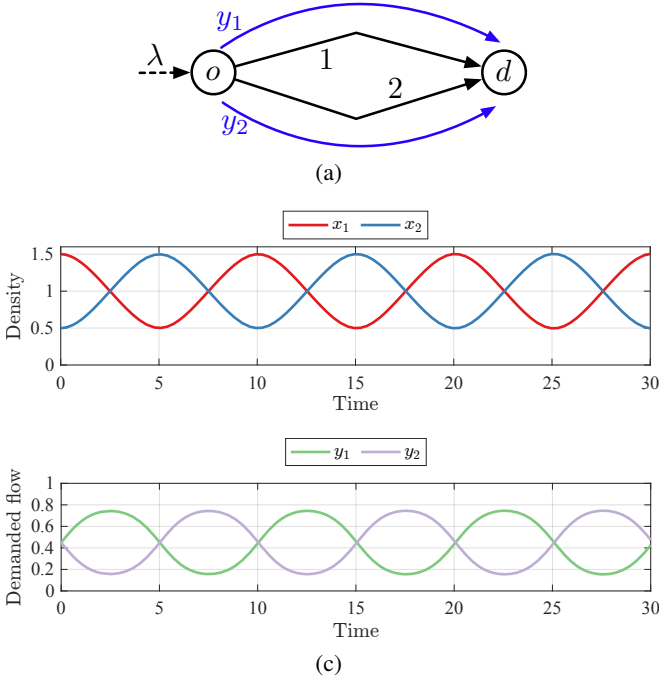


Fig. 5: (a) Network studied in Section IV.A. (b) Simulation of (26)-(27) with  $\lambda = 1$ . The figure illustrates that, in some parameter configurations, this network may admit periodic orbits.

nected as in (14), oscillatory solutions can emerge, as we illustrate through an example next.

Consider the two-link network illustrated in Fig. 5(a). The origin-destination paths are  $p_1 = (1)$  and  $p_2 = (2)$ ; the traffic dynamics (4) are:

$$\dot{x}_1 = -f_1(x_1) + r_{o1}\lambda, \quad \dot{x}_2 = -f_2(x_2) + r_{o2}\lambda. \quad (26)$$

Let the latency functions be  $\ell_1^l(x_1) = x_1$  and  $\ell_1^l(x_2) = x_2$ ; the corresponding path latencies are  $\ell_1(x) = x_1, \ell_2(x) = x_2$ . In this case, the replicator model (9) simplifies to:

$$\begin{aligned} \dot{y}_1 &= y_1(1 - \lambda^{-1}y_1)(x_2 - x_1), \\ \dot{y}_2 &= y_2(1 - \lambda^{-1}y_2)(x_1 - x_2). \end{aligned} \quad (27)$$

It follows from (27) that the unique equilibrium  $(x^*, y^*)$  in  $\text{int}\Delta'$  is given by  $x_1^* = x_2^*$ . Let the flow functions be:

$$f_1(x_1) = \min\left\{x_1, \frac{\lambda}{2}\right\}, \quad f_2(x_2) = \min\left\{x_2, \frac{\lambda}{2}\right\}.$$

From (26), we have that the equilibrium flows are  $y_1^* = f_1(x_1^*) = \lambda/2, y_2^* = f_2(x_2^*) = \lambda/2$ . Hence, (26)-(27) admits an equilibrium given by  $(x_1^*, x_2^*, y_1^*, y_2^*) = (\frac{\lambda}{2}, \frac{\lambda}{2}, \frac{\lambda}{2}, \frac{\lambda}{2})$ .

As a tool to investigate whether the trajectories are periodic orbits, consider the continuously differentiable function:

$$V(x, y) = \frac{1}{2}(x_2 - x_1)^2 + \lambda \ln\left(\frac{\lambda}{y_1}\right) + \lambda \ln\left(\frac{\lambda}{y_2}\right), \quad (28)$$

Defining the compact notation  $\tilde{x} := x_2 - x_1$ , the time-derivative of  $V(x, y)$  along the trajectories of (26)-(27) is:

$$\begin{aligned} \dot{V}(x, y) &= \tilde{x}\dot{\tilde{x}} - \frac{\lambda}{y_1}\dot{y}_1 - \frac{\lambda}{y_2}\dot{y}_2 \\ &= -\tilde{x}(f(x_2) - f(x_1)) + \lambda\tilde{x}(1 - 2r_{o1}) \\ &\quad - \lambda\tilde{x}(1 - \lambda^{-1}y_1) + \lambda\tilde{x}(1 - \lambda^{-1}y_2) \\ &= -\tilde{x}(f(x_2) - f(x_1)) + \lambda\tilde{x}(1 - 2r_{o1}) \\ &\quad - \lambda\tilde{x}(1 - 2\lambda^{-1}y_1) \\ &= -(x_2 - x_1)(f(x_2) - f(x_1)). \end{aligned}$$

Hence, by defining the region

$$M' = \left\{x_1, x_2 \in \mathbb{R}_{\geq 0} : x_1 \geq \frac{\lambda}{2} \text{ and } x_2 \geq \frac{\lambda}{2}\right\},$$

the trajectories of (26)-(27) are trapped inside  $M'$  since at the boundary of  $M'$  we have  $\dot{V}(x, y) = 0$ . Next, let

$$M = M' \cap \{(x, y) : V(x, y) = c\},$$

where  $c \geq 2\lambda \ln(2)$ . Notice that the unique equilibrium point of this system is characterized by  $c = 2\lambda \ln(2)$ . By choosing  $c > 2\lambda \ln(2)$ , any trajectory starting in  $M$  stays in  $M$  for all future times and  $M$  contains no equilibrium point, hence, by application of the Poincaré-Bendixson criterion [36, Lem. 2.1], we conclude that  $M$  contains a periodic orbit. The periodic orbits of this model are illustrated in Fig. 5. We anticipate that the existence of periodic is connected to the failure of our assumptions (cf. (2)); notice also that a modification of (28) will be used as a Lyapunov function shortly below (cf. Section IV.B).

### B. Sufficient conditions for asymptotic stability

Motivated by the findings above, we reinforce Assumption 2 as follows.

#### Assumption 3:

The conditions in Assumption 2 are satisfied. Moreover, the latency functions are strongly monotone, namely, there exists  $\sigma > 0$  such that

$$(x_i - \bar{x}_i)(\ell_i^l(x_i) - \ell_i^l(\bar{x}_i)) \geq \sigma|x_i - \bar{x}_i|^2, \quad (29)$$

for all  $x_i, \bar{x}_i \in [0, C_i)$  and  $i \in \mathcal{L}$ .  $\square$

In words, the assumption asks that the latency functions grow at least linearly with the traffic densities<sup>1</sup>; the parameter  $\sigma$  quantifies the ‘‘steepness’’ of the density-latency maps. In what follows, we will interpret  $\sigma$  as a free parameter, which can be tuned by a system planner to improve the efficiency of a traffic system modeled by (14). The following result characterizes the asymptotic behavior of (14).

#### Theorem 17 (Stability of interconnected system):

Let assumptions 1 and 3 hold and  $\lambda < C_{\text{cut}}$ . Let  $(x(t), y(t))$  denote the solution of (14) with initial conditions

<sup>1</sup>Indeed, strong monotonicity of  $\ell_i^l(x_i)$  is equivalent to imposing that  $\ell_i^l(x_i) - \sigma x_i$  is a monotone function (this follows by rewriting the inequality as  $(x_i - \bar{x}_i)(\ell_i^l(x_i) - \sigma x_i) - (\ell_i^l(\bar{x}_i) - \sigma \bar{x}_i) \geq 0$ ).

$(x(0), y(0))$ ,  $\Delta'$  the restricted simplex induced by  $y(0)$ ,  $\mathcal{R}_{\Delta'}$  the game defined by (16), and  $y^*$  the unique Nash equilibrium of  $\mathcal{R}_{\Delta'}$ . There exists  $\sigma^*, \eta_1, \eta_2 > 0$  such that when  $\sigma$  and  $\eta$  satisfy:

$$\sigma > \sigma^*, \quad \eta \in [\eta_1, \eta_2], \quad (30)$$

then, for any  $(x(0), y(0)) \in \mathbb{R}_{\geq 0}^n \times \Delta'$ ,

$$\lim_{t \rightarrow \infty} \|(x(t), y(t)) - (x^*, y^*)\| = 0,$$

where  $x^* := \varphi(y^*)$ .  $\square$

The following lemma is a minor extension of Lemma 1 under Assumption 3, and is instrumental for the proof.

**Lemma 2 (Strong monotonicity of the flow latencies):**

When Assumptions 1 and 3 hold, the path flow latency functions are strongly monotone, namely, there exists  $\sigma > 0$ :

$$\langle y - \bar{y}, \ell(\varphi(y)) - \ell(\varphi(\bar{y})) \rangle \geq \sigma \|y - \bar{y}\|^2, \quad \forall y, \bar{y} \in \Delta'. \quad (31)$$

$\square$

*Proof of Theorem 17:* Our proof technique relies on showing that the potential function

$$V(x, y) := V_x(x) + V_y(y), \quad (32)$$

where  $V_x(x)$  is a potential function for (14a) and  $V_y(y)$  is a potential function for (14b) strongly decreases along the trajectories of (14) and achieves its minimum at  $(\varphi(y^*), y^*)$ . We will use the following compact notation:

$$A(y) := R(y)^\top - I, \quad \phi(y) := -A(y)^{-1} R_o(y) \lambda,$$

with  $\phi(y) = (\phi_1(y), \dots, \phi_n(y))$ . Since  $\mathcal{G}$  is outflow connected, [39, Thm. 3] guarantees that  $A(y)$  is invertible for any  $y$ . Since  $-A(y^*)$  is a nonsingular M-matrix, [40, Prop. I<sub>25</sub>] guarantees the existence of a positive diagonal matrix  $D = \text{diag}(d_1, \dots, d_n)$  such that

$$\tilde{Q} := -(A(y^*)D + DA(y^*)^\top),$$

is symmetric and positive definite. It follows that the matrix

$$Q := D^{-1} \tilde{Q} D^{-1}, \quad (33)$$

is also positive definite. Let

$$V_x(x) := 2 \sum_{i \in \mathcal{L}} d_i^{-1} \int_0^{x_i} f_i(s) - \phi_i(y^*) ds. \quad (34)$$

The time-derivative of  $V_x(x)$  along the solutions of (14) is:

$$\begin{aligned} \dot{V}_x(x) &= 2(f(x) - \phi(y^*))D^{-1}(A(y)f(x) + R_o(y)\lambda) \\ &= 2(f(x) - \phi(y^*))^\top D^{-1}(A(y^*)f(x) + R_o(y^*)\lambda) \\ &\quad + 2(f(x) - \phi(y^*))^\top D^{-1}(\psi_x(y) - \psi_x(y^*)) \\ &= -(f(x) - \phi(y^*))^\top Q(f(x) - \phi(y^*)) \\ &\quad + 2(f(x) - \phi(y^*))^\top D^{-1}(\psi_x(y) - \psi_x(y^*)) \\ &\leq -\mu \lambda_{\min}(Q) \|x - \varphi(y^*)\|^2 + k \|x - \varphi(y^*)\| \|y - y^*\| \\ &\leq -\frac{\mu \lambda_{\min}(Q)}{2} \|x - \varphi(y^*)\|^2 + \frac{k^2}{2\mu \lambda_{\min}(Q)} \|y - y^*\|^2. \end{aligned} \quad (35)$$

Here, in the second row, we used the compact notation

$$\psi_x(y) := A(y)f(x) + R_o(y)\lambda,$$

the third row follows from (33). The fourth row follows from strong monotonicity of the flow functions (where  $\mu$  is as in Assumption 1) and by using the Cauchy-Schwarz inequality and by noting that  $\psi_x(y)$  is Lipschitz continuous in  $y$ , uniformly in  $x$ , and by letting  $k = 2\|D^{-1}\|L_\psi L_f$ , where  $L_\psi$  and  $L_f$  denote the Lipschitz constants of  $\psi_x(\cdot)$  and  $f(\cdot)$ , respectively (see the proof of Proposition 13 and Remark 14 for a discussion on the Lipschitz property). The fifth row follows from the inequality  $-ax^2 + bx \leq b^2/4a$  for  $a, b > 0, x \in \mathbb{R}$ .

Next, we let

$$V_y(y) = \sum_{p \in \mathcal{P}'} y_p^* \ln \left( \frac{y_p}{y_p^*} \right).$$

The time-derivative of  $V_y(y)$  along the solutions of (14) is given by:

$$\begin{aligned} \dot{V}_y(y) &= -\eta \sum_p y_p^* (\bar{\ell}(x, y) - \ell_p(x)) \\ &= -\eta \lambda \bar{\ell}(x, y) + \eta \sum_p y_p^* \ell_p(x) \\ &= -\eta \sum_p (y_p - y_p^*) \ell_p(x) \\ &= -\eta \sum_p (y_p - y_p^*) \ell_p(\varphi(y)) \\ &\quad - \eta \sum_p (y_p - y_p^*) (\ell_p(x) - \ell_p(\varphi(y))) \\ &\leq -\eta \sigma \|y - y^*\|^2 + \eta L_\ell \|y - y^*\| \|x - \varphi(y)\| \end{aligned} \quad (36)$$

Here, the second row follows from  $\sum_p y_p^* = \lambda$ ; the third row follows from (10); the fourth row from adding and subtracting  $\ell_p(\varphi(y))$ . The fifth row follows by application of the Cauchy-Schwarz inequality, by using continuity of  $\ell(\cdot)$  (where  $L_\ell$  denotes the corresponding Lipschitz constant), and from the following inequality:

$$\langle y - y^*, \ell(\varphi(y)) \rangle \geq \sigma \|y - y^*\|^2. \quad (37)$$

To prove (37), since  $y^*$  is a Nash equilibrium, we have from (17):

$$\langle y^*, \ell(\varphi(y^*)) \rangle + \langle y, \ell(\varphi(y)) \rangle \leq \langle y, \ell(\varphi(y^*)) \rangle + \langle y, \ell(\varphi(y)) \rangle,$$

by re-arranging:

$$\begin{aligned} \langle y, \ell(\varphi(y)) \rangle &\geq \langle y^*, \ell(\varphi(y^*)) \rangle + \langle y, \ell(\varphi(y)) - \ell(\varphi(y^*)) \rangle \\ &> \langle y^*, \ell(\varphi(y^*)) \rangle + c \|y - y^*\|^2 + \langle y^*, \ell(\varphi(y)) - \ell(\varphi(y^*)) \rangle \\ &= \langle y^*, \ell(\varphi(y)) \rangle + c \|y - y^*\|^2, \end{aligned} \quad (38)$$

where the second row follows from Lemma 2. This proves (37). We can further bound (36) as:

$$\begin{aligned} \dot{V}_y(y) &\leq -\eta(\sigma - L_\ell L_\varphi) \|y - y^*\|^2 \\ &\quad + \eta L_\ell \|y - y^*\| \|x - \varphi(y^*)\| \\ &\leq -\eta \left( \frac{\sigma}{2} - L_\ell L_\varphi \right) \|y - y^*\|^2 + \frac{\eta L_\ell^2}{2\sigma} \|x - \varphi(y^*)\|^2, \end{aligned} \quad (39)$$

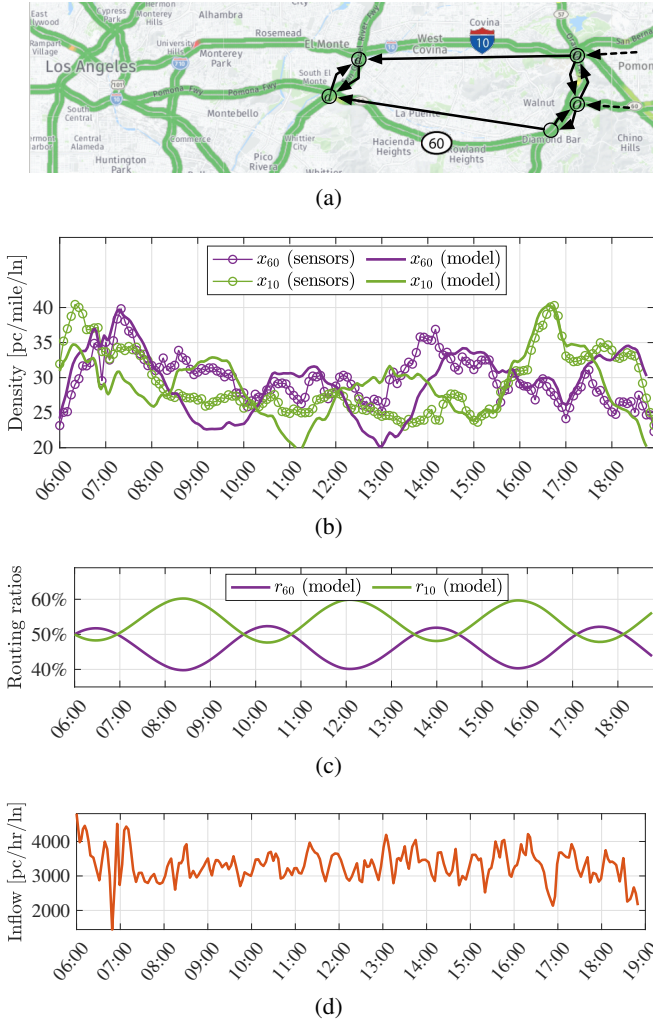


Fig. 6: Time series data for SR60-W and I10-W on March 6, 2020. Results obtained by identifying the parameters of (14) using a prediction-correction algorithm that minimizes the fitting error. (a) Traffic network and graph. (b) Traffic flow data obtained from sensors (continuous lines with circles) and traffic state predicted by our models (continuous lines). (c) Routing predicted by our models. (d) Combined traffic demand entering at the origin, used to simulate the model. The data illustrates a case where the trajectories of (14) oscillate, pointing to the lack of asymptotic stability for the equilibrium.

where the first inequality follows from the Cauchy-Schwarz inequality and by continuity of  $\varphi(\cdot)$  (where  $L_\varphi$  denotes the corresponding Lipschitz constant), and the second row follows from the inequality  $-az^2 + bz \leq b^2/4a$  for  $a, b > 0, x \in \mathbb{R}$ .

By combining (35) and (39) we conclude:

$$\dot{V}(x, y) \leq -c_1 \|x - \varphi(y^*)\|^2 - c_2 \|y - y^*\|^2, \quad (40)$$

where the constants  $c_1$  and  $c_2$  are given by:

$$c_1 := \frac{\mu\lambda_{\min}(Q)}{2} - \frac{\eta L_\ell^2}{2\sigma}, \quad c_2 := \eta \left( \frac{\sigma}{2} - L_\ell L_\varphi \right) - \frac{k^2}{2\mu\lambda_{\min}(Q)}.$$

We thus have that  $c_1 \geq 0$  and  $c_2 \geq 0$  when, respectively,

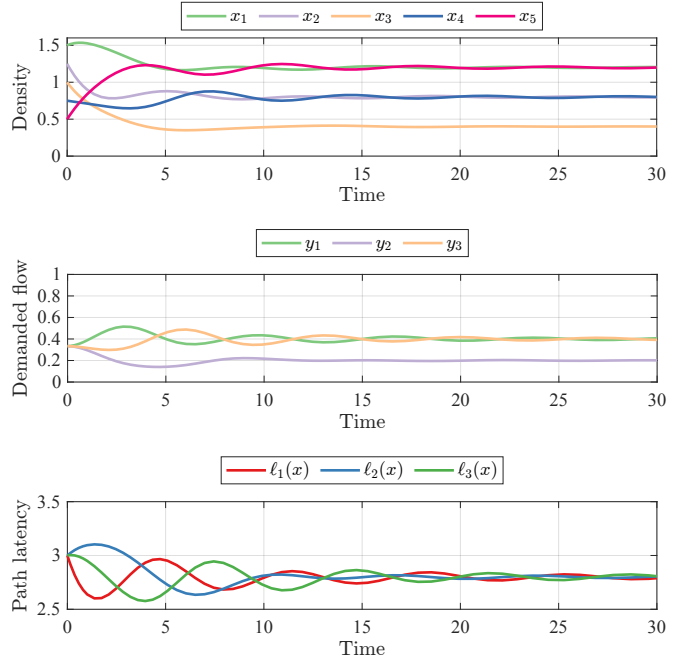


Fig. 7: Time evolution of the state trajectories of the model (14) for the network in Fig. 1 with  $\eta = 1$ . (Top) Evolution of the traffic density state  $x$ . (Middle) Evolution of the demanded path flow state. (Bottom) Evolution of the travel latencies on the paths. The choice  $\eta = 1$  belongs to the range of stabilizing values characterized in Theorem 17, and thus guarantees that the state asymptotically converges to the Nash equilibrium of the underlying game.

$$\eta \leq \eta_2 := \frac{\mu\sigma\lambda_{\min}(Q)}{L_\ell^2}, \quad \eta \geq \eta_1 := \frac{k^2}{\mu\lambda_{\min}(Q)(\sigma - 2L_\ell L_\varphi)}. \quad (41)$$

Thus, there exists a feasible choice of  $\eta$  that guarantees that  $c_1 \geq 0$  and  $c_2 \geq 0$  when  $\sigma > \sigma_1 := 2L_\ell L_\varphi$  and

$$\frac{k^2}{\mu\lambda_{\min}(Q)(\sigma - 2L_\ell L_\varphi)} \leq \frac{\mu\sigma\lambda_{\min}(Q)}{L_\ell^2}. \quad (42)$$

Notice that (42) can always be guaranteed to hold, provided that  $\sigma$  is chosen sufficiently large. To see this, note that  $\sigma$  and  $L_\ell$  are related, such that  $\sigma \leq L_\ell$ . As a worst-case, we consider the case  $\sigma = L_\ell$ . In this case, (42) simplifies to  $\frac{k^2}{\mu\lambda_{\min}(Q)(1-2L_\varphi)} \leq \mu\lambda_{\min}(Q)$ . Since the eigenvalues of  $Q$  can be rescaled by rescaling  $D$  in (33), we conclude that it is always possible to choose  $Q$  such that (42) holds.

Altogether this implies that when  $\sigma > \sigma^*$  – where  $\sigma^* = \max\{\sigma_1, \sigma_2\}$  and  $\sigma_2$  is the smallest value of  $\sigma$  such that (42) holds – and  $\eta \in [\eta_1, \eta_2]$ ,  $V(x, y)$  decreases towards its minimum, given by  $\xi(x, y) = 0$ , which implies  $(x, y) = (\varphi(y^*), y^*)$ . The claim thus follows by application of La Salle's invariance principle [36, Cor. 4.1]. ■

We illustrate in Fig. 4 the relationships between implications. The theorem shows that, provided that the latency functions are sufficiently steep and the imitation rate  $\eta$  is adequately chosen (as in (30)), the trajectories of (14)

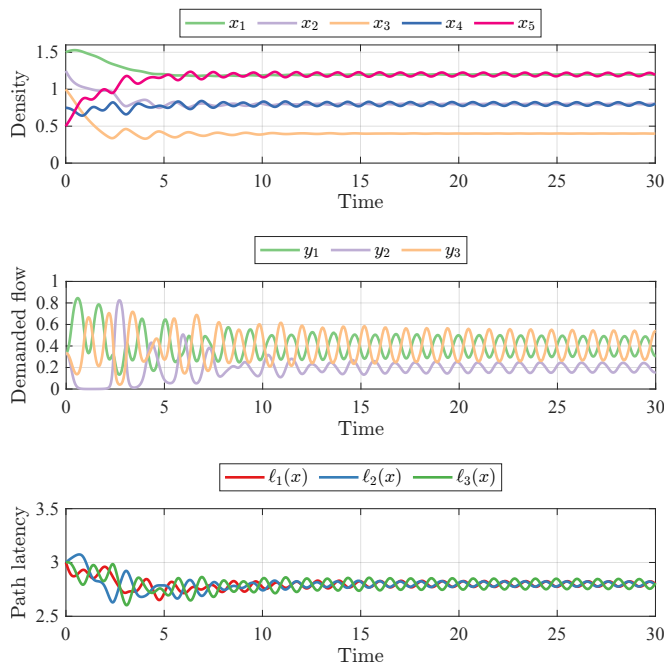


Fig. 8: Time evolution of the state trajectories of the model (14) for the network in Fig. 1 with the choice  $\eta = 1$ . (Top) Evolution of the traffic density state  $x$ . (Middle) Evolution of the demanded path flow state. (Bottom) Evolution of the travel latencies on the paths. The choice  $\eta = 30$  does not belong to the range of stabilizing values characterized in Theorem 17. As illustrated in the simulations, this choice of  $\eta$  originates oscillating trajectories, describing a condition where users repeatedly switch their path preferences.

converge to the unique Nash equilibrium of the game  $\mathcal{R}_{\Delta'}$  from any initial condition. We note that, although the statement provides an existence result for  $\sigma^*$ ,  $\eta_1$ ,  $\eta_2$ , an explicit expression for these quantities is given in the proof in (41) and (42). Intuitively, (42) states that as  $\sigma$  increases, the interval  $[\eta_1, \eta_2]$  becomes wider since  $\eta_1 \rightarrow 0$  and  $\eta_2 \rightarrow +\infty$ . In words, this implies that the steeper the latency functions, the more freedom one has in the choice of  $\eta$ .

Interestingly, the result suggests that asymptotic stability may fail to hold when the latency functions are not sufficiently steep, or the imitation rate is either too small or too large. Intuitively, when  $\sigma$  is small, the path selection process is not sufficiently sensitive to variations of traffic congestion on the links. On the other hand, when  $\eta$  is too large, the population is overreacting to small changes in congestion, and individual users update their preferences without anticipating the strategy of the rest of the population. Finally, a lower bound on  $\eta$  is needed to lower bound the rate of decrease of the replicator model toward its equilibrium.

## V. Simulation Results

This section presents two sets of numerical simulations that illustrate our findings.

### A. Study case from California SR60-W and I10-W

Consider the traffic network in Fig. 6(a), which schematizes the west bounds of the freeways SR60-W and I10-W in Southern California. Let  $x_{60}$  and  $x_{10}$  be the average traffic density in the examined sections of SR60-W (absolute miles 13.1 – 22.4) and in the section of I10-W (absolute miles 24.4 – 36.02), respectively. Moreover, let  $r_{60}$  (resp.  $r_{10} = 1 - r_{60}$ ) be the fraction of travelers choosing freeway SR60-W over I10-W (resp. choosing freeway I10-W over SR60-W) for their commute. Fig. 6(b) illustrates the time-evolution of the recorded traffic densities on the two highways on Friday, March 6, 2020, reconstructed using data from the *Caltrans Freeway Performance Measurement System (PeMS)*; in the same figure, we show the time-evolution of the state of the interconnected model (14). The parameters of the traffic system (4) were derived from the nominal highway characteristics provided by the PeMS. For the routing model (9), the link latency functions are computed by integrating traffic speed data. This data illustrates a case where the trajectories of (14) oscillate over time, implying that the equilibrium points lack to be asymptotically stable; this showcases a scenario where the assumptions of Theorem 17 are not satisfied in practice.

### Remark 18 (Other models could explain Fig. 6):

We remark that several variables affect the behavior of traffic densities in practice (e.g., variable demands, different origin-destination pairs, etc.) and there may exist other viable models that account for these variables and that also explain the data. While all these models are plausible, Fig. 6 shows that there exists a model, with constant inflow, whose state approximately interpolates the available data.  $\square$

### B. Illustrative simulations on synthetic model

Consider the network illustrated in Fig. 1 and discussed in Examples 4-7. Consider a model where  $\lambda = 1$ , for all  $i \in \mathcal{L}$  the outflow functions are linear  $f_i(x_i) = 0.5x_i$ , and the latency functions are given by  $\ell_i(x_i) = x_i, i \in \{1, 3, 5\}$  and  $\ell_i(x_i) = 2x_i, i \in \{2, 4\}$ . Notice that these choices satisfy Assumption 1 and 3. Proposition 15 guarantees that the game  $\mathcal{R}_{\Delta'}$  admits an equilibrium point; by Proposition 13 such equilibrium is unique and evolutionary stable. Solving (20), one obtains the Nash equilibrium  $y^* = (2/5, 1/5, 2/5)$ . It is then possible to use Theorem 17 to determine values of  $\eta$  that guarantee that the trajectories of (14) converge to the Nash equilibrium. For our choices of functions, one can verify by inspection that  $\mu = 0.5, L_f = 0.5, \sigma = 1, L_\ell = 2, \sigma = 1$ . Moreover, we estimated numerically (sampling each variable uniformly in their domain using a Latin Hypercube technique)  $L_\varphi = 0.125, L_\psi = 1.1547$ . We used  $D = 10^2 I$  and obtained matrix  $Q$  (cf. (33)) with  $\lambda_{\min}(Q) = 20$ . This yields  $k = 0.2039$ . With these choices, it is easy to see that (42) is verified, and  $\eta_1 = 2.6667 \cdot 10^{-6}, \eta_2 = 25$ . Fig. 7 illustrates the state trajectories of (14) for  $\eta = 1$ . As guaranteed by Theorem 17, the state trajectories converge to

the Nash equilibrium of the game  $\mathcal{R}_{\Delta'}$ . On the other hand, Fig. 8 illustrates the state trajectories of the interconnected system with the choice  $\eta = 30$ . The simulation demonstrates that an inadequate choice of imitation rate  $\eta$  leads to trajectories that oscillate over time and not approach the Nash equilibrium. The drawbacks of this oscillating phenomenon can be visualized by comparing the path latencies illustrated in the bottom figures of Fig. 7 and Fig. 8. The choice  $\eta = 1$  guarantees that all used paths have the same latency at equilibrium, thus ensuring that all users experience the same travel time. On the other hand, with the choice  $\eta = 30$ , travel latencies are not homogeneous across the three paths, implying that certain users experience a worse travel time and higher congestion. From our simulations, we observed that the amplitude of oscillating trajectories increases with the flow demand  $\lambda$ , thus suggesting that the suboptimality discussed above could deteriorate with increased congestion.

## VI. Conclusions

This paper proposed a dynamic model of traffic and path selection to describe the impact of app-informed travelers in modern traffic networks, where the path selection process occurs at the same timescale as the traffic physics. We studied the properties and stability of the equilibrium points of this model, showing that it is consistent with existing studies in transportation. Our results suggest that the general adoption of navigation systems enables these networks to transfer an amount of flow no smaller than the min-cut capacity, and that the equilibrium points are asymptotically stable provided that the latency functions are sufficiently sensitive and the imitation rate is adequately chosen. Future studies should investigate how our conclusions translate to more general models that account for bounded supply. Our results give rise to several opportunities for future work. By coupling these models with common infrastructure control models (such as variable speed limits and freeway metering), these results may play an important role in designing dynamic controllers for congested infrastructures. Furthermore, our models and stability analysis represent a fundamental framework for future studies on robustness and security analysis.

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