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Polar codes: Encoding/decoding and rate-compatible jointly design for HARQ system

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Abstract: Polar coding are the first class of provable capacity-achieving coding techniques for a wide range of channels. With an ideal recursive structure and many elegant mathematical properties, polar codes are inherently implemented with low complexity encoding and decoding algorithms. Since the block length of the original polar construction is limited to powers of two, rate-compatible polar codes (RCPC) are presented to meet the flexible length/rate transmission requirements in practice. The RCPC codes are well-conditioned to combine with the hybrid automatic repeat request (HARQ) system, providing high throughput efficiency and such RCPC-HAPQ scheme is commonly used in delay-insensitive communication system. This paper first gives a survey of both the classical and state-of-the-art encoding/decoding algorithms for polar codes. Then the RCPC construction methods are discussed, including the puncturing, shortening, multi-kernel construction, etc. Finally, we investigate several RCPC-HARQ jointly design systems and discuss their encoding gain and re-transmission diversity gain.

Key words: polar code; rate compatible; hybrid automatic re-transmission request; polar coding/decoding

1 Introduction

There exist several classes of capacity achieving channel codes, including low-density parity check codes (LDPC)[1−3], turbo codes[4], and polar codes[5]. Among these, polar codes are the first practical scheme which can be proved mathematically to achieve the capacity of a large class of channels, such as the binary input discrete memoryless channel (B-DMC) and the additive white Gaussian noise channel (AWGN). Actually, polar coding is originally conceived as a technique to boost the channel cutoff rate[6], which is equivalently a practical recursive implementation of earlier schemes proposed by Pinsker[7] and Massey[8]. Polar codes have an ideal recursive structure and many attractable mathematical properties, which make them inherently easily implemented in encoding and decoding with low complexity. Polar codes have been adopted as the coding scheme of the control channel for the 5th generation of mobile communication system (5G) and undoubtedly, become one of the competitive potential channel coding candidates for future wireless communications such as the 6G system.

In Ref. [5], Arikan introduced the successive-cancellation (SC) decoding algorithm of polar codes, which successively estimates the transmitted bits in the decoder. However, the SC is a sub-optimal decoding scheme and does not show performance advantages compared to the LDPC and turbo codes, especially for decoding short or moderate block length codes. List decoder can be employed to the SC, called the SCL decoding algorithm[9−11], which stores L-most reliable paths in the decoding tree and can achieve a performance improvement close to that of the maximum likelihood (ML) decoding with a large enough L value. There exist many variations of the SC/SCL decoders, see Refs. [12−15] and references

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therein. Since polar codes are also represented on the factor graph, they can be naturally decoded by the belief propagation (BP) decoding algorithm[16]. In spite of the dissatisfied performance, the BP decoder can achieve fast decoding speed and high throughput due to its inherent parallel decoding structure. Many efforts have been made on BP decoding of polar codes, aiming to improve the performance and reduce the decoding complexity[17–19].

In Arikan's original construction, the length of polar codes is limited to powers of two due to the binary Kronecker matrix, which restricts their flexible applications. To solve this problem, designing polar codes with arbitrary lengths and rates is essential. Such codes are called rate compatible polar codes, which can be generally constructed by puncturing, shortening, repetition, and other methods such as multi-kernel and cascading construction[20–24]. In practice, the wireless communication channels are not idealized. Instead, many channels suffer from time-varying, delay, and fading. Hybrid automatic repeat request (HARQ) is often used for such communication scenarios. It is shown that the rate compatible polar codes are well-conditioned to combine with the HARQ scheme, providing flexible and adaptive transmission. Moreover, polar coding can be also used to ensure secure transmission over relay channels[25].

We give a survey of polar codes in this paper, including their construction, the traditional and the state-of-the-art encoding/decoding algorithms. We outline several classical rate-compatible construction schemes and discuss their limitations. Then, the jointly design of HARQ system with the RCPC codes is further discussed. The remainder of this paper is organized as follows: In Section 2, Arikan's original construction scheme of polar code and several rate-compatible construction schemes are introduced. Decoding algorithms of polar code are demonstrated in Section 3. Section 4 gives several RCPC-HARQ jointly design schemes and Section 5 concludes the paper.

2 Encoding of polar codes

2.1 Arikan's original construction

Polar codes are originally constructed based on the basis idea of channel polarization. For a B-DMC channel $W : C \rightarrow \mathcal{Y}$,where input and output alphabet are denoted as $C \in \{0,1\}$ and $\mathcal{Y}$, respectively. $W(y|c), y \in \mathcal{Y}, c \in C$ denotes the channel transition probabilities. In Ref. [5], Arikan shows that, for N independent copies of a given channel W, after combining and splitting operations, it is possible to synthesize another set of N subchannels $W_N^i (0 \leqslant i \leqslant N-1)$, which are polarized between noiseless channels and full noisy channels for sufficiently large N. Such polarization phenomenon is well-conditioned for error coding scheme: the information bits are sent through those noiseless polarized sub-channels, while the check bits (frozen bits) are placed into the pure noisy sub-channels. The resulting codes constructed with this basic idea are called polar codes.

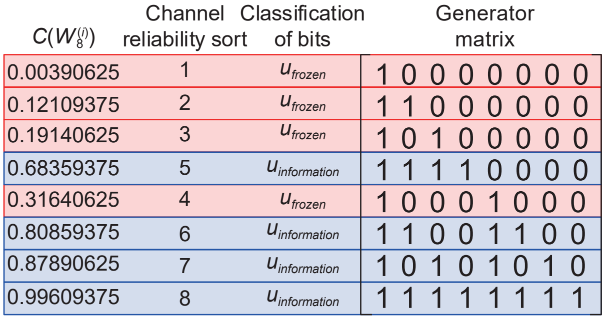
Consider a binary erasure channel (BEC) channel, let $C(W_N^{(i)})$ denote the capacity of $W_N^{(i)}$, which can be calculated recursively by

$$C(W_N^{(2i-1)}) = C(W_{N/2}^{(i)})^2 \tag{1}$$

$$C(W_N^{(2i)}) = 2C(W_{N/2}^{(i)}) - C(W_{N/2}^{(i)})^2 \tag{2}$$

Figure 1 shows the capacity $C(W_N^{(i)})$ of a BEC channel with deletion probability of 0.5, which is sorted in ascending order. When the code rate $R = 0.5$, the least four reliable sub-channels are set as frozen channels, while the remaining four are set as information channels. The information bits $u_{information}$ and frozen bits u_{frozen} are arranged according to such channel polarization result.

For a general B-DMC, let $\underline{u} = (u_0, u_1, \ldots, u_{N-1})$ denote the binary input source and $\underline{c} = (c_0, c_1, \ldots, c_{N-1})$ denote



$C(W_8^{(i)})$	Channel reliability sort	Classification of bits	Generator matrix
0.00390625	1	u_{frozen}	1 0 0 0 0 0 0 0
0.12109375	2	u_{frozen}	1 1 0 0 0 0 0 0
0.19140625	3	u_{frozen}	1 0 1 0 0 0 0 0
0.68359375	5	$u_{information}$	1 1 1 1 0 0 0 0
0.31640625	4	u_{frozen}	1 0 0 0 1 0 0 0
0.80859375	6	$u_{information}$	1 1 0 0 1 1 0 0
0.87890625	7	$u_{information}$	1 0 1 0 1 0 1 0
0.99609375	8	$u_{information}$	1 1 1 1 1 1 1 1

Fig. 1 Example of BEC channel polarization.

the output coded word, where $N=2^n$ is the block length. The input u_0^{N-1} is composed of information bits u_A and frozen bits u_A^c, where $\underline{u}=\{u_A, u_A^c\}$. The polar code then can be defined as

$$\underline{c} = \underline{u} \times \boldsymbol{G}_N \tag{3}$$

where $\boldsymbol{G}_N = \boldsymbol{F}^{\otimes n}$ is the generator matrix and $\boldsymbol{F}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is the binary Kronecker matrix. Figure 2 shows the encoding structure of N=8. Note that, for analysis purpose, polar coding can be implemented alternatively by

$$\underline{c}' = \underline{u}\boldsymbol{G}'_N \tag{4}$$

where $\boldsymbol{G}'_N$ is the generator matrix after permutation of $\boldsymbol{G}_N$ with

$$\boldsymbol{G}'_N = B_N \boldsymbol{G}_N \tag{5}$$

Here, B_N represents the bit-reversal operation. Figure 3 shows the alternative polar encoding for $\boldsymbol{G}'_8$ with reverse shuffle.

2.2 Rate-compatible schemes for polar codes

For the polar construction above, the length of polar codes is limited to powers of two due to the binary Kronecker matrix, which restricts their flexible applications in practice. To solve this problem, researchers have proposed a variety of polar codes with arbitrary lengths and rates, which are known as the rate compatible polar codes. We give a survey of the constructed schemes of the RCPC in this subsection, including the puncturing, shortening, repetition and other methods such as multi-kernel and cascading construction.

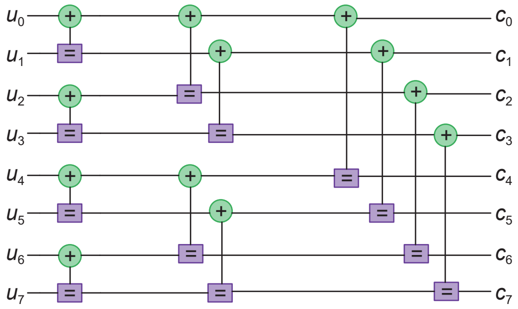


Fig. 2 Factor graph representation for the transformation $\boldsymbol{G}_8$.

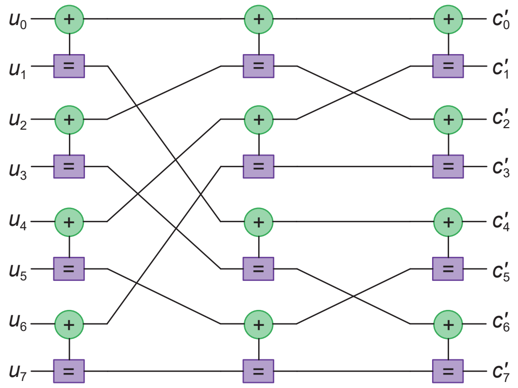


Fig. 3 Alternative representation for $\boldsymbol{G}'_8$ with reverse shuffle.

2.2.1 Multi-kernel construction

In Ref. [5], Arikan pointed out that the polarization phenomenon may not be only limited to binary case, which was soon proved by Korada et al.[20], where the authors show that the polar generator matrix can be defined by kernel matrices with different scales and the resulting codes can achieve more flexible lengths. Other related references can be seen in Refs. [21, 26−28], where the codes can be constructed on a large alphabet and different sizes of kernel matrices, resulting linear/nonlinear, binary/non-binary polar codes with flexible lengths.

Note that, such multi-kernel construction can achieve an improved length flexibility. However, such construction involves Kronecker product, which is non-commutative. Hence, the order of kernels may affect the resulting generator matrix. This limits the corresponding length to the product of two integer's power.

2.2.2 Concatenated polar codes

Another possible RCPC construction scheme is to concatenate polar codes with linear codes. Long codes at any code rate can be obtained by selecting appropriate short component codes as inner/outer codes. In Ref. [29], Reed-Solomon (RS) code is used to concatenate with polar code.

Since the alphabet cardinality of RS is no longer exponential, the concatenated code can achieve rate compatible and arbitrary length can be selected. In Ref. [30], the concatenated scheme using polar code as outer code and LDPC code as inner code is proposed, where the effective code rate is the product of polar code rate and LDPC code rate. Flexible rates could be achieved by fixing polar code length and adjusting the

LDPC length. Similar work can be seen in Ref. [31]. Although the bit error rate (BER) performance of concatenated linear code becomes worse, it provides more flexible in code length selection.

2.2.3 Puncturing schemes

Puncturing is a classical rate compatible scheme[32, 33] usually for low-rate codes. Puncturing scheme initially designs a mother code with lower rate for the worst channel condition, then punctures some bits from the mother code according to a well-designed pattern. Such puncturing pattern is particularly critical, since it has a great impact on the decoding performances. Puncturing scheme for polar is first investigated in Ref. [29], where the author proposed a random and stopping-tree puncturing schemes. In 2013, Niu et al. proposed a puncturing scheme called quasi-uniform puncturing (QUP)[22], which achieves excellent decoding performance[9, 34] compared to the random puncturing. Let $\boldsymbol{p} = (p_0, p_1, \ldots, p_{N-1})$ be the puncturing vector with $p_j \in \{0, 1\}, j = 0, 1, \ldots, N-1$, where the 0 s implies the punctured positions. The QUP scheme can be easily implemented by two steps: (1) initial $\boldsymbol{p}$ to be an all-one vector, then setting the first N_p bits to be zeros; (2) performing the bit-reversal permutation on $\boldsymbol{p}$ to get the puncturing vector. Figure 4 shows an example of the QUP scheme with a length-8 mother code, where $\boldsymbol{p} = (00011111)$. After performing bit-reversal on $\boldsymbol{p}$, we get $\boldsymbol{p} = (01010111)$, implying the index 0, 2, and 4 should be punctured.

It is shown that, the puncturing patterns may affect the split bit channels and cause performance degradation in Ref. [35], then the authors proposed a search algorithm to design the puncturing pattern for output bits, which can achieve a performance to that of the exhaustive search. Based on the incapable bit (IB) and the incapable bit equivalent (IBE), Kim et al. proposed a reduced full search for finding optimal puncturing pattern of short polar codes[36]. Compared to the exhaustive search scheme, the complexity of this scheme increases much slower. Since the generator matrix of the longer polar code can be represented for the shorter one, puncturing bits for a specific shorter code can also be used for the longer code. Based on this property, a new puncturing technique is proposed in Ref. [37], which can outperform the QUP scheme. The authors in Ref. [38] proved that puncturing constrained with the frozen channels is theoretically optimal with respect to the union bound of the block error probability, then a worst quality puncturing (WQP) method was proposed for a fixed-information set, which can outperform many conventional puncturing algorithms. Puncturing position designed for fixed information was also considered in Ref. [39], where an information set approximation puncturing (ISAP) algorithm was proposed. The guard bits are defined before the known information bits, then the puncturing pattern was designed based on the physical code-length through the channel. The ISAP algorithm shows better performance and more stability than the conventional puncturing algorithms, especially for low-code-rate.

2.2.4 Shortening schemes

Length/rate compatible polar codes can also be implemented based on shortening schemes. Actually, shortening scheme can be regarded as a special case of puncturing scheme, where the shortening bits are designed jointly with the frozen bits. A uniform way was given to describe the puncturing and shortening

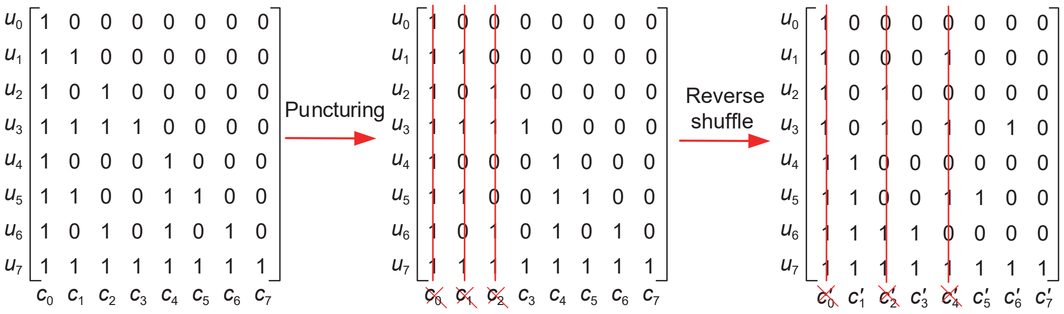


Fig. 4 Generator matrix of polar codes with length 5 constructed by QUP algorithm.

polar code in Ref. [40]. In 2013, Shin et al. obtained the required generator matrix by column reducing (puncturing) based on the parent codes generator matrix, then constructed a suboptimal length-compatible polar code by codeword-puncturing and information refreezing processes. The classical shortening scheme was proposed by Wang and Liu in Ref. [41], where the frozen bit and puncturing bit are jointly designed so that the puncturing bit is exactly linear combination of frozen bit. Hence, the punctured bits are known both by the encoder, and the decoder and the corresponding log-likelihood ratio (LLR) can be set to be $+\infty$. In contrast, the LLR of the conventional puncturing is set to be zero since the puncturing location is unknown. Similarly, Niu et al. proposed an equivalent shortening technique in Ref. [23], called reverse quasi-uniform shortening (RQUS), where the spectral distance (SD) and joint spectral distance (JSD) of C0 (capacity-0) and C1 (Capacity-1) puncturing modes are analyzed. The authors in Ref. [42] proposed an improved version of Ref. [23], which can mitigate the performance degradation for large puncturing bits. To reduce the searching complexity, a new method was proposed to construct the shortening pattern in Ref. [43], where the desired patterns and the frozen symbols are jointly designed. Such method ensures the resulting polar code with sufficiently high minimum distance.

Figure 5 shows an example of shortening scheme of a length-8 mother polar code by shortening 3 bits according to Ref. [41]. The shortening is performed under the weigh-1constraint, indicating that the puncturing set has the same cardinality as that of the frozen set. A simple way to meet this requirement is to shorten the successive last 3 columns from the generator matrix.

2.2.5 Applications in 5G New Radio (NR) schemes

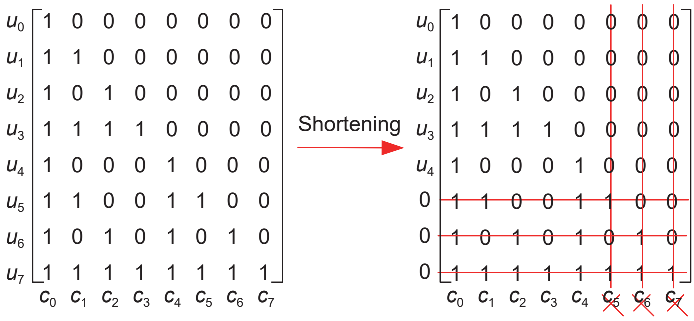
The 5G New Radio (NR) schemes employ three rate-compatible solutions for polar codes, including the puncturing, shortening, and repetition. Repetition scheme is considered only when the physical transmission code length is larger than the mother code length, where repeating bits are selected from the mother code and placed directly into the encoded output bits, as shown in Fig. 6.

Let K be the information bit length and let N and M be the mother code length and the physical transmission code length, respectively. The rate can be calculated by $R = K/M$. The 5G New Radio (NR) schemes select three rate-compatible schemes, according to the relationship of N, M, and R as follows: (1) if $M > N$, the repetition scheme is adopted; (2) if $M < N$ and $R \leqslant 7/16$, the puncturing scheme is adopted; (3) if $M < N$ and $R > 7/16$, the shortening scheme is adopted.

3 Decoding of polar code

3.1 SC and the variations

Arikan firstly proposed successive cancellation (SC) decoding in Ref. [5], where the transmitted bits are estimated successively. The SC algorithm makes the decoding performance easy to be analyzed and can be performed with the recursive structure of the fundamental decoding function. Figure 7 shows SC algorithm on the decoding tree. Let $\hat{u}_1^N$ be the estimation of source block u_1^N. They can be determined with the received signal y_1^N and the first $i-1$ estimation



$$\begin{array}{c|cccccccc} & c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\ \hline u_0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ u_3 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ u_4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ u_5 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ u_6 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ u_7 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \xrightarrow{\text{Shortening}} \begin{array}{c|cccccccc} & c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\ \hline u_0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ u_3 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ u_4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

Fig. 5 Example of shortening scheme for a length-8 mother polar code.

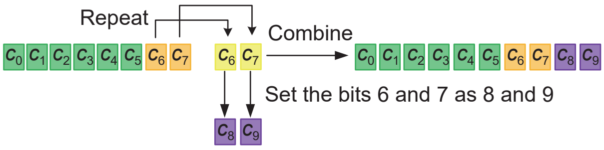


Fig. 6 Example of repetition scheme for the length-8 mother code.

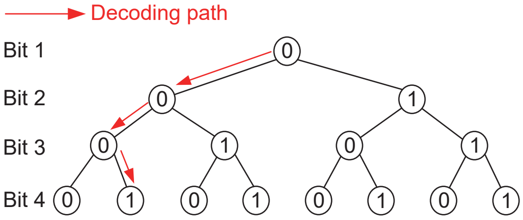


Fig. 7 SC algorithm on the decoding tree.

of source bits $\hat{u}_1^{i-1}$ by

$$\hat{u}_i = \begin{cases} u_i, \ i \in \mathcal{A}^C; \\ h_i\left(y_1^N, \hat{u}_1^{i-1}\right), \ i \in \mathcal{A} \end{cases} \tag{6}$$

where

$$h_i(y_1^N, \hat{u}_1^{i-1}) = \begin{cases} 0, & \dfrac{W_N^{(i)}\left(y_1^N, \hat{u}_1^{i-1}|0\right)}{W_N^{(i)}\left(y_1^N, \hat{u}_1^{i-1}|1\right)} \geqslant 0; \\ 1, & \text{otherwise} \end{cases} \tag{7}$$

The traditional SC decoder suffers from high complexity and latency and is not suitable for real-time applications. In Ref. [44], a simplified SC decoder was proposed for polar codes, where the recursion can be simplified not only by rate-zero (all involved bit are "frozen bits") but also by rate-one (all involved bits are "information bits"). With the simplification in recursion, the latency and computation of SC decoder reduce without performance degradation. A new reformulation for the last stage of SC decoder was proposed in Ref. [24], where two bits are decoded in one clock cycle. Such decoder with this architecture is called the 2b-SC decoder, which results in latency reduction. The SC-Stack (SCS) decoding algorithm for polar codes was proposed in Refs. [12, 13], which has the low computational complexity that is close to SC algorithm in high SNR (signal to noise ratio).

Compared to SC algorithm, SCS has lower decoding complexity. Despite the reduction computation, SCS shows poor performance and large space-complexity when the size of stack is small. In Ref. [45], an improved SC algorithm named SC-Fano was presented, which incorporates the Fano sequential decoding into SC decoding and allows moving back on a code tree during SC decoding procedures. The SC-Fano algorithm achieves an efficient tradeoff between performance and complexity. Note that, both SCS and SC-Fano decoding are sequential polar decoding.

The SC and its variations, SCS and SC-Fano algorithm mentioned above show unsatisfactory performance, especially for the finite code length. List decoder can be employed to the SC, called the SCL decoding algorithm[9, 46, 47], which stores L-most reliable paths in the decoding tree and can achieve a performance improvement close to that of the maximum likelihood (ML) decoding with a large enough L value. Figure 8 shows the SCL algorithm. To break out the high-latency bottleneck of list decoder, a multibit-decision approach was proposed in Ref. [10], which can reduce latency without performance loss by reformulating SCL to perform intermediate decoding. A simplified version was proposed in Ref. [11], which can achieve significant reduction in both latency and complexity by decoding multiple information bits at each decision step intermediately. A reduced latency list decoding (RLLD) algorithm was proposed in Ref. [14], which performs on a binary tree like the existing SCL algorithm, but traverses fewer nodes and thus considers fewer possibility. In Ref. [15], a symbol-decision SC/SCL decoder was proposed to get higher throughput and better error performance by using symbol-wise hard/soft decisions.

The performance of SCL can be further improved with cyclic redundancy check (CRC), which is widely used in practical communication systems for error detection. In Ref. [34], a CRC-aided SCL decoder (CA-SCL) was proposed by outputting multiple candidate-sequences. The one satisfying the CRC check is selected as the estimation, which can achieve

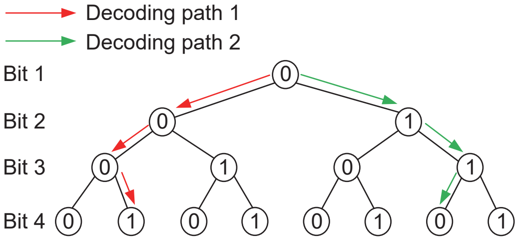


Fig. 8 SCL algorithm on the decoding tree.

notable performance improvement. The CA-SCL algorithm is shown in Fig. 9. For a better balance between performance and complexity, a segmented CRC-aided successive cancellation list (SCA-SCL) decoding scheme was proposed in Ref. [48]. In Ref. [49], three post-processing CA-SCL schemes were proposed, which can improve the decoding performance by alleviate the effects of channel and decision errors.

In Ref. [50], a new SC-based decoding algorithm named SC-Flip was investigated, which provides performance gain but retains the same memory and average computational complexity as the original SC algorithm (at medium to high SNR). Decoding attempts are performed if the initial SC decoding is failed by CRC detecting. Every-attempt consists of one flipping operation for single decision, which is started from the least reliable bit according to the absolute value of the LLR of the initial attempt. The SC flip algorithm is shown in Fig. 10. In Ref. [51], an optimized metric was introduced to determine the flipping positions for SC-Flip decoder, which improves the ability of the first identifying unreliable bit decisions. A progressive bit-flipping decoding algorithm was presented in Ref. [52] based on a search tree. The presented algorithm only considers the most possible error bits included in the modified critical set. To improve the performance of SC flip decoder while reducing complexity, a new SC-Flip decoder was proposed in Ref. [53], which detects, identifies, and flips error bits by both the CRC and distributed parity checks (PCs).

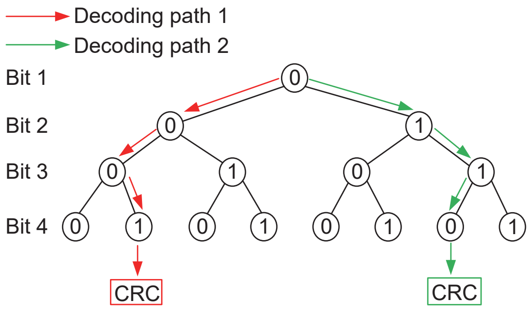


Fig. 9 **CA-SCL algorithm on the decoding tree.**

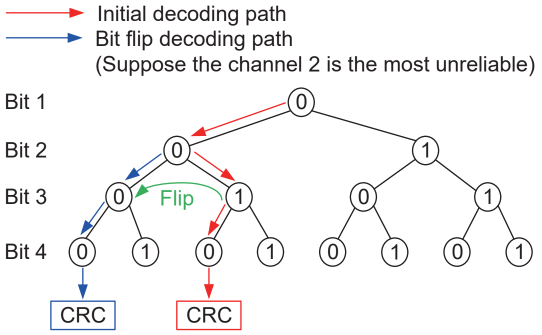


Fig. 10 **SCF algorithm on the decoding tree.**

The complexity and memory of different SC-based decoding algorithms are shown in Table 1.

3.2 BP and the variations

Polar code shares the same generator matrix with RM code, both of them can be represented on the factor graph and inherently, can be decoded with the BP algorithm[54]. Figure 11 gives the Forney factor graph of the polar code.

Each node on the factor graph can be regarded as a 2×2 basic processing element (PE), which is shown in Fig. 12 and to pass the calculated message bidirectionally in the decoder. Let $R_{i,j}$ be the messages passing from left to right of node (i, j), and $L_{i,j}$ be the messages passing from right to left, which can be defined as follows:

$$L_{i,\,j} = f(L_{i+1,\,2j-1},\ L_{i+1,\,2j} + R_{i,\,j+N/2}) \tag{8}$$

$$L_{i,\,j+N/2} = f(R_{i,\,j},\ L_{i+1,\,2j-1}) + L_{i+1,\,2j} \tag{9}$$

$$R_{i+1,\,2j-1} = f(R_{i,\,j},\ L_{i+1,\,2j} + R_{i,\,j+N/2}) \tag{10}$$

$$R_{i+1,\,2j} = f(R_{i,\,j},\ L_{i+1,\,2j-1}) + R_{i,\,j+N/2} \tag{11}$$

The message in the PE is calculated by $f(x,y)$ with

$$f(x,y) = \ln((1+xy)/(X+y)) \approx \\ sign(x)sign(y)\min(|x|,|y|) \tag{12}$$

The message $R_{1,j}$ and $L_{n+1,j}$ are initialized by

$$R_{1,j} = \begin{cases} 0, & \text{if } j \in \mathcal{A}; \\ \infty, & \text{if } j \in \mathcal{A}^c \text{ and } u_j = 0; \\ -\infty, & \text{if } j \in \mathcal{A}^c \text{ and } u_j = 1 \end{cases} \tag{13}$$

Table 1 **Complexity and memory of different SC-based decoding algorithms.**

Decoding algorithm	Computational complexity	Memory complexity
SC	$O(N\log N)$	$O(N)$
SCL	$O(LN\log N)$	$O(LN)$
SCS	$O(DN\log N)$	$O(DN)$
SC Flip	$O(N\log N)$	$O(N)$

Note: L is the list size and D is the stack depth.

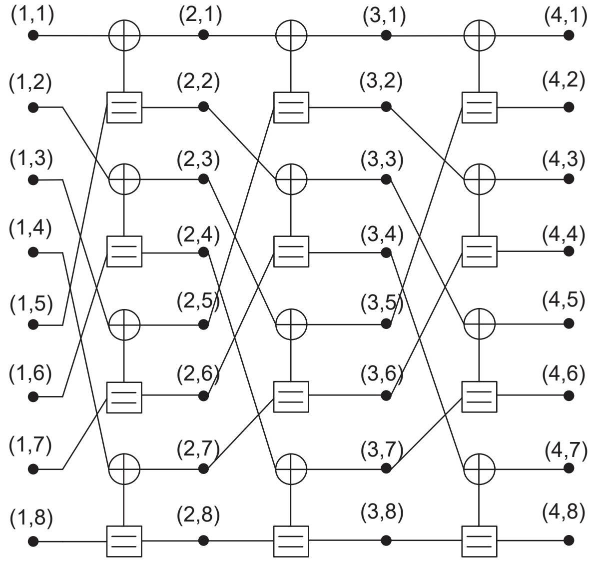


Fig. 11 Forney factor graph of polar code with N=8.

and

$$L_{n+1,j} = \ln\frac{W(y_i|0)}{W(y_i|1)}, 0 \leqslant i < N \qquad (14)$$

Message with respect to other nodes is initialized as zero.

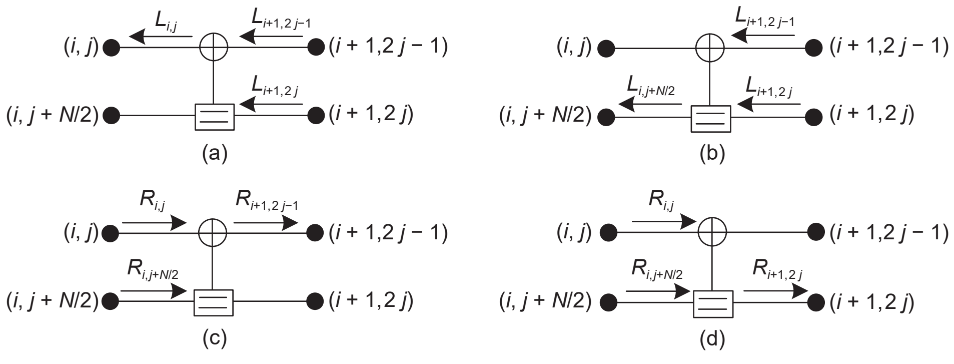
In contrast to the SC-based decoding algorithms, the BP decoding of polar codes has many advantages. More importantly, it can be implemented parallelly and achieve high throughput and low latency. Furthermore, the soft-in/soft-out message-passing decoding with polar codes can be potentially shared the same decoding structure with LDPC codes. However, the BP decoding shows unsatisfied error performance when compared with the SCL algorithm and its modified versions. Many efforts have been made to improve the performance of the BP decoding algorithm. BP list (BPL) decoder was introduced in Refs. [16, 17, 55], which composes of L independent BP decoders based on a set of predetermined permutations. The BPL can achieve comparable performance to that of the SCL. The idea of critical set (CS) in SC-Flip is introduced to the BP decoder[56], resulting in a comparable performance to that of the CA-SCL with a moderate list size. Other variations of the BP decoder can be seen in Refs. [18, 57], where the BLER performances can be improved in different levels.

4 HARQ schemes

In practice, the wireless communication channels are not idealized. Instead, many channels suffer from time-varying, delay and fading. Hybrid automatic repeat request (HARQ) is often used for such communication scenarios. It is shown that the rate compatible polar codes are well-conditioned to combine with the HARQ scheme, providing flexible and adaptive transmission.

The adaptive link transmission based on RCPC and HARQ is an important way to improve the performance, the throughput, and the capacity. The HARQ technology of rate-compatible convolution/ LDPC codes has been researched sufficiently, see Refs. [33, 58] and references therein. Polar codes have also been applied to HARQ system. The chase-combining HARQ (CC-HARQ) scheme based on punctured polar codes was introduced in Ref. [19], which re-transmits the punctured code words after receiving the NAK. The CC-HARQ is easily implemented with acceptable performance and is convenient to combine with other techniques.

An incremental redundancy HARQ (IR-HARQ) scheme for polar codes was investigated in Ref. [59],



(a) (b)

(c) (d)

Fig. 12 Basic processing element (PE) of BP.

which only re-transmits parts of the less reliable bits to aid the received bits in the decoder. Although IR-HARQ has better performance than that of CC-HARQ, the operations such as permuting, re-transmission bits selecting makes larger buffer and higher complexity. To support short packet transmission, an optimized polarizing matrix extension (OPME) method is combined with IR-HARQ, achieving a considerable error-performance improvement[60]. An early termination (ET) technique for IR-HARQ scheme was proposed in Ref. [61], which reduces the latency and power consumption by using the CRC and parity check bit. To improve the throughput, an IR-HARQ scheme with puncturing and extending of polar codes was proposed in Ref. [62], which can reach any required code rate. A polar code extension method by copying information bits to the proper positions of the extend part was proposed in Ref. [63], which has nearly the same performance as the original codes. El-Khamy et al. proposed a construction scheme of RCPC in Ref. [64] by puncturing and extending the short length codes and jointly designed with the CC-HARQ and IR-HARQ schemes, achieving considerable robustness.

An incremental freezing HARQ (IF-HARQ) scheme was introduced in Ref. [65] where the information bits are frozen increasingly until a successful decoding occurs. The IF-HARQ is a capacity-achieving scheme due to the its well-designed structure, which is shown in Fig. 13.

In Ref. [66], Zhao et al. constructed the RCPC by extending the polarizing matrix to make IR-HARQ system more efficient, which is called the adaptive IR-HARQ. The re-transmitted bits are calculated according to the extended matrix and are naturally parts of the coded bits of the longer polar code. Such IR-HARQ scheme is well-conditioned with the QUP polar code[22] and can achieve not only coding gain but also diversity gain with flexible code length and code rate. Hong and Jeong extended the work in Ref. [65] and obtained more puncturing patterns by information-copy operations[67]. Figure 14 gives an example of the adaptive IR-HARQ with the mother code length N=8.

5 Summary

In this survey, we first review the basic concept of polar codes, including the factor graph representation with/without reverse shuffle. Then we give a brief description of both classical and state-of-the-art encoding/decoding algorithms for polar codes. Furthermore, different construction methods for rate-compatible polar codes, as well as their application in 5G NR are discussed, which can ensure the flexible length/rate transmission requirements in practice. Since the RCPC codes are well-conditioned to combine with the hybrid automatic repeat request system, we further give a survey of the RCPC-HARQ combination schemes, which can provide both performance and re-transmission diversity gain.

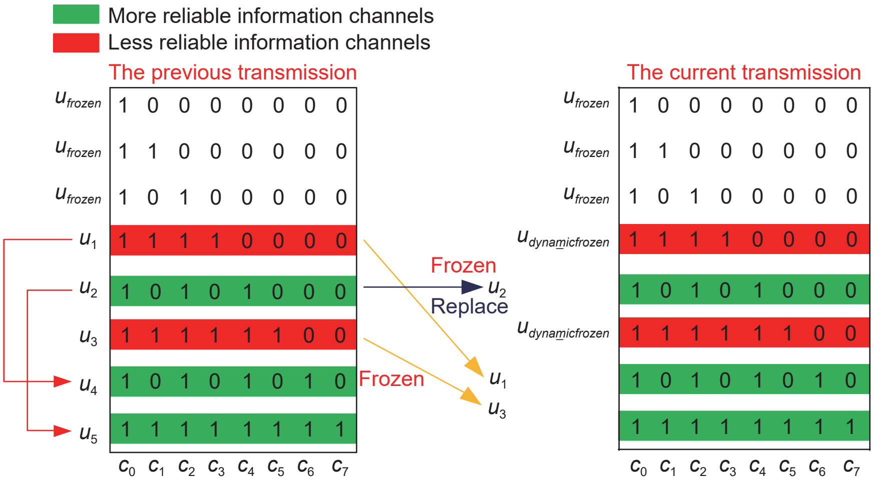
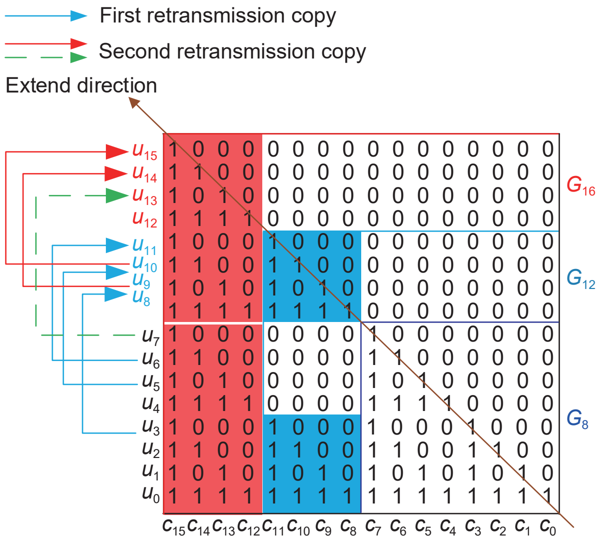


Fig. 13 IF-HARQ scheme.



	c_{15}	c_{14}	c_{13}	c_{12}	c_{11}	c_{10}	c_9	c_8	c_7	c_6	c_5	c_4	c_3	c_2	c_1	c_0
u_{15}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
u_{14}	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
u_{13}	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
u_{12}	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
u_{11}	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
u_{10}	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0
u_9	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0
u_8	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
u_7	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
u_6	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0
u_5	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
u_4	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
u_3	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
u_2	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
u_1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
u_0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Fig. 14 Adaptive IR-HARQ scheme.

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