This article has been accepted for publication in IEEE Transactions on Fuzzy Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2023.3348160

Transition Between TS Fuzzy Models and the Associated Convex Hulls by TS Fuzzy Model Transformation

Péter Baranyi

***Abstract*—One of the primary objectives underlying the extensive 20-year development of the TS Fuzzy model transformation (originally known as TP model transformation) is to establish a framework capable of generating alternative Fuzzy rules for a given TS Fuzzy model, thereby manipulating the associated convex hull to enhance further design outcomes. The existing methods integrated into the TS Fuzzy model transformation offer limited capabilities in deriving only a few types of loose and tight convex hulls. In this article, we propose a radically new approach that enables the derivation of an infinite number of alternative Fuzzy rules and, hence, convex hulls in a systematic and tractable manner. The article encompasses the following key novelties. Firstly, we develop a Fuzzy rule interpolation method, based on the pseudo TS Fuzzy model transformation and the antecedent Fuzzy set rescheduling technique, that leads to a smooth transition between the Fuzzy rules and the corresponding convex hulls. Then, we extend the proposed concept with the antecedent Fuzzy set refinement and reinforcement technique to tackle large-scale problems characterized by a high number of inputs and Fuzzy rules. The paper also includes demonstrative examples to illustrate the theoretical key steps, and concludes with an examination of a real complex engineering problem to showcase the effectiveness and straightforward execution of the proposed convex hull manipulation approach.**

I. Introduction

The development of the TS Fuzzy model transformation, originally known as TP model transformation, commenced approximately two decades ago. The present article is based on the key publications [1]–[7] that have contributed significantly to the developments of the field. Initially, the primary objective of the initial variant of TP model transformation was to numerically reconstruct a TS Fuzzy model representation of a given function or quasi Linear Parameter Varying state-space dynamic model [2]. This approach offered several significant advantages, including the determination of the minimum number of required antecedent Fuzzy sets by dimensions, thereby minimizing the number of Fuzzy rules [6], [8]–[10]. Additionally, it provided the opportunity for further reduction by defining a trade-off between approximation accuracy and the number of Fuzzy rules through the ranking of their importance based on the L_2 norm. These features are a result of the core step of the TP model transformation, which relies on the Higher Order Singular Value Decomposition (HOSVD) where the singular values are associated with the Fuzzy rules. Subsequently, the TP model transformation was expanded to transform a set of functions into a set of TS Fuzzy models with a shared or partially shared antecedent Fuzzy set system [3], [7]. Furthermore, the Pseudo TP model transformation was introduced to derive the TS Fuzzy model representation with a given or partially given antecedent Fuzzy set system, and if an exact representation is not feasible, to identify the best approximation based on the L_2 norm [3], [7].

The subsequent advancements of the TP model transformation primarily focused on ensuring advantageous characteristics of the resulting Ruspini-partitioned antecedent Fuzzy sets. This emphasis stemmed from the understanding that the characteristics of the antecedent Fuzzy sets determine the nature of the convex hull defined by the consequents. It was soon discovered that the TP model transformation could generate various alternatives of TS Fuzzy models with distinct characteristics [1], [6], [11]–[13]. Consequently, design methods that rely on the consequents can be significantly influenced by the TP model transformation. One notable example is the Parallel Distributed Compensation (PDC) framework introduced by Tanaka [14] for control design. Within this framework, the consequents of the controller are derived from the consequents of the TS Fuzzy models, typically through the feasibility of Linear Matrix Inequalities. A comprehensive analysis of the impact of tight or loose convex hulls - derived by TP model transformation - in PDC design has been documented in [7], [15]–[17]. Number of publications have utilized the TP model transformation to derive alternative TS Fuzzy models to be substituted into the new design techniques to achieve improved solutions. Please be referred to some recent ones mostly published in IEEE Transaction on Fuzzy Systems and in Asian Journal of Control [18]–[79].

Recognizing the significance of convex hull manipulation, the TP model transformation has been extended through techniques such as Sum Normalization (SN), Non-Negativeness (NN), Normalised (NO), Close to Normalised (CNO), Inverse Normalised (INO), Relaxed NO and INO (IRNO), and the minimum simplex volume technique [6], [8], [11], [12]. These extensions enable the derivation of different antecedents and loose / tight convex hulls of the consequents.

A. The novel contribution of the article

The article first restructures the TS Fuzzy model transformation based on a novel concepts of TP and TS Fuzzy

P. Baranyi (peter.baranyi@uni-corvinus.hu) is with the Corvinus University of Budapest, Corvinus Institute for Advanced Studies (CIAS) and Institute of Data Analytics and Information Systems, Fővámtér 8, Budapest. Hungary and the Hungarian Research Network (HUN-REN). The author declares that there is no conflict of interest regarding the publication of this paper. Acknowledgement: the paper uses ChatBox App by Florate Ltd. to correct some texts of the paper.

This article has been accepted for publication in IEEE Transactions on Fuzzy Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2023.3348160

grid structures, which serve as the fundamental framework for further derivations. Then, the article proposes the key approach to systematically derive an infinite array of antecedent systems, enabling precise control over the shape and type of the resulting convex hull. This stands in contrast to previous methodologies that only offer a limited range of tight or loose convex hulls (such as SNNN, NO, CNO, INO, IRNO types). The underlying concept revolves around establishing a smooth transition between different TS Fuzzy models and their corresponding convex hulls. About the challenges behind the goal of the article: one potential approach would be to interpolate the consequents, or employing any convex hull determination method, to obtain the convex hull, and subsequently derive the corresponding antecedent Fuzzy sets. However, this approach necessitates the inverse of the Higher Order Singular Value Decomposition (HOSVD), or any similar tensor decomposition, in a manner that the matrices associated with the antecedents should be derived to the interpolated core tensor. Unfortunately, such an inverse operation or tensor decomposition does not exist within tensor algebra and poses a formidable challenge in general. Moreover, ensuring the appropriate number of antecedent Fuzzy sets and their Ruspini-partition characteristics based on the derived matrices further complicates the inverse tensor decomposition. This raises the hypothesis that a viable solution to derive the antecedents to the given convex hull may not exist at all in general.

Therefore the present article proposes the opposite way that leads to a very simple implementation and focuses on the linear interpolation of the antecedents and derives the consequents accordingly. This approach does not lead to the linear interpolation of the consequents, however leads to the monotonic smooth transition between the convex hulls. The proposed approach incorporates the pseudo TP model transformation in a form that aligns with the TS Fuzzy model grid structures. The paper shows that the interpolation of the antecedents may result in a jarring transition of the consequents in many cases. Therefore the paper proposes the rescheduling of the antecedents that guarantees the smooth transition of the consequents, hence, the convex hull. The proposed methodology enables the overall transition to be controlled by a parameter, similar to the linear interpolation.

The present paper also introduces an extension of the convex hull transition methodology to address large-scale problems characterized by a high number of inputs and antecedent Fuzzy sets. In such scenarios, the core step of the TS Fuzzy model transformation, that is based on the HOSVD, entails a significant computational load that severely restricts the applicability of the proposed methodology to complex problems. To overcome this limitation, the study revisits the antecedent Fuzzy set refining technique, termed as enrich technique in the previous work [7]. The paper proves that this technique fails to preserve the Ruspini-partition and may not result in Fuzzy sets in general. To address this issue, the paper proposes a method to reinforce the conditions of the Ruspini-partition of the antecedent Fuzzy sets. The integration of these methods on the bases of the TS Fuzzy grid structures leads to a convex hull manipulation methodology that does not require the execution of the HOSVD on large-sized tensors, but rather simplifies the computation by dimensions.

The paper provides two demonstrative examples to elucidate the convex hull transition and highlight the inadequacy of the refining technique in the absence of the reinforcement method. Additionally, the study presents a real-world complex engineering problem to showcase the efficacy and straightforward applicability of the proposed methodology. This example also underscores the fact that antecedent Fuzzy set interpolation may not result in a smooth transition, but in a jarring transition, without the proposed antecedent Fuzzy set rescheduling.

B. The structure of the paper

Section II of the paper introduces the notation employed throughout the paper. The primary contribution of the study is presented in Sections III, IV and V. Section III proposes the concept of the TP and TS Fuzzy grid structure to facilitate the development of the convex hull manipulation method. Section IV outlines the convex hull transition methodology and provides two demonstrative examples to illustrate the theoretical key points. Section V proposes an extension of the convex hull manipulation methodology to address large-scale problems. Finally, Section VI presents a real-world engineering problem to showcase the straightforward applicability and effectiveness of the proposed methodology. Section VII concludes the paper.

II. Notation

- $i, j, k, l, m, n, g \ldots$ are indices with the upper bounds $I, J, K, L, M, N, G \ldots$ e.g. $i = 1, 2, \ldots, I$ and $i_n = 1, 2, \ldots, I_n$ and so on;
- $a \in \mathbb{R}$, $\mathbf{a} \in \mathbb{R}^N$, $\mathbf{A} \in \mathbb{R}^{I \times J}$, $\mathcal{A} \in \mathbb{R}^{I^N}$ denote, scalars, vectors, matrices and tensors, respectively, where notation $\mathbb{R}^{I^N}$ is equivalent with $\mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N}$.
- $\mathbf{1}$ denotes a vector whose all elements are 1;
- $rank(\mathbf{A})$ denotes the rank of matrix $\mathbf{A}$;
- $\mathbf{I}$ denotes the identity matrix.
- $\{\mathcal{A}\}_{(n)}$ denotes the n-mode layout of $\mathcal{A}$, see [80].
- $rank_n(\mathcal{A})$ denotes the n-mode rank of $\mathcal{A}$ that is $rank_n(\mathcal{A}) = rank(\{\mathcal{A}\}_{(n)})$;
- $[\cdot]_{index}$ addresses elements, e.g. $[\mathcal{A}]_{i_1, i_2, \ldots i_N} = a_{i_1, i_2, \ldots i_N}$ of $\mathcal{A}$;
- $\lambda \in [0, 1]$ is the interpolation and transition parameter;
- $\omega \subset \mathbb{R}$ defines an interval as $\omega = [\omega^{min}, \omega^{max}]$;
- $\Omega \subset \mathbb{R}^N$ is a hyper space as $\Omega = \omega_1 \times \omega_2 \times \ldots \times \omega_N$;
- $\mathcal{A} \in co\{\forall n : \mathcal{B}_n\}$ denotes that $\mathcal{A}$ is within the convex hull defined by the vertices $\mathcal{B}_n$.

The $f(\mathbf{p})$ in the paper denotes a bounded continuosus tensor function $f(\mathbf{p}) \in \mathbb{R}^{J^M}$, where $\mathbf{p} \in \Omega \subset \mathbb{R}^N$, $\Omega = \omega_1 \times \omega_2 \times \cdots \times \omega_N$.

The formula

$$\mathcal{A} = \mathcal{B} \mathop{\boxtimes}_{n=1}^{N} \mathbf{C}_n, \tag{1}$$

denotes the tensor product of the core tensor $\mathcal{B} \in \mathbb{R}^{I^N \times J^M}$ and matrices $\mathbf{C}_n \in \mathbb{R}^{G_n \times I_n}$, that equals $\mathcal{A} \in \mathbb{R}^{G^N \times J^M}$. The tensor product is defined in the works of Lathauwer [80] and in the books [6], [7] about the TP model transformation. In the literature of the TP model transformation the notation $\boxtimes$

is used instead of $\times$ introduced in the works of Lathauwer, to underline that the core tensor may have vector, matrix or even tensor elements.

The more compact tensor algebra-based equivalent variant of the sum operator-based transfer function of the TS Fuzzy model is employed in the TS Fuzzy model transformation related literature. Thus, formula

$$f(\mathbf{p}) = \mathcal{A} \underset{n=1}{\overset{N}{\boxtimes}} \mathbf{w}_n(p_n), \tag{2}$$

with

$$\mathbf{w}_n(p_n) = \begin{bmatrix} w_{n,1}(p_n) & w_{n,1}(p_n) & \dots & w_{n,I_n}(p_n) \end{bmatrix}. \tag{3}$$

is equivalent to

$$f(\mathbf{p}) = \sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \dots \sum_{i_N=1}^{I_N} \prod_{n=1}^{N} w_{n,i_n}(p_n) a_{i_1,i_2,\dots,i_N}. \tag{4}$$

for further details with examples please study [1], [3]–[7], [14].

III. TP and TS Fuzzy grid structure of functions

The current section introduces the HOSVD-based TP and TS Fuzzy grid structures to facilitate further discussions, with a focus on the TS Fuzzy model transformation. Furthermore, this section proposes a conceptual differentiation between the TP and TS Fuzzy model transformation, which deviates from the synonymous use of these terms in the related literature.

Definition 3.1: **Hyper Rectangular Grid Tensor** $\mathcal{D}$

The $N+1$ dimensional hyper rectangular equidistant grid tensor (grid tensor in brief) is defined by $\mathcal{D} \in \mathbb{R}^{G^N \times N}$, where G_n denotes the number of grids on dimension n, in hyperspace $\Omega \subset \mathbb{R}^N$ and constructed by grid vectors $\mathbf{d}_n = \begin{bmatrix} d_{n,1} & d_{n,2} & \dots & d_{n,G_n} \end{bmatrix} \in \mathbb{R}^{G_n}$ defined for each ω_n, which contain a set of equidistantly located grid $d_{n,g} \in \omega_n$ in increasing order, where $d_{n,1} = \omega_n^{min}$ and $d_{n,G} = \omega_n^{max}$. $\mathcal{D}$ contains the coordinates of the N dimensional grid as $[\mathcal{D}]_{g_1,g_2,\dots g_N} = \begin{bmatrix} d_{1,g_1} & d_{2,g_2} & \dots & d_{N,g_N} \end{bmatrix}$.

Remark 3.1: The proposed methods in the paper and the concepts of the TP model transformation in general are not limited to equidistant grid. If there is a priori information about the function's oscillation and rate of change, and they significantly differ in different regions, then could be reasonable to set the density of the grid accordingly via defining non-equidistant grid at specific dimensions or regions. But for the sake of further discussion the paper uses equidistant grid.

Definition 3.2: **Discretised Tensor** $\mathcal{F}^{\mathcal{D}}$ **of function** $f(\mathbf{p})$

The Discretised Tensor $\mathcal{F}^{\mathcal{D}} \in \mathbb{R}^{G^N \times J^M}$ is constructed via the sampling of function $f(\mathbf{p})$ at each grid defined by the grid tensor $\mathcal{D}$ as $[\mathcal{F}^{\mathcal{D}}]_{g_1,g_2,\dots g_N} = f([\mathcal{D}]_{g_1,g_2,\dots g_N})$. Superscript "$D$" of $\mathcal{F}^{\mathcal{D}}$ denotes that the discretisation is based on $\mathcal{D}$.

The paper assigns the notation on a higher conceptual level. The discretised tensor $\mathcal{F}^{\mathcal{D}}$ is always understood in the paper as a result of the discretisation of $f(\mathbf{p})$ over the elements of the grid tensor $\mathcal{D}$ that fits the hyper space Ω of $\mathbf{p}$ according to Definition 3.1.

Method 3.1: **HOSVD based TP Grid Strucutre of function** $f(\mathbf{p})$

The HOSVD based TP grid structure (TP grid structure in brief) of function $f(\mathbf{p})$ over grid $\mathcal{D}$ is derived by the HOSVD [80] of the discretised tensor $\mathcal{F}^{\mathcal{D}}$:

$$\mathcal{F}^{\mathcal{D}} = \mathcal{H} \underset{n=1}{\overset{N}{\boxtimes}} \mathbf{U}_n, \tag{5}$$

where all the zero singular values are discarded, and $\mathcal{H} \in \mathbb{R}^{I^N \times J^M}$ and $\mathbf{U}_n \in \mathbb{R}^{G_n \times I_n}$, where $\forall n : I_n = rank_n(\mathcal{F}^{\mathcal{D}}) \leq G_n$. Note that only the first N dimension of $\mathcal{F}^{\mathcal{D}}$ is decomposed by HOSVD. If non-zero singular values are discarded too, such as $\exists n : I_n < rank_n(\mathcal{F}^{\mathcal{D}})$ then Equ. (5) becomes an approximation:

$$\mathcal{F}^{\mathcal{D}} \approx \mathcal{H} \underset{n=1}{\overset{N}{\boxtimes}} \mathbf{U}_n, \tag{6}$$

where the error is bounded by the sum of the discarded singular values [80].

Remark 3.2: Assume that $f(\mathbf{p})$ is given in the form of

$$f(\mathbf{p}) = \mathcal{A} \underset{n=1}{\overset{N}{\boxtimes}} \mathbf{w}_n(p_n). \tag{7}$$

One can reduce the size of the core tensor via executing HOSVD where all the zero singular values are truncated as:

$$\mathcal{A} = \mathcal{H} \underset{n=1}{\overset{N}{\boxtimes}} \mathbf{U}_n, \tag{8}$$

then

$$f(\mathbf{p}) = \mathcal{H} \underset{n=1}{\overset{N}{\boxtimes}} \mathbf{v}_n(p_n), \tag{9}$$

where

$$\mathbf{v}_n(p_n) = \mathbf{w}_n(p_n)\mathbf{U}_n. \tag{10}$$

Lemma 3.1: Based on Remark 3.2 $\forall n, G_n : rank_n(\mathcal{F}^{\mathcal{D}}) \leq rank_n(\mathcal{H})$. This means that the $rank_n(\mathcal{F}^{\mathcal{D}})$ will never be larger than the $rank_n(\mathcal{H})$ irrespective of $G_n \to \infty$. This practically means that G_n is selected to be large enough to get all the rank, but not necessarily should be extra large. Paper [81] shows how to define the minimum density of the discretisation grid in the TP model transformation, once we know the maximum number of the weighting functions as, for instance, determined in Equ. (7).

Method 3.2: **TS Fuzzy model Grid Structure of function** $f(\mathbf{p})$

The TS Fuzzy model Grid Structure of $f(\mathbf{p})$ over grid $\mathcal{D}$, that is also termed as Convex TP Grid Structure, is derived from the TP grid structure given in Equ. (5):

$$\mathcal{F}^{\mathcal{D}} = \mathcal{S} \underset{n=1}{\overset{N}{\boxtimes}} \mathbf{W}_n. \tag{11}$$

where each element of the discretised tensor is within the convex hull defined by vertices stored in $\mathcal{S}$ as $[\mathcal{F}^{\mathcal{D}}]_{g_1,g_2,\dots g_N} \in co\{\forall i_1, i_2, \dots i_N : [\mathcal{S}]_{i_1,i_2,\dots i_N}\}$, for all $g_1, g_2, \dots g_N$ that is guaranteed by

$$\forall n, g, i : [\mathbf{W}_n]_{g,i} \in [0,1] \subset \mathbb{R} \quad \text{and} \quad \forall n : \mathbf{W}_n \mathbf{1} = \mathbf{1}. \tag{12}$$

Matrices $\mathbf{W}_n$ are transformed from orthonorm matrices $\mathbf{U}_n$ of Equ. (5) by transformations such as SNNN, NO, CNO, NO, INO, RNO or IRNO transformations that guarantee (beside Equ. (12)) various further different proper characteristics of the convex hull defined by the vertices $[\mathcal{S}]_{i_1,i_2,\dots i_N}$, see later

This article has been accepted for publication in IEEE Transactions on Fuzzy Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2023.3348160

and in publications [1], [6], [11]–[13]. The core tensor $\mathcal{S}$ is finally determined by

$$\mathcal{S} = \mathcal{F}^{\mathcal{D}} \boxtimes_{n=1}^{N} (\mathbf{W}_n)^+ . \tag{13}$$

Remark 3.3: Note that while the size of $\mathbf{U}_n$ is $G_n \times I_n$ the size of $\mathbf{W}_n$ may increase by 1 column at some n to guarantee Equ. (12). However, for the sake of simplicity the paper considers the size of $\mathbf{W}_n$ as $G_n \times I_n$.

Lemma 3.2: If both $\mathbf{W}^\alpha$ and $\mathbf{W}^\beta$ hold Equ. (12) then $\mathbf{W}^\lambda = \lambda \mathbf{W}^\alpha + (1-\lambda)\mathbf{W}^\beta$ also holds Equ. (12).

Definition 3.3: **Identity Fuzzy set system $\mathbf{i}(p)$**

Let us introduce the Identity Fuzzy set system denoted by $\mathbf{i}(p)$ and defined by grid vector $\mathbf{d} \in \mathbb{R}^G$. Vector $\mathbf{i}(p) = \begin{bmatrix} i_1(p) & i_2(p) & \dots & i_G(p) \end{bmatrix} \in \mathbb{R}^G$ contains piece-wise linear triangular shaped functions such as

$$\mathbf{i}(p) = \lambda [\mathbf{I}]_g + (1-\lambda)[\mathbf{I}]_{g+1}; \quad \lambda = \frac{d_{g+1} - p}{d_{g+1} - d_g}, \tag{14}$$

where $d_g \leq p \leq d_{g+1}$.

Definition 3.4: **Piece-wise linear Fuzzy set system $\overline{\mathbf{w}}(p)$**

The piece-wise linear Fuzzy set system $\overline{\mathbf{w}}(p) \in \mathbb{R}^I$ - denoted by a bar on top - is defined by a matrix $\mathbf{W}$ and grid vector $\mathbf{d}$ as

$$\overline{\mathbf{w}}(p) = \mathbf{i}(p)\mathbf{W}. \tag{15}$$

Lemma 3.3: Since Equ. (12) the Fuzzy sets in $\overline{\mathbf{w}}(p)$ hold $\forall p : \overline{\mathbf{w}}(p)\mathbf{1} = \mathbf{1}$ and $\forall p, i : [\overline{\mathbf{w}}(p)]_i \in [0,1]$, that property is termed as Ruspinin-partition of the Fuzzy sets. Further, if both $\overline{\mathbf{w}}^\alpha(p)$ and $\overline{\mathbf{w}}^\beta(p)$ defined over the same grid define Ruspini partition then $\overline{\mathbf{w}}^\lambda(p) = \lambda\overline{\mathbf{w}}^\alpha(p) + (1-\lambda)\overline{\mathbf{w}}^\beta(p)$ also defines Ruspini-partition.

The $\overline{\mathbf{w}}_n(p_n)$ is always understood in the paper as it is $\overline{\mathbf{w}}_n(p_n) = \mathbf{i}(p_n)\mathbf{W}_n$ where $\mathbf{W}_n$ is from the TS Fuzzy grid structure of $f(\mathbf{p})$ with grid tensor $\mathcal{D}$ and $\mathbf{i}(p_n)$ is also defined by the same grid.

Method 3.3: **Multi-linear piece-wise approximation $\overline{f}(\mathbf{p})$ of $f(\mathbf{p})$ based on $\mathcal{D}$**

Assume function $f(\mathbf{p})$ and grid tensor $\mathcal{D}$ then

$$f(\mathbf{p}) \approx \overline{f}(\mathbf{p}) = \mathcal{F}^{\mathcal{D}} \boxtimes_{n=1}^{N} \mathbf{i}_n(p_n). \tag{16}$$

Method 3.4: **Multi-linear HOSVD based TS Fuzzy model Approximation**

Based on Equ. (11) and (16) the multi-linear TS Fuzzy model approximation of $f(\mathbf{p})$ via grid $\mathcal{D}$ is determined as

$$f(\mathbf{p}) \approx \overline{f}(\mathbf{p}) = \left(\mathcal{S} \boxtimes_{n=1}^{N} \mathbf{W}_n \right) \boxtimes_{n}^{N} \mathbf{i}(p_n) = \mathcal{S} \boxtimes_{n=1}^{N} \overline{\mathbf{w}}_n(p_n), \tag{17}$$

with $\forall n : \overline{\mathbf{w}}_n(p_n) = \mathbf{i}(p)\mathbf{W}_n$, where $\mathcal{S}$ and $\mathbf{W}$ are defined by the TS Fuzzy grid structure of $f(\mathbf{p})$, see Equ. (11). Equ. (17) is equivalent with the transfer function Equ. (4) of the typical TS Fuzzy model, but in the form of tensor operations. The functions $w_{n,i}(p_n(t))$ of $\mathbf{w}_n(p_n)$ are the membership functions of the antecedent Fuzzy sets and $[\mathcal{S}]_{i_1,i_2,\dots i_N}$ are the consequent structures, e.g. consequent or vertex system matrices in case of Linear Parameter Varying state-space dynamic models. For further details and examples please study [1], [3]–[7], [14].

Lemma 3.4: $\forall \mathbf{p} : \overline{f}(\mathbf{p}) \in co\{\forall i_1, i_2, \dots i_N : [\mathcal{S}]_{i_1,i_2,\dots i_N}\}$ see Lemma 3.3.

Method 3.5: **TS Fuzzy model transformation**

The paper denotes by a hat on top the approximation of $f(\mathbf{p})$ with very high accuracy as $\widehat{f}(\mathbf{p})$, where

$$f(\mathbf{p}) \underset{\epsilon}{\cong} \widehat{f}(\mathbf{p}). \tag{18}$$

If the approximation error ϵ is very small in numerical sense - or is acceptable in the application at hand - then $\widehat{f}(\mathbf{p})$ is considered to be numerically equivalent to $f(\mathbf{p})$. When $\forall n : G_n \to \infty$ and $\epsilon \to 0$ in

$$\mathbf{w}_n(p_n) \underset{\epsilon}{\cong} \widehat{\mathbf{w}}_n(p_n) = \overline{\mathbf{w}}_n(p_n), \tag{19}$$

then Equ. (17) resulted by Method 3.4 turns to be a numerical reconstruction as

$$f(\mathbf{p}) \underset{\epsilon}{\cong} \widehat{f}(\mathbf{p}) = \mathcal{S} \boxtimes_{n=1}^{N} \mathbf{w}_n(p_n) = \overline{f}(\mathbf{p}) = \mathcal{S} \boxtimes_{n=1}^{N} \overline{\mathbf{w}}_n(p_n). \tag{20}$$

The convergence of ϵ in the numerical reconstruction by TP or TS Fuzzy model transformation based on the increasing grid density is investigated in [6], [81]. For further details about the approximation properties of the TP model transformation be kindly referred to key publication [82], [83].

Remark 3.4: Note that if $f(\mathbf{p})$ have

$$f(\mathbf{p}) = \mathcal{S} \boxtimes_{n=1}^{N} \mathbf{w}_n(p_n) \tag{21}$$

representation then $\forall n, G_n : rank_n(\mathcal{F}^{\mathcal{D}}) \leq I_n$ with $G_n \to \infty$, see Lemma 3.1. If $f(\mathbf{p})$ have no

$$f(\mathbf{p}) = \mathcal{S} \boxtimes_{n=1}^{N} \mathbf{w}_n(p_n) \tag{22}$$

representation [82], [83] then $\exists n : rank_n(\mathcal{F}^{\mathcal{D}}) \to \infty$ with $G_n \to \infty$.

IV. Transition between TS Fuzzy models

This section proposes the method for the transition of the convex hulls based on the TS Fuzzy grid structures. The key of the method is the interpolation between $\mathbf{W}_n$ of the TS Fuzzy grid structures. Since it may not lead to smooth transition in general, as demonstrated later, then the proper rescheduling of the columns of $\mathbf{W}_n$ is propsed.

Definition 4.1: **Alternative TS Fuzzy models and grid structures**

The K alternative TS Fuzzy grid structures and TS Fuzzy models are

$$\mathcal{F}^{\mathcal{D}} = \mathcal{S}^k \boxtimes_{n=1}^{N} \mathbf{W}_n^k; \quad \text{and} \quad \overline{f}(\mathbf{p}) = \mathcal{S}^k \boxtimes_{n=1}^{N} \overline{\mathbf{w}}_n^k(p_n), \tag{23}$$

where $\mathcal{S}^k$, $\mathbf{W}_n^k$ and antecedent Fuzzy sets $\overline{\mathbf{w}}_n^k(p_n)$ may differ, but the input-output mappings of the alternative TS Fuzzy models are equivalent.

This article has been accepted for publication in IEEE Transactions on Fuzzy Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2023.3348160

A. Transition between alternative TS Fuzzy models

Assume a given function $f(\mathbf{p})$ and its two alternative TS Fuzzy model grid structures over $\mathcal{D}$ derived by Method 3.2:

$$\mathcal{F}^{\mathcal{D}} = \mathcal{A} \underset{n=1}{\overset{N}{\boxtimes}} \mathbf{W}_n^{\alpha} = \mathcal{B} \underset{n=1}{\overset{N}{\boxtimes}} \mathbf{W}_n^{\beta}, \tag{24}$$

where, for instance, vertices $[\mathcal{A}]_{i_1,i_2,\ldots i_N}$ define a loose convex hull, where $\mathbf{W}_n^{\alpha}$ are derived by SNNN transformation, while the vertices $[\mathcal{B}]_{i_1,i_2,\ldots i_N}$ define a tight convex hull, where $\mathbf{W}_n^{\alpha}$ are derived by CNO or IRNO transformation.

Method 4.1: **Transition between the convex hulls defined by the TS Fuzzy models**

- **Transition between the TS Fuzzy grid structures**
  The transition between tensors $\mathcal{A}$ and $\mathcal{B}$ is based on the interpolation between the matrices for all n as

$$\mathbf{W}_n^{\lambda} = (1-\lambda)\mathbf{W}_n^{\alpha} + \lambda \mathbf{W}_n^{\beta}, \tag{25}$$

$$\mathcal{S}^{\lambda} = \mathcal{F}^{\mathcal{D}} \underset{n=1}{\overset{N}{\boxtimes}} \left(\mathbf{W}_n^{\lambda}\right)^{+}, \tag{26}$$

  that leads to TP grid structure

$$\mathcal{F}^{\mathcal{D}} = \mathcal{S}^{\lambda} \underset{n=1}{\overset{N}{\boxtimes}} \mathbf{W}_n^{\lambda} \tag{27}$$

  if $\mathbf{W}_n^{\lambda}$ define the proper basis. E.g. if $\exists n$: $rank(\mathbf{w}_n^{\lambda}(p_n)) < rank_n(\mathcal{F}^{\mathcal{D}})$ then Equ. (27) does not hold, hence, Equ. (27) becomes a rank reduced approximation of $\mathcal{F}^{\mathcal{D}}$. Furthermore, in such cases, when at least one singular value of any $\mathbf{W}_n^{\lambda}$ converges to zero, the resulting $\mathcal{S}^{\lambda}$ may define a very large convex hull. This leads to a jarring transition of the convex hulls, see the example later in Section VI. Therefore, one has to check if Equ. (27) is acceptable for the problem at hand.
- **Smooth transition - Reschedule the antecedent Fuzzy sets**
  When the rank of $\mathbf{W}_n^{\lambda}$ drops then one can swap the columns of $\mathbf{W}_n^{\alpha}$ or $\mathbf{W}_n^{\beta}$ to find such a pairs of $[\mathbf{W}_n^{\alpha}]_{i_n}$ and $[\mathbf{W}_n^{\beta}]_{i_n}$ which avoid the rank drop of $\mathbf{W}_n^{\lambda}$ and leads to a smooth transition of the convex hulls, see the example later in Section VI.
- **Numerical reconstruction of the transition**
  To achieve an acceptable level of accuracy in the numerical reconstruction of the interpolated antecedent Fuzzy sets and, consequently, the TS Fuzzy model for a given λ, it is suggested to implement the proposed method using a high-density grid.
- **Define the antecedent Fuzzy sets**
  Finally, $\overline{\mathbf{w}}_n^{\lambda}(p_n)$ is derived by $\overline{\mathbf{w}}_n^{\lambda}(p_n) = \mathbf{i}(p_n)\mathbf{W}_n^{\lambda}$ to arrive at

$$f(\mathbf{p}) \cong \mathcal{S}^{\lambda} \underset{n=1}{\overset{N}{\boxtimes}} \overline{\mathbf{w}}_n^{\lambda}(p_n). \tag{28}$$

Lemma 4.1: In Equ. (28) the Ruspini-partition is hold for $\forall n : \overline{\mathbf{w}}_n^{\lambda}(p_n)$, see Lemma 3.3, that guarantees $\forall p : f(\mathbf{p}) \in \{i_1, i_2, \ldots i_N : [\mathcal{S}^{\lambda}]_{i_1,i_2,\ldots i_N}\}$.

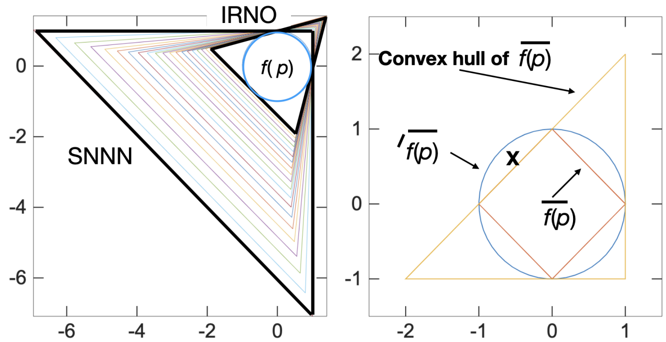

Fig. 1: Convex hulls defined by the vertices. The horizontal axis is assigned to $f_1(p)$ while $f_2(p)$ is assigned to the vertical axis. The left image belongs to Demonstrative example I., while the right one belongs to the Demonstrative example II.

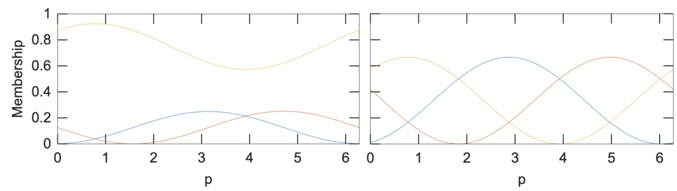

Fig. 2: The IRNO (right) and the SNNN (left) type antecedent Fuzzy membership functions $\overline{\mathbf{w}}^{\alpha}(p)$ and $\overline{\mathbf{w}}^{\beta}(p)$ for $G = 50$.

B. Demonstrative Example I: Smooth transition

To facilitate a simple two-dimensional visualization of the convex hulls, consider the following vector function:

$$\mathbf{f}(p) = \begin{bmatrix} f_1(p) & f_2(p) \end{bmatrix} = \begin{bmatrix} sin(p) & cos(p) \end{bmatrix}, \tag{29}$$

where $p \in [0, 2\pi]$. This vector describes a circle as depicted on Figure 1. Let us derive two alternative TS Fuzzy model grid structures with $G = 50$ by Method 3.1 to arrive at Equ. (24). Let $\mathbf{W}_n^{\alpha}$ and $\mathbf{W}_n^{\beta}$ be derived by IRNO and SNNN transformations to define tight and loose convex hulls, respectively. In the present simple case, it takes a simple matrix form (since $N = 1$) as

$$\mathcal{F}^{\mathcal{D}} = \mathbf{W}^{\alpha}\mathbf{A} = \mathbf{W}^{\beta}\mathbf{B}, \tag{30}$$

where $\mathcal{F}^{\mathcal{D}} \in \mathbb{R}^{G\times 2}$, $\mathbf{W}^{\alpha,\beta} \in \mathbb{R}^{G\times 3}$ and $\mathbf{A}, \mathcal{B} \in \mathbb{R}^{3\times 2}$. The resulting membership functions $\overline{\mathbf{w}}^{\alpha}(p)$ and $\overline{\mathbf{w}}^{\beta}(p)$ of

$$\overline{f}(p) = \overline{\mathbf{w}}^{\alpha}(p)\mathbf{A} = \overline{\mathbf{w}}^{\beta}(p)\mathbf{B} \tag{31}$$

are depicted on Figure 2, and the triangular convex hulls defined by the vertices, namely by the row vectors of $\mathbf{A}$ and $\mathbf{B}$ are depicted on the left image of Figure 1.

Let us define the interpolation by Equ. (25) and then determine $\mathbf{S}^{\lambda}$ by Equ. (26) that leads to Equ. (27) and finally to Equ. (28). Figure 3 depicts the transition of the antecedent Fuzzy membership functions $\overline{\mathbf{w}}^{\lambda}(p)$ for $\lambda = 0 \to 1$. The left image of Figure 1 shows the transition between the loose SNNN type convex hull and the tight IRNO type convex hull defined by $\mathbf{S}^{\lambda}$. Figure 4 shows the smooth transition of the elements of $\mathbf{S}^{\lambda}$ from the tight convex hull to the loose convex hull with $\lambda = 0 \to 1$. Obviously, as demonstrated on

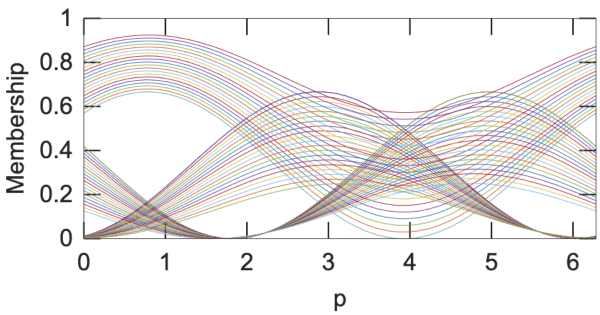

Fig. 3: Transition of the antecedent Fuzzy membership functions for $\lambda = 0 \to 1$.

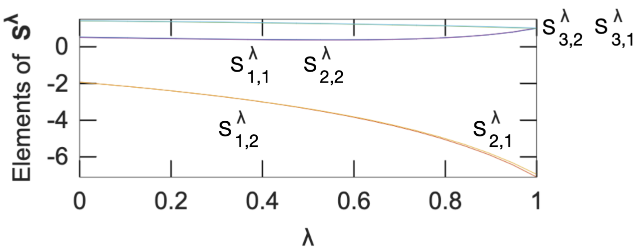

Fig. 4: Transition of the elements of $\mathbf{S}^\lambda$ for $\lambda = 0 \to 1$.

Figure 4, the transition of the convex hull by the proposed method is not equal to the linear interpolation of the vertices $\mathbf{S}^\lambda \approx (1-\lambda)\mathcal{A} + \lambda\mathcal{B}$, except very extreme cases.

V. Extension of the transition Method to large scale problems

The computation complexity of the HOSVD is exploding easily with size of $\mathcal{F}^{\mathcal{D}}$, since the number of elements of $\mathcal{F}^{\mathcal{D}}$ is $\prod_{n=1}^{N} I_n \times \prod_{m=1}^{M} J_m$. Therefore, this section proposes a refinement and a reinforce method that can considerably improve further the approximation accuracy of the TS Fuzzy model transition without executing HOSVD, but break down the computation by dimensions.

A. Refine the resolution

Method 5.1: **Refining the antecedent Fuzzy sets of the TS Fuzzy model**

Assume that a TS Fuzzy model in Equ. (17) is derived by Method 3.2 (or by the transition Method 4.1 for a given λ) over $\mathcal{D}$. The number of rows in $\mathbf{W}_n$ equals the number of grid defined by G_n. The key of the algorithm below is that one may determine the TP grid structure over a grid with lower density first that is enough to define an acceptable $\mathcal{H}$. Then the antecedent Fuzzy sets are refined. Based on Remark 3.4 once all the rank is found by $\mathcal{H}$ then instead of applying considerably larger grid density where upon the TS Fuzzy model transformation is executed, one may apply the refining method proposed below to improve the resolution of the antecedent Fuzzy sets. The same conclusion can be drawn when the acceptable number of the antecedent sets are limited by I_n - for some reason in the application- irrespective of the rank of $\mathcal{F}^{\mathcal{D}}$. In this case instead of defining a considerable larger G_n than I_n one may define a smaller density grid $G_n > I_n$ and once I_n number of non-zero singular values are found then the resolution of the antecedent Fuzzy sets can be improved by the refining method to be introduced below.

- **STEP 1: Additional grid**
  Define additional $'G_k$ number of grid for the k-th dimension by $'\mathbf{d}_k$. Here superscript ' denotes the variable is defined for the increased grid resolution.
- **STEP 2: Discretisation over the new grid**
  Discretise function $f(\mathbf{p})$ over the new grid $'\mathcal{D}$ created from grid vectors $\forall n, n \neq k : d_n$ and $'d_k$, that results in $'\mathcal{F}^{\mathcal{D}}$ that has the same size as $\mathcal{F}^{\mathcal{D}}$ except the k-th dimension, where its size is $'G_k$.
- **STEP 3: Extract the refined $'\mathbf{W}_n$**
  Determine the new row vectors of $'\mathbf{W}_k$ over the new grid $'\mathbf{d}_k$ in the k-th dimension based on the followings

$$'\mathcal{F}^{\mathcal{D}} = \left(\mathcal{S} \mathop{\boxtimes}_{n=1, n\neq k}^{N} \mathbf{W}_n \right) \mathop{\boxtimes}_{k} {}'\mathbf{W}_k, \tag{32}$$

  where $'\mathbf{W}_k$ is to be determined. Since

$$\{'\mathcal{F}^{\mathcal{D}}\}_{(k)} =' \mathbf{W}_k \Big\{ \mathcal{S} \mathop{\boxtimes}_{n=1, n\neq k}^{N} \mathbf{W}_n \Big\}_{(k)}, \tag{33}$$

  then

$$'\mathbf{W}_k = \{'\mathcal{F}^{\mathcal{D}}\}_{(k)} \Big\{ \mathcal{S} \mathop{\boxtimes}_{n=1, n\neq k}^{N} \mathbf{W}_n \Big\}^{+}_{(k)}. \tag{34}$$

  Finally the new grid is merged into $\mathbf{d}_k$ and the assigned rows of $'\mathbf{W}_k \in \mathbb{R}^{'G_K \times I_K}$ is merged to $\mathbf{W}_k$ (according to the merging of the grids) that will have the size of $(G_k +' G_k) \times I_k$.
- **STEP 6: For all dimension**
  Repeat the above steps for all dimensions of $\mathbf{p}$. Then, finally the multi-linear TS Fuzzy model is

$$'\overline{f}(\mathbf{p}) = \mathcal{S} \mathop{\boxtimes}_{n=1}^{N} {}'\overline{\mathbf{W}}_n(p_n) \tag{35}$$

Remark 5.1: It should be noted that if the tensor product is underdetermined, meaning that it has multiple solutions, as expressed by

$$\mathcal{F}^{\mathcal{D}} = \mathcal{S} \mathop{\boxtimes}_{n=1}^{N} \mathbf{W}_n^k, \tag{36}$$

then the refining process may yield antecedent Fuzzy sets that are significantly different in shape from the ones to be refined. This is a relatively common occurrence when $N = 1$ and the initial grid is very sparse. However, it becomes an extreme and particular case when $N > 2$ and the initial grid is not very sparse. If the resulting shape of the antecedent Fuzzy sets are deemed unacceptable for the given problem, the reinforcement method proposed in the subsequent section solve this problem as well and defines the appropriate antecedent Fuzzy sets.

Remark 5.2: The refining method will decrease the approximation error of $\overline{f}(\mathbf{p})$ with the increase of the resolution of the antecedent Fuzzy sets, however, the error caused by $I_n < rank_n('\mathcal{F}^{\mathcal{D}})$ will not be eliminated.

B. Reinforce the Ruspini-partition

There is no guarantee that values of $'\mathbf{W}_n$ hold Equ. (12). Some smaller differences may occur, see the Subsection VI-C of the examples in Section VI. Therefore the Ruspini-partition of the antecedent Fuzzy sets is also not hold in such cases. This is obvious since, for instance, the tight convex hull defined over a sparse grid may not bound the values of the function sampled over the new grid used during refinement, see the Demonstrative example II in Section V-C below and the example in Section VI. Therefore the vertices in $\mathcal{S}$ must be adjusted by the next Method 5.2.

Method 5.2: **Reinforce the Ruspini-partition on the antecedent Fuzzy sets**

This method adjusts the vertices in $\mathcal{S}$ via reinforcing the Ruspini-partion on the antecedent Fuzzy sets $'\overline{\mathbf{w}}_n(p_n)$ without executing HOSVD on a large discretised tensor of $f(\mathbf{p})$. Assume that we have the TS Fuzzy model representation and the grid structure of $f(\mathbf{p})$ derived over $\mathcal{D}$ by Method 3.4:

$$\overline{f}(\mathbf{p}) = \mathcal{S} \mathop{\boxtimes}_{n=1}^{N} \overline{\mathbf{w}}_n(p_n) \quad \text{and} \quad \mathcal{F}^{\mathcal{D}} = \mathcal{S} \mathop{\boxtimes}_{n=1}^{N} \mathbf{W}_n. \tag{37}$$

Further assume that it is refined for new grid $'\mathcal{D}$ as:

$$'\overline{f}(\mathbf{p}) = \mathcal{S} \mathop{\boxtimes}_{n=1}^{N} {'\overline{\mathbf{w}}_n}(p_n) \quad \text{and} \quad {'\mathcal{F}^{\mathcal{D}}} = \mathcal{S} \mathop{\boxtimes}_{n=1}^{N} {'\mathbf{W}_n}. \tag{38}$$

Then the reinforcing is based on the execution of SVD (with discarding all the zero singular values) as

$$'\mathbf{W}_n = \mathbf{U}_n \mathbf{D}_n \mathbf{V}_n^{+}. \tag{39}$$

As a next step the $\mathbf{U}_n$ is transformed by any of the SNNN, CNO, INO or IRNO etc. transformations to $''\mathbf{W}_n$ in order to have tight or loose convex hull. Or simply, one can use the same transformation that was used to derive $\mathbf{W}_n$ in Equ. (37). Note that $\mathbf{U}_n$ is orthonorm resulted by SVD that is required by the transformations. Then, the adjusted core tensor is

$$''\mathcal{S} = {'\mathcal{F}^{\mathcal{D}}} \mathop{\boxtimes}_{n=1}^{N} (''\mathbf{W}_n)^{+} \tag{40}$$

that leads to

$$'\overline{f}(\mathbf{p}) = {''\mathcal{S}} \mathop{\boxtimes}_{n=1}^{N} {''\overline{\mathbf{w}}_n}(p_n) \quad \text{and} \quad {'\mathcal{F}^{\mathcal{D}}} = {''\mathcal{S}} \mathop{\boxtimes}_{n=1}^{N} {''\mathbf{W}_n}, \tag{41}$$

where $''\overline{\mathbf{w}}_n(p_n)$ holds the Ruspini-partition. The superscipt ” denotes that the variable is refined and reinforced.

In conclusion, if the result of the refining Method 5.1 is not acceptable in regard of the Ruspini-partition, then one can execute reinforcing Method 5.2 to derive two alternative TS Fuzzy models for transition, hence, to arrive at Equ. (24) to execute the transition in Method 4.1.

C. Demonstrative example II: Ruspini-partition is lost

Consider Equation (29). Let us use a very sparse grid with $G = 5$ for discretization and define a tight convex hull by IRNO transformation to derive the Multi-linear TS Fuzzy model $\overline{f}(\mathbf{p})$ using Method 3.4. The resulting $\overline{f}(\mathbf{p})$ is rectangular, and the convex hull is triangular, as depicted on the right image of Figure 1. Next, let us define a very high-density grid with $G \to \infty$ and use refining Method 5.1 to derive $f(\mathbf{p}) \cong {'\overline{f}(\mathbf{p})}$. The right image of Figure 1 shows that $'\overline{f}(\mathbf{p})$ is the numerically reconstructed circle defined by $f(\mathbf{p})$. Please compare to the left image. It can be observed that the area denoted by ”X” is not covered by the convex hull of the multi-linear TS Fuzzy model $\overline{f}(\mathbf{p})$ derived over the sparse grid.

The refining Method does not modify $\mathcal{S}$ and, therefore, does not alter the convex hull. However, the refining process will also sample the section of the circle denoted by ”X”, and those values will be included in the refined discretized tensor $'\mathcal{F}^{\mathcal{D}}$. Consequently, both the refined $'\mathcal{F}^{\mathcal{D}}$ and $'\overline{f}(\mathbf{p})$ are not included in $co\{\forall i_1, i_2, \ldots i_N : [\mathcal{S}]_{i_1,i_2,\ldots i_N}\}$ for all sampled grids. This leads to membership functions $'\overline{\mathbf{w}}(p)$ that are not bounded by $[0, 1]$. This is precisely why the reinforcement of the Ruspini-partition is necessary typically in case of tight convex hulls.

VI. Example of a real engineering problem

In the literature related to TP model transformations, the benchmark example of a very complex aeroelastic wing section often appears [3], [5], [15]–[17], [79], [84], that is from a real engineering control problem. For comparability, we also use this example in this paper. Previous papers derive ony SNNN, CNO, IRNO type convex hulls. The present example shows how to derive infinite number of variants of the convex hull. Note that to derive the TS Fuzzy model of the aeroelastic wing section does not need the utilisation of the refining method. However, in this example the refining method is demonstrated as well, and one can compare the result to the result of previous publications.

A. Model of the aeroelastic wing section

The model of the aeroelastic wing section is identified to design a state variable feedback-controller and an observer-based output feedback controller. The challenge in the control design, hence in finding the proper TS Fuzzy model based representation lies in the strong nonlinearities and complexity of the model. The state-space model of the 2D aeroelastic wing section has state vector $\mathbf{x}(t) \in \mathbb{R}^4$ as $\mathbf{x}(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) \end{bmatrix}^T = \begin{bmatrix} h(t) & \alpha(t) & \dot{h}(t) & \dot{\alpha}(t) \end{bmatrix}^T$, where $x_1(t)$ is the plunging displacement and $x_2(t)$ is the pitching displacement. The state-space model has the form of

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \mathbf{y}(t) \end{bmatrix} = \mathbf{S}(\mathbf{p}(t)) \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}, \tag{42}$$

where parameter vector has elements $\mathbf{p}(t) = \begin{bmatrix} U(t) & x_2(t) \end{bmatrix} \in \mathbb{R}^2$. Here free stream velocity $U(t)$ is an external parameter. The entries of the system matrix are

$$\mathbf{S}(\mathbf{p}(t)) = \begin{bmatrix} 0 \quad 0 & 1 \quad 0 \quad 0 \\ 0 \quad 0 & 0 \quad 1 \quad 0 \\ \mathbf{S}_1(\mathbf{p}(t)) & \mathbf{S}_2(\mathbf{p}(t)) \end{bmatrix}, \tag{43}$$

$$\mathbf{S}_1(\mathbf{p}(t)) = \begin{bmatrix} -k_1 & -k_2 U^2(t) - p(k_\alpha(x_2(t))) \\ -k_3 & -k_4 U^2(t) - q(k_\alpha(x_2(t))) \end{bmatrix}, \tag{44}$$

$$\mathbf{S}_2(\mathbf{p}(t)) = \begin{bmatrix} -c_1(U(t)) & -c_2(U(t)) & g_3 U^2(t) \\ -c_3(U(t)) & -c_4(U(t)) & g_4 U^2(t) \end{bmatrix}, \tag{45}$$

This article has been accepted for publication in IEEE Transactions on Fuzzy Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2023.3348160

where

$$p(x_2(t)) = C_p k_\alpha(x_2(t)), \quad q(x_2(t)) = C_q k_\alpha(x_2(t)) \tag{46}$$

$$k_\alpha(x_2(t)) = 2.82(1 - 22.1x_2(t) + 1315.5x_2^2(t) + \\ +8580x_2^3(t) + 17289.7x_2^4(t)), \tag{47}$$

$$c_1(U(t)) = (I_\alpha c_h + U(t)(I_\alpha \rho b c_{l_\alpha} + m x_\alpha \rho c_{m_\alpha}))/d$$

$$c_2(U(t)) = (z\rho U(t)(I_\alpha b^2 c_{l_\alpha} + m x_\alpha b^4 c_{m_\alpha}) - m x_\alpha b c_\alpha)/d$$

$$c_3(U(t)) = -m(x_\alpha b c_h + \rho U(t) b^2 (x_\alpha c_{l_\alpha} + c_{m_\alpha}))/d$$

$$c_4(U(t)) = m(c_\alpha - z\rho U(t) b^3 (x_\alpha c_{l_\alpha} + c_{m_\alpha}))/d. \tag{48}$$

Here $a = -0.673$, $b = 0.135$, $k_h = 2844.4$, $c_h = 27.43$, $c_\alpha = 0.036$, $\rho = 1.225$, $c_{l_\alpha} = 6.28$, $c_{l_\beta} = 3.358$, $c_{m_\alpha} = (0.5 + a) * c_{l_\alpha}$, $m = 12.387$, $c_{m_\beta} = -0.635$, $x_\alpha = -0.3533 - a$, $I_\alpha = 0.065$, $d = 0.5193$, $k_1 = 356$ $k_2 = 0.105$, $k_3 = -2928.1$. $k_4 = -0.4906$,$C_p = -1.0294$, $C_q = 23.851$, $g_3 = -0.054911$, $z = (1/2 - a)$ and $g_4 = 0.2335$.

B. Starting with a sparse grid

In the present example we can start with a dense grid directly as documented in the previous publications. However, in order to demonstrate and compare the effectivenes of the refining method let us execute Method 3.2 on $\mathbf{S}(\mathbf{p}(t))$ over $\Omega = \begin{bmatrix}14 & 25\end{bmatrix} \times \begin{bmatrix}-0.3 & 0.3\end{bmatrix}$ with a low density grid as $G_1 = G_2 = 6$. Then we derive two alternative TS Fuzzy grid structures of $\mathbf{S}(\mathbf{p}(t))$ as in Equ. (24). Here $N = 2$ and $\mathcal{A}$ defines CNO type tight convex hull, while $\mathcal{B}$ defines SNNN type loose convex hull. The antecedent Fuzzy sets $\overline{\mathbf{w}}_n^\alpha(p_n(t))$ and $\overline{\mathbf{w}}_n^\beta(p_n(t))$ are depicted and Figure 5. Note that this sparse grid is enough to find all the ranks of the model.

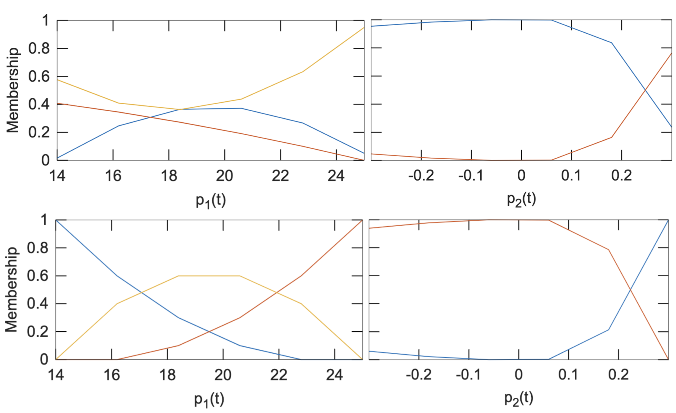

Fig. 5: The SNNN and CNO type antecedent Fuzzy sets of the wing section model, where the grid density is $G = 6$.

C. Refining the TS Fuzzy models

Since, the goal is to determine the transition of the TS Fuzzy model for $\lambda = 0 \to 1$ then we may refine the two alternative TS Fuzzy models instead of refining the interpolated antecedent Fuzzy sets for all $\lambda = 0 \to 1$.

Therefore, let us refine the grid on $p_1(t)$ first. Let the new grid defined by $'G_1 = 50$ ($G_2 = 6$). According to the refining Method 5.1, first define $\mathcal{F}_1^\mathcal{D}$. Then determine

$$'\mathbf{W}_1^\alpha = \{\mathcal{F}_1^\mathcal{D}\}_{(1)} \left(\{\mathcal{A}\}_{(1)}\right)^+, \tag{49}$$

$$'\mathbf{W}_1^\beta = \{\mathcal{F}_1^\mathcal{D}\}_{(1)} \left(\{\mathcal{B}\}_{(1)}\right)^+. \tag{50}$$

For dimension $p_2(t)$ let $'G_2 = 50$. The discretised tensor $\mathcal{F}_2^\mathcal{D}$ is defined over $G_1 = 6$ and $'G_2 = 50$. Then

$$'\mathbf{W}_2^\alpha = \{\mathcal{F}_2^\mathcal{D}\}_{(2)} \left(\{\mathcal{A}\}_{(2)}\right)^+ \tag{51}$$

$$'\mathbf{W}_2^\beta = \{\mathcal{F}_2^\mathcal{D}\}_{(2)} \left(\{\mathcal{B}\}_{(2)}\right)^+. \tag{52}$$

Finally we derived the TS Fuzzy models with resolution of 50×50, however no HOSVD was executed on such a large tensor. In the present example the minimum and the maximum values of $'\mathbf{W}_n^{\alpha,\beta}$ is slightly out of the bound $[0, 1]$ as

$$\min / \max(\forall n :' \mathbf{W}_n^{\alpha/\beta}) = -0.012494/1.0082. \tag{53}$$

If it is not acceptable in the design application, one can execute the reinforcement Method 5.2, see the next subsection.

D. Reinforce the SNNN and the CNO conditions

Because of Equ. (53) let us reinforce the Ruspini-partition. According to Method 5.2 the reinforcemenet is done via executing SVD (with discarding all the zero singular values) on $'\mathbf{W}_n^\alpha$ and $'\mathbf{W}_n^\beta$ as,

$$'\mathbf{W}_n^{\alpha/\beta} = \mathbf{U}_n^{\alpha/\beta} \mathbf{D}_n^{\alpha/\beta} \left(\mathbf{V}_n^{\alpha/\beta}\right)^T, \tag{54}$$

Then the reinforced $''\mathbf{W}_n^\alpha$ and $''\mathbf{W}_n^\beta$ are resulted by executing SNNN and CNO transformation on $\mathbf{U}_n^\alpha$ and $\mathbf{U}_n^\beta$.

E. Transition between TS Fuzzy model alternatives

According to Method 4.1, let us define the linear interpolation as

$$''\mathbf{W}_n^\lambda = (1 - \lambda)''\mathbf{W}_n^\alpha + \lambda''\mathbf{W}_n^\beta. \tag{55}$$

The resulting interpolated refined membership functions $\overline{\mathbf{w}}_n^\lambda(p_n)$ are depicted on Figure 6 for 10 equidistant grid of λ. Then the core tensor is

$$\mathbf{S}^\lambda = ''\mathcal{F}^\mathcal{D} \boxtimes_{n=1}^{N} \left(''\mathbf{W}_n^\lambda\right)^+, \tag{56}$$

where $''\mathcal{F}^\mathcal{D}$ is discretised over grid $'G_1 \times' G_2$. In the present case Equ. (56) does not lead to

$$''\mathcal{F}^\mathcal{D} = \mathcal{S}^\lambda \boxtimes_{n=1}^{N} {}''\mathbf{W}_n^\lambda \tag{57}$$

for all λ, since the rank of $''\mathbf{W}_1^\lambda$ decreases in a certain region of λ. Namely, the third singular value of $''\mathbf{W}_1^\lambda$ gradually converges to zero around $\lambda = 0.58$, see the left image of Figure 7. Figure 8 plots the elements of $\mathcal{S}^\lambda$ for $\lambda = 0 \to 1$. One can observe a jarring transition of the covex hull. Namely, the values of the vertices, hence, the convex hull explodes around $\lambda = 0.58$. If one need smooth transition for all λ then the rescheduling of the antecedent Fuzzy sets leads to solution, see next subsection.

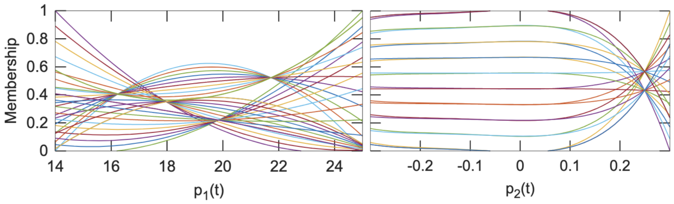

Fig. 6: The interpolated, refined and reinforced antecedent Fuzzy sets for $\lambda = 0 \rightarrow 1$, where $G_1 = G_2 = 50$.

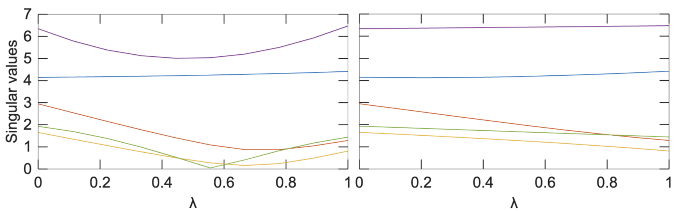

Fig. 7: The singular values of $\forall n : '' \mathbf{W}_n^\lambda$ are plotted on the left side and the singular values of $\forall n : ''' \mathbf{W}_n^\lambda$ resulted after rescheduling are plotted on the right side for $\lambda = 0 \rightarrow 1$.

F. Rescheduling the antecedent Fuzzy sets leads to smooth transition

Let us reshedule the antecedent membership functions via swapping the columns as

$$'''\mathbf{W}_1^\alpha = '' \mathbf{W}_1^\alpha \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad '''\mathbf{W}_2^\alpha = '' \mathbf{W}_2^\alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{58}$$

If we repeat the interpolation from Euq. (55) then we arrive at

$$\mathbf{S}(\mathbf{p}(t)) \cong \overline{\mathbf{S}}(\mathbf{p}(t)) = \mathcal{S}^\lambda \underset{n=1}{\overset{N}{\boxtimes}} {'''}\overline{\mathbf{w}}_n^\lambda(p_n). \tag{59}$$

The interpolated antecedent Fuzzy sets are depicted on Figure 9 (compare to Figure 6). After the rescheduling, none of the singular values of $'''\mathbf{W}_n^\lambda$ get close to zero for any λ, see the right image of Figure 7 and compare to the left image. As a result we have a smooth transition of the convex hull, see the transition of the values of $\mathcal{S}^\lambda$ for $\lambda = 0 \rightarrow 1$ on Figure 10 and compare to Figure 8.

VII. Conclusion

The paper introduces a radically new methodology for manipulating the Fuzzy rules and associated convex hulls of TS Fuzzy models. The proposed solution involves determining the linear interpolation of appropriately paired antecedent Fuzzy sets, followed by the derivation of the corresponding consequents. This approach enables a smooth transition between alternative TS Fuzzy models and their respective convex hulls. The paper highlights the effectiveness of this solution in systematically deriving an infinite number of various types of convex hulls in a controlled manner, in contrast to previous approaches that were limited to a few derivable convex hulls.

The paper also emphasizes the challenges associated with the alternative approach of generating the convex hull first and then deriving the Ruspini-partitioned antecedents. To develop

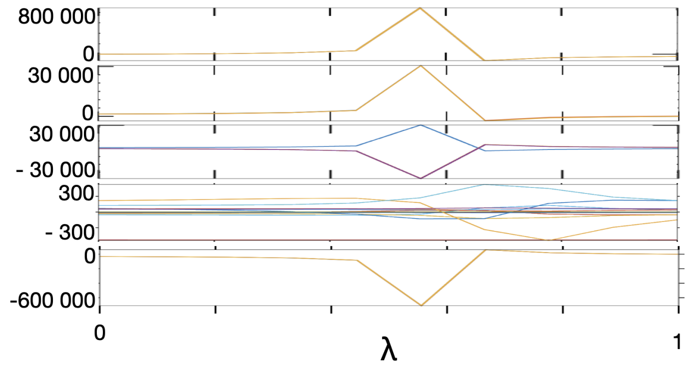

Fig. 8: Elements of $\mathcal{S}^\lambda$ for $\lambda = 0 \rightarrow 1$.The elements are grouped by different patterns for better visualisation

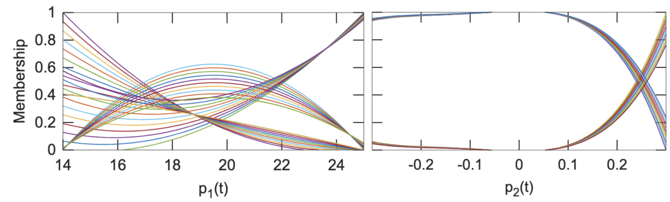

Fig. 9: The rescheduled, interpolated, refined and reinforced antecedent Fuzzy sets for $\lambda = 0 \rightarrow 1$.

such method requires intricate tensor algebraic operations that are currently unavailable, and in many cases, a viable solution does not exist.

Furthermore, the paper presents an extension of the proposed methodology to address large-scale problems. It demonstrates that the refining and reinforcing method proposed can effectively handle the transition between complex alternative TS Fuzzy models without the need for executing the Higher Order Singular Value Decomposition on large-sized tensors. The proposed method is thoroughly validated by numerical and real complex engineering examples, covering all theoretical aspects of the approach.

References

[1] Y. Yam, "Fuzzy approximation via grid point sampling and singular value decomposition," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 27, no. 6, pp. 933–951, December 1997.

[2] P. Baranyi, "TP model transformation as a way to LMI-based controller design," *IEEE Transactions on Industrial Electronics*, vol. 51, no. 2, pp. 387–400, 2004.

[3] ——, "The generalized TP model transformation for T–S fuzzy model manipulation and generalized stability verification," *IEEE Transactions on Fuzzy Systems*, vol. 22, no. 4, pp. 934–948, 2014.

[4] ——, "Extracting LPV and qLPV structures from state-space functions: A TP model transformation based framework," *IEEE Transactions on Fuzzy Systems*, vol. 28, no. 3, pp. 499–509, 2020.

[5] ——, "How to vary the input space of a T–S fuzzy model: A TP model transformation-based approach," *IEEE Transactions on Fuzzy Systems*, vol. 30, no. 2, pp. 345–356, 2022.

[6] P. Baranyi, Y. Yam, and P. Várlaki, *Tensor Product Model Transformation in Polytpic Model Based Control*, ser. Automation and Control Engineering. CRC Press, Taylor & Frances Group, March 2017, ISBN 9781138077782 - CAT K34341.

[7] P. Baranyi, *TP-Model Transformation Based Control Design Frameworks*, ser. Control Engineering. Springer book, July 2016, ISBN 978-3-319-19605-3.

[8] Y. Yam, P. Baranyi, and C. Yang, "Reduction of fuzzy rule base via singular value decomposition," *IEEE Transactions on Fuzzy Systems*, vol. 7, no. 2, pp. 120—132, April 1999.

This article has been accepted for publication in IEEE Transactions on Fuzzy Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2023.3348160

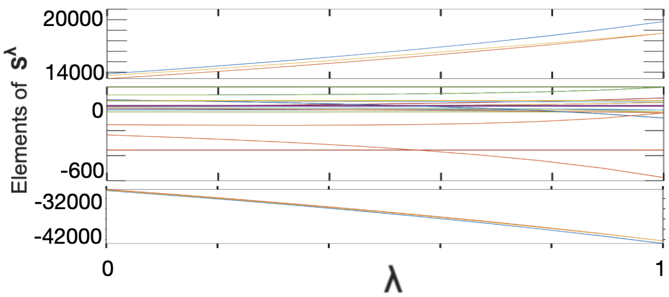

Fig. 10: Elements of $\mathcal{S}^\lambda$ for $\lambda = 0 \to 1$. The elements are grouped for better visualisation.

[9] P. Baranyi, P. Korondi, R. Patton, and H. Hashimoto, "Trade-off between approximation accuracy and complexity for TS fuzzy models," *Asian Journal of Control*, vol. 6, no. 1, pp. 21–33, 2004.

[10] P. Baranyi, Y. Yam, D. Tikk, and R. Patton, "Trade-off between approximation accuracy and complexity: TS controller design via HOSVD based complexity minimization," *Studies in Fuzziness and Soft Computing, Vol. 128. Interpretability Issues in Fuzzy Modeling, J. Casillas, O. Cordón, F. Herrera, L. Magdalena (Eds.), Springer-Verlag*, pp. 249–277, 2003.

[11] P. Várkonyi, D. Tikk, P. Korondi, and P. Baranyi, "A new algorithm for RNO-INO type tensor product model representation," in *Proceedings of the 9th IEEE International Conference on Intelligent Engineering Systems*, 2005, pp. 263–266.

[12] J. Kuti, P. Galambos, and P. Baranyi, "Minimal volume simplex (MVS) polytopic model generation and manipulation methodology for TP model transformation," *Asian Journal of Control*, vol. 19, no. 1, pp. 289–301, 2017.

[13] Y. Yu, Z. Li, X. Liu, K. Hirota, X. Chen, and H. Iu, "A nested tensor product model transformation," *IEEE Transactions on Fuzzy Systems*, vol. 27, no. 1, pp. 1–15, 2019.

[14] K. Tanaka and H. Wang, *Fuzzy control systems design and analysis: a linear matrix inequality approach.* John Wiley and Sons, Inc., U.S.A., 2001.

[15] A. Szollosi and P. Baranyi, "Influence of the tensor product model representation of qLPV models on the feasibility of linear matrix inequality," *Asian Journal of Control*, vol. 18, no. 4, pp. 1328–1342, 2015.

[16] ——, "Improved control performance of the 3-dof aeroelastic wing section: a TP model based 2d parametric control performance optimization," *Asian Journal of Control*, vol. 19, no. 2, pp. 450–466, 2016.

[17] ——, "Influence of the tensor product model representation of qLPV models on the feasibility of linear matrix inequality based stability analysis," *Asian Journal of Control*, vol. 10, no. 1, pp. 531–547, 2017.

[18] Y. Wang, C. Hua, and P. Park, "A generalized reciprocally convex inequality on stability and stabilization for t–s fuzzy systems with time-varying delay," *IEEE Transactions on Fuzzy Systems*, vol. 31, no. 3, pp. 722–733, 2023.

[19] Y. Yu, Z. Li, X. Liu, K. Hirota, X. Chen, T. Fernando, and H. H. C. Iu, "A nested tensor product model transformation," *IEEE Transactions on Fuzzy Systems*, vol. 27, no. 1, pp. 1–15, 2019.

[20] Y. Ren, Q. Li, D.-W. Ding, and X. Xie, "Dissipativity-preserving model reduction for takagi–sugeno fuzzy systems," *IEEE Transactions on Fuzzy Systems*, vol. 27, no. 4, pp. 659–670, 2019.

[21] X. Xie, F. Yang, L. Wan, J. Xia, and K. Shi, "Enhanced local stabilization of constrained n-TS fuzzy systems with lighter computational burden," *IEEE Transactions on Fuzzy Systems*, vol. 31, no. 3, pp. 1064–1070, 2023.

[22] H. Zhang and J. Liu, "Event-triggered fuzzy flight control of a two-degree-of-freedom helicopter system," *IEEE Transactions on Fuzzy Systems*, vol. 29, no. 10, pp. 2949–2962, 2021.

[23] Y. Wang, S. Xu, and C. K. Ahn, "Finite-time composite antidisturbance control for t–s fuzzy nonhomogeneous markovian jump systems via asynchronous disturbance observer," *IEEE Transactions on Fuzzy Systems*, vol. 30, no. 11, pp. 5051–5057, 2022.

[24] X.-H. Chang, Q. Liu, Y.-M. Wang, and J. Xiong, "Fuzzy peak-to-peak filtering for networked nonlinear systems with multipath data packet dropouts," *IEEE Transactions on Fuzzy Systems*, vol. 27, no. 3, pp. 436–446, 2019.

[25] W. Chen, Z. Fei, X. Zhao, and M. V. Basin, "Generic stability criteria for switched nonlinear systems with switching-signal-based lyapunov functions using takagi–sugeno fuzzy model," *IEEE Transactions on Fuzzy Systems*, vol. 30, no. 10, pp. 4239–4248, 2022.

[26] J. Song and X.-H. Chang, "Induced $\mathcal{L}_\infty$ quantized sampled-data control for t–s fuzzy system with bandwidth constraint," *IEEE Transactions on Fuzzy Systems*, vol. 31, no. 3, pp. 1031–1040, 2023.

[27] A. Ait Ladel, A. Benzaouia, R. Outbib, and M. Ouladsine, "Integrated state/fault estimation and fault-tolerant control design for switched t–s fuzzy systems with sensor and actuator faults," *IEEE Transactions on Fuzzy Systems*, vol. 30, no. 8, pp. 3211–3223, 2022.

[28] Y. Kang, L. Yao, and H. Wang, "Fault isolation and fault-tolerant control for takagi–sugeno fuzzy time-varying delay stochastic distribution systems," *IEEE Transactions on Fuzzy Systems*, vol. 30, no. 4, pp. 1185–1195, 2022.

[29] W. Ji, J. Qiu, H. R. Karimi, and Y. Fu, "New results on fuzzy integral sliding mode control of nonlinear singularly perturbed systems," *IEEE Transactions on Fuzzy Systems*, vol. 29, no. 7, pp. 2062–2067, 2021.

[30] Y. Qiu, C. Hua, and Y. Wang, "Nonfragile sampled-data control of t–s fuzzy systems with time delay," *IEEE Transactions on Fuzzy Systems*, vol. 30, no. 8, pp. 3202–3210, 2022.

[31] X. Xie, C. Wei, Z. Gu, and K. Shi, "Relaxed resilient fuzzy stabilization of discrete-time takagi–sugeno systems via a higher order time-variant balanced matrix method," *IEEE Transactions on Fuzzy Systems*, vol. 30, no. 11, pp. 5044–5050, 2022.

[32] Y. Xue, B.-C. Zheng, and X. Yu, "Robust sliding mode control for t-s fuzzy systems via quantized state feedback," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 4, pp. 2261–2272, 2018.

[33] H.-Y. Sun, H.-G. Han, J. Sun, H.-Y. Yang, and J.-F. Qiao, "Security control of sampled-data t–s fuzzy systems subject to cyberattacks and successive packet losses," *IEEE Transactions on Fuzzy Systems*, vol. 31, no. 4, pp. 1178–1188, 2023.

[34] V.-P. Vu and W.-J. Wang, "State/disturbance observer and controller synthesis for the t–s fuzzy system with an enlarged class of disturbances," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 6, pp. 3645–3659, 2018.

[35] Y. Ren, D.-W. Ding, Q. Li, and X. Xie, "Static output feedback control for t–s fuzzy systems via a successive convex optimization algorithm," *IEEE Transactions on Fuzzy Systems*, vol. 30, no. 10, pp. 4298–4309, 2022.

[36] V. C. S. Campos, F. O. Souza, L. A. B. Tôrres, and R. M. Palhares, "New stability conditions based on piecewise fuzzy lyapunov functions and tensor product transformations," *IEEE Transactions on Fuzzy Systems*, vol. 21, no. 4, pp. 748–760, 2013.

[37] X. Liu, Y. Yu, Z. Li, H. H. C. Iu, and T. Fernando, "An efficient algorithm for optimally reshaping the TP model transformation," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 64, no. 10, pp. 1187–1191, 2017.

[38] Y. Zhou, J. Liu, Y. Li, C. Gan, H. Li, and Y. Liu, "A gain scheduling wide-area damping controller for the efficient integration of photovoltaic plant," *IEEE Transactions on Power Systems*, vol. 34, no. 3, pp. 1703–1715, 2019.

[39] A. Hajiloo, M. Keshmiri, W.-F. Xie, and T.-T. Wang, "Robust online model predictive control for a constrained image-based visual servoing," *IEEE Transactions on Industrial Electronics*, vol. 63, no. 4, pp. 2242–2250, 2016.

[40] M. S. Aslam, P. Tiwari, H. M. Pandey, and S. S. Band, "Observer–based control for a new stochastic maximum power point tracking for photovoltaic systems with networked control system," *IEEE Transactions on Fuzzy Systems*, pp. 1–14, 2022.

[41] H. Wang, Y. Kang, L. Yao, H. Wang, and Z. Gao, "Fault diagnosis and fault tolerant control for t–s fuzzy stochastic distribution systems subject to sensor and actuator faults," *IEEE Transactions on Fuzzy Systems*, vol. 29, no. 11, pp. 3561–3569, 2021.

[42] F. Sabbghian-Bidgoli and M. Farrokhi, "Polynomial fuzzy observer-based integrated fault estimation and fault-tolerant control with uncertainty and disturbance," *IEEE Transactions on Fuzzy Systems*, vol. 30, no. 3, pp. 741–754, 2022.

[43] J. Hu, X. Li, Z. Xu, and H. Pan, "Co-design of quantized dynamic output feedback MPC for Takagi-Sugeno model," *IEEE Transactions on Industrial Informatics*, pp. 1–11, 2022.

[44] J. Hu, X. Lv, H. Pan, and M. Zhang, "Handling the constraints in min-max MPC," *IEEE Transactions on Automation Science and Engineering*, pp. 1–9, 2022.

[45] B.-S. Chen, Y.-Y. Tsai, and M.-Y. Lee, "Robust decentralized formation tracking control for stochastic large-scale biped robot team system under external disturbance and communication requirements," *IEEE*

Transactions on Control of Network Systems, vol. 8, no. 2, pp. 654–666, 2021.

[46] S. Tiko and F. Mesquine, "Constrained control for a class of TS fuzzy systems," *IEEE Transactions on Fuzzy Systems*, vol. 31, no. 1, pp. 348–353, 2023.

[47] S. Ijaz, L. Yan, M. T. Hamayun, W. M. Baig, and C. Shi, "An adaptive LPV integral sliding mode FTC of dissimilar redundant actuation system for civil aircraft," *IEEE Access*, vol. 6, pp. 65 960–65 973, 2018.

[48] J. Zaqueros-Martinez, G. Rodriguez-Gomez, E. Tlelo-Cuautle, and F. Orihuela-Espina, "Fuzzy synchronization of chaotic systems with hidden attractors," *Entropy*, vol. 25, no. 3, pp. 495–65 973, 2023.

[49] J. Zhang, X. Xu, L. Yang, and X. Yang, "LPV model-based multivariable indirect adaptive control of damaged asymmetric aircraft," *Aerospace Engineering*, vol. 32, no. 6, pp. 1–16, 2019.

[50] F. Ma, J. Li, L. Wu, and D. Yuan, "Tensor product based polytopic lpv system design of a 6-dof multi-strut platform," *International Journal of Control Automation and Systems.*, vol. 20, pp. 137–146, 2022.

[51] L. Nie, B. Cai, Y. Zhu, J. Yang, and L. Zhang, "Switched linear parameter-varying tracking control for quadrotors with large attitude angles and time-varying inertia," *Optimal Control Applications and Methods*, vol. 42, pp. 1320 – 1336, 2021.

[52] F. Chang, B. Shi, X. Li, G. Zhao, and S. Huan, "Local extrema refinement-based tensor product model transformation controller with problem independent sampling methods," *Asian Journal of Control*, vol. 23, no. 3, pp. 1352–1366, 2021.

[53] E. Hedrea, R. Precup, E. Petriu, C. Bojan-Dragos, and C. Hedrea, "Tensor product-based model transformation approach to cart position modeling and control in pendulum-cart systems," *Asian Journal of Control*, vol. 23, no. 3, pp. 1238–1248, 2021.

[54] M. Kuczmann, "Study of tensor product model alternatives," *Asian Journal of Control*, vol. 23, no. 3, pp. 1249–1261, 2020.

[55] P. Varlaki, L. Palkovics, and A. Rövid, "On modeling and identification of empirical partially intelligible white noise processes," *Asian Journal of Control*, vol. 23, no. 3, pp. 1262–1279, 2020.

[56] Z. Németh and M. Kuczmann, "Tensor product transformation-based modeling of an induction machine," *Asian Journal of Control*, vol. 23, no. 3, pp. 1280–1289, 2020.

[57] B. Takarics and B. Vanek, "Robust control design for the FLEXOP demonstrator aircraft via tensor product models," *Asian Journal of Control*, vol. 23, no. 3, pp. 1290–1300, 2020.

[58] A. Csapo, "Cyclical inverse interpolation: An approach for the inverse interpolation of black-box models using tensor product representations," *Asian Journal of Control*, vol. 23, no. 3, pp. 1301–1312, 2020.

[59] E. Hedrea, R. Precup, R. Roman, and E. Petriu, "Tensor product-based model transformation approach to tower crane systems modeling," *Asian Journal of Control*, vol. 23, no. 3, pp. 1313–1323, 2020.

[60] S. Ile, J. Matusko, and M. Lazar, "Piece-wise ellipsoidal set-based model predictive control of linear parameter varying systems with application to a tower crane," *Asian Journal of Control*, vol. 23, no. 3, pp. 1324–1339, 2020.

[61] A. Boonyaprapasorn, S. Kuntanapreeda, P. Ngiamsunthorn, E. Pengwang, and T. Sangpet, "Tensor product model transformation-based control for fractional-order biological pest control systems," *Asian Journal of Control*, vol. 23, no. 3, pp. 1340–1351, 2020.

[62] E. Precup, E. Petriu, M. Rădac, S. Preitl, L. Fedorovici, and C. Dragoş, "Cascade control system-based cost effective combination of tensor product model transformation and fuzzy control," *Asian Journal of Control*, vol. 17, no. 2, pp. 381–391, 2014.

[63] A. Boonyaprapasorn, S. Kuntanapreeda, T. Sangpet, P. Ngiamsunthorn, and E. Pengwang, "Biological pest control based on tensor product transformation method," *Acta Polytechnica Hungarica*, vol. 17, no. 6, pp. 25–40, 2020.

[64] V. Campos, M. Braga, and L. Frezzatto, "Analytical upper bound for the error on the discretization of uncertain linear systems by using the tensor product model transformation," *Acta Polytechnica Hungarica*, vol. 17, no. 6, pp. 61–74, 2020.

[65] A. Pereira, L. Vianna, N. Keles, and V. Campos, "Tensor product model transformation simplification of Takagi-Sugeno control and estimation laws – an application to a thermoelectric controlled chamber," *Acta Polytechnica Hungarica*, vol. 15, no. 3, pp. 13–29, 2018.

[66] V. L. Campos, V.C.S. and M. Braga, "A tensor product model transformation approach to the discretization of uncertain linear systems," *Acta Polytechnica Hungarica*, vol. 15, no. 3, pp. 31–43, 2018.

[67] Y. Kan, Z. He1, and J. Zhao, "Tensor product model-based control design with relaxed stability conditions for perching maneuvers," *Acta Polytechnica Hungarica*, vol. 15, no. 3, pp. 45–61, 2018.

[68] H. Gong, H. Sun, B. Wang, Y. Yu, Z. Li, X. Liao, and X. Liu, "Tensor product model-based control for space-craft with fuel slosh dynamics," *Acta Polytechnica Hungarica*, vol. 15, no. 3, pp. 63–80, 2018.

[69] H. Du, J. Yan, and F. Fan, "A state and input constrained control method for air-breathing hypersonic vehicles," *Acta Polytechnica Hungarica*, vol. 15, no. 3, pp. 81–99, 2018.

[70] L. Kovács and G. Eigner, "Tensor product model transformation-based parallel distributed control of tumor growth," *Acta Polytechnica Hungarica*, vol. 15, no. 3, pp. 101–123, 2018.

[71] E. Hedrea, R. Precup, and C. Bojan-Dragos, "Results on tensor product-based model transformation of magnetic levitation systems," *Acta Polytechnica Hungarica*, vol. 16, no. 9, pp. 93–111, 2019.

[72] G. Zhao, H. Li, and Z. Song, "Tensor product model transformation based decoupled terminal sliding mode control," *International Journal of Systems Science*, vol. 47, no. 8, pp. 1791–1803, 2016.

[73] F. Ma, J. Li, and L. Wu, "Tensor product model HOSVD based polytopic LPV controller for suspension anti-vibration system," *Journal of Vibration and Control*, vol. 29, no. 1-2, pp. 5–20, 2022.

[74] H. Xing, J. Wei, and J. Caisheng, "Robust controller designing for an air-breathing hypersonic vehicle with an HOSVD-based LPV model," *International Journal of Aerospace Engineering*, pp. 1–12, 2021.

[75] P. Korondi, C. Budai, H. Hashimoto, and F. Harashima, *Tensor Product Model Transformation Based Sliding Mode Design for LPV Systems*. Springer International Publishing, 2015, pp. 277–298.

[76] A. Carlos and S. Antonio, "Relaxed LMI conditions for closed-loop fuzzy systems with tensor-product structure," *Engineering Applications of Artificial Intelligence*, vol. 20, no. 8, pp. 1036–1046, 2007.

[77] X. Liu, X. Xin, Z. Li, and Z. Chen, "Near optimal control based on the tensor-product technique," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 64, no. 5, pp. 560–564, 2017.

[78] P. Baranyi and B. Takarics, "Aeroelastic wing section control via relaxed tensor product model transformation framework," *Journal of Guidance, Control and Dynamics*, vol. 37, no. 5, pp. 1671–1678, 2014.

[79] P. Baranyi, "Tensor product model based control of two dimensional aeroelastic system," *Journal of Guidance, Control and Dynamics*, vol. 29, no. 2, pp. 391–400, 2006.

[80] L. De Lathauwer, B. De Moor, and J. Vandewalle, "A multilinear singular value decomposition," *SIAM Journal on Matrix Analysis*, vol. 21, no. 4, pp. 1253–1278, 2000.

[81] L. Szeidl and P. Varlaki, "HOSVD based canonical form for polytopic models of dynamic systems," *Journal of Advanced Computational Intelligence and Intelligent Informatics*, vol. 13, no. 1, pp. 52–60, 2009.

[82] D. Tikk, P. Baranyi, and R. Patton, "Approximation properties of TP model forms and its consequences to TPDC design framework," *Asian Journal of Control*, vol. 9, no. 3, pp. 221–231, 2008.

[83] D. Tikk, P. Baranyi, R. Patton, and J. Tar, "Approximation capability of TP model forms," *Australian Journal of Intelligent Information Processing Systems*, vol. 8, pp. 155–163, 2004.

[84] P. Baranyi, Z. Petres, V. P.L., K. P., and Y. Yam, "Determination of different polytopic models of the prototypical aeroelastic wing section by TP model transformation," *Journal of Advanced Computational Intelligence and Intelligent Informatics*, vol. 10, no. 4, pp. 486–493, 2006.

Peter Baranyi obtained his M.Sc. and Ph.D. degrees from Budapest University of Technology and Economics, Hungary, in 1995 and 1999, respectively. He further received the D.Sc. degree from the Hungarian Academy of Sciences in 2006. He is a member of the Hungarian Academy of Engineering. He initiated the core theory of the TP model transformation. His research interests encompass various areas, including quasi-linear parameter-varying systems, linear matrix inequalities, and control design based on TP / TS Fuzzy model transformation.