Υ-Values: Power Indices à La Orness for Nonadditive Measures

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***Abstract*—Fuzzy measures, also known as capacities, nonadditive measures, and monotonic games, are increasingly used in all kind of applications. Fuzzy measures are set functions. So, for a given set X, we need to define $2^{|X|} - 2$ parameters (excluding the measure on the empty set and on X itself). Because of that they are difficult to visualize, and indices and metrics have been defined. The Shapley value is an example. It permits us to determine weights of importance of each element in X. In this article, we introduce an alternative index. We call it Υ-values. We provide an axiomatic characterization. These values are inspired on the Shapley values, and they are associated to set size, or position (order statistics) in a chain. Thus, also position when the measure is used in combination with a fuzzy integral. Andness and orness are measures that permit to evaluate the degree of simultaneity (conjunction) and substitutability (disjunction) of an aggregation function. We show the connection between our value and these concepts. In a way, Υ-values define a power index à la orness.**

***Index Terms*—Aggregation operators, andness, fuzzy measures, orness, Shapley values.**

I. Introduction

SHAPLEY values [1] are one of the existing power indices that were introduced for studying the worth of coalitions. A set of winning coalitions is an example of games. Games are set functions that for each set assign a value of its worth. Fuzzy measures and capacities are equivalent names to denote monotonic games. In this case, the set function needs to be monotonic with respect to set inclusion.

Shapley values are extensively used to study these mathematical objects (winning coalitions, games, fuzzy measures, and capacities). Given a set function, they assign a value to each of the elements in the reference set in a way that the value represents the importance of an element taking into account the full measure. The Shapley value is characterized by a set of properties. One of them is that the sum of the Shapley values add to one (when the set function on the reference set X is one). So, they distribute the measure on X over all the elements in X.

Fuzzy measures and fuzzy integrals have been used for data aggregation [2], [3], [5] in different types of applications [6], [7], [8]. The aggregation of a set of values (i.e., a function) with a fuzzy integral requires a fuzzy measure [9], [10], [11] (or more than one when hierarchical models are used [12]). As fuzzy measures μ are set functions, for their definition on a reference set X, we need $2^{|X|}$ values. As $\mu(\emptyset) = 0$ and, if μ is normalized, $\mu(X) = 1$, this means that $2^{|X|} - 2$ arbitrary values are required. This large number of values makes measures difficult to grasp by users and practitioners. Indices have been proposed to help users to understand the measures.

The Shapley value is one of these indices. It is a way to determine the importance of each $x \in X$ taking into account the whole measure. Shapley values add to $\mu(X)$, so, they are a distribution of $\mu(X)$ among the elements in X. They are useful to define measures in applications.

In the context of data aggregation, other indices have been defined as well. For example, interaction [13] between the elements in X for a given measure μ. Interaction indices permit to know which are the positive and the negative interactions between elements in X. Andness and orness [14], [15] are other proposed indices. Andness permits to evaluate the degree of simultaneity or conjunction of an aggregation function. It is defined in terms of the similarity between this function and the minimum. Orness permits to evaluate the degree of substitutability (disjunction) of an aggregation function. In this case, we calculate the similarity between the function and the maximum. Orness and andness have been computed for different aggregation functions [5], [16], [17]. In particular, for OWA note that andness and orness is not limited to means between minimum and maximum, but they can be computed for t-norms and t-conorms [18] (i.e., conjunctions and disjunction in fuzzy logic).

The concept of andness and orness, as well as importance weights are key [5], [19], [20] in data aggregation. When modeling decisions taking into account different criteria, we need to use an aggregation that takes into account that different criteria have different importances (i.e., weights), as well as implement a certain level of conjunction / disjunction (i.e., in what degree we can compensate a criteria with a bad score with a criteria with a good score). Andness-directedness is about selecting an appropriate aggregation function given the andness level.

In this article, we introduce Shapley-like values. These new values are associated to set size. In the context of data aggregation, when a Choquet integral [21] of a function with respect to a measure is applied, these new values are associated to the position of the smallest input, the second smallest,..., the largest input of the function. As a consequence, these values can be seen as power indices à la orness, and can naturally be linked to

Manuscript received 14 December 2023; revised 15 March 2024; accepted 11 April 2024. Date of publication 22 April 2024; date of current version 2 July 2024. This work was supported in part by the Wallenberg AI, Autonomous Systems and Software Program (WASP), in part by the Knut and Alice Wallenberg Foundation, and in part by the KEMPE foundation. Recommended by Associate Editor D. Wu.

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Digital Object Identifier 10.1109/TFUZZ.2024.3392268

disjunction and conjunction. The larger the indices associated to larger inputs, the larger the disjunction and orness. On the contrary, the larger the indices associated to lower inputs, the lower the disjunction and orness, the larger the conjunction and andness.

For a given measure μ, we will denote the power indices by Υ_μ. While Shapley power indices can be seen as similar to weights for a weighted mean, our power indices can be seen as weights for an OWA operator. Because of that, we also introduce in this article the computation of the orness associated to such indices. This is based on Yager's expression [22] for orness of OWA. Our definitions are illustrated with some examples.

The rest of this article is organized as follows. In Section II, we provide some preliminaries that are needed later on. This includes some concepts related to aggregation, OWA, fuzzy measures, and power indices. Then, in Section III we introduce the main definitions and results. We prove properties of the index and a characterization, and also include an example. Section IV provides some experiments and analysis, the experiments also give insight of this index. Finally, Section V concludes this article.

II. Preliminaries

We begin this section reviewing some definitions related to aggregation operators. We review the OWA operator introduced by Yager [23]. It corresponds to a linear combination of order statistics. We provide its definition in the following. To do so, we begin defining weighting vector.

Definition 1: A vector $v = (v_1, \ldots, v_n)$ is a weighting vector of dimension n if and only if weights are positive and add to one. That is, $0 \leq v_i$ and $\sum_{i=1}^n v_i = 1$.

Definition 2 (See [23]): Let w be a weighting vector of dimension n. Then, a mapping OWA : $\mathbb{R}^n \to \mathbb{R}$ is an ordered weighting averaging operator of dimension n if

$$\mathrm{OWA}_w(a_1, \ldots, a_n) = \sum_{i=1}^{n} w_i a_{\sigma(i)}$$

where σ defines a permutation of the inputs a_i such that $a_{\sigma(i-1)} \geq a_{\sigma(i)}$ for all $i = 2, \ldots, n$.

Observe that $a_{\sigma(i)}$ corresponds to the ith largest element in the collection $a_1, \ldots, a_n$. So, $a_{\sigma(i)}$ is the $n - i + 1$ order statistic.

Given a weighting vector p, the weighted mean of $(a_1, \ldots, a_n)$ with respect to p corresponds to $\sum_{i=1}^n p_i a_i$. The weighted OWA (WOWA) operator is an operator that can be seen as a generalization of both the OWA and the weighted mean. It aggregates $(a_1, \ldots, a_n)$ with respect to two weighting vectors. One corresponding to the one of the OWA, say w as previous, and another corresponding to the one of the weighted mean, say p as previous. We denote this operator by $\mathrm{WOWA}_{p,w}$. A WOWA with weights p and w is equivalent to a Choquet integral with a distorted probability. The definition of the distortion w^* from the weights w was provided [24], [25] in the original definition of the WOWA. We will not provide the definition of WOWA, but refer the reader to, e.g., Torra and Narukawa [24], [26] for details and properties.

Andness and orness were introduced by Dujmović [14], [15] as a way to evaluate the level of conjunction and level of disjunction of an operator. Andness is the similarity to the minimum and orness the similarity to the maximum. Yager [22] introduced an expression for OWA based on the OWA weights. The expression follows. This expression is equivalent to Dujmović's orness when we apply it to the OWA operator with weights w.

Definition 3 (See [22]): Let w be a weighting vector. Then, the orness for OWA_w is defined in terms of w as follows:

$$\mathrm{orness}(\mathrm{OWA}_w) = \frac{1}{n-1} \sum_{i=1}^{n} (n-i) w_i.$$

It was shown [27], [28] that the orness of the $\mathrm{WOWA}_{p,w}$ corresponds to the orness of OWA_w. In other words, the orness only depends on the OWA weights but not on the importance of the inputs. Andness-directedness for the OWA and WOWA families has also been studied [29].

A. Fuzzy Measures

Measures are set functions that are additive and can be seen as generalizations of the idea of length and volume. Fuzzy measures, also known as nonadditive measures, monotonic games, simple games, and capacities, are also set functions but the additivity condition is replaced by a monotonicity one. The formal definition follows.

Definition 4: Let X be a reference set. Then, a set function μ on X is a fuzzy measure if it satisfies the following conditions.
1) $\mu(\emptyset) = 0$.
2) $\mu(X) = 1$.
3) $\mu(A) \leq \mu(B)$ if $A \subseteq B \subseteq X$.

We will use the term just measure or game when the monotonicity condition is dropped. Although this condition is relevant in aggregation, it is not so in other contexts, and indices have been defined for set functions.

The condition $\mu(X) = 1$ is a normalization. We call measures satisfying this condition normalized measures. We assume that measures are normalized unless stated otherwise.

There are different types of measures. In this article, we are interested in the following ones.
1) A measure is additive when $\mu(A \cup B) = \mu(A) + \mu(B)$ for $A \cap B = \emptyset$. Naturally, a probability is an example of additive measure. Given a weighting vector p where p_i corresponds to the weight of x_i, we can define $\mu(A) = \sum_{x_i \in A} p_i$.
2) A measure is symmetric when $\mu(A) = \mu(B)$ if the cardinality of both sets is the same. That is, $|A| = |B|$. If we have a weighting vector with weights w_i, the measure $\mu(A) = \sum_{i=1}^{|A|} w_i$ is symmetric.
3) Given a nonempty subset T of X (i.e., $\emptyset \subset T \subseteq X$), the unanimity measure μ_T is the measure $\mu_T(A) = 1$ if $T \subseteq A$ and 0 otherwise.

It can be proven that an arbitrary fuzzy measure can be expressed in terms of unanimity measures.

Proposition 1 (See [4], [30]): Any fuzzy measure can be expressed as a linear combination of unanimity measures. In other words, let μ_T denote the unanimity measure for a set $\emptyset \subset$

$T \subseteq X$. Then, for any fuzzy measure μ there are coefficients c_T such that

$$\mu = \sum_{\emptyset \subset T \subseteq X} c_T \mu_T.$$

The coefficients c_T are the Möbius transform. There are equivalent representations of fuzzy measures by means of transforms. The Möbius transform is the most well-known one. Other transforms exist. In this article we will use the $(\max, +)$-transform.

Definition 5 (See [31]): Let μ be a nonadditive measure on X. Then, we define the $(\text{Max}, +)$-transform as the set function $m_\mu : 2^X \to \mathbb{R}^+$ such that

$$m_\mu(B) = \mu(B) - \max_{A \subset B} \mu(A). \quad (1)$$

This definition provides a transform that is always positive. More particularly, for normalized fuzzy measures, we have that $m_\mu(A) \in [0, 1]$ for any $A \subseteq X$. Compare with Möbius transform that can take values arbitrarily large (positive and negative).

Data can be aggregated using fuzzy measures. In this case, measures represent our background information. Aggregation is done using fuzzy integrals [26], [32], [33], [34], [35]. Choquet and Sugeno integrals [21], [36] are the most well-known ones. It is relevant here to mention that a Choquet integral with respect to an additive measure corresponds to a weighted mean with respect to the corresponding weights. Similarly, a Choquet integral with respect to a symmetric measure corresponds to an OWA operator. Finally, a Choquet integral with respect to a distorted probability corresponds to the WOWA operator. In particular, WOWA has a weighting vector p that corresponds to the probability, or weights of the weighted mean; and a weighting vector w that represents the distortion function (say w^*), or weights of the OWA. Then, the associated fuzzy measure μ is $w^* \circ p$.

B. Chains

We will use chains on a reference set X. Let us first consider two sets A and B subsets of X with $A \neq B$. Then, we say that A is covered by B when for all C such that $A \subseteq C \subsetneq B$ with $C \neq B$, then $C = A$. We will denote A covered by B by $A \prec B$ or $B \succ A$. In other words, when $A \prec B$ it means that there are no other subsets of X between A and B.

Definition 6: Let $\mathcal{C} = (C_0, C_1, \ldots, C_n)$ with $C_i \subseteq X$ for $i = 1, \ldots, n$. Then, we say that $\mathcal{C}$ is a maximal chain of subsets of X if it satisfies the following:

$$\emptyset = C_0 \prec C_1 \prec \cdots \prec C_{n-1} \prec C_n = X.$$

We denote by $\mathcal{M}(X)$ the set of maximal chains defined on the reference set X. We will use also $\mathcal{M}$ for the sake of conciseness when the set X is clear.

We will need to denote the ith set of a chain $\mathcal{C}$. We will denote it by C_i. For any maximal chain $|C_i| = i$.

C. Power Indices

Given a fuzzy measure μ on a reference set X, a power index assigns a value to each X. This index is a value of the power or relevance for each $x \in X$. The Shapley value [1] is one of the most well-known and used power index. When $\mu(X) = 1$, it assigns values in [0,1] such that add to one. Let ϕ denote the Shapley power index, then $\sum_{x \in X} \phi(x) = 1$, and $\phi(x) \geq 0$.

It is relevant to underline that when μ is an additive measure, and $\mu(A) = \sum_{x \in A} p(x)$ for some probabilities p, then $\phi(x) = p(x)$. More formally, the Shapley value is defined as

$$\varphi_i(\mu) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\,(n - |S| - 1)!}{n!} (\mu(S \cup \{i\}) - \mu(S)).$$

There is an equivalent definition of Shapley value in terms of chains. In this case, $\mathcal{C}_{x_i}$ represents the largest set in the chain $\mathcal{C}$ where x_i is not present, and, thus, $\mathcal{C}_{x_i} \cup \{x_i\}$ is the smallest set in the chain with x_i. The equivalent expression for the Shapley value is

$$\varphi_i(\mu) = \frac{1}{n!} \sum_{\mathcal{C} \in \mathcal{M}(X)} (\mu(\mathcal{C}_{x_i} \cup \{x_i\}) - \mu(\mathcal{C}_{x_i})).$$

III. Beyond Power Indices

Let X be a reference set and μ a measure on X. Then, let us consider values related to set size, or order position in $\{1, \ldots, |X|\}$. We call it Υ-values and use Υ to denote them. Then, Υ_i corresponds to power for including an ith element into a set, or the given ith position in a chain of sets. In the context of aggregation, the values give information about the tradeoff between minimum and maximum when data is aggregated using an integral with respect to μ. Therefore, they can be seen as tradeoff values.

Definition 7: Let X be a reference set, let μ be a measure on X, let $\mathcal{M}(X)$ be the set of maximal chains defined on the reference set X. Then, we define the values $\Upsilon = (\Upsilon_1, \ldots, \Upsilon_n)$ as follows:

$$\begin{aligned}\Upsilon_i &= \frac{1}{|\mathcal{M}(X)|} \sum_{\mathcal{C} \in \mathcal{M}(X)} (\mu(\mathcal{C}_i) - \mu(\mathcal{C}_{i-1})) \\ &= \frac{1}{n!} \sum_{\mathcal{C} \in \mathcal{M}(X)} (\mu(\mathcal{C}_i) - \mu(\mathcal{C}_{i-1})).\end{aligned}$$

A. Properties

We begin proving an equivalent expression for Υ-values.

Proposition 2: The following expression is equivalent to Υ_i, as defined previously:

$$\Upsilon_i(\mu) = \frac{1}{\binom{n}{i}} \sum_{|S|=i} \mu(S) - \frac{1}{\binom{n}{i-1}} \sum_{|S'|=i-1} \mu(S').$$

Proof: Let us consider the original expression for Υ_i

$$\Upsilon_i = \frac{1}{n!} \sum_{\mathcal{C} \in \mathcal{M}(X)} (\mu(\mathcal{C}_i) - \mu(\mathcal{C}_{i-1})).$$

Now, we group all appearances of sets S of cardinality i and all appearances of sets S' of cardinality $i - 1$. There are $\binom{n}{i}$ different sets of cardinality i. Therefore, any of the sets appears in $n!/\binom{n}{i}$ chains. Similarly, any of the sets of cardinality $i - 1$

appears in $n!/\binom{n}{i-1}$ chains. Therefore, we can rewrite the above-mentioned expression as

$$\Upsilon_i = \frac{1}{n!}\left(\sum_{|S|=i} \frac{n!}{\binom{n}{i}}\mu(S) - \sum_{|S'|=i-1} \frac{n!}{\binom{n}{i-1}}\mu(S')\right).$$

As this expression is equivalent to the one in the proposition, the proof is completed.

Now, we prove some basic properties. Note that the property below that $\sum_i \Upsilon_i = \mu(X)$ is known as efficiency.

Proposition 3: Let μ be a normalized fuzzy measure on X, let $n = |X|$, and let Υ be defined as previous. Then:

- $\Upsilon_i \geq 0$ for all $i = 1, \ldots, n$; and
- $\sum_{i=1}^{n} \Upsilon_i = 1$.

When the measure is not normalized, we have $\sum_{i=1}^{n} \Upsilon_i = \mu(X)$, and if $\mu(\emptyset)$ is not zero, then the total is $\mu(X) - \mu(\emptyset)$.

Proof: The first property follows from the monotonicity condition of measures. As differences are always positive, the values are positive. The second property corresponds to

$$\sum_{i=1}^{n} \Upsilon_i = \sum_{i=1}^{n} \frac{1}{n!} \sum_{\mathcal{C} \in \mathcal{M}(X)} \mu(\mathcal{C}_i) - \mu(\mathcal{C}_{i-1}).$$

Let us swap the two summatories on the right

$$\sum_{i=1}^{n} \Upsilon_i = \frac{1}{n!} \sum_{\mathcal{C} \in \mathcal{M}(X)} \sum_{i=1}^{n} \mu(\mathcal{C}_i) - \mu(\mathcal{C}_{i-1}).$$

It is easy to see that for a given chain, the total contribution is $\mu(\mathcal{C}_n = X) - \mu(\mathcal{C}_0 = \emptyset) = \mu(\mathcal{C}_n) = \mu(X)$. If the measure is normalized, this is of course equal to one. So

$$\begin{aligned}\sum_{i=1}^{n} \Upsilon_i = & \frac{1}{n!} \sum_{\mathcal{C} \in \mathcal{M}(X)} \mu(\mathcal{C}_n) - \mu(\mathcal{C}_0) \\ = & \frac{1}{n!} \sum_{\mathcal{C} \in \mathcal{M}(X)} \mu(X) = \frac{1}{n!} \sum_{\mathcal{C} \in \mathcal{M}(X)} 1.\end{aligned}$$

As there are $n!$ chains, this result is one, and the second property is also proven. Therefore, the theorem is proven.

We will now prove a result about symmetric measures. They are measures μ of the form $\mu(A) = w(|A|)$ for a certain function w. Symmetric measures can be expressed in terms of a weighted vector $w = (w_1, \ldots, w_n)$. In particular, for any set A such that $|A| = i$, $\mu(A) = \sum_{k=1}^{i} w_k$. Therefore, it is easy to see that for a chain $\mathcal{C}$

$$\mu(\mathcal{C}_i) - \mu(\mathcal{C}_{i-1}) = w_i.$$

This implies the following result.

Corollary 1: Let w be a weighting vector, and let μ be a symmetric measure defined from w. Then, for all i in $1, \ldots, n$ we have that the following equation holds:

$$\Upsilon_i = w_i.$$

Let us consider the case of additive measures. That is, the measure of a set is the addition of the weights of elements in the set. Let these weights be p_i (i.e., p_i is the weight of x_i). Then, $\mu(A) = \sum_{x_i \in A} p_i$ for any set A.

It is easy to see that

$$\mu(\mathcal{C}_i) - \mu(\mathcal{C}_{i-1}) = p_i$$

when $x_i = \mathcal{C}_i \setminus \mathcal{C}_{i-1}$.

Then, for any i, we have that we need to consider all elements x_i and their weights p_i. As there are n of such elements, the average of all weights p_i will be $\sum_{i=1}^{n} p_i/n$, and as $\sum_{i=1}^{n} p_i = 1$. This is of course $1/n$. Therefore, the following proposition follows.

Proposition 4: Let p be a weighting vector, and let μ be an additive measure built from p. Then, for all i in $1, \ldots, n$ we have that the following equation holds:

$$\Upsilon_i = 1/n.$$

B. Relation With Aggregation Functions

Proposition 4 shows that for additive measures Υ_i is $1/n$, and Corollary 1 shows that for symmetric measures Υ_i corresponds to weights w_i. Let us rewrite this sentence in terms of aggregation functions. Corollary 1 establishes that Υ corresponds to OWA weights, and Proposition 4 establishes that Υ is not affected by the weighted mean weights.

This can be contrasted with Shapley values. It is known that for additive measures built from weights p, Shapley values correspond to weights. In contrast, Shapley values of symmetric measures are $1/n$.

These results show that both types of indices are complementary and in short, Shapley mainly corresponds to weighted mean weights, and Υ to OWA weights. We then can observe the case of WOWA operators (i.e., a Choquet integral with a distorted probability) and the associated measure. We have that, in general, both Shapley and Υ-values will be different than $1/n$. One that approximates the probability (i.e., Shapley values) and another that approximates the distortion or OWA weights (i.e., Υ-values). We provide an example in the following.

Example 1: Let $X = \{x_1, x_2, x_3, x_4\}$. Then, let us consider a distorted probability on X defined by the quantifier $q(x) = 1$ if $x \geq 0.75$ and $x/0.75$ otherwise, and the probability $p = (0.5, 0.3, 0.15, 0.05)$. That is, $\mu = q \circ p$ or, equivalently, $\mu(A) = g(P(A))$ with $P(A) = \sum_{x \in A} p(x)$. This produces the following measure (see [26], Table 6.4):

- $\mu(\emptyset) = 0.0$, $\mu(\{x_4\}) = 0.0666$
- $\mu(\{x_3\}) = 0.2$, $\mu(\{x_3, x_4\}) = 0.2666$
- $\mu(\{x_2\}) = 0.4$, $\mu(\{x_2, x_4\}) = 0.4666$
- $\mu(\{x_2, x_3\}) = 0.6$, $\mu(\{x_2, x_3, x_4\}) = 0.6666$
- $\mu(\{x_1\}) = 0.6666$, $\mu(\{x_1, x_4\}) = 0.7333$
- $\mu(\{x_1, x_3\}) = 0.8666$, $\mu(\{x_1, x_3, x_4\}) = 0.9333$
- $\mu(\{x_1, x_2\}) = 1.0$, $\mu(\{x_1, x_2, x_4\}) = 1.0$
- $\mu(\{x_1, x_2, x_3\}) = 1.0$, $\mu(X) = 1.0$.

For this measure we have that the Shapley value is

$$\phi = (0.5445, 0.2778, 0.1333, 0.0444)$$

and the Υ-values are

$$\Upsilon = (0.3333, 0.3222, 0.24446, 0.10003)$$

with an orness equal to 0.6296.

We want to underline that this distorted probability below used in combination with a Choquet integral is mainly equivalent to a WOWA operator with weights $p = (0.5, 0.3, 0.15, 0.05)$ and $w = (1/3, 1/3, 1/3, 0)$.

C. Additional Properties and Axiomatic Characterization

In this section we review additional properties for power indices. We show that some of them which are satisfied by Shapley values are also satisfied by Υ-values but not all of them. Then, we introduce new definitions which are satisfied by Υ-values. We conclude the section providing an axiomatic characterization of the new values.

We begin reviewing and discussing the definitions of dummy and null player, which are important properties for the Shapley value, and used in its characterization.

Definition 8: Let X be a reference set and μ be a fuzzy measure. Then, an element x is a dummy player when $\mu(S \cup \{x\}) = \mu(S) + \mu(\{x\})$ for all $S \subseteq (X \setminus \{x\})$.

Definition 9: Let X be a reference set and μ be a fuzzy measure. Then, an element x is a null player when $\mu(S \cup \{x\}) = \mu(S)$ for all $S \subseteq (X \setminus \{x\})$.

There are two properties associated to dummy player and null player. We define the one associated to dummy player. The one for null player is analogous.

Definition 10: An index f satisfies the dummy player property when for all dummy players $x \in X$, $f_{i(x)}(\mu) = \mu(\{x\})$ for all measures μ. Here, $i(x)$ refers to the position in f associated to the element x.

This property holds for the Shapley value but cannot be satisfied by the Υ-values because there is no index associated to any particular x. As an alternative, we define the concept of null cardinality and dummy cardinality and the null (and dummy) cardinality position property. They are analogous to the previous ones but associated to set sizes instead of particular elements.

Definition 11: Let X be a reference set and μ be a measure. Then:

- a cardinality i is a null cardinality when $\mu(S \cup \{x\}) = \mu(S)$ for all $|S \cup \{x\}| = i$, such that $S \subseteq (X \setminus \{x\})$;
- a cardinality i is a dummy cardinality when $\mu(S \cup \{x\}) = \mu(S) + h_i$ for all $|S \cup \{x\}| = i$, such that $S \subseteq (X \setminus \{x\})$, and h_i a certain constant.

Definition 12: An index f satisfies the dummy cardinality property when for all dummy cardinality i, $f_i(\mu) = h_i$ for all measures μ. Similarly, an index f satisfies the null cardinality property when for all null cardinality i, $f_i(\mu) = 0$ for all measures μ.

Proposition 5: The Υ-values satisfy the null cardinality and dummy cardinality properties.

Proof: It is clear that the dummy cardinality property is stronger than the null cardinality. Therefore, we prove the latter.

Let us consider the Υ-values of a measure μ with a dummy cardinality i. That is

$$\Upsilon_i = (1/n!) \sum_{\mathcal{C} \in \mathcal{M}} (\mu(\mathcal{C}_i) - \mu(\mathcal{C}_{i-1})).$$

Then, the definition of a dummy cardinality i implies, for all $\mathcal{C}$, that $\mu(\mathcal{C}_i) = \mu(\mathcal{C}_{i-1}) + h_i$ for some constant h_i. Therefore

$$(\mu(\mathcal{C}_i) - \mu(\mathcal{C}_{i-1})) = \mu(\mathcal{C}_{i-1}) + h_i - \mu(\mathcal{C}_{i-1}) = h_i$$

it follows that:

$$\Upsilon_i = (1/n!) \sum_{\mathcal{C} \in \mathcal{M}} (\mu(\mathcal{C}_i) - \mu(\mathcal{C}_{i-1})) = (1/n!) \sum_{\mathcal{C} \in \mathcal{M}} h_i = h_i.$$

Therefore, the proposition is proven.

Proposition 6: Let μ_T be the unanimity measure (unanimity game) for $T \subset X$ and $T \neq \emptyset$. Let $n = |X|$, $t = |T|$. Then:

1) if $i < |T|$ then, $\Upsilon_i = 0$;
2) if $i = |T|$ then
$$\Upsilon_i = \frac{1}{n!} t!(n-t)!;$$
3) if $i > |T|$ then
$$\Upsilon_i = \frac{1}{n!} \binom{t}{1} \binom{n-t}{i-t} (i-1)!(n-i)!.$$

We have considered the case of $i = |T|$ as independent of $i > |T|$ but note that the expression of the latter case reduces to the one of the former when $i = |T|$.

Proof: We consider the proof of each case separately. Then, for each i we need to consider the chains for which $\mu_T(C_i) = 1$ and $\mu_T(C_{i-1}) = 0$, as they are the only ones that influence the result.

Let us begin with the case $i < t$. It is clear that in this case, for any chain, we have both $\mu(C_i) = 0$ and $\mu(C_{i-1}) = 0$. We need to have at least t elements to have a measure equal to one. Therefore, the index will be always zero. So, $\Upsilon_i = 0$.

Let us consider the case $i = |T|$. We can build all the relevant chains as follows. The first t sets of a chain will be formed adding elements of T in an arbitrary order. There are $t!$ of these possible chains. Then, for the remaining part of a chain, we will use the elements from $X \setminus T$ in any arbitrary order. Therefore, we have $(n-t)!$ of such chains. So, the total number of chains is $t!(n-t)!$ and, therefore

$$\Upsilon_i(\mu_T) = \frac{1}{n} t!(n-t)!.$$

Let us complete the proof with the case $i > |T|$. We will use $k = i - t$. As we need to count the chains with $\mu(C_i) = 1$ while $\mu(C_{i-1}) = 0$ it means that the last element to be added to the chain at position i (i.e., $C_i \setminus C_{i-1}$) is one of the elements in T. This means that in C_{i-1} we have $t-1$ elements of T and the rest k elements are from $X \setminus T$. Then, in C_i, we will have all the t elements of T and the k elements from $X \setminus T$. So, to build the chains we proceed as follows:

1) Select an element t_0 from T that is the one to be added in position i.
2) Consider k elements from $n - t$ elements in $X \setminus T$.
3) Construct the $(t + k - 1)!$ possible chains with the available $t + k - 1$ elements (the $t - 1$ from $T \setminus \{t_0\}$ and the other k selected).
4) Add t_0 to each of the chains.
5) Complete the chains with any of the $(n - t - k)!$ chains obtained with the remaining $(n - t - k)$ objects.

This results into the following number of chains:

$$\binom{t}{1}\binom{n-t}{k}(t+k-1)!1(n-t-k)!.$$

Expressing this equation in terms of i (instead of k) and dividing by $n!$ be obtain

$$\Upsilon_i(\mu_T) = \frac{1}{n!}\binom{t}{1}\binom{n-t}{i-t}(i-1)!(n-i)!.$$

Definition 13: Let π be a permutation of X. We define the measure μ^π as $\mu^\pi(\pi(A)) = \mu(A)$ for all $A \subseteq X$. Equivalently, we can define $\mu^\pi(B) = \mu(\pi^{-1}(B))$.

Definition 14: Let X be a set, μ a measure on this set, and f a measure index. Then, f satisfies the anonymity property when given a permutation π on X, we have

$$f(\mu^\pi) = \pi^*(f(\mu))$$

where $\pi^*(x) = (\pi^*_{\pi(1)}(x), \ldots, \pi^*_{\pi(|X|)}(x))$ and $\pi^*_{\pi(k)}(x) = x_K$.

This property holds for the Shapley value. In contrast, it is easy to prove that the Υ-values do not satisfy the anonymity property. In contrast, when we apply a permutation to the elements in X, the Υ-values do not change. We call this property absolute anonymity. We define it below and prove that it holds for the definition of Υ-values.

Definition 15: Let X, μ and f be defined as previous. Then, an index f satisfies the absolute anonymity property when given a permutation π on X, we have

$$f(\mu^\pi) = f(\mu).$$

Proposition 7: The Υ-values satisfy absolute anonymity.

Proof: The definition of Υ considers all maximal chains on X. Given a permutation π on X, the set $\mathcal{M}^\pi$ of maximal chains on the permutation π of X will result into the same chains as $\mathcal{M}$. Therefore, for each $\mathcal{C}$ in $\mathcal{M}$ there will be a chain $\mathcal{C}'$ in $\mathcal{M}^\pi$ such that $\mu^\pi(\mathcal{C}'_i)$ and $\mu^\pi(\mathcal{C}'_{i-1})$ are precisely $\mu(\mathcal{C}_i)$ and $\mu(\mathcal{C}_{i-1})$.

Definition 16: Let X be a reference set. Then, an index f satisfies the additivity property when for any pair of measures μ and ν on X it holds that

$$f(\mu+\nu) = f(\mu) + f(\nu).$$

Proposition 8: The Υ-values satisfy the additivity property.

Proof: Let μ' be $\mu+\nu$. We will denote the index for μ' by Υ'.

Let us begin considering a chain $\mathcal{C}$ on X, and the computation of the ith index. Then, for $\mu+\nu$ we have that:

- $(\mu+\nu)(\mathcal{C}_i) = \mu(\mathcal{C}_i) + \nu(\mathcal{C}_i)$;
- $(\mu+\nu)(\mathcal{C}_{i-1}) = \mu(\mathcal{C}_{i-1}) + \nu(\mathcal{C}_{i-1})$.

Therefore

$$\begin{aligned}&(\mu+\nu)(\mathcal{C}_i) - (\mu+\nu)(\mathcal{C}_{i-1})\\ &= \mu(\mathcal{C}_i) + \nu(\mathcal{C}_i) - \mu(\mathcal{C}_{i-1}) - \nu(\mathcal{C}_{i-1}).\end{aligned}$$

We can use this expression to compute Υ'_i, so

$$\begin{aligned}\Upsilon'_i &= \frac{1}{n!}\sum_{\mathcal{C}\in\mathcal{M}}((\mu+\nu)(C_i) - (\mu+\nu)(C_{i-1}))\\
&= \frac{1}{n!}\sum_{\mathcal{C}\in\mathcal{M}}(\mu(\mathcal{C}_i) + \nu(\mathcal{C}_i) - \mu(\mathcal{C}_{i-1}) - \nu(\mathcal{C}_{i-1}))\\
&= \frac{1}{n!}\sum_{\mathcal{C}\in\mathcal{M}}((\mu(\mathcal{C}_i) - \mu(\mathcal{C}_{i-1})) + (\nu(\mathcal{C}_i) - \nu(\mathcal{C}_{i-1})))\\
&= \frac{1}{n!}\sum_{\mathcal{C}\in\mathcal{M}}(\mu(\mathcal{C}_i) - \mu(\mathcal{C}_{i-1})) + \frac{1}{n!}\sum_{\mathcal{C}\in\mathcal{M}}(\nu(\mathcal{C}_i) - \nu(\mathcal{C}_{i-1}))\\
&= \Upsilon_i(\mu) + \Upsilon_i(\nu).\end{aligned}$$

This completes the proof.

Theorem 1: An index $f : G^N \to \mathbb{R}^N$ satisfies additivity, absolute anonymity, dummy cardinality, and efficiency if and only if f correspond to the Υ-values.

The proof of this theorem is based on functional equations. In particular, we will use a generalization of the Cauchy's equation. This is in its simplest form $g(x+y) = g(x) + g(y)$ which under appropriate conditions can only correspond to a function of the form $g(x) = \alpha x$ for a constant α. In our case, we apply Cauchy's equation to a measure (i.e., a vector in $\mathbb{R}^{2^n}$) and the function is bounded assuming e.g. efficiency. Then, a similar result applies (see Proposition 1 and Corollary 2 of Chapter 4 in Aczél and Dhombres [37]) and the function $g : \mathbb{R}^m \to \mathbb{R}$ needs to be $g(x) = \sum_{j=1}^m \alpha_i x_i$.

In the proof we need to consider the 2^n values of the measure. We will assume that we have ordered them in a given particular order. This order is arbitrary, we assume that s_j is the set associated to the jth position, and m_S will be the value of the measure associated to the position for set S.

Proof: We have already proven in previous propositions that Γ-values satisfy the properties in the theorem. Therefore, we only need to prove here that if these properties are satisfied for an index f then the index f exactly corresponds to the Υ-values.

Let f be a power index that is additive for any measure. Then, this means that for any pair of arbitrary measures μ_1 and μ_2 we have $f(\mu_1+\mu_2) = f(\mu_1) + f(\mu_2)$. The index is such that $f : \mathbb{R}^{2^n} \to \mathbb{R}^n$. We denote by f_i the ith component of f. Naturally, f_i is also additive.

Let us focus on one of the components. This component f_i is also additive, otherwise f would not be so. Then, it satisfies Cauchy's equation. Because of that, f_i is a linear combination of the values in the measure, and it should be of the form

$$f_i(\mu) = \sum_{j=0}^{2^n-1} c_{i,j} m_j$$

where m_j corresponds to the jth coefficient of the measure for set s_j, and $c_{i,j}$ is an arbitrary constant for the jth coefficient.

Note that the coefficients of f_i are not necessarily the ones for $f_{i'}$ for $i \neq i'$.

Let us consider the property of dummy cardinality. This means that for all $|S \cup \{x\}| = i$ such that $S \subseteq (X \setminus \{x\})$, if $\mu(S \cup \{x\}) = \mu(S) + h_i$ we need to have $f_i(\mu) = h_i$.

As our index is valid for any measure it also applies to measures that satisfy the property of dummy cardinality. Let us consider now one of these measures. In particular, let us consider a measure μ that satisfies dummy cardinality for cardinality

i, and a value h_i, and let μ_0 be a measure that is zero in all sets except for an arbitrary set s_{j_0} for which is one. This set is required to have a cardinality different to i and to $i-1$. Then, if $\mu' = \mu + \mu_0$ we need to have

$$f_i(\mu') = f_i(\mu) + c_{i,j_0} \cdot 1.$$

Nevertheless as μ satisfies the property of dummy cardinality for i, this is also the case for μ'. Therefore, $f_i(\mu') = f_i(\mu) = h_i$, which implies $c_{i,j_0} = 0$. As the selection of s_{j_0} is arbitrary, we can infer that $c_{i,j_0} = 0$ for all cardinalities j_0 different to i and $i-1$.

This means, that the only coefficients for f_i that can be different to zero are the ones for cardinalities i and $i-1$. So, we can express f_i as follows:

$$f_i(\mu) = \sum_{|S|=i} c_{iS} m_S + \sum_{|S'|=i-1} c_{iS'} m_{S'} \tag{2}$$

where m_S just corresponds to $\mu(S)$.

As the index applies to any measure, we can consider again its application to another particular measure. Let us take the measure μ that satisfies the property of dummy cardinality for i and as such that $m_{S'} = 0$ for the sets S' of cardinality $i-1$ and $m_S = h_i$ for the sets S of cardinality i. Then, the above-mentioned expression should be equal to h_i and removing the values equal to zero reduces to the following:

$$f_i(\mu) = \sum_{|S|=i} c_{iS} h_i = h_i.$$

So, $\sum_{|S|=i} c_{iS} = 1$.

In addition, the condition of absolute anonymity implies that all these coefficients should be equal. To prove this, take a set S_0 of cardinality i, and define a measure zero everywhere except one at S_0 where it is one. Then, the index f_i will be c_{iS_0}. If we consider a permutation so $S'_0 = \pi(S_0)$, then, the index will be $c_{iS'_0}$. So, we need $c_{iS_0} = c_{iS'_0}$.

Let us denote this equal value as c_i. So, $c_{iS} = c_i$. Then, as the number of sets of cardinality i are $\binom{n}{i}$ we have $\sum_{|S|=i} c_{iS} = \sum_{|S|=i} c_i = c_i \sum_{|S|=i} 1 = c_i \binom{n}{i} = 1$, from which we infer $c_{iS} = c_i = 1/\binom{n}{i}$.

We can do similarly to determine the coefficients for sets S' of cardinality $i-1$. In this case we take a measure μ that satisfies the property of dummy cardinality for i and as such that $m_{S'} = -h_i$ for the sets S' of cardinality $i-1$ and $m_S = 0$ for the sets S of cardinality i. In this case, the $f_i(\mu)$ should be h_i and, as previous, removing the values equals to zero we obtain the following expression:

$$f_i(\mu) = \sum_{|S'|=i-1} c_{iS'}(-h_i) = h_i$$

which implies $\sum_{|S'|=i-1} c_{iS'} = -1$. As there are $\binom{n}{i-1}$ sets of cardinality $i-1$ and using, as previous, the condition of absolute anonymity, we obtain $c_{iS'} = 1/\binom{n}{i-1}$.

Now, we just need to replace in (2) the coefficients by the expressions we have just determined. This, naturally, corresponds to

$$f_i(\mu) = \sum_{|S|=i} \frac{1}{\binom{n}{i}} \mu(S) + \sum_{|S'|=i-1} \frac{1}{\binom{n}{i-1}} \mu(S').$$

As this expression corresponds to the one in Proposition 2, the theorem is proven.

D. Orness From Υ-Values

We have reviewed the definition of orness for OWA. Let us introduce an orness-like measure for measures based on OWA orness and the Υ-values.

Definition 17: Let X be a reference set, μ be a fuzzy measure, and Υ be the Υ-values of μ. Then, we define Υ-orness for μ using Definition 3 as follows:

$$\text{orness}(\Upsilon_\mu) = \frac{1}{n-1} \sum_{i=1}^{n} (n-i) \Upsilon_i.$$

Observe that this definition follows the structure of $\text{orness}(\text{OWA}_w) = \frac{1}{n-1} \sum_{i=1}^{n} (N-i) w_i$ replacing $\Upsilon_i = w_i$.

Let us consider the following fuzzy measures.

- $\mu_n(A) = 0$ for all $A \neq X$, and $\mu_n(X) = 1$.
- $\mu_x(A) = 1$ for all $A \neq \emptyset$, and $\mu_x(\emptyset) = 0$.

When the Choquet integral aggregates values with respect to the measure μ_n, it results into the minimum of these values. In contrast, when the Choquet integral aggregates values with respect to the measure μ_x, it results into the maximum of these values.

We can observe (or prove) that for the first measure μ_n, we obtain $\Upsilon = (0, \ldots, 0, 1)$. That is, $\Upsilon_i = 0$ for all $i \neq |X|$ and $\Upsilon_{|X|} = 1$. In contrast, for the second measure μ_x we obtain $\Upsilon = (1, 0, \ldots, 0)$. That is, $\Upsilon_1 = 1$ and $\Upsilon_i = 0$ for all $i \neq 1$.

Then, $\text{orness}(\Upsilon(\mu_n)) = 0$ and $\text{orness}(\Upsilon(\mu_x)) = 1$, as expected.

In Proposition 8 we have proven that Υ satisfies the additivity property. If Υ is linear with respect to Υ, Υ-orness is also linear with respect to additivity of μ.

Proposition 9: Let X be a reference set. Then, for any pair of measures μ and ν, Υ-orness satisfies

$$\text{orness}(\Upsilon_{\mu+\nu}) = \text{orness}(\Upsilon_\mu) + \text{orness}(\Upsilon_\nu).$$

IV. Effect of a Value in a Measure in its Orness

In this section, we study how changing the value of a measure for a set affects the orness of this measure. For this, we use the definition of orness in terms of the Υ-values.

Let us denote the fuzzy measure by μ and the particular set of our study by A. We use the $(\max, +)$-transform [31] (see Definition 5). We know that this transform is always positive, and that increasing the transform for A will increase the value of $\mu(A)$ and will not decrease the value of $\mu(B)$ for all $B \neq A$. That is, the resulting measure is nondecreasing with respect to positive changes to the transform of A.

The process is as follows for a given measure μ and a set A.

1) $m_\mu :=$ Compute the $(\max, +)$-transform of measure μ.
2) Define a new transform $m'_{m,A,\alpha}$ as follows:

- $m'_{m,A,\alpha}(B) := m_\mu(B)$ for all $B \neq A$;

- $m'_{m,A,\alpha}(A) := \alpha$.

3) $\mu' :=$ the measure associated to transform $m'_{m,A,\alpha}$.
4) Compute orness of μ'.

Note that given a fuzzy measure μ, all values α will produce a fuzzy measure μ' but not all these measures will be normalized. Say, if we use a value $\alpha = 1$ we may produce $\mu'(X) > 1$, and if we use a value $\alpha = 0$ we may produce $\mu'(X) < 1$. We will restrict α in the range that produces a normalized fuzzy measure.

As the measure μ' is nondecreasing with respect to α, this range of values is an interval. We will use I_μ to denote the interval of values of α that produce a normalized fuzzy measure.

The following example illustrates this fact.

Example 2: Let $X = \{x_1, x_2\}$. Then, let μ be a fuzzy measure on the reference set X defined as follows:

$$\mu(\emptyset) = 0, \mu(\{x_1\}) = 0.7, \mu(\{x_2\}) = 0.2, \mu(\{x_1, x_2\}) = 1.$$

The $(\max, +)$-transform of this measure is

$$m_\mu(\emptyset) = 0, m_\mu(\{x_1\}) = 0.7$$
$$m_\mu(\{x_2\}) = 0.2, m_\mu(\{x_1, x_2\}) = 0.3.$$

It is easy to see that if we consider the set $A = \{x_1\}$ for any $\alpha < 0.7$ we will have $\mu'(X) < 1.0$ because

$$\mu(X) = \max(m(\{x_1\}), m(\{x_2\})) + m(\{x_1, x_2\}).$$

Similarly, for any $\alpha > 0.7$, we have $\mu'(X) > 1.0$.

Therefore, for this measure, only the value $\alpha = 0.7$ provides a normalized fuzzy measure. That is, $I_\mu = [0.7, 0.7]$.

To study the effects of changing a value in the orness we will consider several fuzzy measures defined using the reference set X. We consider the effects for element x_1 in X.

We begin considering μ_n and μ_x, as defined previously. It is easy to see that for μ_n, its transform coincides with the measure as $m_{\mu_n}(A) = 0$ for all $A \neq X$, and $m_{\mu_n}(X) = 1$. Then, any $\alpha \neq 0$ will produce a nonnormalized fuzzy measure. Therefore, $I_{\mu_n} = [0.0, 0.0]$. That is, the only possible value for $m'(A)$ is precisely $m_{\mu_n}(A)$. In this case, the orness for the measure is 0.0, as already discussed previously.

In contrast, for μ_x, we have $I_{\mu_x} = [0.0, 1.0]$ as any value is possible. In this case, the range of orness depends on the cardinality of the set X. For $m'(\{x_1\}) = 1$, we have that orness is 1. Nevertheless, for $m'(\{x_1\}) = 0$, we have that orness depends on the cardinality of X. When X contains four elements, the orness ranges from 0.916 to 1.0 when α goes from 0.0 to 1.0. The more elements we have, the largest the minimal orness when $m'(\{x_1\}) = 0$.

In Fig. 1 (top), we show the value of orness for different values of α when X contains 4, 5, and 6 elements. We see that for any α the orness is high, larger than 0.9. In Fig. 1 (bottom), we illustrate the value of minimal orness when the cardinality of X increases from 4 to 10. The larger the dimension of X, the largest the orness as the relevance of a single object is less and less. That is, the larger the dimension, the less orness is changed, and the lesser the influence of the measure of a single object.

These two measures are the most extreme ones. So, we have also considered some other more realistic measures. They confirm the findings of these extreme cases. We have considered two sets of three measures. First, they are the measures μ_G, μ_T and μ_4. μ_G is defined on a set X with three elements. It is the Grabisch measure in Example 5.12 in Torra and Narukawa's [26] work. μ_T is a measure on a set X with five elements. It corresponds to the one in Example 5.66 (Table 5.9) in Torra and Narukawa's [26] work. The measure μ_4 is a measure defined by the values in the following. The other two measures are provided in the appendix

- $\mu(\emptyset) = 0.0$, $\mu(\{x_1\}) = 0.1$
- $\mu(\{x_2\}) = 0.2$, $\mu(\{x_1, x_2\}) = 0.2$
- $\mu(\{x_3\}) = 0.3$, $\mu(\{x_3, x_1\}) = 0.31$
- $\mu(\{x_3, x_2\}) = 0.32$, $\mu(\{x_3, x_1, x_2\}) = 0.4$
- $\mu(\{x_4\}) = 0.4$, $\mu(\{x_4, x_1\}) = 0.41$
- $\mu(\{x_4, x_2\}) = 0.42$, $\mu(\{x_4, x_1, x_2\}) = 0.42$
- $\mu(\{x_4, x_3\}) = 0.7$, $\mu(\{x_4, x_3, x_1\}) = 0.70$
- $\mu(\{x_4, x_3, x_2\}) = 0.70$, $\mu(\{X\}) = 1.00$.

Then, we also consider the three measures μ_{CI}, μ_{SI}, and μ_{Li}. The latter three measures were learned from examples. Each corresponds to the solution of a measure identification

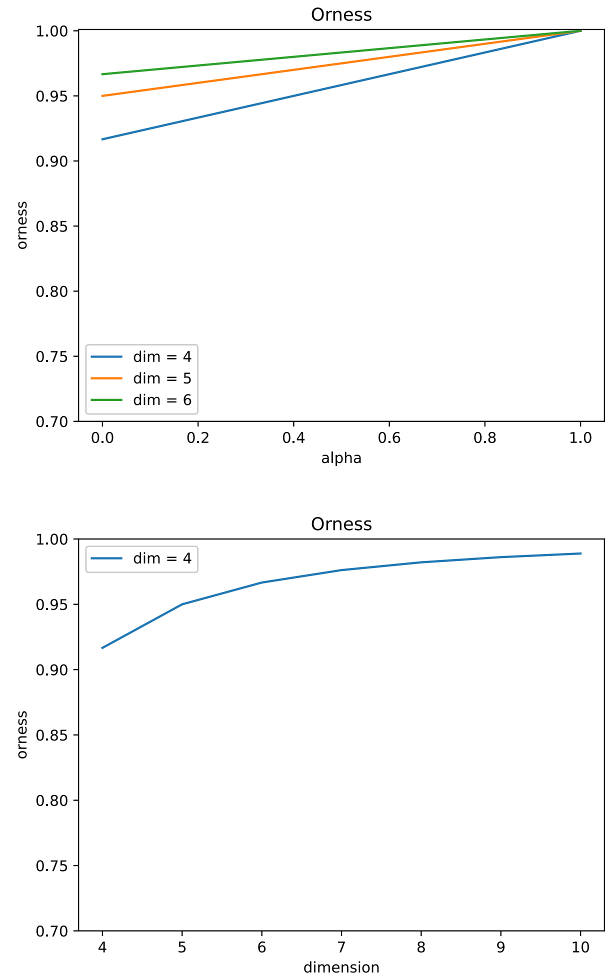

Fig. 1. Orness for the fuzzy measure μ_x for different cardinalities of X and different values of α for $m(\{x_1\})$. In the top figure, the case of dimensions 4, 5, and 6 and α in [0,1]. In the bottom figure, the minimum orness achieved for dimensions 4–10.

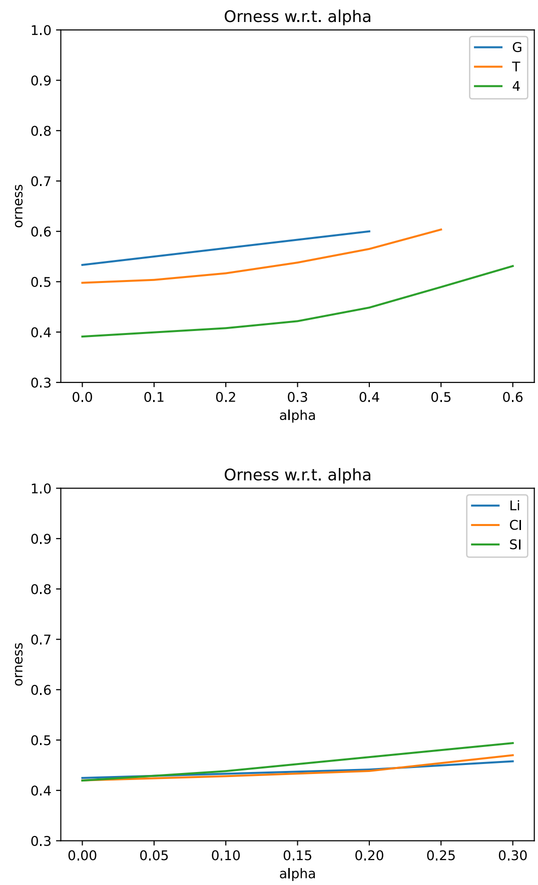

Fig. 2. Orness for several fuzzy measures for different values of α. In the top figure, measures μ_G, μ_T, and μ_4 are considered, and in the right measures μ_{CI}, μ_{SI}, and μ_{Li} are considered.

problem using the same input data. In other words, they are data-driven solutions of the same data. The data-driven problem is established by Li et al. [38]. Then, μ_{Li} corresponds to the solution using the means square error on a model based on the Choquet integral. Quadratic optimization software is used to find the solution. Therefore, this corresponds to the best possible solution, as the software finds the optimal solution of the problem. In contrast, μ_{CI} and μ_{SI} correspond to the best solution found using genetic algorithms when we use either a model based on the Choquet integral or the Sugeno integral. So, in this case, global optimum is not ensured. The algorithms and solutions are described in Torra's [31] work. The measures are provided in the appendix.

For these measures, we have considered again the modification of $\mu(\{x_1\})$, and observe how the global orness changes. We observe in all cases that the influence on the global orness of changing the measure of a single singleton is limited. In the case of μ_{CI}, μ_{SI}, and μ_{Li}, the orness of the three measures are quite similar. As explained previously, all three measures solve the same problem. Nevertheless, although the measures are similar they are not equal and some sets differ with a value up to 0.2.

In all these figures we illustrate how changes in the measure associated to $\mu(\{x_1\})$ influence the Υ-orness. We have proven in Proposition 9 that Υ-orness satisfies the additivity property. In the figures we see linearity (see Fig. 1) when we increase $\mu(\{x_1\})$. Nevertheless, we also see piecewise linearity (see Fig. 2). The latter is because increasing $\mu(\{x_1\})$ can be expressed as addition of two measures when for all $x_1 \in A$ we have $\mu(\{x_1\}) \leq \mu(A)$. After this point, our algorithm, to update the measure, will also increase $\mu(A)$. This is the point in which we have in the figure that linearity changes. It is easy to see that the slope of the orness should increase at this point (as we are increasing the measure for additional sets). Fig. 2 shows clearly this property.

V. Conclusion

In this article, we have proposed a power index for fuzzy measures related to set size. We called it Υ_μ. We have shown its relationship with orness and andness, and also to OWA. We have studied some properties, introduced a characterization, and provided some examples. As future work we plan to study additional properties, in particular, Υ and Υ-orness when there are communication structures, and also games with community controls [39].

In relation to aggregation functions, we have shown that the index corresponds to OWA weights when the fuzzy measure is the one associated to OWA. Then, we have discussed that this index and the Shapley index are complementary. We have seen that in the case of WOWA and distorted probabilities, both indices Shapley and Υ have a different role. This is also inferred from the characterization. In general, for arbitrary fuzzy measures, both indices have a role. This permits to consider both indices useful for analysis of fuzzy measures. As future work, we will consider the approximation of arbitrary fuzzy measures μ in terms of distorted probabilities constructed from both Shapley and Υ indices from μ.

We have also defined the orness of a measure in terms of the indices Υ, using Yager expression of orness. Then, it is of relevance to compare the orness in these terms (i.e., only considering the measure and the Υ indices) and orness in terms of the volume of the fuzzy integral. Note that for this, we need to know which is the fuzzy integral used. Choquet and Sugeno integrals would lead to different results. The experiments show that the value obtained for our orness is similar to the one of the Choquet integral. This needs further analysis.

Acknowledgment

The author wants to specially thank one of the reviewers for realizing that some of the proofs in the submitted manuscript were wrong.

Appendix

We reproduce here for convenience the measures used in this article. The values of the measure are given representing the sets in dyadic form. That is position 0 corresponds to the empty set, position 1 corresponds to the set $\{x_1\}$, position 2 (with binary representation 10) corresponds to the set $\{x_2\}$, position 3 (with binary representation 11) corresponds to the set $\{x_2, x_1\}$, etc. The position $2^n - 1$ with $n = |X|$ (with a binary representation of n values of 1) corresponds to the set X

1) μ_G=[0,0.3,0.45,0.9,0.45,0.9,0.5,1] (three elements in X);
2) $\mu_T = [0.0, 0.04296875, 0.1375, 0.2375, 0.06909722, 0.1375, 0.3, 0.5, 0.18333333, 0.3, 0.61666666, 0.7625, 0.38333333, 0.61666666, 0.81666666, 0.9, 0.1, 0.18333333, 0.38333333, 0.61666666, 0.2375, 0.38333333, 0.7, 0.81666666, 0.5, 0.7, 0.8625, 0.93090277, 0.7625, 0.8625, 0.95703125, 1.0]$ (five elements in X);
3) $\mu_{Li} = [0, 0.11902721152246414, 0.2192511179972149, 0.36944035435601363, 0.278254585089656, 0.4216047035265328, 0.38662134767666456, 0.4859785348532103, 0.40336076705004414, 0.5130985615950798, 0.5008868475489031, 0.6440459784023589, 0.5477764557651374, 0.6801695626428382, 0.5592789623500151, 1.0]$;
4) $\mu_{CI} = [0.0, 0.13274336283185842, 0.17699115044247787, 0.336283185840708, 0.19469026548672566, 0.4247787610619469, 0.415929203539823, 0.48672566371681414, 0.2743362831858407, 0.5132743362831859, 0.49557522123893805, 0.6814159292035398, 0.5398230088495575, 0.7522123893805309, 0.6548672566371682, 1.0]$;
5) $\mu_{SI} = [0.0, 0.11688311688311688, 0.06493506493506493, 0.16883116883116883, 0.03896103896103896, 0.5064935064935064, 0.19480519480519481, 0.5064935064935064, 0.3116883116883117, 0.6883116883116883, 0.6233766233766234, 0.7142857142857143, 0.4935064935064935, 0.8831168831168831, 0.8961038961038961, 1.0]$.

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