Extracting Concepts From Fuzzy Relational Context Families

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***Abstract*—Fuzzy relational formal concept analysis (FRCA) mines collections of fuzzy concept lattices from *fuzzy relational context families*, which are special datasets made of fuzzy formal contexts and fuzzy relations between objects of different types. Mainly, FRCA consists of the following procedures: first, an initial fuzzy relational context family is transformed into a collection of fuzzy formal contexts; second, a fuzzy concept lattice is generated from each fuzzy formal context by using one of the techniques existing in the literature. The principal tools to transform a fuzzy context family into a set of fuzzy formal contexts are the so-called *fuzzy scaling quantifiers*, which are particular fuzzy quantifiers based on the concept of *evaluative linguistic expression*. FRCA can be applied whenever information needs to be extracted from multirelational datasets including vagueness, and it can be viewed as an extension of both *relational concept analysis* and *fuzzy formal concept analysis*. This article contributes to the development of fuzzy relational concept analysis by achieving the following goals. First of all, we present and study a new class of fuzzy quantifiers, called *t-scaling quantifiers*, to extract fuzzy concepts from fuzzy relational context families. Subsequently, we provide an algorithm to generate, given a t-scaling quantifier, a collection of fuzzy concept lattices from a special fuzzy relational context family, which is composed of a pair of fuzzy formal contexts and a fuzzy relation between their objects. After that, we introduce an ordered relation on the set of all t-scaling quantifiers, which allows us to discover a correspondence among fuzzy concept lattices deriving from different t-scaling quantifiers. Finally, we discuss how the results obtained for t-scaling quantifiers can be extended to the class of fuzzy scaling quantifies. Therefore, this analysis highlights the main differences between t-scaling and fuzzy quantifiers.**

***Index Terms*—Fuzzy concept lattices, fuzzy concepts, fuzzy formal contexts, fuzzy quantifiers, fuzzy relational context families.**

I. Introduction

FORMAL concept analysis (FCA) is a mathematical theory created to produce a conceptual hierarchy called *concept lattice*, starting from a formal context, which is a triple composed of a set of objects, a set of attributes, and a relation between objects and attributes [1], [2], [3]. Mathematically, a *concept lattice* is a particular lattice having *formal concepts* as elements. Given a formal context (X, Y, I), a formal concept is a pair (A, B), where the components A and B determine each other: *A is the set of all objects of X having all attributes of B*, and *B is the set of all attributes of Y being satisfied by all objects of A*. According to the philosophical tradition, A and B are, respectively, called *extent* and *intent* of the concept. Furthermore, formal concepts are ordered with the *subconcept–superconcept relation* capturing that a concept can be more specific, or more general, than another (for example, the concept "tiger" is more specific than the concept "feline").

FCA is an appealing research topic from a theoretical perspective [4], [5], [6] and finds applications in different areas of computer science, such as information retrieval, machine learning, and knowledge discovery [7], [8], [9], [10], [11].

A large group of scholars, motivated by the need to solve real-life problems, has extended FCA in several ways (for some examples, see [12], [13], [14], and [15]). In this article, we are interested in *fuzzy formal concept analysis* (FFCA) and *relational concept analysis* (RCA). Both are theories proposed to broaden the scope of FCA as follows.

FFCA extends FCA, using fuzzy logic, to also deal with vague information. Shortly speaking, FFCA mines concept hierarchies from datasets called *fuzzy formal contexts*, where attributes are satisfied by objects with truth degrees belonging to a graded scale, which is usually the real interval [0,1]. Among all the existing FFCA approaches, we focus on the one developed by Bělohlávek [16] and independently by Pollandt [17], where each concept is uniquely determined by a fuzzy set of objects A and a fuzzy set of attributes B connected to each other as follows: given an object x and an attribute y, $A(x)$ is the degree to which *x has all attributes of B* and $B(y)$ is the degree to which *y is shared by all objects of A*. Such concepts are constructed by considering *complete residuated lattices* as algebraic structures of truth degrees [18].

RCA combines FCA with description logic to extract concept hierarchies from multirelational datasets. The RCA input is a *relational context family*, which is composed of several formal contexts and intercontext relations, namely, relations between objects of different formal contexts. First, the RCA process transforms the initial relational context family in a collection of formal contexts by using the so-called *scaling quantifiers*. After that, it generates a set of concept lattices (the RCA output) by employing the classical FCA techniques [19], [20], [21].

Scaling quantifiers are binary relations on the power set 2^X of a given universe X and measure how large the intersection of two subsets A and B of X is w.r.t. the size of A. Their definitions carry an *existential import*, also called *presupposition*, corresponding to the assumption that the universe of quantification must be nonempty. An example of scaling

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quantifier is $\mathcal{Q}_{30} : 2^X \times 2^X \to \{0,1\}$ such that $\mathcal{Q}_{30}(A,B) = 1$ if and only if at least 30% of elements in A belong to B, and the intersection between A and B contains at least an element of X (the latter condition represents the existential import).[1] The choice of scaling quantifiers, during the RCA process, is up to one or more users according to the initial dataset and the final classification that they would like to obtain.

Unfortunately, we cannot employ RCA to extract concept hierarchies from vague datasets because RCA only deals with crisp sets and relations. This limit has motivated Boffa et al. [23] to propose a first RCA generalization based on fuzzy logic. Thus, FFCA and RCA have recently been unified to create *fuzzy relational concept analysis* (FRCA).

FRCA has the purpose of extrapolating information (i.e., collections of fuzzy concept lattices) from multirelational datasets involving vagueness (i.e., fuzzy relational context families). A *fuzzy relational context family* extends the notion of relational context family by taking into account fuzzy (instead of crisp) relations.

The extraction of fuzzy concept lattices is obtained by employing the so-called *fuzzy scaling quantifiers*, which are generalizations of RCA scaling quantifiers. Mathematically, fuzzy scaling quantifiers are special fuzzy quantifiers defined on the *standard Łukasiewic MV-algebra* and based on the concept of *evaluative linguistic expressions*. These are expressions of natural language having the form $\langle hedge\rangle\langle big\rangle$, where an hedge is an adverbial modification like *very*, *extremely*, and *roughly*, and their theory is constructed in a formal system of higher order fuzzy logic (fuzzy type theory) [24], [25], [26]. In addition, fuzzy scaling quantifiers are interpretations in a model of intermediate quantifiers, which are special formulas of the formal *theory of intermediate generalized quantifiers* presented in [27] and elsewhere.

Let $[0,1]^X$ be the collection of all fuzzy sets on a universe X; an example of fuzzy scaling quantifier is $\mathcal{S}_{\mathsf{Very}} : [0,1]^X \times [0,1]^X \to [0,1]$, where $\mathcal{S}_{\mathsf{Very}}(A,B)$ is the truth degree of the sentence "a *very big* part of A is included in B."[2]

In the formula of fuzzy scaling quantifiers, each linguistic expression $\langle hedge\rangle\langle big\rangle$ is modeled by a function $Bi_\nu : [0,1] \to [0,1]$, which is normal and increasing.

In this article, we present and study a new class of FRCA quantifiers called *t-scaling quantifiers*, which are also extensions of RCA scaling quantifiers. A t-scaling quantifier $\mathcal{S}_t$ is uniquely determined by a threshold $t \in [0,1]$. Formally, $\mathcal{S}_t$ is a function assigning a value of [0,1] to each pair of fuzzy sets of a universe X, where $\mathcal{S}_t(A,B)$ is the truth degree of the sentence "a part of A being *at least as big as t (in the scale [0,1])* is included in B." As for fuzzy scaling quantifiers given in [23], the formula of $\mathcal{S}_t(A,B)$ includes the subformula $\bigvee_{x\in X} A(x)$ representing the existential import and capturing that the universe of quantification A must not be empty, i.e., $\bigvee_{x\in X} A(x)$ is the truth degree of the sentence "there exists at least one element in A."

[1] In [22], $\mathcal{Q}_{30}$ is called *general universal-percent quantifier*.

[2] In the theory of intermediate quantifiers, S_{Very} corresponds to the quantifier "most," i.e., $\mathcal{S}_{\mathsf{Very}}(A,B)$ is the truth degree of the sentence "*most* elements of A are in B."

The existential import is a philosophical concept discussed in several publications, especially in those concerning the study of Aristotle square (see [28], [29], [30], [31], and [32] for some examples) Traditionally, it refers to the consideration that the sentence "All As are B" has sense if "As exist."

The algebraic structures of truth degrees, chosen to obtain t-scaling quantifiers and the related fuzzy concepts, are complete residuated lattices having [0,1] as support [18]. These are the most used structures in FFCA applications and include the *standard Łukasiewicz MV-algebra* (already considered in [23]) and the *standard Gödel algebra*.

In this article, t-scaling quantifiers clearly play a fundamental role. However, generalized quantifiers have recently been introduced in FFCA to extend the definition of concept-forming operators, which are based on the *universal quantifier* "all" [33], [34], [35], [36].

The main motivations to introduce t-scaling quantifiers in FRCA are explained in what follows. The definition of t-scaling quantifiers is based on a complete residuated lattice having [0,1] as support, which is more general than the standard Łukasiewicz MV-algebra used to define fuzzy scaling quantifiers in [23]. Hence, the concept extraction with t-scaling quantifiers could be realized in future application not necessarily considering the standard Łukasiewicz MV-algebra, but selecting the most appropriate complete residuated lattice $\langle [0,1], \wedge, \vee, \otimes, \to, 0, 1\rangle$, according to the situation to analyze.

Additionally, during the FRCA process, one or more experts in the given domain, who usually do not have mathematical skills, must select the most suitable quantifiers to produce the final concept extraction. Therefore, using t-scaling (instead of fuzzy scaling) quantifiers is convenient for the following reasons.

1) Each t-scaling quantifier is uniquely determined by a threshold belonging to [0,1], while each fuzzy scaling quantifiers by a function from [0,1] to [0,1], which models an evaluative linguistic expression. Therefore, for experts, it is certainly easier to determine thresholds than functions.
2) The meaning of t-scaling quantifiers can be better understood by experts because it can be traced back to the meaning of percentage. Indeed, Theorem 3.4 proves the existence of a one-to-one correspondence between t-scaling quantifiers and scaling quantifiers presented in [22]: for each $t \in [0,1]$, the t-scaling quantifier $\mathcal{S}_t(A,B)$ is the generalization of the quantifier $\mathcal{Q}_{t*100}(A,B)$ expressing that *at least* $t * 100\%$ *of the elements of A belong to B (the existential import previously described must be included)*. On the other hand, infinite fuzzy scaling quantifiers forming the class $\tilde{\mathbf{S}}_{t*100}$ can be viewed as generalizations of $\mathcal{Q}_{t*100}$ (see Section V-A). Also, $\mathcal{S}_t$ belongs to the class $\tilde{\mathbf{S}}_{t*100}$, when $t \geq 0.5$ and we confine to the standard Łukasiewicz MV-algebra (see Remark 3.5). Thus, according to the previous considerations, experts could use $\mathcal{S}_t$ instead of any quantifier in $\tilde{\mathbf{S}}_t$.
3) Experts select the most suitable quantifiers also evaluating how their choice affects the final concept classification. Theorem 4.5 provides a way to compare concept lattices deriving from each pair of different t-scaling quantifiers. Such result helps experts to make the selection according

to the final classification that they would like to obtain. Unfortunately, as explained by Remark 5.3, the same is not always possible when we consider a pair of fuzzy scaling quantifiers, and therefore, this makes it more difficult for experts to understand what the best quantifiers to employ are.

Although in this article we introduce t-scaling quantifiers, their main results that consist in proposing FRCA algorithms and comparing their corresponding fuzzy concepts are also provided for fuzzy scaling quantifiers.

Essentially, this article extends the study on FRCA started in [23] and aims to provide new tools for data analysis and knowledge discovery in the FCA framework. Furthermore, it responds to the need stated in [22] and other papers to broaden the RCA scope to analyze datasets that involve vagueness. Therefore, the algorithms and results proposed in this article can be applied anytime information needs to be extracted from datasets having the form of fuzzy relational context families. The following is an example. We can consider the fuzzy relations $I : X \times Y \to [0,1]$, $J : Z \times W \to [0,1]$, and $r : X \times Z \to [0,1]$, where X is a set of individuals; Y is made of personality characteristics like *sociable* and *impulsive*; Z is a set of sports like *volleyball*, *yoga*, and *football*; W is made of sport attributes like *creative*, *funny*, and *aerobic*; finally, r expresses how much a given person in X is interested in a given sport in Z. Then, using FRCA and choosing the t-quantifiers with the threshold $t = 0.5$, we can discover, for instance, that *all individuals that are sociable with a degree of at least 0.7* and *are interested in at least 50% of sports being both funny and aerobic with a degree of at least 0.8.* Furthermore, FRCA can be used to solve all the problems already considered in the RCA applications, but involving datasets characterized by fuzzy relations; for example, the extraction of *link key candidate* from fuzzy resource description framework (RDF) graphs [37] instead of the classical ones considered in [38] or the construction of fuzzy ontology by extending the results in [39] . The rest of this article is organized as follows. Section II reviews some basic notions and results regarding fuzzy logic and FRCA. Section III defines and studies t-scaling quantifiers, and presents FRCA algorithms. Section IV is devoted to introduce a total order on t-scaling quantifiers and show a correspondence among fuzzy concepts deriving from different t-quantifiers. Then, in Section V, we describe how the results obtained for t-scaling quantifiers in Sections III and IV can be extended to the class of fuzzy scaling quantifies. Therefore, the main differences between t-scaling and fuzzy scaling quantifiers are highlighted. Finally, Section VI concludes this article.

II. Preliminaries

This section focuses on preliminary notions and results we need in this article. Let us underline that all the concepts will be provided by assuming that the initial universe is finite.

A. Mathematical Tools for Fuzzy Logic

Definition II.1: A *fuzzy set* A of a universe X is a function $A : X \to [0,1]$, and we write $A \underset{\sim}{\subset} X$ in symbols.

Let $x \in X$; $A(x)$ is the truth degree of the statement "x belongs to A."

In the following, we use the symbol $[0,1]^X$ to denote the collection of all the fuzzy sets of X. Moreover, let $A \underset{\sim}{\subset} X$; we write $A = \emptyset$ to indicate that $A(x) = 0$ for each $x \in X$.

We now review the notion of residuated lattice, which is a general truth structure for fuzzy logic.

Definition II.2 (see [18]): A *residuated lattice* is an algebra $\langle \mathsf{L}, \wedge, \vee, \otimes, \to, \mathbf{0}, \mathbf{1} \rangle$, where:

i) $\langle \mathsf{L}, \wedge, \vee, \mathbf{0}, \mathbf{1} \rangle$ is a bounded lattice;
ii) $\langle \mathsf{L}, \otimes, \mathbf{1} \rangle$ is a commutative monoid, i.e., $\otimes$ is a binary operation that is commutative, associative, and $a \otimes \mathbf{1} = a$ for each $a \in \mathsf{L}$;
iii) $a \otimes b \le c$ if and only if $a \le b \to c$, for each $a, b, c \in \mathsf{L}$ (adjunction property).

A residuated lattice $\langle \mathsf{L}, \wedge, \vee, \otimes, \to, \mathbf{0}, \mathbf{1} \rangle$ is complete if its reduct $\langle \mathsf{L}, \wedge, \vee \rangle$ is a complete lattice.

The following proposition lists some properties satisfied by every complete residuated lattice.

Proposition II.3: Let $\langle \mathsf{L}, \wedge, \vee, \otimes, \to, \mathbf{0}, \mathbf{1} \rangle$ be a complete residuated lattice; then, the following properties hold: let $I = \{1, \ldots, n\}$.

a) If $a_i \le b_i$ for each $i \in I$, then $\bigwedge_{i \in I} a_i \le \bigwedge_{i \in I} b_i$.
b) If $a_i \le b_i$ for each $i \in I$, then $\bigvee_{i \in I} a_i \le \bigvee_{i \in I} b_i$.
c) $\bigwedge_{i \in I} a_i = 1$ if and only if $a_i = 1$ for each $i \in I$.
d) $\bigwedge_{i \in I} a_i = 0$ if and only if there exists $i \in I$ such that $a_i = 0$.
e) $\bigvee_{i \in I} a_i = 1$ if and only if there exists $i \in I$ such that $a_i = 1$.
f) $\bigvee_{i \in I} a_i = 0$ if and only if $a_i = 0$ for each $i \in I$.
g) If $J \subseteq I$, then $\bigvee_{i \in J} a_i \le \bigvee_{i \in I} a_i$.
h) $a \to b = 1$ if and only if $a \le b$.
i) If $a \le b$, then $k \to a \le k \to b$.
j) If $a \le b$ and $c \le d$, then $a \otimes c \le b \otimes d$.

Example II.4 (see [40]): A special complete residuated lattice is the *standard Łukasiewicz MV-algebra*$\langle [0,1], \wedge, \vee, \otimes, \to, 0, 1 \rangle$, where $a \wedge b = \min(a,b)$, $a \vee b = \max(a,b)$, $a \otimes b = \max(0, a + b - 1)$, $a \to b = \min(1, 1 - a + b)$, for each $a, b \in [0,1]$.

In this article, we choose complete residuated lattices with support $\mathsf{L} = [0,1]$ as basic structures of truth values.

The inclusion relation between classical sets is generalized as follows.

Definition II.5: Let $A, B \underset{\sim}{\subset} X$. Then, B *includes* A if and only if $A(x) \le B(x)$ for each $x \in X$, and we write $A \subseteq B$ in symbols.

Then, we deal with particular cases of fuzzy measures on fuzzy sets.

Definition II.6 (see [41]): A *fuzzy measure* on fuzzy sets is a function $\mu : [0,1]^X \to [0,1]$ such that $\mu(X) = 1$ and $\mu(\emptyset) = 0$, and if $A \subseteq B$, then $\mu(A) \le \mu(B)$, i.e., μ is a monotone function.

Examples of fuzzy measures are defined as follows.

Definition II.7 (see [42]): Let AX. Then, the *cardinality* $|A|$ of A is given by

$$|A| = \sum_{x \in X} A(x). \tag{1}$$

Definition II.8 (see [25]): Let $A \underset{\sim}{\subset} X$. Then, the measure $\mu_A : [0,1]^X \to [0,1]$ is defined as follows: let $B \underset{\sim}{\subset} X$:

$$\mu_A(B) = \begin{cases} 1, & \text{if } A = \emptyset \text{ or } B = A \\ \frac{|B|}{|A|}, & \text{if } A \neq \emptyset \text{ and } B \subseteq A. \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Moreover, $\mu_A(B)$ expresses "how large the size of B is w.r.t. the size of A."

We require a special operation to form a new fuzzy set from a given one by extracting several elements together with their membership degrees and putting the other membership degrees equal to 0.

Definition II.9 (see [27]): Let $A, B \underset{\sim}{\subset} X$; the *cut of A with respect to B* is a fuzzy set $A|B \underset{\sim}{\subset} X$ given by

$$(A|B)(x) = \begin{cases} A(x), & \text{if } A(x) = B(x) \\ 0, & \text{otherwise} \end{cases}. \tag{3}$$

Example II.10: Let $X = \{x_1, \ldots, x_5\}$ be a universe; we consider $A, B \underset{\sim}{\subset} X$ such that $A = \{0.25/x_1, 0.5/x_2, x_3, x_4, 0.6/x_5\}$ and $B = \{0.3/x_1, 0.5/x_2, 0.2/x_3, x_4, 0.5/x_5\}$. Then, according to the previous definition, the cut of A w.r.t. B is a new fuzzy set of X, exactly $A|B = \{0.5/x_2, x_4\}$.

Why do we need the notion of fuzzy set cuts? In order to provide the formula of the t-scaling quantifier $\mathcal{S}_t(A, B)$ (see Definition 3.1), we should have considered universes of quantification smaller than A, which correspond to the fuzzy sets included in A according to Definition 2.5. However, the properties of implication suggest considering only the fuzzy sets with membership degrees significantly smaller than those of A. Therefore, a satisfactory solution was to consider the cuts of A, namely the collection $\{A|Z \mid Z \underset{\sim}{\subset} X\}$. For example, let $A = \{0.5/x_1, 0.2/x_2, 0.8/x_3\}$; then, the cuts of A are the following ones: $\emptyset$, $\{0.5/x_1\}$, $\{0.2/x_2\}$, $\{0.8/x_3\}$, $\{0.5/x_1, 0.2/x_2\}$, $\{0.5/x_1, 0.8/x_3\}$, $\{0.2/x_2, 0.8/x_3\}$, and A.

Moreover, we evaluate the size of $A|Z$ w.r.t. A by using the operator $\Delta_t : [0,1] \to [0,1]$ that transforms each element of [0,1] being greater than or equal to t in 1 and the remaining ones in 0. Namely, let $x \in [0,1]$;

$$\Delta_t(x) = \begin{cases} x, & \text{if } x \geq t \\ 0, & \text{otherwise} \end{cases}. \tag{4}$$

The concept of inclusion given in Definition 2.5 is generalized as follows.

Definition II.11: Let $A, B \in [0,1]^X$; we set

$$\mathcal{S}_X(A, B) = \bigwedge_{x \in X} (A(x) \to B(x)) \tag{5}$$

where $\mathcal{S}_X(A, B)$ represents the degree of inclusion of A in B.

Observe that if $\mathcal{S}_X(A, B) = 1$, then A *is included in* B according to Definition 2.5.

Fuzzy Galois connections and fuzzy closure operators are fundamental notions in fuzzy logic.

Definition II.12 (see [43]): Let $\langle [0,1], \wedge, \vee, \otimes, \to, \mathbf{0}, \mathbf{1} \rangle$ be a complete residuated lattice, and let X and Y be universes. A *fuzzy Galois connection* between X and Y is a pair $\langle f, g \rangle$ of mappings $f : [0,1]^X \to [0,1]^Y$ and $g : [0,1]^Y \to [0,1]^X$ satisfying the following conditions for each $A, A_i, A_j \in [0,1]^X$ and $B, B_i, B_j \in [0,1]^Y$.

i) $\mathcal{S}_X(A_i, A_j) \leq \mathcal{S}_Y(f(A_j), f(A_i))$.
ii) $\mathcal{S}_Y(B_i, B_j) \leq \mathcal{S}_X(g(B_j), g(B_i))$.
iii) $A \subseteq g(f(A))$.
iv) $B \subseteq f(g(B))$.

Definition II.13 (see [44]): Let $\langle [0,1], \wedge, \vee, \otimes, \to, \mathbf{0}, \mathbf{1} \rangle$ be a complete residuated lattice, and let X be a universe. A *fuzzy closure operator* on X is a mapping $C : [0,1]^X \to [0,1]^X$ satisfying the following conditions for each $A, B \in [0,1]^X$.

i) $A \subseteq C(A)$.
ii) If $A \subseteq B$, then $C(A) \subseteq C(B)$.
iii) $C(A) = C(C(A))$.

B. Fuzzy Formal Concept Analysis

Let $\langle [0,1], \wedge, \vee, \otimes, \to, 0, 1 \rangle$ be a complete residuated lattice.[3]

Definition II.14: A *fuzzy formal context* is a triple (X, Y, I), where X is a set of objects, Y is a set of attributes, and I is a fuzzy relation on $X \times Y$, i.e., $I : X \times Y \to [0,1]$.

Definition II.15 (see [16] and [17]): Let (X, Y, I) be a fuzzy formal context. If $A \underset{\sim}{\subset} X$ and $B \underset{\sim}{\subset} Y$, then

$$A^{\uparrow_I}(y) = \bigwedge\nolimits_{x \in X} (A(x) \to I(x,y))$$
$$B^{\downarrow_I}(x) = \bigwedge\nolimits_{y \in Y} (B(y) \to I(x,y))$$

for all $x \in X$ and $y \in Y$.

$A^{\uparrow_I}(y)$ and $B^{\downarrow_I}(x)$ are the truth degrees of the statements "y is shared by all objects of A" and "x has all attributes of B," respectively.

The following results regarding the operators of Definition 2.15 have been proved in [43], [45], and [46].

Theorem II.16: Let (X, Y, I) be a fuzzy formal context. Then, the pair made of $^{\uparrow_I} : [0,1]^X \to [0,1]^Y$ and $^{\downarrow_I} : [0,1]^Y \to [0,1]^X$ is a Galois connection.

Theorem II.17: Let (X, Y, I) and (X, Y, J) be fuzzy formal contexts. Then, $I \subseteq J$ if and only if $A^{\uparrow_I} \subseteq A^{\uparrow_J}$ and $B^{\downarrow_I} \subseteq B^{\downarrow_J}$ for all $A \in [0,1]^X$ and $B \in [0,1]^Y$.

As shown below, operators of Definition 2.15 are employed to extract fuzzy concepts from every fuzzy formal context.

Definition II.18: Let (X, Y, I) be a fuzzy formal context, and let $A \underset{\sim}{\subset} X$ and $B \underset{\sim}{\subset} Y$. Then, (A, B) is a *fuzzy concept* of (X, Y, I) if and only if $A^{\uparrow_I} = B$ and $B^{\downarrow_I} = A$.

Algorithms for generating all the fuzzy concepts from a fuzzy formal context are provided in [47] and elsewhere.

We denote the set of all fuzzy concepts of (X, Y, I) with $\mathcal{B}(X, Y, I)$.

$(\mathcal{B}(X, Y, I), \mathcal{R})$ is a complete fuzzy lattice,[4] called the *fuzzy concept lattice* of (X, Y, I), where the relation $\mathcal{R}$

[3]The notions of this subsection hold for complete residuated lattices having a generic set as support as well.

[4]The notion of complete fuzzy lattice is provided in [48].

is defined by $\mathcal{R}((A_1, B_1), (A_2, B_2)) = \mathcal{S}_X(A_1, A_2)$ for all $(A_1, B_1), (A_2, B_2) \in \mathcal{B}(X, Y, I)$ [2], [48].

Theorem II.19: Let (X, Y, I) be a fuzzy formal context, let $A \underset{\sim}{\subset} X$, and let $B \underset{\sim}{\subset} Y$. Then, $A^{\uparrow_I\downarrow_I}$ and $B^{\downarrow_I\uparrow_I}$ are, respectively, the extent and the intent of concepts of $\mathcal{B}((X, Y, I), \mathcal{R})$.

C. Fuzzy Relational Concept Analysis

In FRCA, a significant role is played by the so-called *fuzzy scaling quantifiers*, which are generalizations of standard scaling quantifiers by using fuzzy logic.

Among all the RCA scaling quantifiers considered in [22], we are interested in the following.

Definition II.20:

Let X be a universe; we put $\mathcal{P}(X)^2 = \{(A, B) \mid A, B \subseteq X\}$. Let $n \in [0, 100]$; the *universal-percent scaling quantifier* on X is a function $\mathcal{Q}_n : \mathcal{P}(X)^2 \to \{0, 1\}$ such that, given $A, B \subseteq X$,

$$\mathcal{Q}_n(A, B) = 1 \quad \text{iff} \quad |A \cap B| \geq \frac{n}{100}|A| \text{ and } |A \cap B| > 0.$$

Scaling quantifiers given by Definition 2.20 have been extended in the fuzzy logic framework as follows (see [23] for more details).

Definition II.21: Let $\langle [0, 1], \wedge, \vee, \otimes, \to, 0, 1\rangle$ be the standard Łukasiewic MV-algebra; let Bi_ν be a function modeling an evaluative linguistic expression with the form $\langle hedge \rangle$ $\langle big \rangle$ in the context [0,1], and let X be a universe. Then, the *fuzzy ν-universal scaling quantifier* on X is a function $\mathcal{S}_\nu : [0, 1]^X \times [0, 1]^X \to [0, 1]$ such that, given $A, B \underset{\sim}{\subset} X$,

$$\mathcal{S}_\nu(A, B) = \bigvee_{Z \underset{\sim}{\subset} X} \left(\left(\bigwedge_{x \in X} ((A|Z)(x) \to B(x)) \otimes \bigvee_{x \in X} (A|Z)(x) \right) \wedge Bi_\nu(\mu_A(A|Z)) \right). \tag{6}$$

Remark II.22: Mathematically, Bi_ν is a function from [0,1] to [0,1], which is normal (i.e., there exists at least an element x of [0,1] such that $Bi_\nu(x) = 1$) and increasing (i.e., if $x \leq y$, then $Bi_\nu(x) \leq Bi_\nu(x)$, for each $x \in [0, 1]$). Bi_ν is obtained by composing two functions: Bi modeling the expression *Big* and ν modeling an adverbial modification called hedge like `Very`. The role of Bi_ν in the previous definition is to evaluate $\mu_A(A|Z)$. Then, if ν models `Very`, $Bi_\nu(\mu_A(A|Z))$ is the degree to which the size of $A|Z$ is *Very Big* w.r.t. the size of A. More explanations are found in [23] and [49].

FRCA analyzes data organized as a fuzzy relational context family.

Definition II.23: A *fuzzy relational context family* is a pair $(\mathbf{K}, \mathbf{R})$, where:

i) $\mathbf{K}$ is a set of fuzzy formal contexts $\{(X_1, Y_1, I_1), \ldots, (X_n, Y_n, I_n)\}$;
ii) $\mathbf{R}$ is a set of fuzzy binary relations $\{r_1, \ldots, r_m\}$ with domain and range in $\{X_1, \ldots, X_n\}$.

A set of fuzzy concept lattices is extracted from a fuzzy relational context family $(\mathbf{K}, \mathbf{R})$ in two fundamental steps.

1) $(\mathbf{K}, \mathbf{R})$ is transformed into a set $\mathbf{K}'$ of fuzzy formal contexts by means of selected fuzzy scaling quantifiers.
2) A new fuzzy concept lattice is extracted from each fuzzy formal context of $\mathbf{K}'$, by using the existing fuzzy FCA techniques.

Mainly, step 1 is realized as follows.

a) Let $\mathbf{SQ}$ be the collection of all fuzzy scaling quantifiers; we consider the functions $s : \mathbf{R} \to \mathbf{SQ}$, $k_{\text{dom}} : \mathbf{R} \to \mathbf{K}$, and $k_{\text{cod}} : \mathbf{R} \to \mathbf{K}$ such that for each fuzzy relation $r : A \times B \to [0, 1]$; and $k_{\text{dom}}(r)$ and $k_{\text{cod}}(r)$ are two fuzzy formal contexts of $\mathbf{K}$ having A and B as sets of objects, respectively.[5]

b) For each $(X, Y, I) \in \mathbf{K}$, we consider the set of relations

$$\{r_1, \ldots, r_n\} = \{r \in \mathbf{R} \mid k_{\text{dom}}(r) = (X, Y, I)\}$$

and let $i \in \{1, \ldots, n\}$; we denote the fuzzy concept lattice extracted by $k_{\text{cod}}(r_i)$ with $\mathcal{L}_i$. Moreover, given $r_i : X \times Z \to [0, 1]$ and $x \in X$, we use the symbol $r_i(x)$ to indicate a fuzzy set of Z such that $(r_i(x))(z) = r_i(x, z)$ for each $z \in Z$, and the symbols E^i_C to indicate the extent of the concept C of $\mathcal{L}_i$.

Then, we construct a new fuzzy formal context (X, Y^*, I^*) such that:

1) $Y^* = Y \cup Y_1 \cup \ldots \cup Y_n$, where $Y_i = \{y^i_C \mid C \in \mathcal{L}_i\}$;
2) let $(x, y) \in X \times Y^*$; then

$$I^*(x, y) = \begin{cases} I(x, y), & \text{if } y \in Y \\ \mathcal{S}(r_i(x), E^i_C), \text{with } \mathcal{S} = s(r_i), & \text{if } y = y^i_C \end{cases}. \tag{7}$$

Given $i \in \{1, \ldots, n\}$ and $C \in \mathcal{L}_i$, y^i_C is called *fuzzy relational attribute*.

Therefore, a new family of fuzzy formal context is given by

$$\mathbf{K}' = \{(X, Y^*, I^*) \mid (X, Y, I) \in \mathbf{K}\}.$$

Of course, $(X, Y^*, I^*) = (X, Y, I)$ when the set $\{r \in \mathbf{R} \mid k_{\text{dom}}(r) = (X, Y, I)\}$ is empty.

Eventually, observe that (X, Y^*, I^*) contains both information of (X, Y, Z) and that of the fuzzy relations of $\{r \in \mathbf{R} \mid k_{\text{dom}}(r) = (X, Y, I)\}$.

Then, step 2 can be realized by employing one of the several algorithms introduced in the literature (for example, see [47] and [50]).

III. FRCA With t-Scaling Quantifiers

In this section, we first present a new family of fuzzy scaling quantifiers called t-scaling quantifiers (see Section III-A). Subsequently, we show a procedure to mine a collection of fuzzy concept lattices from a special fuzzy relational context family by using a fixed t-scaling quantifier (see Section III-B).

In the following, we consider a universe X and a complete residuated lattice $\langle [0, 1], \wedge, \vee, \otimes, \to, 0, 1\rangle$.

A. T-Scaling Quantifiers

Definition III.1: Let $t \in [0, 1]$. Then, the *fuzzy t-scaling quantifier* on X is a function $\mathcal{S}_t : [0, 1]^X \times [0, 1]^X \to [0, 1]$ such that,

[5] s, k_{dom}, and k_{cod} can be determined by experts or users during the RCA process.

given $A, B \underset{\sim}{\subset} X$,

$$\mathcal{S}_t(A,B) = \bigvee_{Z \underset{\sim}{\subset} X} \left(\left(\bigwedge_{x\in X} ((A|Z)(x) \to B(x)) \otimes \bigvee_{x\in X} (A|Z)(x) \right) \wedge \Delta_t(\mu_A(A|Z)) \right). \quad (8)$$

Moreover, $\mathcal{S}_t(A,B)$ is the truth degree of the following statement:

> There exists a cut $A|Z$ of A such that "all elements of $A|Z$ belong to B," "there exists at least one element in $A|Z$," and "the size of $A|Z$ is at least as large as t (in the scale [0,1]) w.r.t. the size of A".

A fundamental role in the definition of t-scaling quantifiers is played by $\bigvee_{x\in X}(A|Z)(x)$ interpreting the logical formula $(\exists x)(A|Z)(x)$ in a model of fuzzy predicate logic. The latter captures that *there exists at least one element of* X *in* $A|Z$ and speaks about the *existential import* (or presupposition). Let us underline that in fuzzy logic, the existential import is included into the formula of quantifiers by the strong conjunction, in order to guarantee the validity of some syllogisms needing the adjunction property [27]. T-scaling quantifiers satisfy the properties shown in the next proposition.

Proposition III.2: Let $t \in [0,1]$, and let $A, B \underset{\sim}{\subset} X$. If $A = B$ or $A = \emptyset$, then $\mathcal{S}_t(A,B) = \bigvee_{x\in X} A(x)$.

Proof: Let $A = B$. By Definition 2.9 together with Proposition 2.3(h), if $A = B$, then $A|Z(x) \to B(x) = 1$ for each $x \in X$. Consequently, let $Z \underset{\sim}{\subset} X$; we get $\bigwedge_{x\in X} A|Z(x) \to B(x) = 1$ from Proposition 2.3(c). Since $a \otimes 1 = a$ in every complete residuated lattice, $\mathcal{S}_t(A,B) = \bigvee_{Z \underset{\sim}{\subset} X}(\bigvee_{x\in X}(A|Z)(x) \wedge \Delta_t(\mu_A(A|Z)))$. Moreover, by Definition 2.9 and (4), we obtain

$$\bigvee_{x\in X} (A|Z)(x) \leq \bigvee_{x\in X} A(x) \text{ and } \Delta_t(\mu_A(A|Z)) \leq \Delta_t(\mu_A(A))$$

for each $Z \underset{\sim}{\subset} X$. Then, $\bigvee_{x\in X}(A|Z)(x) \wedge \Delta_t(\mu_A(A|Z)) \leq \bigvee_{x\in X} A(x) \wedge \Delta_t(\mu_A(A))$ from Proposition 2.3(a). Hence, the thesis clearly follows.

Let $A = \emptyset$. By Definitions 2.8 and 2.9 together with (4), $\Delta_t(\mu_\emptyset(\emptyset|Z)) = \Delta_t(1) = 1$ and $(\emptyset|Z)(x) \to B(x) = 1$ for each $Z \underset{\sim}{\subset} X$. Thus, the thesis derives from the properties of complete residuated lattices (see Proposition 2.3). ■

In the following theorem, we show another way to obtain the t-scaling quantifier corresponding to $t = 1$.

Theorem III.3: Let $A, B \underset{\sim}{\subset} X$; then

$$\mathcal{S}_1(A,B) = \bigwedge_{x\in X} (A(x) \to B(x)) \otimes \bigvee_{x\in X} A(x). \quad (9)$$

Proof: By (4), $\Delta_1(\mu_A(A|Z)) = 1$ if and only if $\mu_A(A|Z) = 1$, namely, $A|Z = A$ or $A = \emptyset$ from Definition 2.8.

If $A = \emptyset$, then $\mathcal{S}_1(A,B) = 0$ and $\bigwedge_{x\in X}(A(x) \to B(x)) \otimes \bigvee_{x\in X} A(x) = 0$.

Suppose that $A \neq \emptyset$. If $A|Z = A$, then $(\mathcal{S}(A|Z,B) \otimes \bigvee_{x\in X}(A|Z)(x)) \wedge \Delta_1(\mu_A(A|Z)) = (\mathcal{S}(A|Z,B) \otimes \bigvee_{x\in X}(A|Z)(x)) \wedge 1$.[6] The latter equals $\mathcal{S}(A,B) \otimes \bigvee_{x\in X} A(x)$ from the property $a \wedge 1 = a$. Otherwise, if $A|Z \neq A$, then $(\mathcal{S}(A|Z,B) \otimes \bigvee_{x\in X}(A|Z)(x)) \wedge \Delta_1(\mu_A(A|Z)) = \mathcal{S}(A|Z, B) \wedge 0$. The latter equals 0 from the property $a \wedge 0 = 0$. Moreover, since $a \vee 0 = a$ is satisfied in every bounded lattice, we can conclude that (9) holds. ■

By Theorem 3.3, (8) can be rewritten as follows:

$$\mathcal{S}_t(A,B) = \bigvee_{Z \underset{\sim}{\subset} X} \left(\mathcal{S}_1(A|Z,B) \wedge \Delta_t(\mu_A(A|Z)) \right). \quad (10)$$

Then, $\mathcal{S}_t(A,B)$ is constructed by applying $\mathcal{S}_1$ to all pairs as $(A|Z, B)$, where $A|Z$ represents a universe of quantification smaller than A, and by using Δ_t to evaluate the size of $A|Z$ w.r.t. the size of A.

We can prove that each t-quantifier equals a special RCA scaling quantifier given by Definition 2.20, when both apply to pairs of classical sets of the initial universe.

Theorem III.4: Let $A, B \subseteq X$ and $n \in [0, 100]$; then, $\mathcal{Q}_n(A,B) = \mathcal{S}_{n/100}(A,B)$.

Proof: Let $n \in [0,100]$. We consider $A, B \subseteq X$ such that $\mathcal{Q}_n(A,B) = 1$. Then, we intend to prove that $\mathcal{S}_t(A,B) = 1$, where $t = n/100$.

Since both A and B are classical sets of X, $\mathcal{S}_t(A,B) \in \{0,1\}$ and $\{A|Z \text{ with } Z \underset{\sim}{\subset} X\}$ coincides with the collection of all subsets of A.

By Definition 2.20, we get $A \cap B \neq \emptyset$, and therefore, $\bigvee_{x\in X}(A \cap B)(x) = 1$ from Proposition 2.3(e). Moreover, for each $x \in X$, $(A \cap B)(x) \leq B(x)$, and hence, $(A \cap B)(x) \to B(x) = 1$ from Proposition 2.3(h). Then, by Proposition 2.3(c), $S_1(A \cap B, B) = 1$. In addition, (4) implies that $\Delta_t(\mu_A(A \cap B)) = 1$. Consequently, we obtain $\mathcal{S}_1(A \cap B, B) \wedge \Delta_t(\mu_A(A \cap B)) = 1$.

Finally, $\mathcal{S}_1(A \cap B, B) \wedge \Delta_t(\mu_A(A \cap B)) \leq \mathcal{S}_t(A,B)$. Thus, $\mathcal{S}_t(A,B) = 1$.

Now, let $A, B \subseteq X$ such that $\mathcal{Q}_n(A,B) = 0$; we want to prove that $\mathcal{S}_t(A,B) = 0$, where $t = n/100$. Therefore, let $Z \underset{\sim}{\subset} X$; as underlined above, we have $A|Z \subseteq A$. If $A|Z \subseteq A \cap B$, then $\Delta_t(\mu_A(A \cap B)) = 0$. Otherwise, there exists $x \in X$ such that $A|Z(x) = 1$ and $B(x) = 0$. Hence, $A|Z(x) \to B(x) = 0$, and by Proposition 2.3(d), $\mathcal{S}_1(A|Z,B) = 0$. Therefore, using Proposition 2.3(f) together with (10), we have $\mathcal{S}_t(A,B) = 0$. ■

Remark III.5: If $\langle [0,1], \wedge, \vee, \otimes, \to, 0, 1 \rangle$ is the standard Łukasiewicz MV-algebra and $t \in [0.5, 1]$, then $\mathcal{S}_t$ belongs to the family of fuzzy scaling quantifiers given by Definition 2.21 and introduced in [23].

B. Algorithms in FRCA

This subsection principally provides two algorithms in FRCA. The first one, given $t \in [0,1]$ and $A, B \underset{\sim}{\subset} X$, computes $\mathcal{S}_t(A,B)$. The second one generates fuzzy concept lattices from a fuzzy relational context family composed of two fuzzy formal contexts (X, Y, I) and (Z, W, J), and a fuzzy relation between

[6] Recall that $\mathcal{S}$ is defined by (5).

X and Z. These algorithms are based on the results presented below.

In the following theorem, we rewrite the formula of $\mathcal{S}_t(A, B)$ by considering not all, but only specific cuts of A, namely all those whose size w.r.t. A is at least large t.

Theorem III.6: Let $A, B \underset{\sim}{\subset} X$, and let $t \in [0, 1]$; we put

$$\mathcal{H}_t(A) = \{Z \underset{\sim}{\subset} X \mid \mu_A(A|Z) \geq t\}.$$

Then

$$\mathcal{S}_t(A, B) = \bigvee_{Z \in \mathcal{H}_t(A)} \mathcal{S}_1(A|Z, B). \tag{11}$$

Proof: We can rewrite (10) in the following equivalent form:

$$\mathcal{S}_t(A, B) = \bigvee_{Z \in \mathcal{H}_t(A)} (\, \mathcal{S}_1(A|Z, B) \wedge \Delta_t(\mu_A(A|Z))\,) \vee \times \bigvee_{Z \notin \mathcal{H}_t(A)} (\, \mathcal{S}_1(A|Z, B) \wedge \Delta_t(\mu_A(A|Z))\,). \tag{12}$$

Let $Z \underset{\sim}{\subset} X$. By (4), if $Z \in \mathcal{H}_t(A)$, then $\Delta_t(\mu_A(A|Z)) = 1$. Thus, $\mathcal{S}_1(A|Z, B) \wedge \Delta_t(\mu_A(A|Z)) = \mathcal{S}_1(A|Z, B) \wedge 1 = \mathcal{S}_1(A|Z, B)$.

Otherwise, if $Z \notin \mathcal{H}_t(A)$, then $\Delta_t(\mu_A(A|Z)) = 0$. Hence, $\mathcal{S}_1(A|Z, B) \wedge \Delta_t(\mu_A(A|Z)) = \mathcal{S}_1(A|Z, B) \wedge 0 = 0$.

Therefore, the thesis follows from the properties of complete residuated lattices (see Proposition 2.3). ■

We can find $\mathcal{S}_t(A, B)$ also by considering in (11) only some of the fuzzy sets in $\mathcal{H}_t(A)$. To achieve this goal, we need to define and study a family of cuts of A.

Definition III.7: Let $A, B \underset{\sim}{\subset} X$, and let $k \in \mathsf{K}_{(A,B)}$, where

$$\mathsf{K}_{(A,B)} = \{k \in [0, 1] \mid A(x) \to B(x) = k, \text{ for some } x \in X\}. \tag{13}$$

Then, we put

$$A_k(x) = \begin{cases} A(x), & \text{if } A(x) \to B(x) \geq k \\ 0, & \text{otherwise} \end{cases}. \tag{14}$$

Remark III.8: It is easy to verify that A_k is a cut of A, for each $k \in \mathsf{K}_{(A,B)}$. Moreover, $A_k = A$, when $\bigwedge_{x\in X} A(x) \to B(x) = k$.

The following proposition states that given $k \in \mathsf{K}_{(A,B)}$, A_k is the maximum on the set of all cuts $A|Z$ of A satisfying a special condition.

Proposition III.9: Let $A, B, Z \underset{\sim}{\subset} X$ such that $\bigwedge_{x\in X}(A|Z)(x) \to B(x) = k$. Then, $(A|Z)(x) \leq A_k(x)$ for each $x \in X$.

Proof: Suppose that $\bigwedge_{x\in X}(A|Z)(x) \to B(x) = k$. Then, for each $x \in X$, $(A|Z)(x) \to B(x) \geq k$. Hence, let $x \in X$; if $A(x) \to B(x) < k$, then both $(A|Z)(x)$ and $A_k(x)$ must be equal to 0. Otherwise, if $A(x) \to B(x) \geq k$, then $A_k(x)$ is equal to $A(x)$, and $(A|Z)(x)$ equals 0 or $A(x)$. Consequently, we get $(A|Z)(x) \leq A_k(x)$. ■

The next theorem rewrites the expression of $\mathcal{S}_t(A, B)$ considering a subset $\mathcal{H}^*_t(A, B)$ of $\mathcal{H}_t(A)$ given by

$$\mathcal{H}^*_t(A, B) = \{Z \in \mathcal{H}_t(A) \mid \exists k \in \mathsf{K}_{(A,B)} \text{ with } A|Z = A_k\}.$$

Theorem III.10: Let $A, B \underset{\sim}{\subset} X$, and let $t \in [0, 1]$. Then

$$\mathcal{S}_t(A, B) = \bigvee_{Z \in \mathcal{H}^*_t(A,B)} \mathcal{S}_1(A|Z, B).$$

Proof: Let $Z \in \mathcal{H}_t(A)$. We intend to prove that there exists $\tilde{Z} \in \mathcal{H}^*_t(A, B)$ such that $\mathcal{S}_1(A|Z, B) \leq \mathcal{S}_1(A|\tilde{Z}, B)$.

If $A|Z = \emptyset$, then $\mathcal{S}_1(A|Z, B) = 0$. Consequently, $\mathcal{S}(A|Z, B) \leq \mathcal{S}(A|\tilde{Z}, B)$ for each $\tilde{Z} \in \mathcal{H}^*_t(A, B)$.

If $A|Z \neq \emptyset$, we consider $k \in \mathsf{K}_{(A,B)}$ such that $\bigwedge_{x\in X}(A|Z)(x) \to B(x) = k$. Then, we can consider $\tilde{Z}X$ such that $A_k = A|\tilde{Z}$.

By Proposition 3.9, $(A|Z)(x) \leq A_k(x)$ for each $x \in X$. Thus, by Proposition 2.3(b), we get $\bigvee_{x\in X}(A|Z)(x) \leq \bigvee_{x\in X} A_k(x)$.

Therefore, by Proposition 2.3(j), we have

$$\bigwedge_{x\in X} (A|Z)(x) \to B(x) \otimes \bigvee_{x\in X} (A|Z)(x) \leq \bigwedge_{x\in X} A_k(x) \to B(x) \otimes \bigvee_{x\in X} A_k(x). \tag{15}$$

Thus, we have shown that $\mathcal{S}_1(A|Z, B) \leq \mathcal{S}_1(A|\tilde{Z}, B)$, where $A|\tilde{Z}$ belongs to $\mathcal{H}^*_t(A, B)$.

Hence, using Proposition 2.3(b) again,

$$\bigvee_{Z \in \mathcal{H}_t(A)} \mathcal{S}_1(A|Z, B) \leq \bigvee_{Z \in \mathcal{H}^*_t(A,B)} \mathcal{S}_1(A|Z, B)$$

namely $\mathcal{S}_t(A, B) \leq \bigvee_{Z\in\mathcal{H}^*_t(A,B)} \mathcal{S}_1(A|Z, B)$ from Theorem 3.6.

Of course, by Proposition 2.3(g), $\mathcal{H}^*_t(A, B) \subseteq \mathcal{H}_t(A)$ implies that

$$\bigvee_{Z \in \mathcal{H}^*_t(A,B)} \mathcal{S}_1(A|Z, B) \leq \bigvee_{Z \in \mathcal{H}_t(A)} \mathcal{S}_1(A|Z, B).$$

Then, $\bigvee_{Z\in\mathcal{H}^*_t(A,B)} \mathcal{S}_1(A|Z, B) \leq \mathcal{S}_t(A, B)$, by using Theorem 3.6 again. ■

Now, employing the previous results, we propose the procedure P_1, which takes as input a pair of fuzzy sets A and B of a universe X, and a threshold $t \in [0, 1]$, and finds the value $\mathcal{S}_t(A, B)$.

In detail, P_1 is based on Theorem 3.10: it computes the supremum of the values corresponding to $\mathcal{S}_1(A_k, B)$, where A_k is a cut of A given by Definition 3.7 such that $\mu_A(A_k) \geq t$.

Example III.11: Consider $A = \{0.5/x_1, 0.3/x_2, 0.4/x_3, x_4, x_5, x_6\}$ and $B = \{x_1, x_2, 0.5/x_3, 0.2/x_4, 0.5/x_5, x_6\}$, and assume that the standard Łukasiewicz MV-algebra is our structure of truth values. Then, $\mathsf{K} = \{0.4, 0.5, 1\}$ because $A(x_i) \to B(x_i) = 1$ if $i \in \{1, 2, 3, 6\}$, $A(x_4) \to B(x_4) = 0.4$ and $A(x_5) \to B(x_5) = 0.5$. Also, we choose $t = 0.6$. Then, $\mu_A(A_{0.4}) = 1$, $\mu_A(A_{0.5}) = 0.76$, and $\mu_A(A_1) = 0.42$.

Since $\mu_A(A_{0.4}), \mu_A(A_{0.5}) \geq 0.7$, P_1 returns 0.5, which is the maximum between $0.4 \otimes 1 = 0.4$ and $0.5 \otimes 1 = 0.5$.

Algorithm 1: The Algorithm for Finding the Values Assumed by a t-Scaling Quantifier.

```
procedure P_1(A, B, t)
K ← {k ∈ [0,1] | A(x) → B(x) = k, with x ∈ X}
  for all k ∈ K do
    if μ_A(A_k) ≥ t then
      n ← ⋀_{x∈X} A_k(x) → B(x)
      m ← ⋁_{x∈X} A_k(x)
      S ← S ∪ {n ⊗ m}
    end if
end for
s* ← ⋁_{s∈S} s
return s*
end procedure
```

We currently have enough tools to present the procedure P_2. Its input consists of a fuzzy relational context family $(\mathbf{K}, \mathbf{R})$, where $\mathbf{K} = \{(X, Y, I), (Z, W, J)\}$ and $\mathbf{R} = \{(X, Z, r)\}$, and a threshold $t \in [0, 1]$, and its output is a pair of fuzzy concept lattices $\{\mathcal{L}_1, \mathcal{L}_2\}$ associated with $(\mathbf{K}, \mathbf{R})$ through $\mathcal{S}_t$.

Let us point out that P_2 recalls, in addition to P_1, the procedures P_3 and P_4. These, given a fuzzy formal context (X, Y, I), respectively, compute the fuzzy concept lattice of (X, Y, I) and the extent of all fuzzy concepts of (X, Y, I) by using one of the existing FFCA techniques (for example, see [47] and [51]).

Algorithm 2: The Algorithm for Extracting a Collection of Fuzzy Concept Lattices From a Fuzzy Relational Context Family, Which Is Composed of Two Fuzzy Formal Contexts and a Fuzzy Relation Between Their Objects.

```
procedure P_2((X, Y, I), (Z, W, J), (X, Z, r), t)
Y* ← Y
for all x ∈ X do
  for all y ∈ Y do
    I*(x, y) ← I(x, y)
  end for
end for
L_1 ← P_3(Z, W, J)
E ← P_4(L_1)
for all E ∈ E do
  Y* ← Y* ∪ {y_E}
  for all x ∈ X do
    I*(x, y_E) ← P_1(r(x), E, t) {As explained in
    Section II-C, r(x) is a fuzzy set such that
    r(x)(z) = r(x, z)}.
  end for
  L_2 ← P_4(X, Y*, I*)
end for
return {L_1, L_2}
end procedure
```

Eventually, the concept lattices related to a general fuzzy context family $(\mathbf{K}, \mathbf{R})$ such that $|\mathbf{K}| \geq 2$ and $|\mathbf{R}| \geq 1$ can be obtained by applying the procedure $P2$ to $(K, K,' r, t)$ for each relation $r \in \mathbf{R}$, where t, K, and K' are selected, as described in Section II-C.

IV. Comparing Concept Lattices Deriving From Different t-Scaling Quantifiers

In this section, we first introduce a total order on t-scaling quantifiers. Then, we compare fuzzy concept lattices deriving from different t-scaling quantifiers.

An ordered relation on t-scaling quantifiers can be defined as follows.

Definition IV.1: Let $\mathbf{S} = \{\mathcal{S}_t \mid t \in [0, 1]\}$, and let $\mathcal{S}, \mathcal{S}' \in \mathbf{S}$. Then

$$\mathcal{S} \preceq_{\mathbf{S}} \mathcal{S}' \text{ iff } \mathcal{S}(A, B) \leq \mathcal{S}'(A, B) \text{ for each } A, B \underset{\sim}{\subset} X. \quad (16)$$

The next theorem shows that $\preceq_{\mathbf{S}}$ is a total order on $\mathbf{S}$, i.e., $\mathcal{S} \preceq_{\mathbf{S}} \mathcal{S}'$ or $\mathcal{S}' \preceq_{\mathbf{S}} \mathcal{S}$, for each $\mathcal{S}, \mathcal{S}' \in \mathbf{S}$.

Theorem IV.2: Let $s, t \in [0, 1]$ such that $s \leq t$. Then, $\mathcal{S}_t \preceq_{\mathbf{S}} \mathcal{S}_s$.

Proof: Let $A, B \underset{\sim}{\subset} X$. By (4), $\Delta_t(\mu_A(A|Z)) \leq \Delta_s(\mu_A(A|Z))$ for each $Z \underset{\sim}{\subset} X$. Then, by Proposition 2.3(a), we have

$$\mathcal{S}_1(A|Z, B) \wedge \Delta_t(\mu_A(A|Z)) \leq \mathcal{S}_1(A|Z, B) \wedge \Delta_s(\mu_A(A|Z))$$

for each $Z \underset{\sim}{\subset} X$. Thus, by Proposition 2.3(b), we have

$$\bigvee_{Z \underset{\sim}{\subset} X} \mathcal{S}_1(A|Z, B) \wedge \Delta_t(\mu_A(A|Z)) \leq \bigvee_{Z \underset{\sim}{\subset} X} \mathcal{S}_1(A|Z, B) \wedge \Delta_s(\mu_A(A|Z)).$$

Namely, $\mathcal{S}_t(A, B) \leq \mathcal{S}_s(A, B)$ from (10). ■

In the following, we consider a fuzzy relational context family

$$(\mathbf{K}, \mathbf{R}) = (\{(X, Y, I), (Z, W, J)\}, \{(X, Z, r)\})$$

and we denote with (X, Y^*, I_t) the fuzzy formal context obtained from $\mathcal{B}(Z, W, J)$ and (X, Z, r), by using the quantifiers $\mathcal{S}_t$.[7] For convenience, we can write $^{\uparrow_t}$ instead of $^{\uparrow_{I_t}}$ (dually, $^{\downarrow_t}$ instead of $^{\downarrow_{I_t}}$). Moreover, given $C \in \mathcal{B}(Z, W, J)$, the symbol y_C indicates the relational attribute associated with C.

Remark IV.3: By (7), Theorem 4.2 implies that, given $s, t \in [0, 1]$ such that $s \leq t$, $I_t \subseteq I_s$ (i.e., $I_t(x, y) \leq I_s(x, y)$ for all $x \in X$ and $y \in Y^*$).

Therefore, using Theorem 4.2, we can compare particular fuzzy sets deriving from different t-scaling quantifiers. More precisely, the following proposition holds.

Proposition IV.4: Let $C \in \mathcal{B}(Z, W, J)$, and let $s, t \in [0, 1]$ such that $s \leq t$. Then, $\{k/y_C\}^{\downarrow_t} \subseteq \{k/y_C\}^{\downarrow_s}$ for each $k \in [0, 1]$.

Proof: Let $x \in X$. Since $s \leq t$, we have $I_t(x, y_C) \leq I_s(x, y_C)$ from Remark 4.3. Consequently, by Proposition 2.3(i), $k \to I_t(x, y_C) \leq k \to I_s(x, y_C)$. Therefore, the thesis follows from Definition 2.15.

[7] Let us notice that Y^* does not depend on the t-scaling quantifier choice.

TABLE I
FUZZY RELATIONS I, J, AND r

I	y_1	y_2
x_1	0	0.5
x_2	1	0
x_3	0.75	1

J	w_1	w_2
z_1	1	0.5
z_2	1	1
z_3	0	0
z_4	0.75	1

r	z_1	z_2	z_3	z_4
x_1	0.5	0.5	1	1
x_2	0.75	0	0.25	0
x_3	1	0	0.5	0.75

The next theorem exhibits a connection among fuzzy concepts that are generated by different t-scaling quantifiers. In particular, let s and t be thresholds in [0,1] such that $s \leq t$; each fuzzy concept corresponding to t is less than or equal to at least one corresponding to s.

To compare concepts of different lattices, we use an ordered relation $\preceq$ on the set $[0,1]^X \times [0,1]^{Y^*} = \{(A,B) \mid A \underset{\sim}{\subseteq} X \text{ and } B \underset{\sim}{\subseteq} Y^*\}$, where let $(A_i, B_i), (A_j, B_j) \in [0,1]^X \times [0,1]^{Y^*}$,

$(A_i, B_i) \preceq (A_j, B_j)$ if and only if $A_i \subseteq A_j$ and $B_i \subseteq B_j$.

Theorem IV.5: Let $s, t \in [0,1]$ such that $s \leq t$. Then, for each $(A,B) \in \mathcal{B}(X, Y^*, I_t)$, there exists $(A^*, B^*) \in \mathcal{B}(X, Y^*, I_s)$ such that $(A,B) \preceq (A^*, B^*)$.

Proof: Let $(A,B) \in \mathcal{B}(X, Y^*, I_t)$, and let $B^* = B^{\downarrow_s \uparrow_s}$. Then, by Theorem 2.19, B^* is the intent of a concept of $\mathcal{B}(X, Y^*, I_s)$. Moreover, by Theorem 2.16, ${}^{\downarrow_s \uparrow_s}$ is a closure operator. Hence, we get $B \subseteq B^*$ from Definition 2.13(i).

We now intend to prove that $A \subseteq A^*$, where $A^* = (B^*)^{\downarrow_s}$ and $A = B^{\downarrow_t}$. By Remark 4.3, $I_t \subseteq I_s$. Then, by Theorem 2.17, $B^{\downarrow_t} \subseteq B^{\downarrow_s}$. Since ${}^{\uparrow_s \downarrow_s}$ is a closure operator [see Definition 2.13(i)], we get $B^{\downarrow_s} \subseteq (B^{\downarrow_s})^{\uparrow_s \downarrow_s}$.

Thus, we can conclude that $B^{\downarrow_t} \subseteq (B^{\downarrow_s \uparrow_s})^{\downarrow_s}$, namely, $A \subseteq A^*$. ■

Let us provide an illustrative example, where concepts arising from different quantifiers are compared through $\preceq$.

Example IV.6: Consider a fuzzy relational context family

$$(\mathbf{K}, \mathbf{R}) = (\{(X, Y, I), (Z, W, J)\}, \{r\})$$

such that $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$, $Z = \{z_1, z_2\}$, $W = \{w_1, w_2, w_3\}$, and $I : X \times Y \to Ł_3$, $J : Z \times W \to Ł_3$ and $r : X \times Z \to Ł_3$ are provided by Table I .[8] We aim:

1) to find fuzzy concepts hidden in $(\mathbf{K}, \mathbf{R})$ using $\mathcal{S}_{0.25}$ and $\mathcal{S}_{0.75}$;
2) to compare, employing $\preceq$, each fuzzy concept deriving from $\mathcal{S}_{0.75}$ with at least one deriving from $\mathcal{S}_{0.25}$.

To achieve these goals, we consider $\mathcal{B}(Z, W, J) = \{C_1, \ldots, C_7\}$, where

$$C_1 = (\{z_1, z_2, z_3, z_4\}, \emptyset)$$
$$C_2 = (\{z_1, z_2, 0.75/z_3, z_4\}, \{0.25/w_1, 0.25/w_2\})$$
$$C_3 = (\{z_1, z_2, 0.5/z_3, z_4\}, \{0.5/w_1, 0.5/w_2\})$$
$$C_4 = (\{z_1, z_2, 0.75/z_4\}, \{w_1, 0.5/w_2\})$$
$$C_5 = (\{0.5/z_1, z_2, z_4\}, \{0.75/w_1, w_2\})$$
$$C_6 = (\{0.75/z_1, z_2, 0.75/z_4\}, \{w_1, 0.75/w_2\})$$
$$C_7 = (\{0.5/z_1, z_2, 0.75/z_4\}, \{w_1, w_2\}).$$

[8] $Ł_3$ is the support of the *three-element Łukasiewicz algebra*, namely, $Ł_3 = \{0, 0.5, 1\}$ [16].

Then, we need to find $I_{0.75}$ and $I_{0.25}$, which are fuzzy relations on $X \times Y^*$, where $Y^* = Y \cup \{y_{C_1}, \ldots, y_{C_7}\}$, determined by $\mathcal{S}_{0.75}$ and $\mathcal{S}_{0.25}$, respectively. $I_{0.75}$ and $I_{0.25}$ are defined by Table II and are obtained from $\mathcal{B}(Z, W, J)$ and r as follows: given $t \in \{0.25, 0.75\}$ and $x \in X$,

1) $I_t(x, y_i) = I(x, y_i)$ for each $i \in \{1, 2\}$;
2) $I_t(x, y_{C_i}) = \mathcal{S}_t(x, y_{C_i})$ for each $i \in \{1, \ldots, 7\}$.

Therefore, we can compute the fuzzy concepts of $\mathcal{B}(X, Y^*, I_{0.25})$ and $\mathcal{B}(X, Y^*, I_{0.75})$, which are listed in Tables III and IV .

Finally, according to Theorem 4.2, we can verify that $C^1_{0.75} \preceq C^1_{0.25}$, $C^2_{0.75} \preceq C^5_{0.25}$, $C^3_{0.75} \preceq C^6_{0.25}$, $C^4_{0.75} \preceq C^2_{0.25}, C^3_{0.25}$, $C^5_{0.75} \preceq C^8_{0.25}, C^{12}_{0.25}$, $C^6_{0.75} \preceq C^{10}_{0.25}$, $C^7_{0.75} = C^3_{0.25}$, $C^8_{0.75} = C^4_{0.25}$, $C^9_{0.75} \preceq C^9_{0.25}$, $C^{10}_{0.75} \preceq C^{11}_{0.25}$, $C^{11}_{0.75} \preceq C^{14}_{0.25}$, $C^{12}_{0.75} = C^{12}_{0.25}$, $C^{13}_{0.75} \preceq C^{16}_{0.25}$, $C^{14}_{0.75} = C^{16}_{0.25}$, $C^{15}_{0.75} \preceq C^{18}_{0.25}$, and $C^{16}_{0.75} \preceq C^{18}_{0.25}$.

V. COMPARISON OF T-SCALING AND FUZZY SCALING QUANTIFIERS

Let $\tilde{\mathbf{S}}$ be the collection of all the fuzzy scaling quantifiers introduced in [23]. We intend to answer the questions: Can the results obtained for $\mathbf{S}$ in the previous sections be extended to $\tilde{\mathbf{S}}$? If so, how?

Let us recall that we need to confine to the standard Łukasiewicz MV-algebra, in order to consider $\tilde{\mathbf{S}}$. Moreover, fuzzy scaling and t-scaling quantifiers substantially differ in their formula: $\mu_A(A|Z)$ is evaluated by $Bi_\nu : [0,1] \to [0,1]$ in (6), while $\mu_A(A|Z)$ is evaluated by $\Delta_t : [0,1] \to \{0,1\}$ in (8).

A. Extending Results of Section III to Fuzzy Scaling Quantifiers

1) Proposition 3.2 also holds for the quantifiers of $\tilde{\mathbf{S}}$. The demonstration can be obtained by substituting Δ_t with Bi_ν into the proof of Proposition 3.2. This is possible because by Remark 2.22, $Bi_\nu(\mu_A(A|Z)) \leq Bi_\nu(\mu_A(A))$ (i.e., Bi_ν is increasing) and $Bi_\nu(\mu_\emptyset(\emptyset|Z)) = 1$ (i.e., Bi_ν is normal).
2) Regarding Theorem 3.3, we can notice that $\mathcal{S}_1 \in \tilde{\mathbf{S}}$. In [23], $\mathcal{S}_1$ coincides with the quantifier "all," which is based on the evaluative linguistic expression "utmost" (indicated with Δ_1), and it is defined by either (8) or (9).
3) Theorem 3.4 leads to a one-to-one correspondence between Boolean scaling quantifiers given by Definition 2.20 and t-scaling quantifies. In particular, we can consider a bijective function such that $\mathcal{Q}_n \mapsto \mathcal{S}_{\frac{n}{100}}$ for each $n \in [0, 100]$ or equivalently its inverse such that $\mathcal{S}_t \mapsto \mathcal{Q}_{t*100}$ for each $t \in [0,1]$, where by Theorem 3.4, $\mathcal{Q}_n(A,B) = \mathcal{S}_{\frac{n}{100}}(A,B)$ and $\mathcal{S}_t(A,B) = \mathcal{Q}_{t*100}(A,B)$ for each $A, B \subseteq X$.
   Such correspondence cannot be replied for the quantifiers of $\tilde{\mathbf{S}}$ by considering that, in general, Theorem 3.3 does not hold for fuzzy scaling quantifiers. Namely, there exists $\mathcal{S} \in \tilde{\mathbf{S}} \setminus \mathbf{S}$ that applied on classical sets does not equal any $\mathcal{Q}_n$ with $n \in [0, 100]$. However, we have proved in [23] that given $n \in [0, 100]$, we can find a class of quantifiers $\tilde{\mathbf{S}}_n \subset \tilde{\mathbf{S}}$, which is connected with $\mathcal{Q}_n$ by the following

TABLE II
FUZZY RELATIONS $I_{0.75}$ AND $I_{0.25}$ ARISING FROM (Z, W, J) AND r

$I_{0.75}$	y_1	y_2	y_{C_1}	y_{C_2}	y_{C_3}	y_{C_4}	y_{C_5}	y_{C_6}	y_{C_7}	$I_{0.25}$	y_1	y_2	y_{C_1}	y_{C_2}	y_{C_3}	y_{C_4}	y_{C_5}	y_{C_6}	y_{C_7}
x_1	0	0.5	1	0.75	0.5	0	0	0	0	x_1	0	0.5	1	1	1	0.75	1	0.5	0.75
x_2	1	0	0.75	0.75	0.75	0.75	0.5	0.5	0.5	x_2	1	0	0.75	0.75	0.75	0.75	0.5	0.5	0.5
x_3	0.75	1	1	1	1	0.75	0.5	0.75	0.5	x_3	0.75	1	1	1	1	1	0.75	0.75	0.75

TABLE III
FUZZY CONCEPTS OF $\mathcal{B}(X, Y_{0.75}, I_{0.75})$

$C^1_{0.75}$	$(\{x_1, x_2, x_3\}, \{0.75/y_{C_1}, 0.75/y_{C_2}, 0.5/y_{C_3}\})$
$C^2_{0.75}$	$(\{x_1, 0.75/x_2, x_3\}, \{0.25/y_2, y_{C_1}, 0.75/y_{C_2}, 0.5/y_{C_3}\})$
$C^3_{0.75}$	$(\{x_1, 0.5/x_2, x_3\}, \{0.5/y_2, y_{C_1}, 0.75/y_{C_2}, 0.5/y_{C_3}\})$
$C^4_{0.75}$	$(\{0.75/x_1, x_2, x_3\}, \{0.25/y_1, 0.75/y_{C_1}, 0.75/y_{C_2}, 0.75/y_{C_3}, 0.25/y_{C_4}, 0.25/y_{C_5}, 0.25/y_{C_6}, 0.25/y_{C_7}\})$
$C^5_{0.75}$	$(\{0.75/x_1, 0.75/x_2, x_3\}, \{0.25/y_1, 0.25/y_2, y_{C_1}, y_{C_2}, 0.75/y_{C_3}, 0.25/y_{C_4}, 0.25/y_{C_5}, 0.25/y_{C_6}, 0.25/y_{C_7}\})$
$C^6_{0.75}$	$(\{0.75/x_1, 0.5/x_2, x_3\}, \{0.25/y_1, 0.5/y_2, y_{C_1}, y_{C_2}, 0.75/y_{C_3}, 0.25/y_{C_4}, 0.25/y_{C_5}, 0.25/y_{C_6}, 0.25/y_{C_7}\})$
$C^7_{0.75}$	$(\{0.5/x_1, x_2, x_3\}, \{0.5/y_1, 0.75/y_{C_1}, 0.75/y_{C_2}, 0.75/y_{C_3}, 0.5/y_{C_4}, 0.5/y_{C_5}, 0.5/y_{C_6}, 0.5/y_{C_7}\})$
$C^8_{0.75}$	$(\{x_2, 0.75/x_3\}, \{y_1, 0.75/y_{C_1}, 0.75/y_{C_2}, 0.75/y_{C_3}, 0.75/y_{C_4}, 0.5/y_{C_5}, 0.5/y_{C_6}, 0.5/y_{C_7}\})$
$C^9_{0.75}$	$(\{0.5/x_1, 0.75/x_2, x_3\}, \{0.5/y_1, 0.25/y_2, y_{C_1}, y_{C_2}, y_{C_3}, 0.5/y_{C_4}, 0.5/y_{C_5}, 0.5/y_{C_6}, 0.5/y_{C_7}\})$
$C^{10}_{0.75}$	$(\{0.5/x_1, 0.5/x_2, x_3\}, \{0.5/y_1, 0.5/y_2, y_{C_1}, y_{C_2}, y_{C_3}, 0.5/y_{C_4}, 0.5/y_{C_5}, 0.5/y_{C_6}, 0.5/y_{C_7}\})$
$C^{11}_{0.75}$	$(\{0.5/x_1, x_3\}, \{0.5/y_1, y_2, y_{C_1}, y_{C_2}, y_{C_3}, 0.5/y_{C_4}, 0.5/y_{C_5}, 0.5/y_{C_6}, 0.5/y_{C_7}\})$
$C^{12}_{0.75}$	$(\{0.75/x_2, 0.75/x_3\}, \{y_1, 0.25/y_2, y_{C_1}, y_{C_2}, y_{C_3}, y_{C_4}, 0.75/y_{C_5}, 0.75/y_{C_6}, 0.75/y_{C_7}\})$
$C^{13}_{0.75}$	$(\{0.5/x_2, 0.75/x_3\}, \{y_1, 0.5/y_2, y_{C_1}, y_{C_2}, y_{C_3}, y_{C_4}, 0.75/y_{C_5}, y_{C_6}, 0.75/y_{C_7}\})$
$C^{14}_{0.75}$	$(\{0.5/x_2, 0.5/x_3\}, \{y_1, 0.5/y_2, y_{C_1}, y_{C_2}, y_{C_3}, y_{C_4}, y_{C_5}, y_{C_6}, y_{C_7}\})$
$C^{15}_{0.75}$	$(\{0.75/x_3\}, \{y_1, y_2, y_{C_1}, y_{C_2}, y_{C_3}, y_{C_4}, 0.75/y_{C_5}, y_{C_6}, 0.75/y_{C_7}\})$
$C^{16}_{0.75}$	$(\{0.5/x_3\}, \{y_1, y_2, y_{C_1}, y_{C_2}, y_{C_3}, y_{C_4}, y_{C_5}, y_{C_6}, y_{C_7}\})$

TABLE IV
FUZZY CONCEPTS OF $\mathcal{B}(X, Y_{0.25}, I_{0.25})$

$C^1_{0.25}$	$(\{x_1, x_2, x_3\}, \{0.75/y_{C_1}, 0.75/y_{C_2}, 0.75/y_{C_3}, 0.75/y_{C_4}, 0.5/y_{C_5}, 0.5/y_{C_6}, 0.5/y_{C_7}\})$
$C^2_{0.25}$	$(\{0.75/x_1, x_2, x_3\}, \{0.25/y_1, 0.75/y_{C_1}, 0.75/y_{C_2}, 0.75/y_{C_3}, 0.75/y_{C_4}, 0.5/y_{C_5}, 0.5/y_{C_6}, 0.5/y_{C_7}\})$
$C^3_{0.25}$	$(\{0.5/x_1, x_2, x_3\}, \{0.5/y_1, 0.75/y_{C_1}, 0.75/y_{C_2}, 0.75/y_{C_3}, 0.75/y_{C_4}, 0.5/y_{C_5}, 0.5/y_{C_6}, 0.5/y_{C_7}\})$
$C^4_{0.25}$	$(\{x_2, 0.75/x_3\}, \{y_1, 0.75/y_{C_1}, 0.75/y_{C_2}, 0.75/y_{C_3}, 0.75/y_{C_4}, 0.5/y_{C_5}, 0.5/y_{C_6}, 0.5/y_{C_7}\})$
$C^5_{0.25}$	$(\{x_1, 0.75/x_2, x_3\}, \{0.25/y_2, y_{C_1}, y_{C_2}, y_{C_3}, 0.75/y_{C_4}, 0.75/y_{C_5}, 0.5/y_{C_6}, 0.75/y_{C_7}\})$
$C^6_{0.25}$	$(\{x_1, 0.5/x_2, x_3\}, \{0.5/y_2, y_{C_1}, y_{C_2}, y_{C_3}, 0.75/y_{C_4}, 0.75/y_{C_5}, 0.5/y_{C_6}, 0.75/y_{C_7}\})$
$C^7_{0.25}$	$(\{x_1, 0.5/x_2, 0.75/x_3\}, \{0.5/y_2, y_{C_1}, y_{C_2}, y_{C_3}, 0.75/y_{C_4}, y_{C_5}, 0.5/y_{C_6} 0.75/y_{C_7}\})$
$C^8_{0.25}$	$(\{0.75/x_1, 0.75/x_2, x_3\}, \{0.25/y_1, 0.25/y_2, y_{C_1}, y_{C_2}, y_{C_3}, y_{C_4}, 0.75/y_{C_5}, 0.75/y_{C_6}, 0.75/y_{C_7}\})$
$C^9_{0.25}$	$(\{0.5/x_1, 0.75/x_2, x_3\}, \{0.5/y_1, 0.25/y_2, y_{C_1}, y_{C_2}, y_{C_3}, y_{C_4}, 0.75/y_{C_5}, 0.75/y_{C_6}, 0.75/y_{C_7}\})$
$C^{10}_{0.25}$	$(\{0.75/x_1, 0.5/x_2, x_3\}, \{0.25/y_1, 0.5/y_2, y_{C_1}, y_{C_2}, y_{C_3}, y_{C_4}, 0.75/y_{C_5}, 0.75/y_{C_6}, 0.75/y_{C_7}\})$
$C^{11}_{0.25}$	$(\{0.5/x_1, 0.5/x_2, x_3\}, \{0.5/y_1, 0.5/y_2, y_{C_1}, y_{C_2}, y_{C_3}, y_{C_4}, 0.75/y_{C_5}, 0.75/y_{C_6}, 0.75/y_{C_7}\})$
$C^{12}_{0.25}$	$(\{0.75/x_2, 0.75/x_3\}, \{y_1, 0.25/y_2, y_{C_1}, y_{C_2}, y_{C_3}, y_{C_4}, 0.75/y_{C_5}, 0.75/y_{C_6}, 0.75/y_{C_7}\})$
$C^{13}_{0.25}$	$(\{0.75/x_1, 0.5/x_2, 0.75/x_3\}, \{0.25/y_1, 0.5/y_2, y_{C_1}, y_{C_2}, y_{C_3}, y_{C_4}, y_{C_5}, 0.75/y_{C_6}, y_{C_7}\})$
$C^{14}_{0.25}$	$(\{0.5/x_1, x_3\}, \{0.5/y_1, y_2, y_{C_1}, y_{C_2}, y_{C_3}, y_{C_4}, 0.75/y_{C_5}, 0.75/y_{C_6}, 0.75/y_{C_7}\})$
$C^{15}_{0.25}$	$(\{0.5/x_1, 0.5/x_2, 0.75/x_3\}, \{0.5/y_1, 0.5/y_2, y_{C_1}, y_{C_2}, y_{C_3}, y_{C_4}, y_{C_5}, y_{C_6}, y_{C_7}\})$
$C^{16}_{0.25}$	$(\{0.5/x_2, 0.75/x_3\}, \{y_1, 0.5/y_2, y_{C_1}, y_{C_2}, y_{C_3}, y_{C_4}, y_{C_5}, y_{C_6}, y_{C_7}\})$
$C^{17}_{0.25}$	$(\{0.5/x_1, 0.75/x_3\}, \{0.5/y_1, y_2, y_{C_1}, y_{C_2}, y_{C_3}, y_{C_4}, y_{C_5}, y_{C_6}, y_{C_7}\})$
$C^{18}_{0.25}$	$(\{0.75/x_3\}, \{y_1, y_2, y_{C_1}, y_{C_2}, y_{C_3}, y_{C_4}, y_{C_5}, y_{C_6}, y_{C_7}\})$

relations: let $\mathcal{S} \in \tilde{\mathbf{S}}_n$ and $\mathcal{Q}_n \leq \mathcal{S}$, and if $\mathcal{S}(A, B) = 1$, then $\mathcal{Q}_n(A, B) = 1$. Hence, also quantifiers of $\tilde{\mathbf{S}}_n$ can be considered generalizations of $\mathcal{Q}_n$.

Therefore, the previous considerations suggest us to partition the set of all fuzzy scaling quantifiers as follows: $\tilde{\mathbf{S}} = \bigcup_{n \in [0,100]} \tilde{\mathbf{S}}_n$.

Furthermore, since both $\mathcal{S}_t$ and quantifiers of $\tilde{\mathbf{S}}_{t*100}$ are generalizations of the Boolean scaling quantifier $\mathcal{Q}_{t*100}$, we can identify $\mathcal{S}_t$ with $\tilde{\mathbf{S}}_{t*100}$.

4) Theorem 3.6 can be rewritten for fuzzy scaling quantifiers as follows.

Theorem V.1: Let $\mathcal{S}_\nu \in \tilde{\mathbf{S}}$ and let $A, B \underset{\sim}{\subset} X$, we put

$$\mathcal{H}_\nu(A) = \{Z \underset{\sim}{\subset} X \mid Bi_\nu(\mu_A(A|Z)) \neq 0\}.$$

Then

$$\mathcal{S}_\nu(A, B) = \bigvee_{Z \in \mathcal{H}_\nu(A)} \mathcal{S}_1(A|Z, B) \wedge Bi_\nu(\mu_A(A|Z)). \quad (17)$$

Proof: The proof is analogous to that of Theorem 3.6. Indeed, we can rewrite (12) by substituting $\mathcal{H}_t(A)$ with $\mathcal{H}_\nu(A)$, and Δ_t with Bi_ν. Then, since $\bigvee_{Z \notin \mathcal{H}_\nu(A)} (\mathcal{S}_1(A|Z, B) \wedge Bi_\nu (\mu_A(A|Z))) = \bigvee_{Z \notin \mathcal{H}_\nu(A)} (\mathcal{S}_1(A|Z, B) \wedge 0) = 0 \vee \cdots \vee 0 = 0$, we have $\mathcal{S}_\nu(A, B) = (\bigvee_{Z \in \mathcal{H}_\nu(A)} (\mathcal{S}_1(A|Z, B) \wedge Bi_\nu (\mu_A(A|Z))) \vee 0 = \bigvee_{Z \in \mathcal{H}_\nu(A)} (\mathcal{S}_1(A|Z, B) \wedge Bi_\nu(\mu_A(A| Z)))$. ■

5) Theorem 3.10 can be rewritten for fuzzy scaling quantifiers as follows.

Theorem V.2: Let $\mathcal{S}_\nu \in \tilde{\mathbf{S}}$ and let $A, B \underset{\sim}{\subset} X$, we put

$$\mathcal{H}^*_\nu(A,B)=\{Z \in \mathcal{H}_\nu(A) | \exists k \in \mathsf{K}_{(A,B)} \text{ with } A|Z = A_k\}.$$

Then

$$\mathcal{S}_\nu(A,B) = \bigvee_{Z\in\mathcal{H}^*_\nu(A,B)} (\mathcal{S}_1(A|Z,B) \wedge Bi_\nu(\mu_A(A|Z))).$$

Proof: The proof can be obtained from that of Theorem 3.10 by using the properties of complete residuated lattices. First, we need to substitute everywhere $\mathcal{H}_t(A)$ and $\mathcal{H}^*_t(A,B)$ with $\mathcal{H}_\nu(A)$ and $\mathcal{H}^*_\nu(A,B)$, respectively. Second, in order to prove the inequality $\bigvee_{Z\in\mathcal{H}_\nu(A)} \mathcal{S}_1(A|Z,B) \otimes Bi_\nu(\mu_A(A|Z)) \leq \bigvee_{Z\in\mathcal{H}^*_\nu(A,B)} \mathcal{S}_1(A|Z,B) \otimes Bi_\nu(\mu_A(A|Z))$, the following further sentences must be added after (15). Since $(A|Z)(x) \leq (A|\tilde{Z})(x)$ for each $x \in X$ and μ_A is an increasing function, we get $\mu_A(A|Z) \leq \mu_A(A|\tilde{Z})$. Moreover, it is true that $Bi_\nu(\mu_A(A|Z)) \leq Bi_\nu(\mu_A(A|\tilde{Z}))$ because Bi_ν is an increasing function too. Thus, the inequalities $Bi_\nu(\mu_A(A|Z)) \leq Bi_\nu(\mu_A(A|\tilde{Z}))$ and $\mathcal{S}_1(A|Z,B) \leq \mathcal{S}_1(A|\tilde{Z},B)$ imply that $\mathcal{S}_1(A|Z,B) \otimes Bi_\nu(\mu_A(A|Z)) \leq \mathcal{S}_1(A|\tilde{Z},B) \otimes Bi_\nu(\mu_A(A|\tilde{Z}))$. Finally, considering that $A|\tilde{Z} \in \mathcal{H}^*_\nu(A,B)$, we can conclude that $\bigvee_{Z\in\mathcal{H}_\nu(A)} \mathcal{S}_1(A|Z,B) \otimes Bi_\nu(\mu_A(A|Z)) \leq \bigvee_{Z\in\mathcal{H}^*_\nu(A,B)} \mathcal{S}_1(A|Z,B) \otimes Bi_\nu(\mu_A(A|Z))$. ■

6) Algorithm 1 can be modified to work with fuzzy scaling quantifiers. Indeed, the procedure P_1 must have the function Bi_ν instead of the threshold t as input. Moreover, concerning the *if/then statement*, we need to substitute the condition $\mu_A(A_k) \geq t$ with $Bi_\nu(\mu_A(A_k)) \neq 0$, add $l \to Bi_\nu(\mu_A(A_k))$ as statement to execute, and write $S \to S \cup \{(n \times m) \otimes l\}$ instead of $S \to S \cup \{n \times m\}$.
7) Algorithm 2 can be used for fuzzy scaling quantifiers only by changing the input t of P_2 with Bi_ν and the procedure P_1 as explained in the previous point.

B. Extending Results of Section IV to Fuzzy Scaling Quantifiers

The relation given by Definition 4.1 can be extended to the class of fuzzy scaling quantifiers: let $\mathcal{S}, \mathcal{S}' \in \tilde{\mathbf{S}}$:

$$\mathcal{S} \preceq_{\tilde{\mathbf{S}}} \mathcal{S}' \text{ iff } \mathcal{S}(A,B) \leq \mathcal{S}'(A,B) \text{ for each } A, B \underset{\sim}{\subset} X.$$

The results proved in Section IV can be extended for fuzzy scaling quantifiers by take into account $\preceq_{\tilde{\mathbf{S}}}$ and a specific pair of evaluative linguistic expressions: let $\mathcal{S}_{\nu_1}, \mathcal{S}_{\nu_2} \in \tilde{\mathbf{S}}$ such that $Bi_{\nu_1} \subseteq Bi_{\nu_2}$; then, we have the following.

1) $\mathcal{S}_{\nu_1} \preceq_{\tilde{\mathbf{S}}} \mathcal{S}_{\nu_2}$ (see Theorem 4.2).
2) $I_{\nu_1} \subseteq I_{\nu_2}$, where I_{ν_1} and I_{ν_2} are, respectively, related to $\mathcal{S}_{\nu_1}$ and $\mathcal{S}_{\nu_2}$ by means of (7) (see Remark 4.3).
3) $\{k/y_C\}^{I_{\nu_1}} \subseteq \{k/y_C\}^{I_{\nu_2}}$ for each $k \in [0,1]$ (see Proposition 4.4).
4) For each $(A,B) \in \mathcal{B}(X,Y^*,I_{\nu_1})$, there exists $(A^*,B^*) \in \mathcal{B}(X,Y^*,I_{\nu_2})$ such that $(A,B) \preceq (A^*,B^*)$ (see Theorem 4.5).

Remark V.3: By Theorem 4.2, we can easily consider a total order $\preceq_{\mathbf{S}}$ on $\mathbf{S}$, namely, $(\mathbf{S}, \preceq_{\mathbf{S}})$ is a chain. Then, we can compare the concepts deriving from any pairs of t-scaling quantifiers by using Theorem 4.5. Unfortunately, the same is not possible for the class fuzzy scaling quantifiers by considering that $\preceq_{\tilde{\mathbf{S}}}$ is not a total order on $\tilde{\mathbf{S}}$. Indeed, let $\mathcal{S}_{\nu_1}, \mathcal{S}_{\nu_2} \in \tilde{\mathbf{S}}$; it can happen that $\mathcal{S}_{\nu_1} \not\preceq_{\tilde{\mathbf{S}}} \mathcal{S}_{\nu_2}$ and $\mathcal{S}_{\nu_2} \not\preceq_{\tilde{\mathbf{S}}} \mathcal{S}_{\nu_1}$. Consequently, we cannot always compare concepts deriving from two different fuzzy scaling quantifiers.

VI. Conclusion

In this article, we focused on deriving information (i.e., collections of fuzzy concept lattices) from particular datasets (i.e., fuzzy relational context families) by employing t-scaling and fuzzy scaling quantifiers. As a future project, we intend to introduce and study new quantifiers in fuzzy relation concept analysis. For example, quantifiers extracting negative information from data, i.e., information based on the absence of a certain amount of properties in objects. We would also like to consider and study t-scaling quantifiers as generalized fuzzy subsethood measures by extending the definitions given in [52].

In addition, we will organize special FRCA quantifiers in structures of opposition, similarly to those constructed in [33], [34], and [36]. Moreover, by understanding relationships between FRCA quantifiers of different types, we could discover connections between their derived fuzzy concept lattices.

Finally, we plan to implement the algorithms presented in this article using real datasets and apply our theoretical results to solve concrete problems in other research domains. After that, it would be very interesting to compare, given $t \in \underset{\sim}{[}0,1]$, the concept lattices obtained by using the quantifiers of $\tilde{\mathbf{S}}_{t*100}$ and the t-scaling quantifiers $\mathcal{S}_t$.

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