Global Stabilization for Stochastic Continuous Cascade Nonlinear Systems Subject to SISS Inverse Dynamics and Time-Delay: A Dynamic Gain Approach

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***Abstract*—This article is devoted to the global continuous control for stochastic low-order cascade nonlinear systems with time-varying delay and stochastic inverse dynamics. Compared with existing results, the nature of only continuous, but nonsmooth, is unfolded since the power of the stochastic cascade system is of low order; and all the traditional growth conditions on unknown drift and diffusion nonlinearities and local Lipschitz condition are quitted, which largely extends the scope of application. Combining with stochastic input-to-state stability, two new lemmas are developed with rigorous proofs to deal with uncertain nonlinear terms and unmeasurable stochastic inverse dynamics. A continuous control scheme consisting of a delay-independent partial state feedback controller and a serial of dynamic update laws is proposed to guarantee the globally asymptotical stability of the closed-loop system.**

***Index Terms*—Continuous system, dynamic gain, stochastic low-order cascade nonlinear system, stochastic inverse dynamics, time-varying delay.**

I. Introduction

COMPARED with linear systems, nonlinear systems can better characterize the nature of real systems. It is widely recognized that almost all systems met in practical applications can be described by nonlinear physical models, such as aerospace, intelligent robotics, chemical engineering, and so on [1], [2]. Further, stochastic noises and time-delay extensively occur in the actual engineering, so the research on stochastic time-delay nonlinear systems has been extensively investigated during the past few decades [3]–[14]. In particular, stochastic inverse dynamics exist largely in all kinds of practical plants. Accordingly, the nonlinear control for stochastic cascade systems has stirred plenty of research interests by characterizing unmeasured stochastic inverse dynamics with various conditions [18]–[20].

On the other hand, for the stochastic low-order nonlinear system, which means the system with power less than one, there are few results so far because it is only continuous but not smooth and all the control methods requiring any smoothness will fail. Fortunately, Tzamtzi and Tsinias [15] provided the explicit formulas of feedback stabilizers for a class of uncontrollable linearization triangular systems and Čelikovský and Aranda-Bricaire [16] studied the constructive nonsmooth stabilization of triangular systems. Moreover, the dynamic uncertainties were considered in [17]. The authors in [21]–[23] proposed the non-Lipschitz continuous approach to nonlinear systems. Further, based on a dynamic gain approach, Zhang *et al.* [24] studied the nonsmooth feedback control of time-delay cascade nonlinear systems. Then, for stochastic low-order nonlinear systems in the absence of time-delay, the authors in [25]–[28] established and enriched the finite-time theoretical framework. Liu *et al.* and Shao *et al.* [25], [26] addressed the finite-time control for stochastic low-order nonlinear systems with lower-triangular and upper-triangular forms, respectively. The authors in [27] and [28] solved the finite-time stabilization problem of stochastic low-order nonlinear systems with time-varying powers and stochastic inverse dynamics. Obviously, when time-delay is taken into account, the control problem becomes more difficult for stochastic low-order nonlinear systems, where the smoothness has already disappeared. Therefore, *a natural motivation* of this article is how to generalize the existing stochastic theory to a class of low-order stochastic cascade nonlinear systems with time-delay and stochastic inverse dynamics.

It should be noted that most of the above outcomes are based on the assumption of local Lipschitz and certain growth conditions on nonlinear functions, which greatly limits its scope of application. Actually, it is difficult to maintain these growth conditions in the actual system, especially in the presence of time-delay and stochastic inverse dynamics. What is more, considering the fact that the local Lipschitz condition is hard

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to meet in many stochastic nonlinear systems, hence *another intuitive motivation* of this article is how to relax or even remove such strong restrictions for nonlinear functions, and broaden the classical stochastic stability theory to cover the case of only continuous rather than local Lipschitz.

Motivated by the above discussions, when both time-delay and unmeasured stochastic inverse dynamics exist, an interesting and challenging problem arises immediately: *if all the additional growth conditions of nonlinear functions are abandoned, where the local Lipschitz condition is reduced to continuous, is it possible to design a delay-independent controller for stochastic cascade low-order nonlinear systems in a continuous fashion?* An affirmative solution is proposed, and such a continuous criteria will be shown in this article by the feat of stochastic input-to-state stability (SISS) condition, generalized small gain conditions, Lyapunov–Krasoviskii (L–K) functional, and dynamic control gain laws [29]–[34]. Different from the full-state feedback control by use of the information of whole states of the cascade nonlinear systems, the partial-state feedback control adopted in this article is that only using the information of driving subsystems, while the information of driven subsystems is just needed for the theoretical analysis of the whole cascade system.

The main contributions of this article are highlighted threefold as follows.

1) For the first time, this article aims at the global stabilization for stochastic low-order cascade nonlinear systems in the presence of SISS inverse dynamics and time-varying delay. With the introduction of two key lemmas, the traditional growth conditions on unknown drift and diffusion terms are removed, and the strong limitation of locally Lipschitz on nonlinear functions is relaxed to be continuous, which is rather important in controller design analysis and enlarges the practical application range.
2) Due to the nature of only continuous but nonsmoothness of low-order systems, thereby a new idea of continuous control is proposed to overcome the problem that all existing control strategies requiring arbitrary smoothness are no longer applicable to low-order systems, which expands the stochastic cascade time-delay results to a continuous fashion.
3) Quadratic L–K functionals are designed elaborately to replace the quartic L–K functionals used in traditional stochastic control results, which saves complicated calculations caused by recursive controller design. Then, a delay-independent controller and a series of dynamic gain laws are designed simultaneously to guarantee the boundedness of all signals and the globally asymptotically stability of the closed-loop system.

Notations: $\mathbb{R}^n$ is the n-dimensional Euclidean space. $\mathcal{C}^i$ is the the family of all the functions with continuous ith partial derivations. $|\cdot|$ is the the absolute value. Z^T is the the transpose of matrix Z. For a given real matrix $A=(a_{ij})_{n\times m}$, $\|A\|_F=(\Sigma_{i=1}^n\Sigma_{j=1}^m a_{ij}^2)^{\frac{1}{2}}$ denotes Frobenius norm. $\bar{z}_i=[z_1,\ldots,z_i]^T$. $\mathrm{T}r\{Z\}$ is the trace of square Z. $\|Z\|^p=(\sum_{i=1}^n Z_i^2)^{\frac{p}{2}}$ with real number $p>0$ for real vector $Z=[Z_1,\ldots,Z_n]^T$, in particular, when $p=1$, it denotes Euclidean norm, $\|Z\|^1$ is simplified as $\|Z\|$. δ is positive definite: $\delta:\mathbb{R}^n\to\mathbb{R}^+$ is a continuously differentiable function, which satisfies $\delta(\eta)\geq 0$ and $\delta(\eta)=0$ if and only if $\eta=0$. δ is negative definite: $-\delta(\eta)$ is positive definite. The continuous mapping $\beta(z)$ is said to belong to class $\mathcal{K}$ if $\beta:[0,a^*)\to[0,\infty)$ is strictly increasing and $\beta(0)=0$, moreover, $\beta(z)\in\mathcal{K}_\infty$ for all $z\geq 0$ and $\beta(z)\to\infty, z\to\infty$. For $z\in[0,a^*)$, $s\in[0,\infty)$, the continuous mapping $\beta^*(z,s)\in\mathcal{KL}$ if, for each fixed s, $\beta^*(z,s)\in\mathcal{K}$ with respect to z and, for each fixed z, $\beta^*(z,s)$ is decreasing with respect to s and $\beta^*(z,s)\to 0, s\to\infty$. The notation of a function is sometimes predigested, for example, a function $g(x(t))$ is denoted by $g(x)$, $g(\cdot)$, or g.

II. Preliminaries and Problem Formulation

A. Preliminaries

The following stochastic time-delay nonlinear system are considered:

$$dx(t)=f(t,x(t),x(t-\tau(t)))dt+g^T(t,x(t),x(t-\tau(t)))d\omega \tag{1}$$

with initial data $\{x(\varsigma):-\tau\leq\varsigma\leq 0\}=\xi\in\mathcal{C}^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$ and $\forall t\geq 0$, where $\tau(t):\mathbb{R}^+\to[0,\tau]$ is a Borel measurable function; ω is an $\imath$-dimensional standard Wiener process defined on a complete probability space $\{\Omega,\mathcal{F},P\}$, where Ω is a sample space, $\mathcal{F}$ is a σ-field, P is the probability measure; $f:\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ and $g^T:\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{n\times r}$ are continuous functions with $f(t,0,0)\equiv 0$ and $g(t,0,0)\equiv 0$.

Definition 1 [1]: For a $\mathcal{C}^{2,1}$ function $V(x(t),t)$ of system (1), the differential operator $\mathcal{L}$ is defined as $\mathcal{L}V=\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f+\frac{1}{2}\mathrm{T}r\{g\frac{\partial^2 V}{\partial x^2}g^T\}$, where $\frac{1}{2}\mathrm{T}r\{g\frac{\partial^2 V}{\partial x^2}g^T\}$ is known as the Hessian term of $\mathcal{L}$.

Definition 2 [1]: The equilibrium $x(t)=0$ of system (1) is said to be globally stable in probability, if for any $\varepsilon>0$, there holds $\lim_{x_0\to 0}P\{\sup_{t>0}\|x(t)\|>\varepsilon\}=0$. Moreover, the equilibrium $x(t)=0$ of system (1) is said to be globally asymptotically stable in probability, if for any $\varepsilon>0$, there exists a function $\mu(\cdot,\cdot)\in\mathcal{KL}$ such that $P\{\|x(t)\|\leq\mu(\|\xi\|,t)\}\geq 1-\varepsilon$ for any $t\geq 0$, $\psi\in\mathcal{C}^n_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)\setminus\{0\}$, where $\|\xi\|=\sup_{\varsigma\in[-\tau,0]}\|x(\varsigma)\|$.

Lemma 1 [31]: For $m,n\in\mathbb{R}$, $p\geq 1$, one has

$$|m-n|^p\leq 2^{p-1}|m^p-n^p|,\ |m^{\frac{1}{p}}-n^{\frac{1}{p}}|\leq 2^{\frac{p-1}{p}}|m-n|^{\frac{1}{p}}$$
$$(|m|+|n|)^{\frac{1}{p}}\leq|m|^{\frac{1}{p}}+|n|^{\frac{1}{p}}\leq 2^{\frac{p-1}{p}}(|m|+|n|)^{\frac{1}{p}}$$

and for any $p>0$ and $m_1,\ldots,m_k\in\mathbb{R}$, then

$$(|m_1|+\cdots+|m_k|)^p\leq\max\{k^{p-1},1\}(|m_1|^p+\cdots+|m_k|^p).$$

Lemma 2 [31]: For given positive real numbers m, n, and a function $a(x,y)$, there holds for all $x\in\mathbb{R}$, $y\in\mathbb{R}$

$$|a(x,y)x^m y^n|\leq c(x,y)|x|^{m+n}+\frac{n}{m+n}\left(\frac{m}{(m+n)c(x,y)}\right)^{\frac{m}{n}}\cdot|a(x,y)|^{\frac{m+n}{n}}|y|^{m+n}$$

where $c(x,y)>0$.

Lemma 3 [24]: For a function $f(\xi_1,\xi_2)\in\mathcal{C}^0,\xi_1\in\mathbb{R}^m$, $\xi_2\in\mathbb{R}^n$, there exist smooth functions $\zeta_1(\xi_1)\geq 0,\zeta_2(\xi_2)\geq 0,y_1(\xi_1)\geq 1,y_2(\xi_2)\geq 1$ that satisfy $|f(\xi_1,\xi_2)|\leq\zeta_1(\xi_1)+\zeta_2(\xi_2),|f(\xi_1,\xi_2)|\leq y_1(\xi_1)y_2(\xi_2)$.

Lemma 4 [24]: If $h(z_1, z_2)$ is a real-valued continuous function, there exist nonnegative smooth scalar functions $x(z_1), y(z_2)$ satisfying $h(z_1, z_2)(\|z_1\| + \|z_2\|) \leq x(z_1)\|z_1\| + y(z_2)\|z_2\|$.

It is worth noting that the drift term and the diffusion term of system (2) are merely *continuous*. And we know that local Lipschitz condition can ensure the existence of a unique strong solution for a given initial condition. However, due to the inherent difficulty that many stochastic nonlinear systems do not satisfy the local Lipschitz condition, we need to broaden the classical stochastic stability theory to cover the scope of having at least one weak solution. A generalized lemma is presented below by integrating the ideas in [10] and [35]–[37]. It can be proved in a similar way as [35, Th. 2.1] and [37, Th. 2.1]. Hence the details are omitted.

Lemma 5: Suppose that f and g are locally bounded in $[x, x(t-\tau)]$ (uniformly for t), and there exist $V(x(t), t) \in \mathcal{C}^{2,1}$, $W(x) \geq 0$, and $\vartheta_1(\cdot) \in \mathcal{K}_\infty, \vartheta_2(\cdot) \in \mathcal{K}_\infty$ such that $\vartheta_1(\|x(t)\|) \leq V(x(t), t) \leq \vartheta_2(\sup_{-\tau \leq t \leq 0} \|x(t+s)\|), \mathcal{L}V(x(t), t) \leq -W(x)$, then the trivial solution of system (1) is globally stable in probability, and moreover, $P\{\lim_{t\to\infty} W(x(t)) = 0\} = 1$. Particularly, if W is positive definite, then the trivial solution of system (1) is globally asymptotically stable in probability.

B. Problem Formulation

Consider the stochastic cascade time-delay nonlinear system

$$\begin{cases} dz_i = f_{0i}(\bar{z}_i, \bar{z}_i(t-\tau(t)), \bar{\zeta}_i, \bar{\zeta}_i(t-\tau(t)))dt \\ \qquad + g_{0i}^T(\bar{z}_i, \bar{z}_i(t-\tau(t)), \bar{\zeta}_i, \bar{\zeta}_i(t-\tau(t)))d\omega \\ d\zeta_i = \zeta_{i+1}^r dt + f_i(\bar{z}_i, \bar{z}_i(t-\tau(t)), \bar{\zeta}_i, \bar{\zeta}_i(t-\tau(t)))dt \\ \qquad + g_i^T(\bar{z}_i, \bar{z}_i(t-\tau(t)), \bar{\zeta}_i, \bar{\zeta}_i(t-\tau(t)))d\omega \end{cases} \tag{2}$$

where $\zeta = [\zeta_1, \ldots, \zeta_n]^T \in \mathbb{R}^n$ is measurable state, $z_i \in \mathbb{R}^{n_i}$, $i = 1, \ldots, n$, represent unmeasured stochastic inverse dynamics, $u := \zeta_{n+1} \in \mathbb{R}$ is control input. $r \in \mathbb{R}^{\leq 1}_{\text{odd}} \triangleq \{\frac{\rho_1}{\rho_2} \mid \rho_1$ and ρ_2 are positive odd integers satisfying $\rho_1 < \rho_2\}$. The time-varying delay $\tau(t)$ is bounded and $0 < \dot{\tau}(t) \leq \tilde{\tau} < 1$ with a constant $\tilde{\tau}$. $f_{0i} : \mathbb{R}^{2i+2n_1+\cdots+2n_i} \to \mathbb{R}^{n_i}$, $g_{0i} : \mathbb{R}^{2i+2n_1+\cdots+2n_i} \to \mathbb{R}^{m\times n_i}$, $f_i : \mathbb{R}^{2i+2n_1+\cdots+2n_i} \to \mathbb{R}$, $g_i : \mathbb{R}^{2i+2n_1+\cdots+2n_i} \to \mathbb{R}^m$ are the same as (1), so are ω and the initial state, moreover, $f_i(t, 0, 0, 0, 0) = 0$ and $g_i(t, 0, 0, 0, 0) = 0$, $i = 1, \ldots, n$.

The control goal is to design a delay-independent controller to ensure that the trivial solution of the closed-loop system is globally asymptotically stable in probability in a continuous fashion. To this aim, the following assumptions are needed.

Assumption 1: For each z_i-subsystem in (2), there is a scalar function $U_{0i}(z_i) \in \mathcal{C}^2$ which is positive definite and radially unbounded, and $\mathcal{K}_\infty$ functions $\Delta_{0i}, \Delta_{1i}, \Delta_{2i}$, and a smooth function $\Delta_i(X_i) \geq 0, \Delta_i(0) = 0, X_i = [\bar{z}_i, \bar{z}_i(t-\tau(t)), \bar{\zeta}_i, \bar{\zeta}_i(t-\tau(t))]$, such that

$$\Delta_{1i}(\|z_i\|) \leq U_{0i}(z_i) \leq \Delta_{2i}(\|z_i\|)$$

$$\mathcal{L}U_{0i}(z_i) \leq -\Delta_{0i}(\|z_i\|) + \Delta_i(\bar{z}_i, \bar{z}_i(t-\tau(t)), \bar{\zeta}_i, \bar{\zeta}_i(t-\tau(t))). \tag{3}$$

Equation (3) is called SISS condition.

Assumption 2: There exists a nonnegative smooth function ψ_i that satisfies $\|g_{0i}(\bar{z}_i, \bar{z}_i(t-\tau(t)), \bar{\zeta}_i, \bar{\zeta}_i(t-\tau(t)))\|_F \leq \psi_i(\|z_i\|)$.

Remark 1: Assumption 1 indicates that system (2) satisfies the SISS condition, which is common and often encountered in existing results on stochastic inverse dynamics, such as [19], [20], and [29]. Moreover, (3) is also considered in [19], and it is worth noting that the restrictive constraint of $\frac{\partial^l \Delta_i}{\partial X^l(0)} = 0, l = 1, 2, 3$ on functions $\Delta_i(X)$ is relaxed in this article. Assumption 2 means that the diffusion vector field of inverse dynamics is confined by the dynamics itself. Moreover, there exists a nonnegative smooth function ϕ_i to ensure $\frac{\partial U_{0i}(z_i)}{\partial z_i} \leq \phi_i(\|z_i\|)$ since any continuous function can be bounded by a smooth function. The control design in this article is continuous considering that the studied system (2) not only has the power less than one but also, for wider application, its drift and diffusion terms are only continuous rather than smooth, which can be guaranteed in most practical situations. The difficulties in the control design in continuous case are mainly in two parts. 1) It is hard to extend the existing results to continuous fashion as all existing control strategies requiring arbitrary smoothness are no longer applicable. 2) The appearance of $\zeta_{i+1}^r (0 < r < 1)$ leads to a completely nondifferentiable situation. This makes the selection of the Lyapunov function and the design of the controller very difficult since the basic idea of the Lyapunov stability theory is the derivative operation. Therefore, the control of low-order systems is entirely different and more difficult than those in high-order or strict-feedback systems in literature [18], [19] where the traditional backstepping method can be directly used. □

Lemma 6 [33]: If z_i-subsystem satisfies Assumption 1 and

$$\limsup_{s\to 0^+} \frac{\chi_i(s)}{\Delta_{0i}(s)} < \infty, \quad \limsup_{s\to 0^+} \frac{\phi_i^2(s)\psi_i^2(s)}{\Delta_{0i}(s)} < \infty$$
$$\int_0^\infty [\gamma_i(\Delta_1^{-1}(s))]' e^{-\int_0^s [\nu_i(\Delta_1^{-1}(v))]^{-1} dv} ds < +\infty \tag{4}$$

with a smooth function $\chi_i(s)$ and two continuous increasing functions $\gamma_i(s) \geq 0$ and $\nu_i(s) > 0$ for $s \geq 0$ and satisfying

$$\gamma_i(s)\Delta_{0i}(s) \geq 4\chi_i(s), \quad \nu_i(s)\Delta_{0i}(s) \geq 2\phi_i^2(s)\psi_i^2(s) \tag{5}$$

then there exists a continuous differentiable function $\varrho_i(s) > 0$ $\forall s \in [0, \infty)$ which is nondecreasing and satisfies

$$\varrho_i(U_{0i}(z_i))\Delta_{0i}(\|z_i\|)$$
$$\geq 2\varrho_i'(U_{0i}(z_i))\phi_i^2(\|z_i\|)\psi_i^2(\|z_i\|) + 4\chi_i(\|z_i\|) \tag{6}$$

for any $z_i \in \mathbb{R}^{n_i}$.

In view of all the traditional restrictive conditions on unknown drift and diffusion terms being removed, the following two crucial lemmas are introduced to accelerate the controller design and theoretical analysis, which are rigorously proved in the Appendix.

Lemma 7: For each nonlinear function $f_i, g_i, i=1, \ldots, n$ of (2), there exist nonnegative smooth functions $\gamma_{ilj}, \gamma_{ilk}^0$, $\Upsilon_{ilj}, \Upsilon_{ilk}^0, l = 1, 2$, such that

$$|f_i(\bar{z}_i, \bar{z}_i(t-\tau(t)), \bar{\zeta}_i, \bar{\zeta}_i(t-\tau(t)))|$$
$$\leq \sum_{j=1}^{i} (\|z_j\|^r \gamma_{i1j}(z_j) + \|z_j(t-\tau(t))\|^r \gamma_{i2j}(z_j(t-\tau(t)))$$
$$+ |\zeta_j|^r \Upsilon_{i1j}(\zeta_j) + |\zeta_j(t-\tau(t))|^r \Upsilon_{i2j}(\zeta_j(t-\tau(t))))$$

$$\begin{aligned}&\left\|g_i^T(\bar{z}_i,\bar{z}_i(t-\tau(t)),\bar{\zeta}_i,\bar{\zeta}_i(t-\tau(t)))g_j^T(\bar{z}_j,\bar{z}_j(t-\tau(t)),\bar{\zeta}_j,\bar{\zeta}_j(t-\tau(t)))\right\|\\&\leq\sum_{k=1}^{i}\big(\|z_k\|^{1+r}\gamma_{i1k}^0(z_k)+\|z_k(t-\tau(t))\|^{1+r}\gamma_{i2k}^0(z_k(t-\tau(t)))\\&\quad+\zeta_k^{1+r}\Upsilon_{i1k}^0(\zeta_k)+\zeta_k^{1+r}(t-\tau(t))\Upsilon_{i2k}^0(\zeta_k(t-\tau(t)))\big),\ i\leq j.\end{aligned}\tag{7}$$

Lemma 8: Let $Y=[\zeta_1,\ldots,\zeta_n]^T\in\mathbb{R}^n$ and $\Delta:\mathbb{R}^n\to\mathbb{R}$ be a $\mathcal{C}^2$ nonnegative smooth function with $\Delta(0)=0$. Then, there are smooth scalar functions $b_i(\zeta_i)\geq 0, i=1,\ldots,n$, such that $\Delta(Y)\leq\sum_{i=1}^n\zeta_i^{1+r}b_i(\zeta_i)$, where r is the system power defined as in (2).

Remark 2: From Lemma 8, the nonlinear functions $\Delta_i, i=1,\ldots,n$ of Assumption 1 indicate that

$$\begin{aligned}&\Delta_i(\bar{z}_i,\bar{z}_i(t-\tau(t)),\bar{\zeta}_i,\bar{\zeta}_i(t-\tau(t)))\\&\leq\sum_{j=1}^{i-1}\left(\|z_j\|^{1+r}\gamma_{i1j}^*(z_j)+\|z_j(t-\tau(t))\|^{1+r}\gamma_{i2j}^*(z_j(t-\tau(t)))\right)\\&\quad+\sum_{j=1}^{i}\left(\zeta_j^{1+r}\Upsilon_{i1j}^*(\zeta_j)+\zeta_j(t-\tau(t))^{1+r}\Upsilon_{i2j}^*(\zeta_j(t-\tau(t)))\right)\end{aligned}\tag{8}$$

with nonnegative smooth functions $\gamma_{ilj}^*(\cdot),\Upsilon_{ilj}^*(\cdot), l=1,2$. □

III. Main Results

A. Dynamic State Feedback Controller Design

The recursive steps for designing the virtual controllers are presented as follows.

Step 1: For $(z_1,\zeta_1)-$subsystem of (2), let $\eta_1=\zeta_1$ and $U_1(z_1,\zeta_1,\ell_1)=\int_0^{U_{01}(z_1)}\varrho_1(s)ds+\frac{1}{2}(1+\frac{1}{\ell_1})W_1, W_1=\eta_1^2$, where $\varrho:[0,\infty)\to(0,\infty)$ is a smooth nondecreasing function and $\ell_1(t)\geq 1$ is a dynamic gain to be designed in the next step. According to Assumption 2, Definition 1, [33, eq. (34)], (7), and (8), there holds

$$\begin{aligned}\mathcal{L}U_1&=\varrho_1(U_{01}(z_1))\mathcal{L}U_{01}(z_1)\\&\quad+\frac{1}{2}\varrho_1'(U_{01}(z_1))\left\|\left(\frac{\partial U_{01}(z_1)}{\partial z_1}\right)^Tg_0\right\|_F^2\\&\quad+\left(1+\frac{1}{\ell_1}\right)\eta_1(\zeta_2^r+f_1)\\&\quad+\frac{1}{2}\left(1+\frac{1}{\ell_1}\right)\|g_1^Tg_1\|-\frac{\dot{\ell}_1}{2\ell_1^2}\eta_1^2\\&\leq-\varrho_1(U_{01}(z_1))\Delta_{01}(\|z_1\|)+\varrho_1(U_{01}(z_1))(\zeta_1^{1+r}\Upsilon_{111}^*(\zeta_1)\\&\quad+\zeta_1^{1+r}(t-\tau)\Upsilon_{121}^*(\zeta_1(t-\tau)))+\frac{1}{2}\varrho_1'(U_{01}(z_1))\phi_1^2\psi_1^2\\&\quad+\left(1+\frac{1}{\ell_1}\right)\eta_1(\zeta_2^r+f_1)\\&\quad+\frac{1}{2}\left(1+\frac{1}{\ell_1}\right)\|g_1^Tg_1\|-\frac{\dot{\ell}_1}{2\ell_1^2}\eta_1^2.\end{aligned}\tag{9}$$

First of all, by means of changing supply rate in [32], it derives

$$\begin{aligned}&-\frac{1}{2}\varrho_1(U_{01}(z_1))\Delta_{01}(\|z_1\|)+\varrho_1(U_{01}(z_1))(\zeta_1^{1+r}\Upsilon_{111}^*(\zeta_1)\\&\quad+\zeta_1^{1+r}(t-\tau)\Upsilon_{121}^*(\zeta_1(t-\tau)))\\&\leq\eta_1^{1+r}\Upsilon_{111}^{0*}(\zeta_1)+\eta_1^{1+r}(t-\tau)\Upsilon_{121}^{0*}(\zeta_1(t-\tau)))\end{aligned}\tag{10}$$

with nonnegative smooth functions $\Upsilon_{111}^{0*},\Upsilon_{121}^{0*}$. Then, from Lemmas 2 and 7, it shows

$$\begin{aligned}&\left(1+\frac{1}{\ell_1}\right)|\eta_1f_1|\\&\leq 16\eta_1^{1+r}+\|z_1\|^{1+r}\bar{\gamma}_{111}(z_1)\\&\quad+\|z_1((t-\tau))\|^{1+r}\bar{\gamma}_{121}(z_1(t-\tau))\\&\quad+\zeta_1^{1+r}\bar{\Upsilon}_{111}(\zeta_1)+\zeta_1^{1+r}(t-\tau)\bar{\Upsilon}_{111}(\zeta_1(t-\tau)),\\&\frac{1}{2}\left(1+\frac{1}{\ell_1}\right)\|g_1^Tg_1\|\\&\leq\|z_1\|^{1+r}\gamma_{111}^0(z_1)+\|z_1((t-\tau))\|^{1+r}\gamma_{121}^0(z_1(t-\tau))\\&\quad+\zeta_1^{1+r}\Upsilon_{111}^0(\zeta_1)+\zeta_1^{1+r}(t-\tau)\Upsilon_{111}^0(\zeta_1(t-\tau)).\end{aligned}\tag{11}$$

Substituting (10) and (11) into (9), it yields

$$\begin{aligned}\mathcal{L}U_1\leq&-\frac{1}{2}\varrho_1(U_{01}(z_1))\Delta_{01}(\|z_1\|)\\&+\frac{1}{2}\varrho_1'(U_{01}(z_1))\phi_1^2(z_1)\psi_1^2(z_1)\\&+\left(1+\frac{1}{\ell_1}\right)\eta_1(\zeta_2^{*r}+\zeta_2^r-\zeta_2^{*r})+16\eta_1^{1+r}\\&+\|z_1\|^{1+r}(\bar{\gamma}_{111}+\gamma_{111}^0)(z_1)\\&+\|z_1(t-\tau)\|^{1+r}(\bar{\gamma}_{121}+\gamma_{121}^0)(z_1(t-\tau))\\&+\eta_1^{1+r}(\Upsilon_{111}^{0*}+\bar{\Upsilon}_{111}+\Upsilon_{111}^0)\\&+\eta_1^{1+r}(t-\tau)(\Upsilon_{121}^{0*}+\bar{\Upsilon}_{111}+\Upsilon_{121}^0)-\frac{\dot{\ell}_1}{2\ell_1^2}\eta_1^2.\end{aligned}\tag{12}$$

Consider the L–K functional

$$\begin{aligned}U_{1LK}=U_1+\int_{t-\tau}^t&\left(\|z_1(s)\|^{1+r}(\bar{\gamma}_{121}+\gamma_{121}^0)(z_1(s))\right.\\&\left.+\eta_1^{1+r}(s)(\Upsilon_{121}^{0*}+\bar{\Upsilon}_{111}+\Upsilon_{121}^0)(\zeta_1(s))\right)ds.\end{aligned}\tag{13}$$

By means of Lemmas 1 and 2, it follows that

$$\begin{aligned}&\mathcal{L}U_{1LK}\\&\leq-\frac{1}{2}\varrho_1(U_{01}(z_1))\Delta_{01}(\|z_1\|)+\frac{1}{2}\varrho_1'(U_{01}(z_1))\phi_1^2(z_1)\psi_1^2(z_1)\\&+\left(1+\frac{1}{\ell_1}\right)\eta_1\zeta_2^{*r}+\eta_2^{1+r}+\|z_1\|^{1+r}(\bar{\gamma}_{111}+\gamma_{111}^0+\bar{\gamma}_{121}+\gamma_{121}^0)\\&+16\eta_1^{1+r}+\eta_1^{1+r}(\Upsilon_{111}^{0*}+\bar{\Upsilon}_{111}+\Upsilon_{111}^0+\Upsilon_{121}^{0*}+\bar{\Upsilon}_{111}+\Upsilon_{121}^0)\\&-\frac{\dot{\ell}_1}{2\ell_1^2}\eta_1^2.\end{aligned}\tag{14}$$

Take $\eta_2 = \zeta_2 - \zeta_2^*$. With (14) and $\ell_1 \geq 1$, the virtual controller

$$\begin{aligned}\zeta_2^* = &-\eta_1\left(n+21+\Upsilon_{111}^{0*}+\bar{\Upsilon}_{111}+\Upsilon_{111}^{0}+\Upsilon_{121}^{0*}+\bar{\Upsilon}_{111}+\Upsilon_{121}^{0}\right)^{\frac{1}{r}}\\ := &-\eta_1\epsilon_1(\zeta_1)\end{aligned} \tag{15}$$

leads to

$$\begin{aligned}&\mathcal{L}U_{1LK}\\ &\leq -\frac{1}{2}\varrho_1(U_{01}(z_1))\Delta_{01}(\|z_1\|)+\frac{1}{2}\varrho_1'(U_{01}(z_1))\phi_1^2(z_1)\psi_1^2(z_1)\\ &\quad+\|z_1\|^{1+r}(\bar{\gamma}_{111}+\gamma_{111}^0+\bar{\gamma}_{121}+\gamma_{121}^0)\\ &\quad-(n+1)\eta_1^{1+r}-\frac{\dot{\ell}_1}{2\ell_1^2}\eta_1^2+\eta_2^{1+r}.\end{aligned} \tag{16}$$

Step 2: For $(z_1, z_2, \zeta_1, \zeta_2)-$subsystem, consider the L–K functional

$$\begin{aligned}U_2 &= U_{1LK}+\frac{1}{\ell_1}\int_0^{U_{02}(z_2)}\varrho_2(s)ds+\frac{1}{2}\left(\frac{1}{\ell_1}+\frac{1}{\ell_1\ell_2}\right)W_2\\ W_2 &= \eta_2^2\end{aligned} \tag{17}$$

with a smooth nondecreasing function $\varrho_2 : [0,\infty) \to (0,\infty)$ and dynamic gain $\ell_2(t) \geq 1$ will be determined in the next step. Then

$$\begin{aligned}\mathcal{L}U_2 \leq &-\frac{1}{2}\varrho_1(U_{01}(z_1))\Delta_{01}(\|z_1\|)+\frac{1}{2}\varrho_1'(U_{01}(z_1))\phi_1^2(z_1)\psi_1^2(z_1)\\ &-\frac{\dot{\ell}_1}{2\ell_1^2}\eta_1^2-(n+1)\eta_1^{1+r}+\eta_2^{1+r}+\|z_1\|^{1+r}(\bar{\gamma}_{111}+\gamma_{111}^0\\ &+\bar{\gamma}_{121}+\gamma_{121}^0)-\frac{\dot{\ell}_1}{\ell_1^2}\int_0^{U_{02}(z_2)}\varrho_2(s)ds\\ &-\frac{1}{\ell_1}\varrho_2(U_{02}(z_2))\Delta_{02}(\|z_2\|)+\frac{1}{\ell_1}\varrho_2(U_{02}(z_2))\\ &\times\big(\|z_1\|^{1+r}\ \gamma_{211}^*(z_1)+\|z_1(t-\tau)\|^{1+r}\gamma_{221}^*(z_1(t-\tau))\\ &+\zeta_1^{1+r}\Upsilon_{211}^*(\zeta_1)+\zeta_1^{1+r}(t-\tau)\Upsilon_{221}^*(\zeta_1(t-\tau))\\ &+\zeta_2^{1+r}\Upsilon_{212}^*(\zeta_2)+\zeta_2^{1+r}(t-\tau)\Upsilon_{222}^*(\zeta_2(t-\tau))\big)\\ &+\frac{1}{2\ell_1}\varrho_2'(U_{02}(z_2))\phi_2^2(z_2)\psi_2^2(z_2)+\frac{1}{\ell_1}\left(1+\frac{1}{\ell_2}\right)\eta_2\zeta_3^r\\ &+\frac{1}{\ell_1}\left(1+\frac{1}{\ell_2}\right)\eta_2\left(f_2-\frac{\partial W_2}{\partial\zeta_1}(\zeta_2^r+f_1)\right)\\ &+\frac{1}{\ell_1}\left(1+\frac{1}{\ell_2}\right)\Big|\frac{\partial^2 W_2}{\partial\zeta_1^2}g_1^Tg_1\\ &+\frac{\partial^2 W_2}{\partial\zeta_2^2}g_2^Tg_2+2\frac{\partial^2 W_2}{\partial\zeta_1\partial\zeta_2}g_1^Tg_2\Big|\\ &-\frac{\dot{\ell}_1}{2\ell_1^2}\eta_2^2-\frac{\dot{\ell}_1\ell_2+\ell_1\dot{\ell}_2}{2\ell_1^2\ell_2^2}\eta_2^2.\end{aligned} \tag{18}$$

Similarly, by the method of changing supply rate, it shows

$$\begin{aligned}&-\frac{1}{2\ell_1}\varrho_2(U_{02}(z_2))\Delta_{02}(\|z_2\|)\\ &+\frac{1}{\ell_1}\varrho_2(U_{02}(z_2))\big(\|z_1\|^{1+r}\gamma_{211}^*(z_1)\\ &\quad+\|z_1(t-\tau)\|^{1+r}\gamma_{221}^*(z_1(t-\tau))\\ &\quad+\zeta_1^{1+r}\Upsilon_{211}^*(\zeta_1)+\zeta_1^{1+r}(t-\tau)\Upsilon_{221}^*(\zeta_1(t-\tau))\\ &\quad+\zeta_2^{1+r}\Upsilon_{212}^*(\zeta_2)+\zeta_2^{1+r}(t-\tau)\Upsilon_{222}^*(\zeta_2(t-\tau))\big)\\ &\leq\|z_1\|^{1+r}\gamma_{211}^{0*}(z_1)+\|z_1(t-\tau)\|^{1+r}\gamma_{221}^{0*}(z_1(t-\tau))\\ &\quad+\frac{1}{\ell_1}\eta_1^{1+r}\Upsilon_{211}^{0*}(\zeta_1)+\frac{1}{\ell_1}\eta_1^{1+r}(t-\tau)\Upsilon_{221}^{0*}(\zeta_1(t-\tau))\\ &\quad+\frac{1}{\ell_1}\eta_2^{1+r}\Upsilon_{212}^{0*}(\zeta_2)+\frac{1}{\ell_1}\eta_2^{1+r}(t-\tau)\Upsilon_{222}^{0*}(\zeta_2(t-\tau))\end{aligned} \tag{19}$$

with nonnegative smooth functions $\gamma_{2k1}^{0*}, \Upsilon_{2k1}^{0*}, \Upsilon_{2k2}^{0*}, k = 1, 2$. From Lemmas 1 and 2 and the definition of η_2

$$\begin{aligned}&\eta_2^{1+r}\Upsilon_{212}^{0*}(\zeta_2)\leq\eta_1^{1+r}\Gamma_{211}(\zeta_1)+\eta_2^{1+r}\Gamma_{212}(\zeta_1,\zeta_2)\\ &\eta_2^{1+r}(t-\tau)\Upsilon_{222}^{0*}(\zeta_2(t-\tau))\leq\eta_1^{1+r}(t-\tau)\Gamma_{221}(\zeta_1(t-\tau))\\ &\qquad+\eta_2^{1+r}(t-\tau)\Gamma_{222}(\zeta_1(t-\tau),\zeta_2(t-\tau))\end{aligned} \tag{20}$$

with nonnegative smooth functions $\Gamma_{211}, \Gamma_{212}, \Gamma_{221}$, and Γ_{222}. According to Lemmas 1, 2, and 7, there holds

$$\begin{aligned}&\eta_2\left(f_2-\frac{\partial W_2}{\partial\zeta_1}(\zeta_2^r+f_1)\right)\\ &\leq\|z_1\|^{1+r}\delta_{211}(z_1)+\|z_1(t-\tau)\|^{1+r}\delta_{221}(z_1(t-\tau))\\ &\quad+\|z_2\|^{1+r}\delta_{212}(z_2)+\|z_2(t-\tau)\|^{1+r}\delta_{222}(z_2(t-\tau))\\ &\quad+\eta_2^{1+r}\bar{\delta}_{212}(\zeta_1,\zeta_2)+\frac{1}{2}\eta_1^{1+r}+\eta_1^{1+r}(t-\tau)\bar{\delta}_{221}(\zeta_1(t-\tau))\\ &\quad+\eta_2^{1+r}(t-\tau)\bar{\delta}_{222}(\zeta_1(t-\tau),\zeta_2(t-\tau))\end{aligned} \tag{21}$$

with nonnegative smooth functions $\delta_{2\iota1}, \delta_{2\iota2}, \bar{\delta}_{2\iota2}, \bar{\delta}_{221}, \iota = 1, 2$. From Lemmas 1 and 2 and (8)

$$\begin{aligned}&\Big|\frac{\partial^2 W_2}{\partial\zeta_1^2}g_1^Tg_1+\frac{\partial^2 W_2}{\partial\zeta_2^2}g_2^Tg_2+2\frac{\partial^2 W_2}{\partial\zeta_1\partial\zeta_2}g_1^Tg_2\Big|\\ &\leq\|z_1\|^{1+r}\lambda_{211}(z_1)+\|z_1(t-\tau)\|^{1+r}\lambda_{221}(z_1(t-\tau))\\ &\quad+\|z_2\|^{1+r}\lambda_{212}(z_2)+\|z_2(t-\tau)\|^{1+r}\lambda_{222}(z_2(t-\tau))\\ &\quad+\eta_2^{1+r}\bar{\lambda}_{212}(\zeta_1,\zeta_2)+\frac{1}{2}\eta_1^{1+r}+\eta_1^{1+r}(t-\tau)\bar{\lambda}_{221}(\zeta_1(t-\tau))\\ &\quad+\eta_2^{1+r}(t-\tau)\bar{\lambda}_{222}(\zeta_1(t-\tau),\zeta_2(t-\tau))\end{aligned} \tag{22}$$

with nonnegative smooth functions $\lambda_{2\iota1}, \lambda_{2\iota2}, \bar{\lambda}_{2\iota2}, \bar{\lambda}_{221}, \iota = 1, 2$.

Construct the L–K functional

$$\begin{aligned}U_{2LK} = U_2+\int_{t-\tau}^{t}&\big(\|z_1(s)\|^{1+r}(\gamma_{221}^{0*}+\delta_{221}+\lambda_{221})(z_1(s))\\ &+\frac{1}{\ell_1(s)}\|z_2(s)\|^{1+r}(\delta_{222}+\lambda_{222})(z_2(s))\\ &+\frac{1}{\ell_1(s)}\eta_1^{1+r}(s)(\Upsilon_{221}^{0*}+\Upsilon_{221}+\bar{\delta}_{221}+\bar{\lambda}_{221})(\zeta_1(s))\\ &+\eta_2^{1+r}(s)(\Upsilon_{222}+\bar{\delta}_{222}+\bar{\lambda}_{222})(\zeta_1(s),\zeta_2(s))\big)\,ds.\end{aligned} \tag{23}$$

Then, it follows from (18)–(23) that

$$\begin{aligned}
&\mathcal{L}U_{2LK}\\
&\le -\frac{1}{2}\varrho_1(U_{01}(z_1))\Delta_{01}(\|z_1\|)+\|z_1\|^{1+r}(\bar{\gamma}_{111}+\gamma^0_{111}+\bar{\gamma}_{121}\\
&\quad+\gamma^0_{121}+\gamma^{0*}_{211}+\delta_{211}+\lambda_{211}\\
&\quad+\gamma^{0*}_{221}+\delta_{221}+\lambda_{221})(z_1)+\frac{1}{2}\varrho_1'(U_{01}(z_1))\phi_1^2(z_1)\psi_1^2(z_1)\\
&\quad-\frac{1}{2\ell_1}\varrho_2(U_{02}(z_2))\Delta_{02}(\|z_2\|)\\
&\quad+\frac{1}{\ell_1}\|z_2\|^{1+r}\left(\delta_{212}+\lambda_{212}+\delta_{222}+\lambda_{222}\right)(z_2)\\
&\quad+\frac{1}{2\ell_1}\varrho_2'(U_{02}(z_2))\phi_2^2(z_2)\psi_2^2(z_2)-n\eta_1^{1+r}-\frac{\dot{\ell}_1}{2\ell_1^2}\eta_1^2\\
&\quad+\left(\frac{1}{\ell_1}-\frac{1}{\ell_1(t-\tau)}\right)\|z_2(t-\tau)\|^{1+r}\left(\delta_{222}+\lambda_{222}\right)(z_2(t-\tau))\\
&\quad+\frac{1}{\ell_1}\eta_1^{1+r}\left(\Upsilon^{0*}_{211}+\Gamma_{211}+\Upsilon^{0*}_{221}+\Upsilon_{221}+\bar{\delta}_{221}+\bar{\lambda}_{221}\right)(\zeta_1)\\
&\quad+\left(\frac{1}{\ell_1}-\frac{1}{\ell_1(t-\tau)}\right)\eta_1^{1+r}(t-\tau)\left(\Upsilon^{0*}_{221}+\Gamma_{221}+\bar{\delta}_{221}+\bar{\lambda}_{221}\right)(\zeta_1)\\
&\quad+\frac{1}{\ell_1}\left(1+\frac{1}{\ell_2}\right)\eta_2\zeta_3^{*r}+2\eta_2^{1+r}\\
&\quad+\eta_2^{1+r}\left(\Gamma_{212}+\bar{\delta}_{212}+\bar{\lambda}_{212}+\Gamma_{222}+\bar{\delta}_{222}+\bar{\lambda}_{222}\right)(\zeta_1,\zeta_2)\\
&\quad+\eta_3^{1+r}-\frac{\dot{\ell}_1}{\ell_1^2}\int_0^{U_{02}(z_2)}\varrho_2(s)ds-\frac{\dot{\ell}_1}{2\ell_1^2}\eta_2^2-\frac{\dot{\ell}_1\ell_2+\ell_1\dot{\ell}_2}{2\ell_1^2\ell_2^2}\eta_2^2
\end{aligned}\tag{24}$$

with $\eta_3=\zeta_3-\zeta_3^*$. Design the dynamic gain update law

$$\begin{aligned}
\dot{\ell}_1&=\max\{-2\ell_1^2\eta_1^{-1+r}+\ell_1\varrho_1(\zeta_1)\eta_1^{-1+r},0\},\ \ell_1(0)=1\\
\varrho_1(\zeta_1)&=2\sum_{i=2}^{n}\left(\Upsilon^{0*}_{i11}+\Gamma_{i11}+\Upsilon^{0*}_{i21}+\Upsilon_{i21}+\bar{\delta}_{i21}+\bar{\lambda}_{i21}\right)(\zeta_1)
\end{aligned}\tag{25}$$

where nonnegative smooth functions $\Upsilon^{0*}_{i11},\Gamma_{i11},\Upsilon^{0*}_{i21},\Upsilon_{i21}$, $\bar{\delta}_{i21}$, and $\bar{\lambda}_{i21}$ will be determined in the corresponding next step. In view of (25) and $r\in\mathbb{R}^{\le 1}_{\rm odd}$, the following properties are given:

$$\begin{aligned}
&-\frac{\dot{\ell}_1}{2\ell_1^2}\eta_1^2\le\eta_1^{1+r}-\frac{1}{2\ell_1}\eta_1^{1+r}\varrho_1(\zeta_1)\\
&\frac{1}{\ell_1}-\frac{1}{\ell_1(t-\tau)}\le 0\\
&-\frac{\dot{\ell}_1}{\ell_1^2}\int_0^{U_{02}(z_2)}\varrho_2(s)ds-\frac{\dot{\ell}_1}{2\ell_1^2}\eta_2^2-\frac{\dot{\ell}_1\ell_2+\ell_1\dot{\ell}_2}{2\ell_1^2\ell_2^2}\eta_2^2\le-\frac{\dot{\ell}_2}{2\ell_1\ell_2^2}\eta_2^2.
\end{aligned}\tag{26}$$

Substituting (25) and (26) into (24) yields

$$\begin{aligned}
&\mathcal{L}U_{2LK}\\
&\le-\frac{1}{2}\varrho_1(U_{01}(z_1))\Delta_{01}(\|z_1\|)+\|z_1\|^{1+r}(\bar{\gamma}_{111}+\gamma^0_{111}+\bar{\gamma}_{121}\\
&\quad+\gamma^0_{121}+\gamma^{0*}_{211}+\delta_{211}+\lambda_{211}+\gamma^{0*}_{221}+\delta_{221}+\lambda_{221})(z_1)\\
&\quad+\frac{1}{2}\varrho_1'(U_{01}(z_1))\phi_1^2(z_1)\psi_1^2(z_1)-\frac{1}{2\ell_1}\varrho_2(U_{02}(z_2))\Delta_{02}(\|z_2\|)\\
&\quad+\frac{1}{\ell_1}\|z_2\|^{1+r}\left(\delta_{212}+\lambda_{212}+\delta_{222}+\lambda_{222}\right)(z_2)-(n-1)\eta_1^{1+r}\\
&\quad+\frac{1}{2\ell_1}\varrho_2'(U_{02}(z_2))\phi_2^2(z_2)\psi_2^2(z_2)+\frac{1}{\ell_1}\left(1+\frac{1}{\ell_2}\right)\eta_2\zeta_3^{*r}+2\eta_2^{1+r}\\
&\quad+\frac{1}{\ell_1}\eta_1^{1+r}\sum_{m=3}^{n}\left(\Upsilon^{0*}_{m11}+\Gamma_{m11}+\Upsilon^{0*}_{m21}+\Upsilon_{m21}+\bar{\delta}_{m21}+\bar{\lambda}_{m21}\right)(\zeta_1)\\
&\quad+\eta_2^{1+r}\left(\Gamma_{212}+\bar{\delta}_{212}+\bar{\lambda}_{212}+\Gamma_{222}+\bar{\delta}_{222}+\bar{\lambda}_{222}\right)(\zeta_1,\zeta_2)\\
&\quad+\eta_3^{1+r}-\frac{\dot{\ell}_2}{2\ell_1\ell_2^2}\eta_2^2.
\end{aligned}\tag{27}$$

Design a virtual controller as

$$\begin{aligned}
\zeta_3^*&=-\ell_1^{\frac{1}{r}}\eta_2\big(n+2+(\Gamma_{212}+\bar{\delta}_{212}+\bar{\lambda}_{212}+\Gamma_{222}+\bar{\delta}_{222}\\
&\qquad+\bar{\lambda}_{222})(\zeta_1,\zeta_2)\big)^{\frac{1}{r}}\\
&:=-\ell_1^{\frac{1}{r}}\eta_2\epsilon_2(\zeta_1,\zeta_2).
\end{aligned}\tag{28}$$

From (27) and (28), it follows that

$$\begin{aligned}
\mathcal{L}U_{2LK}&\le-\frac{1}{2}\varrho_1(U_{01}(z_1))\Delta_{01}(\|z_1\|)+\|z_1\|^{1+r}(\bar{\gamma}_{111}+\gamma^0_{111}\\
&\quad+\bar{\gamma}_{121}+\gamma^0_{121}+\gamma^{0*}_{211}+\delta_{211}+\lambda_{211}+\gamma^{0*}_{221}+\delta_{221}\\
&\quad+\lambda_{221})(z_1)+\frac{1}{2}\varrho_1'(U_{01}(z_1))\phi_1^2(z_1)\psi_1^2(z_1)\\
&\quad-\frac{1}{2\ell_1}\varrho_2(U_{02}(z_2))\Delta_{02}(\|z_2\|)\\
&\quad+\frac{1}{\ell_1}\|z_2\|^{1+r}\big(\delta_{212}+\lambda_{212}+\delta_{222}+\lambda_{222}\big)(z_2)\\
&\quad+\frac{1}{2\ell_1}\varrho_2'(U_{02}(z_2))\phi_2^2(z_2)\psi_2^2(z_2)-(n-1)\eta_1^{1+r}\\
&\quad-n\eta_2^{1+r}+\eta_3^{1+r}-\frac{1}{\ell_1}\eta_1^{1+r}\sum_{m=3}^{n}\big(\Upsilon^{0*}_{m11}+\Gamma_{m11}\\
&\quad+\Upsilon^{0*}_{m21}+\Upsilon_{m21}+\bar{\delta}_{m21}+\bar{\lambda}_{m21}\big)(\zeta_1)-\frac{\dot{\ell}_2}{2\ell_1\ell_2^2}\eta_2^2.
\end{aligned}\tag{29}$$

Inductive Step: Suppose that at Step $i-1$ there exists an L–K functional $U_{(i-1)LK}$ and a set of virtual controllers $\zeta_1^*,\ldots,\zeta_i^*$ described as

$$\begin{aligned}
&\zeta_1^*=0,\qquad \zeta_2^*=-\eta_1\epsilon_1(\zeta_1)\\
&\zeta_j^*=-\ell_1^{\frac{1}{r}}\cdots\ell_{j-2}^{\frac{1}{r}}\eta_{j-1}\epsilon_{j-1}(\bar{\ell}_{j-3},\bar{\zeta}_{j-2}),\quad j=3,\ldots,i\\
&\eta_j=\zeta_j-\zeta_j^*,\quad j=1,\ldots,i
\end{aligned}\tag{30}$$

with nonnegative smooth functions ϱ_j,ϵ_j, and a series of dynamic gains $\ell_j(t)\ge 1=\ell_j(0), j=1,\ldots,i-1$ chosen as

$$\begin{aligned}
\dot{\ell}_1&=\max\{-2\ell_1^2\eta_1^{-1+r}+\ell_1\varrho_1(\zeta_1)\eta_1^{-1+r},0\}\\
\dot{\ell}_2&=\max\{-2\ell_2^2\eta_2^{-1+r}+\ell_2\varrho_2(\ell_1,\zeta_1,\zeta_2)\eta_2^{-1+r},0\}
\end{aligned}$$

$$\vdots$$

$$\dot{\ell}_{i-2}= \max\{-2\ell_{i-2}^2\eta_{i-2}^{-1+r}+\ell_{i-2}\varrho_{i-2}(\bar{\ell}_{i-3},\bar{\zeta}_{i-2})\eta_{i-2}^{-1+r},0\} \tag{31}$$

such that

$$\begin{aligned}
&\mathcal{L}U_{(i-1)LK}\\
&\leq -\frac{1}{2}\varrho_1(U_{01}(z_1))\Delta_{01}(\|z_1\|)+\|z_1\|^{1+r}\big(\bar{\gamma}_{111}+\gamma_{111}^0+\bar{\gamma}_{121}\\
&\quad+\gamma_{121}^0+\sum_{j=2}^{i-1}(\gamma_{j11}^{0*}+\delta_{j11}+\lambda_{j11}+\gamma_{j21}^{0*}+\delta_{j21}+\lambda_{j21})\big)(z_1)\\
&\quad+\frac{1}{2}\varrho_1'(U_{01}(z_1))\phi_1^2(z_1)\psi_1^2(z_1)-\frac{1}{2\ell_1}\varrho_2(U_{02}(z_2))\Delta_{02}(\|z_2\|)\\
&\quad+\frac{1}{\ell_1}\|z_2\|^{1+r}\left(\sum_{j=3}^{i-1}(\gamma_{j12}^{0*}+\gamma_{j22}^{0*})+\sum_{j=2}^{i-1}(\delta_{j12}++\lambda_{j12}+\delta_{j22}\right.\\
&\quad\left.+\lambda_{j22})\right)(z_2)+\frac{1}{2\ell_1}\varrho_2'(U_{02}(z_2))\phi_2^2(z_2)\psi_2^2(z_2)\\
&\quad-\cdots-\frac{1}{2\ell_1\cdots\ell_{i-2}}\varrho_{i-1}(U_{0(i-1)}(z_{i-1}))\Delta_{0(i-1)}(\|z_{i-1}\|)\\
&\quad+\frac{1}{\ell_1\cdots\ell_{i-2}}\|z_{i-1}\|^{1+r}\big(\delta_{(i-1)1(i-1)}+\lambda_{(i-1)1(i-1)}\\
&\quad+\delta_{(i-1)2(i-1)}+\lambda_{(i-1)2(i-1)}\big)(z_{i-1})\\
&\quad+\frac{1}{2\ell_1\cdots\ell_{i-1}}\varrho_{i-1}'(U_{0(i-1)}(z_{i-1}))\phi_{i-1}^2(z_{i-1})\psi_{i-1}^2(z_{i-1})\\
&\quad-(n-i+2)\sum_{j=1}^{i-1}\eta_j^{1+r}-\eta_{i-1}^{1+r}+\eta_i^{1+r}-\frac{\dot{\ell}_{i-1}}{2\ell_1\cdots\ell_{i-2}\ell_{i-1}^2}\eta_{i-1}^2\\
&\quad-\sum_{j=1}^{i-2}\frac{1}{2\ell_1\cdots\ell_j}\eta_j^{1+r}\sum_{m=j+2}^{n}(\Upsilon_{m1j}^{0*}+\Gamma_{m1j}+\Upsilon_{m2j}^{0*}+\Upsilon_{m2j}\\
&\quad+\bar{\delta}_{m2j}+\bar{\lambda}_{m2j})(\zeta_1).
\end{aligned}\tag{32}$$

Now, we claim that (32) also holds at step i. To show this, consider

$$\begin{aligned}
U_i &= U_{(i-1)LK}+\frac{1}{\ell_1\cdots\ell_{i-1}}\int_0^{U_{0i}(z_i)}\varrho_i(s)ds+W_i(\bar{\ell}_{i-2},\bar{\zeta}_i),\\
W_i &= \frac{1}{2\ell_1\cdots\ell_{i-1}}(1+\frac{1}{\ell_i})\eta_i^2
\end{aligned}\tag{33}$$

where dynamic gain $\ell_i(t)\geq 1$ will be designed later and ϱ_i is a smooth nondecreasing function. From $\ell_j\geq 1$, (32), (33), and the manner of changing supply rate, there holds

$$\begin{aligned}
&\mathcal{L}U_{iLK}\\
&\quad\leq \mathcal{L}U_{(i-1)LK}-\frac{1}{2\ell_1\cdots\ell_{i-1}}\varrho_i(U_{0i}(z_i))\Delta_{0i}(\|z_i\|)\\
&\qquad+\sum_{j=1}^{i-1}\frac{1}{\ell_1\cdots\ell_{j-1}}\left(\|z_j\|^{1+r}\gamma_{i1j}^{0*}(z_j)\right.\\
&\qquad\left.+\|z_j(t-\tau)\|^{1+r}\gamma_{i2j}^{0*}(z_j(t-\tau))\right)\\
&\qquad+\sum_{j=1}^{i}\frac{1}{\ell_1\cdots\ell_j}\left(\zeta_j^{1+r}\hat{\Upsilon}_{i1j}^0(\zeta_j)\right.\\
&\qquad\left.+\zeta_j^{1+r}(t-\tau)\hat{\Upsilon}_{i2j}^0(\zeta_j(t-\tau))\right)\\
&\qquad+\frac{1}{2\ell_1\cdots\ell_{i-1}}\varrho_i'(U_{0i}(z_i))\phi_i^2(z_i)\psi_i^2(z_i)\\
&\qquad+\frac{1}{\ell_1\cdots\ell_{i-1}}\left(1+\frac{1}{\ell_i}\right)\eta_i(\zeta_{i+1}^{*r}+\zeta_{i+1}^r-\zeta_{i+1}^{*r})\\
&\qquad+\frac{2}{\ell_1\cdots\ell_{i-1}}\eta_if_i+\Big|\sum_{j=1}^{i-1}\frac{\partial W_i}{\partial\zeta_j}(\zeta_{j+1}^r+f_j)\Big|\\
&\qquad+\Big|\sum_{j=1}^{i-2}\frac{1}{\ell_1\cdots\ell_{i-1}}\left(1+\frac{1}{\ell_i}\right)\eta_i\frac{\partial\zeta_i^*}{\partial\ell_j}\dot{\ell}_j\Big|\\
&\qquad+\frac{1}{2}\Big|\sum_{j=1}^{i-1}\frac{\partial^2W_i}{\partial\zeta_j^2}g_j^Tg_j+\sum_{q,j=1,q\neq j}^{i-1}\frac{\partial^2W_i}{\partial\zeta_q\partial\zeta_j}g_q^Tg_j\\
&\qquad+2\sum_{q=1}^{i-1}\frac{\partial^2W_i}{\partial\zeta_i\partial\zeta_q}g_i^Tg_q+\frac{\partial^2W_i}{\partial\zeta_i^2}g_i^Tg_i\Big|\\
&\qquad-\sum_{j=1}^{i-1}\frac{\ell_1\cdots\dot{\ell}_j\cdots\ell_{i-1}}{\ell_1^2\cdots\ell_{i-1}^2}\int_0^{U_{0i}(z_i)}\varrho_i(s)ds\\
&\qquad-\sum_{j=1}^{i-1}\frac{\ell_1\cdots\dot{\ell}_j\cdots\ell_{i-1}}{2\ell_1^2\cdots\ell_{i-1}^2}\left(1+\frac{1}{\ell_i}\right)\eta_i^2-\frac{\dot{\ell}_i}{2\ell_1\cdots\ell_{i-1}\ell_i^2}\eta_i^2
\end{aligned}\tag{34}$$

with nonnegative smooth functions $\gamma_{i\iota j}^{0*}$, $\hat{\Upsilon}_{i\iota j}^0$, $\iota=1,2$. From (8), (30), and Lemmas 2 and 3, one has

$$\begin{aligned}
&\zeta_j^{1+r}\hat{\Upsilon}_{i1j}^0(\zeta_j)\leq\eta_{j-1}^{1+r}\Gamma_{i1(j-1)}^{0*}(\bar{\ell}_{j-2},\bar{\zeta}_{j-1})+\eta_j^{1+r}\Gamma_{i1j}^{0*}(\bar{\ell}_{j-2},\bar{\zeta}_j),\\
&\zeta_j^{1+r}(t-\tau)\hat{\Upsilon}_{i2j}^0(\zeta_j(t-\tau))\\
&\leq\eta_{j-1}^{1+r}(t-\tau)\Gamma_{i2(j-1)}^{0*}(\bar{\ell}_{j-2}(t-\tau),\bar{\zeta}_{j-1}(t-\tau))\\
&\quad+\eta_j^{1+r}(t-\tau)\Gamma_{i2j}^{0*}(\bar{\ell}_{j-2}(t-\tau),\bar{\zeta}_j(t-\tau))
\end{aligned}\tag{35}$$

with nonnegative smooth functions $\Gamma_{i\iota(j-1)}^{0*}$, $\Gamma_{i\iota j}^{0*}$, $\iota=1,2$. And by Lemmas 2, 3, 7, and (30), one can obtain

$$\begin{aligned}
&\frac{2}{\ell_1\cdots\ell_{i-1}}\eta_if_i+\Big|\sum_{j=1}^{i-1}\frac{\partial W_i}{\partial\zeta_j}(\zeta_{j+1}^r+f_j)\Big|\\
&\quad+\Big|\sum_{j=1}^{i-2}\frac{1}{\ell_1\cdots\ell_{i-1}}\left(1+\frac{1}{\ell_i}\right)\eta_i\frac{\partial\zeta_i^*}{\partial\ell_j}\dot{\ell}_j\Big|\\
&\leq\frac{2}{\ell_1\cdots\ell_{i-1}}\bigg(|\eta_if_i|+\sum_{j=1}^{i-1}\Big|\eta_i\frac{\partial\zeta_i^*}{\partial\zeta_j}\Big|\big(|\eta_{j+1}|^r\\
&\quad+|\ell_1\cdots\ell_{j-1}\eta_j\epsilon_j(\bar{\ell}_{j-2},\bar{\zeta}_{j-1})|^r+|f_j|\big)+\sum_{j=1}^{i-2}\Big|\eta_i\frac{\partial\zeta_i^*}{\partial\ell_j}\dot{\ell}_j\Big|\bigg)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\ell_1\cdots\ell_{i-1}}\Bigg(\sum_{j=1}^{i}\big(\|z_j\|^{1+r}\delta_{i1j}(z_j)\\
&\quad+\|z_j(t-\tau)\|^{1+r}\delta_{i2j}(z_j(t-\tau))\big)\\
&\quad+\eta_i^{1+r}\bar\delta_{i1i}(\bar\ell_{i-2},\bar\zeta_i)\\
&\quad+\eta_i^{1+r}(t-\tau)\bar\delta_{i2i}(\bar\ell_{i-2}(t-\tau),\bar\zeta_i(t-\tau))\\
&\quad+\sum_{j=1}^{i-1}\eta_j^{1+r}+\sum_{j=1}^{i-1}\eta_j^{1+r}(t-\tau)\bar\delta_{i2j}(\bar\ell_{j-1}(t-\tau),\bar\zeta_j(t-\tau))\Bigg)
\end{aligned}\tag{36}$$

with nonnegative smooth functions $\delta_{i\iota j},\bar\delta_{i\iota j},\bar\delta_{i2j},\iota=1,2$. By Lemmas 2, 3, 7, (30), (33), and $\ell_j\geq 1$, it follows that

$$\begin{aligned}
&\frac{1}{2}\Bigg|\sum_{j=1}^{i-1}\frac{\partial^2 W_i}{\partial\zeta_j^2}g_j^Tg_j+\sum_{q,j=1,q\neq j}^{i-1}\frac{\partial^2 W_i}{\partial\zeta_q\partial\zeta_j}g_q^Tg_j\\
&\quad+2\sum_{q=1}^{i-1}\frac{\partial^2 W_i}{\partial\zeta_i\partial\zeta_q}g_i^Tg_q+\frac{\partial^2 W_i}{\partial\zeta_i^2}g_i^Tg_i\Bigg|\\
&\leq\frac{\left(1+\frac{1}{\ell_i}\right)}{2\ell_1\cdots\ell_{i-1}}\Bigg|\sum_{j=1}^{i-1}\Bigg(\left(\frac{\partial(-\zeta_i^*)}{\partial\zeta_j}\right)^2+\eta_i\left(\frac{\partial^2(-\zeta_i^*)}{\partial\zeta_j^2}\right)\Bigg)g_j^Tg_j\\
&\quad+\sum_{q,j=1,q\neq j}^{i-1}\left(\left(\frac{\partial(-\zeta_i^*)}{\partial\zeta_j}\right)\left(\frac{\partial(-\zeta_i^*)}{\partial\zeta_q}\right)+\eta_i\frac{\partial^2(-\zeta_i^*)}{\partial\zeta_j\partial\zeta_q}\right)g_q^Tg_j\\
&\quad+2\sum_{q=1}^{i-1}\frac{\partial(-\zeta_i^*)}{\partial\zeta_q}g_i^Tg_q+g_i^Tg_i\Bigg|\\
&\leq \frac{1}{\ell_1\cdots\ell_{i-1}}\Bigg(\sum_{j=1}^{i}\big(\|z_j\|^{1+r}\lambda_{i1j}(z_j)\\
&\quad+\|z_j(t-\tau)\|^{1+r}\lambda_{i2j}(z_j(t-\tau))\big)\\
&\quad+\eta_i^{1+r}\bar\lambda_{i1i}(\bar\ell_{i-2},\bar\zeta_i)\\
&\quad+\eta_i^{1+r}(t-\tau)\bar\lambda_{i2i}(\bar\ell_{i-2}(t-\tau),\bar\zeta_i(t-\tau))\\
&\quad+\sum_{j=1}^{i-1}\eta_j^{1+r}+\sum_{j=1}^{i-1}\eta_j^{1+r}(t-\tau)\bar\lambda_{i2j}(\bar\ell_{j-1}(t-\tau),\bar\zeta_j(t-\tau))\Bigg)
\end{aligned}\tag{37}$$

with nonnegative smooth functions $\lambda_{i\iota j},\bar\lambda_{i\iota j},\bar\lambda_{i2j},\iota=1,2$. Construct the L–K functional

$$\begin{aligned}
&U_{iLK}\\
&=U_i+\int_{t-\tau}^{t}\Bigg(\sum_{j=1}^{i-1}\frac{1}{\ell_1(s)\cdots\ell_{j-1}(s)}\|z_j(s)\|^{1+r}(\gamma_{i2j}^{0*}+\delta_{i2j}\\
&\quad+\lambda_{i2j})(z_j(s))\\
&\quad+\frac{1}{\ell_1(s)\cdots\ell_{i-1}(s)}\|z_i(s)\|^{1+r}(\delta_{i2i}+\lambda_{i2i})(z_i(s))\\
&\quad+\sum_{j=1}^{i-1}\frac{1}{\ell_1(s)\cdots\ell_{j-1}(s)}\eta_j^{1+r}(s)(\Upsilon_{i2j}^{0*}\\
&\quad+\Upsilon_{i2j}^{0}+\bar\delta_{i2j}+\bar\lambda_{i2j})(\bar\ell_{j-1}(s),\bar\zeta_j(s))\\
&\quad+\eta_i^{1+r}(s)(\Upsilon_{i2i}^{0*}+\bar\delta_{i2i}+\bar\lambda_{i2i})(\bar\ell_{i-2}(s),\bar\zeta_i(s))\Bigg)ds.
\end{aligned}\tag{38}$$

From (34)–(38), a delay-independent gain update law is designed as

$$\begin{aligned}
\dot\ell_{i-1}&=\max\{-2\ell_{i-1}^2\eta_{i-1}^{-1+r}\\
&\quad+\ell_{i-1}\varrho_{i-1}(\bar\ell_{i-2}(s),\bar\zeta_{i-1}(s))\eta_{i-1}^{-1+r},0\}\\
\varrho_{i-1}(\bar\ell_{i-2},\bar\zeta_{i-1})&=2\sum_{m=i}^{n}\big(\Upsilon_{m2(i-1)}^{0*}+\Gamma_{m2(i-1)}^{0}+\bar\delta_{m2(i-1)}\\
&\quad+\bar\lambda_{m2(i-1)}\big)(\bar\ell_{i-2},\bar\zeta_i)
\end{aligned}\tag{39}$$

with $\ell_{i-1}(0)=1$, from which, a virtual controller is devised by

$$\begin{aligned}
\zeta_{i+1}^*&=-\ell_1^{\frac{1}{r}}\cdots\ell_{i-1}^{\frac{1}{r}}\eta_i\big(n-i+4+(\bar\delta_{i1i}+\bar\lambda_{i1i}+\Upsilon_{i2i}^{0*}\\
&\quad+\bar\delta_{i2i}+\bar\lambda_{i2i}+\Upsilon_{i1i}^{0*})(\bar\ell_{i-2},\bar\zeta_i)\big)^{\frac{1}{r}}\\
&:=-\ell_1^{\frac{1}{r}}\cdots\ell_{i-1}^{\frac{1}{r}}\eta_i\epsilon_i(\bar\ell_{i-2},\bar\zeta_i).
\end{aligned}\tag{40}$$

Then

$$\begin{aligned}
&-\frac{\dot\ell_{i-1}}{2\ell_1\cdots\ell_{i-2}\ell_{i-1}^2}\eta_{i-1}^2\\
&\leq\frac{1}{\ell_1\cdots\ell_{i-2}}\eta_{i-1}^{1+r}-\frac{1}{2\ell_1\cdots\ell_{i-1}}\eta_{i-1}^{1+r}\varrho_{i-1}(\bar\ell_{i-2},\bar\zeta_{i-1})\\
&\quad\frac{1}{\ell_j}-\frac{1}{\ell_j(t-\tau)}\leq 0,\ j=1,\cdots i-1\\
&\quad-\sum_{j=1}^{i-1}\frac{\ell_1\cdots\dot\ell_j\cdots\ell_{i-1}}{\ell_1^2\cdots\ell_{i-1}^2}\int_0^{U_{0i}(z_i)}\varrho_i(s)ds\\
&\quad-\sum_{j=1}^{i-1}\frac{\ell_1\cdots\dot\ell_j\cdots\ell_{i-1}}{2\ell_1^2\cdots\ell_{i-1}^2}\\
&\quad\cdot\left(1+\frac{1}{\ell_i}\right)\eta_i^2-\frac{\dot\ell_i}{2\ell_1\cdots\ell_{i-1}\ell_i^2}\eta_i^2\leq-\frac{\dot\ell_i}{2\ell_1\cdots\ell_{i-1}\ell_i^2}\eta_i^2.
\end{aligned}\tag{41}$$

With the aid of (38)–(41), one obtains

$$\begin{aligned}
&\mathcal{L}U_{iLK}\\
&\leq\mathcal{L}U_{(i-1)LK}-\frac{1}{2\ell_1\cdots\ell_{i-1}}\varrho_i(U_{0i}(z_i))\Delta_{0i}(\|z_i\|)\\
&\quad+\sum_{j=1}^{i-1}\frac{1}{\ell_1\cdots\ell_{j-1}}\|z_j\|^{1+r}(\gamma_{i1j}^{0*}\\
&\quad+\gamma_{i2j}^{0*}+\delta_{i1j}+\delta_{i2j}+\lambda_{i1j}+\lambda_{i2j})\\
&\quad+\|z_i\|^{1+r}(\delta_{i1i}+\delta_{i2i}+\lambda_{i1i}+\lambda_{i2i})
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{i-1}\frac{1}{\ell_1\cdots\ell_{j-1}}\eta_j^{1+r}\big(\Gamma_{i1j}^{0*}\\
&+\Upsilon_{i1j}^{0*}+\Gamma_{i2j}^{0*}+\Upsilon_{i2j}^{0*}+\bar{\delta}_{i2j}+\bar{\lambda}_{i2j}\big)\\
&+\eta_i^{1+r}\big(\Upsilon_{i1i}^{0*}+\Upsilon_{i2i}^{0*}+\bar{\delta}_{i1i}+\bar{\delta}_{i2i}+\bar{\lambda}_{i1i}+\bar{\lambda}_{i2i}+2\big)+\eta_{i+1}^{1+r}\\
&+\frac{1}{\ell_1\cdots\ell_{i-2}}\eta_{i-1}^{1+r}-\frac{1}{2\ell_1\cdots\ell_{i-1}}\eta_{i-1}^{1+r}\varrho_{i-1}-\frac{\dot{\ell}_i}{2\ell_1\cdots\ell_{i-1}\ell_i^2}\eta_i^2\\
&\le-\frac{1}{2}\varrho_1\Delta_{01}+\|z_1\|^{1+r}(\bar{\gamma}_{111}+\gamma_{111}^0+\bar{\gamma}_{121}+\gamma_{121}^0\\
&+\sum_{j=2}^{i}(\gamma_{j11}^{0*}+\gamma_{j21}^{0*}+\delta_{j11}+\delta_{j21}+\lambda_{j11}+\lambda_{j21}))(z_1)\\
&+\frac{1}{2}\varrho_1'(U_{01}(z_1))\phi_1^2(z_1)\psi_1^2(z_1)-\frac{1}{2\ell_1}\varrho_2\Delta_{02}\\
&+\frac{1}{\ell_1}\|z_2\|^{1+r}\Bigg(\sum_{j=3}^{i}(\gamma_{j12}^{0*}+\gamma_{j22}^{0*})+\sum_{j=2}^{i}(\delta_{j12}+\delta_{j22}+\lambda_{j121}\\
&+\frac{1}{2\ell_1}\varrho_2'\phi_2^2\psi_2^2+\lambda_{j22})\Bigg)(z_2)-\cdots-\frac{1}{2\ell_1\cdots\ell_{i-1}}\varrho_i\Delta_{0i}\\
&+\frac{1}{\ell_1\cdots\ell_{i-1}}\|z_i\|^{1+r}(\delta_{i1i}+\delta_{i2i}+\lambda_{i1i}+\lambda_{i2i})\\
&+\frac{1}{2\ell_1\cdots\ell_{i-1}}\varrho_i'\phi_i^2\psi_i^2-(n-i+1)\sum_{j=1}^{i}\eta_j^{1+r}-\eta_i^{1+r}+\eta_{i+1}^{1+r}\\
&-\frac{\dot{\ell}_i}{2\ell_1\cdots\ell_{i-1}\ell_i^2}\eta_i^2\\
&-\sum_{j=1}^{i-1}\frac{1}{2\ell_1\cdots\ell_j}\eta_j^{1+r}\sum_{m=j+2}^{n}(\Gamma_{m1j}^{0*}+\Upsilon_{m1j}^{0*}\\
&+\Gamma_{m2j}^{0*}+\Upsilon_{m2j}^{0*}+\bar{\delta}_{m2j}+\bar{\lambda}_{m2j}).
\end{aligned}\tag{42}
$$

At step n, for $i=n+1$ and recalling $u=\zeta_{n+1}$, there holds

$$
\begin{aligned}
&\mathcal{L}U_{nLK}\\
&\le-\frac{1}{2}\varrho_1\Delta_{01}+\frac{1}{2}\varrho_1'(U_{01}(z_1))\phi_1^2(z_1)\psi_1^2(z_1)\\
&\quad+\|z_1\|^{1+r}(\bar{\gamma}_{111}+\gamma_{111}^0+\bar{\gamma}_{121}+\gamma_{121}^0\\
&\quad+\sum_{j=2}^{n}(\gamma_{j11}^{0*}+\gamma_{j21}^{0*}+\delta_{j11}\\
&\quad+\delta_{j21}+\lambda_{j11}+\lambda_{j21}))(z_1)-\frac{1}{2\ell_1}\varrho_2\Delta_{02}\\
&\quad+\frac{1}{\ell_1}\|z_2\|^{1+r}\Bigg(\sum_{j=3}^{n}(\gamma_{j12}^{0*}\\
&\quad+\gamma_{j22}^{0*})+\sum_{j=2}^{n}(\delta_{j12}+\delta_{j22}+\lambda_{j121}+\lambda_{j22})\Bigg)(z_2)\\
&\quad+\frac{1}{2\ell_1}\varrho_2'\phi_2^2\psi_2^2\\
&\quad-\cdots-\frac{1}{2\ell_1\cdots\ell_{n-1}}\varrho_n\Delta_{0n}+\frac{1}{\ell_1\cdots\ell_{n-1}}\|z_n\|^{1+r}(\delta_{n1n}\\
&\quad+\delta_{n2n}+\lambda_{n1n}+\lambda_{n2n})+\frac{1}{2\ell_1\cdots\ell_{n-1}}\varrho_n'\phi_n^2\psi_n^2-\sum_{j=1}^{n}\eta_j^{1+r}.
\end{aligned}\tag{43}
$$

From Lemma 6, if ϱ_j are chosen to satisfy

$$
\begin{aligned}
&\frac{1}{4}\varrho_1(U_{01}(z_1))\Delta_{01}(\|z_1\|)\\
&\ge\frac{1}{2}\varrho_1'(U_{01}(z_1))\phi_1^2(z_1)\psi_1^2(z_1)+\|z_1\|^{1+r}\big(\bar{\gamma}_{111}+\gamma_{111}^0+\bar{\gamma}_{121}\\
&\quad+\gamma_{121}^0+\sum_{j=2}^{n}(\gamma_{j11}^{0*}+\gamma_{j21}^{0*}+\delta_{j11}\\
&\quad+\delta_{j21}+\lambda_{j11}+\lambda_{j21})\big)(z_1),\\
&\frac{1}{4\ell_1}\varrho_2(U_{02}(z_2))\Delta_{02}(\|z_2\|)\\
&\ge\frac{1}{2\ell_1}\varrho_2'(U_{02}(z_2))\phi_2^2(z_2)\psi_2^2(z_2)+\frac{1}{\ell_1}\|z_2\|^{1+r}\Bigg(\sum_{j=3}^{n}(\gamma_{j12}^{0*}\\
&\quad+\gamma_{j22}^{0*})+\sum_{j=2}^{n}(\delta_{j12}+\delta_{j22}+\lambda_{j121}+\lambda_{j22})\Bigg)(z_2)\\
&\vdots\\
&\frac{1}{4\ell_1\cdots\ell_{n-1}}\varrho_n(U_{0n}(z_n))\Delta_{0n}(\|z_n\|)\\
&\ge\frac{1}{2\ell_1\cdots\ell_{n-1}}\varrho_n'(U_{0n}(z_n))\phi_n^2(z_n)\psi_n^2(z_n)\\
&\quad+\frac{1}{\ell_1\cdots\ell_{n-1}}\|z_n\|^{1+r}(\delta_{n1n}+\delta_{n2n}+\lambda_{n1n}+\lambda_{n2n})(z_n)
\end{aligned}\tag{44}
$$

and the dynamic partial state feedback controller is designed as

$$
\begin{aligned}
\dot{\ell}_1&=\max\{-2\ell_1^2\eta_1^{-1+r}+\ell_1\varrho_1(\zeta_1)\eta_1^{-1+r},0\}\\
\dot{\ell}_2&=\max\{-2\ell_2^2\eta_2^{-1+r}+\ell_2\varrho_{i-1}(\ell_1,\zeta_1,\zeta_2)\eta_2^{-1+r},0\}\\
&\vdots\\
\dot{\ell}_{n-1}&=\max\{-2\ell_{n-1}^2\eta_{n-1}^{-1+r}+\ell_{n-1}\varrho_{i-1}(\bar{\ell}_{n-1},\bar{\zeta}_{n-1})\eta_{n-1}^{-1+r},0\}\\
u&=-\ell_1^{\frac{1}{r}}\cdots\ell_{n-1}^{\frac{1}{r}}\eta_n\epsilon_n(\bar{\ell}_{n-2},\bar{\zeta}_n)
\end{aligned}\tag{45}
$$

with $\ell_1(0)=\cdots=\ell_{n-1}(0)=1$ and

$$
\begin{aligned}
&\zeta_1^*=0,\ \ \zeta_2^*=-\eta_1\epsilon_1(\zeta_1)\\
&\zeta_k^*=-\ell_1^{\frac{1}{r}}\cdots\ell_{k-2}^{\frac{1}{r}}\eta_{k-1}\epsilon_{k-1}(\bar{\ell}_{k-3},\bar{\zeta}_{k-2}),\ k=3,\ldots,n\\
&\eta_k=\zeta_k-\zeta_k^*,\ \ k=1,\ldots,n
\end{aligned}\tag{46}
$$

one has

$$
\begin{aligned}
\mathcal{L}U_{nLK}\le\ &-\sum_{j=1}^{n}\frac{1}{4\ell_1\cdots\ell_{j-1}}\varrho_j(U_{0j}(z_j))\Delta_{0j}(\|z_j\|)\\
&-\sum_{j=1}^{n}\eta_j^{1+r}.
\end{aligned}\tag{47}
$$

This completes the recursive design.

B. Stability Analysis

The main results are stated here.

Theorem 1: Considering the stochastic time-delay cascade nonlinear system (2) under Assumptions 1 and 2 and (4)–(6), there exists a delay-independent dynamic controller (45) such that: 1) the closed-loop system has an almost surely continuous solution on $[-\tau, \infty)$ for any initial value $[z^T(0), \zeta^T(0)]$; 2) all the signals of closed-loop system are almost surely bounded; and 3) the trivial solution is globally asymptotically stable in probability.

Proof: First, it is noted that the nonlinear functions f and g in system (2) are continuous, so the existence of a weak solution can be obtained for any initial condition according to [36, Th. 1]. Furthermore, for the closed-loop system (2)–(45), it derives that $dX(t) = F(X(t))dt + G(X(t))d\omega$ with $F(X) = [f_{01}, \ldots, f_{0n}, \zeta_2^r + f_1, \ldots, \zeta_{n+1}^r + f_n]^T$, $G(X) = [g_{01}^T, \ldots, g_{0n}^T, g_1^T, \ldots, g_n^T]^T$. With a similar mind of [28], by means of truncation functions and the continuity of nonlinear functions, the closed-loop cascade system (2) and (45) has an almost surely continuous solution on $[-\tau, \infty)$.

Second, according to Itô's formula and (47), there holds

$$\begin{aligned}
&E\{U_{nLK}(z(\aleph_c \wedge t), \zeta(\aleph_c \wedge t), \ell(\aleph_c \wedge t), \aleph_c \wedge t)\}\\
&\quad = EU_{nLK}(z(0), \zeta(0), \ell(0), 0)\\
&\qquad + E\int_0^{\aleph_c \wedge t} \mathcal{L}U_{nLK}(z(s), \zeta(s), \ell(s), s)ds\\
&\qquad + E\int_0^{\aleph_c \wedge t} \frac{\partial U_{nLK}}{\partial z}(s) g_0(s) d\omega(s)\\
&\qquad + E\int_0^{\aleph_c \wedge t} \frac{\partial U_{nLK}}{\partial \zeta}(s) g(s) d\omega(s)\\
&\quad = U_{nLK}(z(0), \zeta(0), \ell(0), 0)\\
&\qquad + E\int_0^{\aleph_c \wedge t} \mathcal{L}U_{nLK}(z(s), \zeta(s), \ell(s), s)ds\\
&\quad \le U_{nLK}(z(0), \zeta(0), \ell(0), 0)
\end{aligned} \tag{48}$$

$\aleph_c := \inf\{t \ge 0 : |\zeta(t)| \ge c, c \ge 0\}$, $g_0 = [g_{01}, \ldots, g_{0n}]^T$, $g = [g_1, \ldots, g_n]^T$. From (38), it shows that

$$\begin{aligned}
&E\{U_{nLK}(z(\aleph_c \wedge t), \zeta(\aleph_c \wedge t), \ell(\aleph_c \wedge t), \aleph_c \wedge t)\}\\
&\ge E\Bigg(\Bigg(\int_0^{U_{01}(z_1)} \varrho_1(s)ds + \frac{1}{\ell_1(s)}\int_0^{U_{02}(z_2)} \varrho_2(s)ds\\
&\quad + \cdots + \frac{1}{\ell_1(s)\cdots\ell_{n-1}(s)}\int_0^{U_{0n}(z_n)} \varrho_n(s)ds\\
&\quad + \frac{1}{2}\zeta_1^2(s) + \frac{1}{2\ell_1(s)}(\zeta_2(s) - \zeta_2^*(s))^2\\
&\quad + \cdots + \frac{1}{2\ell_1(s)\cdots\ell_{n-1}(s)}(\zeta_n(s) - \zeta_n^*(s))^2\Bigg)\Bigg|_{s=\aleph_c \wedge t}\Bigg).
\end{aligned} \tag{49}$$

Letting $c \to \infty$, (48) and (49) indicate that $z_1(t), \cdots, \frac{1}{\ell_1(t)\cdots\ell_{n-1}(t)}\int_0^{U_{0n}(z_n)} \varrho_n(s)ds$ and $\zeta_1(t)$, $\frac{1}{2\ell_1(t)}(\zeta_2(t) - \zeta_2^*(t))^2, \cdots, \frac{1}{2\ell_1(t)\cdots\ell_{n-1}(t)} \cdot (\zeta_n(t) - \zeta_n^*(t))^2$ are bounded almost surely on $[-\tau, \aleph_\infty)$. Equation (25) means that the $\ell_1(t)$ is monotone nondecreasing. We prove that $\ell_1(t)$ is almost surely bounded on $[-\tau, \aleph_\infty)$. If not, assume $\lim_{t\to\aleph_\infty} \ell_1(t) = \infty$. By the almost sure boundedness of ζ_1 and the nature of continuity, $\varrho_1(\zeta_1)$ is almost surely bounded. Hence, there exists a $0 < T_1 < \aleph_\infty$ to ensure $-2\ell_1^2\eta_1^{-1+r} + \ell_1\varrho_1(\zeta_1)\eta_1^{-1+r} \le 0$ on $[T_1, \aleph_\infty)$. From (25), it leads to a contradiction to $\lim_{t\to\aleph_\infty} \ell_1(t) = \infty$. Thus, $\ell_1(t)$ is almost surely bounded on $[-\tau, \aleph_\infty)$. Since $\frac{1}{\ell_1(s)}\int_0^{U_{02}(z_2)} \varrho_2(s)ds$ and $\frac{1}{2\ell_1(s)}(\zeta_2(s) - \zeta_2^*(s))^2$ are almost surely bounded, $\int_0^{U_{02}(z_2)} \varrho_2(s)ds$ and $\zeta_2(s) - \zeta_2^*(s)$ are almost surely bounded on $[-\tau, \aleph_\infty)$, and so are $z_2(t)$ and $\zeta_2(t)$ with the help of the definition of ϱ_2, U_{02}, and ζ_2^* in (30). Recursively, it can be concluded that $\ell_2(t), z_3(t), \zeta_3(t), \ldots, \ell_{n-1}(t), z_n(t), \zeta_n(t)$ and $u(t)$ are almost surely bounded on $[-\tau, \aleph_\infty)$.

Now, we begin to prove $\aleph_\infty = \infty$ almost surely. If not, there exists a pair of positive constants ε and T_* that satisfy

$$P\{\aleph_\infty \le T_*\} > 2\varepsilon. \tag{50}$$

From $\aleph_c \to \aleph_\infty$ almost surely as $c \to \infty$, there exists a sufficiently large constant $R^* > 0$ to maintain

$$P\{\aleph_c \le T_*\} > \varepsilon \quad \forall\, c \ge R^*. \tag{51}$$

Choosing $\bar{U} = \frac{1}{2}\zeta_1^2 + \frac{1}{2\ell_1}(\zeta_2 - \zeta_2^*)^2 + \cdots + \frac{1}{2\ell_1\cdots\ell_{n-1}}(\zeta_n - \zeta_n^*)^2$, and considering the fact that ℓ_i is almost surely bounded and $\ell_i \ge 1$, for $R > 0$, it derives that

$$\inf_{\|\zeta\| \ge R} \bar{U}(\zeta) \to \infty \text{ as } R \to \infty. \tag{52}$$

For any fixed $c \ge R^*$, it follows from the definition of $\aleph_c$ and (48) and (49) that

$$\begin{aligned}
P\{\aleph_c \le T_*\} \inf_{\|\zeta\| \ge c} U(\zeta) &\le E\{I_{\aleph_c \le T_*}\bar{U}(\zeta(\aleph_c)\}\\
&= E\{I_{\aleph_c \le T_*}\bar{U}(\zeta(\aleph_c \wedge T_*))\}\\
&\le E\{\bar{U}(\zeta(\aleph_c \wedge T_*))\}\\
&\le U_{nLK}(\zeta(0), \ell(0), 0).
\end{aligned} \tag{53}$$

By (51) and (53), it derives

$$\varepsilon \inf_{\|\zeta\| \ge c} \bar{U}(\zeta) \le U_{nLK}(\zeta(0), \ell(0), 0) \tag{54}$$

when $c \to \infty$, it brings a contradiction to (52). Thus, $\aleph_\infty = \infty$ almost surely.

Finally, from (38), it indicates that $U_{nLK} \ge \int_0^{U_{01}(z_1)} \varrho_1(s)ds + \frac{1}{\ell_1(t)}\int_0^{U_{02}(z_2)} \varrho_2(s)ds + \cdots + \frac{1}{\ell_1(t)\cdots\ell_{n-1}(t)}\int_0^{U_{0n}(z_n)} \varrho_n(s)ds + \frac{1}{2}\eta_1^2 + \frac{1}{2\ell_1(t)}\eta_2^2 + \cdots + \frac{1}{2\ell_1(t)\cdots\ell_{n-1}(t)}\eta_n^2$. Then, with an eye to the construction of ϱ_i, U_{0i} and $\ell_i \ge 1$, there is a function $\varpi_1 \in \mathcal{K}_\infty$ such that

$$U_{nLK} \ge \varpi_1(\|z(t), \zeta(t)\|). \tag{55}$$

Moreover, there exists a constant $D > 0$ and class $\mathcal{K}_\infty$ functions $\bar{\varrho}_1(\cdot), \ldots, \bar{\varrho}_n(\cdot), \Xi_1(\cdot), \ldots, \Xi_n(\cdot), \Lambda_1(\cdot), \ldots, \Lambda_n(\cdot)$ such that

U_{nLK}

$$
\begin{aligned}
&\leq D\sum_{j=1}^{n}(\bar{\varrho}_j(\Delta_{2j}(t))+\eta_j^2(t))\\
&\quad+\sum_{j=1}^{n}\bigg(\int_{t-\tau(t)}^{t}\|z_j(\iota)\|^{1+r}\Xi_j(\zeta(\iota))d\iota\\
&\quad+\int_{t-\tau(t)}^{t}\eta_j^2(\iota)\Lambda_j(\zeta(\iota))d\iota\bigg)\\
&\overset{\iota=t+s}{\leq} D\sum_{j=1}^{n}(\bar{\varrho}_j(\Delta_{2j}(t))+\eta_j^2(t))\\
&\quad+\sum_{j=1}^{n}\bigg(\int_{-\tau(t)}^{0}\|z_j(t+s)\|^{1+r}\Xi_j(\zeta(t+s))d(t+s)\\
&\quad+\int_{-\tau(t)}^{0}\eta_j^2(t+s)\Lambda_j(\zeta(t+s))d(t+s)\bigg)\\
&\leq D\sum_{j=1}^{n}\sup_{-\tau\leq s\leq 0}(\bar{\varrho}_j(\Delta_{2j}(t+s))+\eta_j^2(t+s))\\
&\quad+D\sum_{j=1}^{n}\sup_{-\tau\leq s\leq 0}\big(\|z_j(t+s)\|^{1+r}\Xi_j(\zeta(t+s))\\
&\quad+\eta_j^2(t+s)\Lambda_j(\zeta(t+s))\big)\\
&\leq \varpi_2(\sup_{-\tau\leq s\leq 0}\|z(t+s),\zeta(t+s)\|).
\end{aligned}
\tag{56}
$$

It is easy to see that ϖ_2 is also a $\mathcal{K}_\infty$ function. From Lemma 5, it can be deduced from (47), (55), and (56) that the trivial solution of the closed-loop system (2) and (45) is globally asymptotically stable in probability. □

Remark 3: Compared with [20] and [28] in which the global stabilization and finite-time stabilization problems of stochastic low-order and high-order SISS nonlinear systems have been solved, respectively, the innovation of this article is that the growth conditions of nonlinear functions are removed; and by designing a series of dynamic gain laws, a delay-independent controller is proposed to extend the continuous domain of stochastic low-order cascade nonlinear system with time-varying delay. In addition, compared with [19], the local Lipschitz condition is reduced to continuous and the system power is low-order which produces the hardship of completely nondifferentiable of this article; moreover, the restrictive constraint of $\frac{\partial^l \Delta_i}{\partial X^l(0)}=0, l=1,2,3$ on functions $\Delta_i(X)$ in Assumption 1 is relaxed and quadratic L–K functionals replace traditional quadric Lyapunov functions in the stochastic control, which saves a great deal of calculation. □

Remark 4: It should be noted that this article only requires existence of solution to the stochastic nonlinear system (2), but it does not require uniqueness. For a stochastic nonlinear system, the requirement of a unique solution is very restrictive, i.e., the linear growth condition and the local Lipschitz condition need to be imposed on the system as assumptions, which greatly limits its applications. Therefore, the existence of a solution for a stochastic system is more important than the uniqueness from a viewpoint of stabilization. And it is sufficient to assure the existence of a solution to study global stability of a stochastic nonlinear system. □

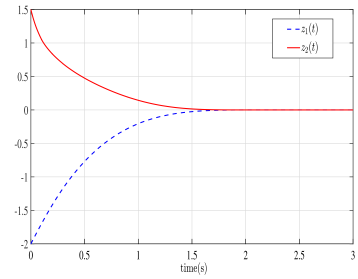

Fig. 1. Trajectories of the state z_1 and z_2.

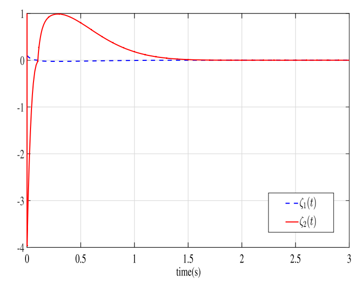

Fig. 2. Trajectories of the state ζ_1 and ζ_2.

IV. Simulation Examples

Consider the stochastic continuous cascade nonlinear system

$$
\begin{cases}
dz_1=\left(-2z_1^{\frac{5}{7}}-\zeta_1^{\frac{5}{7}}\right)dt+0.5z_1^{\frac{6}{7}}d\omega\\
dz_2=\left(-3z_2^{\frac{5}{7}}-\zeta_2^{\frac{5}{7}}\right)dt+\sin(\zeta_1(t-\tau))d\omega\\
d\zeta_1=\left(\zeta_2^{\frac{5}{7}}+\sin(z_1(t-\tau))\right)dt+\zeta_1^{\frac{6}{7}}(t-\tau)d\omega\\
d\zeta_2=u^{\frac{5}{7}}dt+\zeta_2^{\frac{6}{7}}d\omega
\end{cases}
\tag{57}
$$

where $\tau=0.1\sin(t)+1$. Apparently, Assumptions 1 and 2 are satisfied with $U_{01}(s)=U_{02}(s)=s^2$. By means of the control design in the previous section, $r=\frac{5}{7}$, $\eta_1=\zeta_1, U_{1LK}=\int_0^{z_1^2}(6+4s^2)ds+\frac{1}{2}(1+\frac{1}{\ell_1})\eta_1^2+\int_{t-\tau}^{t}(\eta_1^{\frac{12}{7}}(s)(\zeta_1(s))+z_1^{\frac{12}{7}}(s)(z_1(s)))ds$, the first virtual controller $\zeta_2^*=-\eta_1\epsilon_1, \epsilon_1=(13.8+2.4\zeta_1^{\frac{12}{7}})^{\frac{7}{5}}$ leads to $\mathcal{L}U_{1LK}\leq -2.75z_1^{\frac{12}{7}}\varrho_1+\frac{1}{2}\varrho_1'\phi_1^2\psi_1^2+2.4z_1^{\frac{12}{7}}z_1^{\frac{44}{7}}+\eta_2^{\frac{12}{7}}-3\eta_1^{\frac{12}{7}}-\frac{\dot{\ell}_1}{2\ell_1^2}\eta_1^2$ with $\eta_2=\zeta_2-\zeta_2^*$ and $\varrho_1(s)=\varrho_2(s)=6+4s^2$. Take $U_{2LK}=U_{1LK}+\frac{1}{\ell_1}\int_0^{z_2^2}(6+4s^2)ds+\frac{1}{2}(\frac{1}{\ell_1}+\frac{1}{\ell_1\ell_2})\eta_2^2+\int_{t-\tau}^{t}\frac{1}{\ell_1(s)}\eta_1^{\frac{12}{7}}(s)(43320+5345.2\eta_1^{\frac{12}{7}}(s))ds$ and

$$
\begin{aligned}
\dot{\ell}_1&=\max\{-2\ell_1^2\eta_1^{\frac{-2}{7}}+\ell_1\varrho_1(\zeta_1)\eta_1^{\frac{-2}{7}},0\}\\
u&=-\ell_1^{\frac{7}{5}}\eta_2\big(21.5+712\eta_2^{\frac{12}{7}}+3128\eta_2^{\frac{108}{35}}\\
&\quad+2(71061+1410\zeta_1^{\frac{24}{5}}+3556\zeta_1^{\frac{24}{7}})\\
&\quad+2\eta_2^{\frac{2}{7}}(\epsilon_1+5.8\epsilon_1^{\frac{2}{7}}\zeta_1^{\frac{12}{7}})((\eta_2-\epsilon_1\zeta_1)^{\frac{5}{7}}+\sin(z_1(t-\tau)))\big)^{\frac{7}{5}}
\end{aligned}
\tag{58}
$$

results in $\mathcal{L}U_{2LK}\leq-\eta_1^{\frac{12}{7}}-\eta_2^{\frac{12}{7}}$.

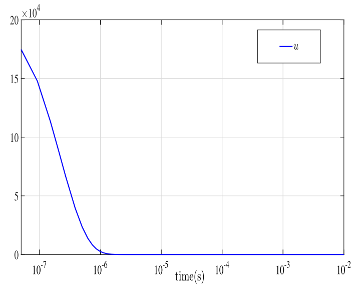

Fig. 3. Trajectory of the control u.

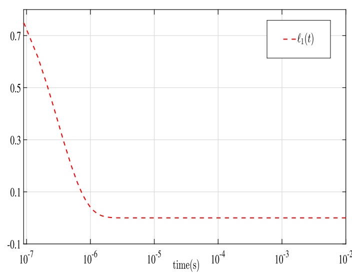

Fig. 4. Trajectory of the dynamic gain ℓ_1.

For demonstration, the initial condition is chosen as $[z_1(0),z_2(0),\zeta_1(0),\zeta_2(0)]^T=[-2,1.5,0.1,1]^T$. Figs. 1–4 indicate that the controller (58) can stabilize the closed-loop system and the effectiveness of the provided control strategy is clarified.

V. Conclusion

This article has studied the continuous control for stochastic low-order cascade nonlinear systems with time-varying delay and stochastic inverse dynamics. The restrictive conditions on unknown drift and diffusion nonlinearities have been removed. For low-order power situation, a continuous control scheme has been proposed by means of the dynamic gain manner and SISS. However, considering that the low-order powers of the system inevitably leads to exponential inflation in the design of the controller, there is a pity that the magnitude of controller and dynamic gain might be large, so how to reduce the cost of control is an important issue to be solved in the future. What is more, the stabilization problems of "polynomial integrators, general triangular form systems, fractional power integrators," "different low-order powers," and "output-feedback controller" are still open.

Appendix

Proof of Lemma 7: Because f_i and g_i are continuous and vanish at the origin, combining the fact that $r\in\mathbb{R}_{\text{odd}}^{\triangleleft}$ with Lemmas 3 and 4, we obtain the existence of nonnegative smooth functions $\tilde{\gamma}_{ilj}, \tilde{\Upsilon}_{ilj}, \tilde{\gamma}^0_{ilk}, \tilde{\Upsilon}^0_{ilk}, l=1,2$, such that

$$
\begin{aligned}
&\left|f_i(\bar{z}_i,\bar{z}_i(t-\tau(t)),\bar{\zeta}_i,\bar{\zeta}_i(t-\tau(t)))\right|\\
&\le\sum_{j=1}^{i}\big(\|z_j\|\tilde{\gamma}_{i1j}(z_j)+\|z_j(t-\tau(t))\|\tilde{\gamma}_{i2j}(z_j(t-\tau(t)))\\
&\quad+|\zeta_j|\tilde{\Upsilon}_{i1j}(\zeta_j)+|\zeta_j(t-\tau(t))|\tilde{\Upsilon}_{i2j}(\zeta_j(t-\tau(t)))\big)\\
&=\|z_j(t-\tau(t))\|^r\|z_j(t-\tau(t))\|^{1-r}\tilde{\gamma}_{i2j}(z_j(t-\tau(t)))\\
&\quad+\sum_{j=1}^{i}\big(\|z_j\|^r\|z_j\|^{1-r}\tilde{\gamma}_{i1j}(z_j)+|\zeta_j|^r\cdot|\zeta_j|^{1-r}\tilde{\Upsilon}_{i1j}(\zeta_j)\\
&\quad+|\zeta_j(t-\tau(t))|^r\cdot|\zeta_j(t-\tau(t))|^{1-r}\tilde{\Upsilon}_{i2j}(\zeta_j(t-\tau(t)))\big)\\
&\le\sum_{j=1}^{i}\big(\|z_j\|^r\gamma_{i1j}(z_j)+\|z_j(t-\tau(t))\|^r\gamma_{i2j}(z_j(t-\tau(t)))\\
&\quad+|\zeta_j|^r\Upsilon_{i1j}(\zeta_j)+|\zeta_j(t-\tau(t))|^r\Upsilon_{i2j}(\zeta_j(t-\tau(t)))\big)\\
&\Big\|g_i^T(\bar{z}_i,\bar{z}_i(t-\tau(t)),\bar{\zeta}_i,\bar{\zeta}_i(t-\tau(t)))g_j^T(\bar{z}_j,\\
&\bar{z}_j(t-\tau(t)),\bar{\zeta}_j,\bar{\zeta}_j(t-\tau(t)))\Big\|\\
&\le\sum_{k=1}^{i}\big(\|z_k\|^2\tilde{\gamma}^0_{i1k}(z_k)+\|z_k(t-\tau(t))\|^2\tilde{\gamma}^0_{i2k}(z_k(t-\tau(t)))\\
&\quad+\zeta_k^2\tilde{\Upsilon}^0_{i1k}(\zeta_k)+\zeta_k^2(t-\tau(t))\tilde{\Upsilon}^0_{i2k}(\zeta_k(t-\tau(t)))\big)\\
&=\sum_{k=1}^{i}\big(\|z_k\|^{1+r}\|z_k\|^{1-r}\tilde{\gamma}^0_{i1k}(z_k)+\zeta_k^{1+r}\zeta_k^{1-r}\tilde{\Upsilon}^0_{i1k}(\zeta_k)\\
&\quad+\|z_k(t-\tau(t))\|^{1+r}\|z_k(t-\tau(t))\|^{1-r}\tilde{\gamma}^0_{i2k}(z_k(t-\tau(t)))\\
&\quad+\zeta_k^{1+r}(t-\tau(t))\zeta_k^{1-r}(t-\tau(t))\tilde{\Upsilon}^0_{i2k}(\zeta_k(t-\tau(t)))\big)\\
&\le\sum_{k=1}^{i}\big(\|z_k\|^{1+r}\gamma^0_{i1k}(z_k)\\
&\quad+\|z_k(t-\tau(t))\|^{1+r}\gamma^0_{i2k}(z_k(t-\tau(t)))\\
&\quad+\zeta_k^{1+r}\Upsilon^0_{i1k}(\zeta_k)\\
&\quad+\zeta_k^{1+r}(t-\tau(t))\Upsilon^0_{i2k}(\zeta_k(t-\tau(t)))\big),i\le j
\end{aligned}
\tag{59}
$$

where $\gamma_{ilj}(\cdot)\ge\|z_j\|^{1-r}\tilde{\gamma}_{ilj}(\cdot),\gamma^0_{ilk}(\cdot)\ge\|z_k\|^{1-r}\tilde{\gamma}^0_{ilk}(\cdot),\Upsilon_{ilj}(\cdot)\ge|\zeta_j|^{1-r}\tilde{\Upsilon}^0_{ilj}(\cdot),\Upsilon^0_{ilk}(\cdot)\ge|\zeta_k|^{1-r}\tilde{\Upsilon}^0_{ilk}(\cdot)$, $l=1,2$ are smooth functions. □

Proof of Lemma 8: Inspired by the proof of Hadamard's lemma in [38, p. 17], it derives that

$$\Delta(Y)-\Delta(0)=\int_0^1 d\Delta(\theta Y)=R(Y)Y=Y^TR^T(Y)\tag{60}$$

where $R(Y)=\int_0^1\frac{\partial\Delta}{\partial\beta}|_{\beta=\theta Y}d\theta$ is a n-dimensional covector. Since $\Delta(0)=0$ and $\Delta(Y)\ge0$, it can be seen directly that $\Delta(Y)$ is at its minimum at $Y=0$, which means that $\frac{\partial\Delta}{\partial Y}(0)=0$ and $R^T(0)=0$. Similarly, it shows that

$$R^T(Y)-R^T(0)=\left(\int_0^1\frac{\partial R^T}{\partial\beta}|_{\beta=\theta Y}d\theta\right)Y:=H(Y)Y\tag{61}$$

where $H(Y)$ is a $n\times n$ matrix. From (60) and (61) and Lemmas 3 and 4, there exist appropriate smooth functions $\bar{b}_i(\zeta_i)\ge0$ such that

$$\Delta(Y) \leq \|H(Y)\| \cdot \|Y\|^2 \leq \sum_{i=1}^{n} \zeta_i^2 \bar{b}_i(\zeta_i). \tag{62}$$

Then, for each $l = 1, \ldots, n$, it can be deduced from the definition of r that ζ_l^{1-r} is a continuous function. Lemma 4 shows that there exists a smooth scalar function $b_l(\zeta_l)$ satisfying $\zeta_l^{1-r}\bar{b}_l(\zeta_l) \leq b_l(\zeta_l)$. Thus, $\Delta(Y) \leq \sum_{i=1}^{n} \zeta_i^{1+r} b_i(\zeta_i)$. □

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