Stability and L_1-Gain Analysis of Periodic Piecewise Positive Systems With Constant Time Delay

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***Abstract*—This article is concerned with the stability and L_1-gain analysis of periodic piecewise positive systems with constant time delay. λ-exponential stability, which is applied to characterize the decay rates of the considered systems, is investigated first. A copositive Lyapunov–Krasovskii functional is used to obtain a sufficient stability condition. The stability condition characterizes the convergent speed of the state by the system matrices and the size of the time delay. One can also apply the Lyapunov–Krasovskii functional to characterize the L_1-gain of the systems. By taking advantage of the periodic property of the system, linear inequalities are employed to characterize the L_1-gain, and an unweighted upper bound of the L_1-gain of the system is given.**

***Index Terms*—L_1-gain analysis, periodic systems, positive systems, stability analysis, time-delay systems.**

I. Introduction

The stability and input–output gain analysis for positive systems with time delays has been a hot topic in recent years not only because of its practical application in the areas of economics, biology, chemistry, and transportation [1]–[3], but also due to the nice properties inherited from the positivity of their system dynamics [4], [5]. For general systems with time delays, the Lyapunov–Krasovskii functional approach is commonly used for analyzing the stability and designing the H_∞ control schemes [6]–[9]. Since Lyapunov–Krasovskii functional candidates usually consist of inner product terms with vectors $x(t)$ and $x(t-d)$, some matrix inequalities are commonly given to decouple them [10], [11]. Due to the nonnegativity of the state in positive systems, copositive Lyapunov–Krasovskii functionals can be applied to analyze the stability and input–output gain performance of positive time-delay systems. The early result on the stability of linear positive systems with constant delay was proposed by Haddad and Chellaboina [12]. By applying copositive functionals, the delay term $x(t-d)$ can be eliminated and a delay independent equivalent stability condition can be given. The result was extended to the case of time-varying delays in [13]. Except for stability, L_1- and L_∞-gain analysis of positive systems is also one of the hot topics. The analysis of those gains could trace back to [14], where the authors characterized the input–output gains by linear inequalities. By unifying all the existing results for positive time-delay systems into a single framework, Briat summarized the stability and performance analysis of positive systems with delays in [15]. Furthermore, the results were extended to different performance analysis including state estimation [16], [17], decay rate characterization [18], and different kinds of positive systems, including positive periodic systems [19], 2-D positive Roesser systems [20], switched positive systems [21], [22]. For the switched positive systems with time delays, the stability and input–output gain are analyzed. By choosing a constant vector for the copositive Lyapunov–Krasovskii functional, a sufficient exponential stability condition of discrete-time switched positive linear systems with time delay was given in [21]. The diagonal Lyapunov function was given for quadratic Lyapunov functionals in [23], and a less conservative stability criterion was obtained by extending the Lyapunov matrices to time-varying Lyapunov matrix functions. By applying copositive Lyapunov–Krasovskii functionals, the ℓ_1-gain and L_1-gain characterization of both discrete-time and continuous-time switched positive systems with time delays were investigated in [24]–[26], respectively.

As a special kind of switched systems, periodic piecewise systems can be regarded as switched systems with a fixed switching sequence and a fixed dwell time for each subsystem. Such systems can be found in electrical circuits with switches [27], crossroad models with signal lamps [28], and ac–dc converters with reduced dc-link capacitance [29]. To our best knowledge, there are a few results on such kinds of systems. For periodic piecewise systems without time delays, the stability and L_2-gain were analyzed by using the approaches in analyzing the switched systems with dwell-time constraint [30]. A reduced conservative stability condition was then given in [31] by using a matrix polynomial approach. Along this line, the above results were extended to the time delay case. Xie *et al.* [32] applied time-varying Lyapunov–Krasovskii functionals to facilitate H_∞ controller design. Furthermore, the guaranteed cost control of the periodic piecewise linear time-delay systems was given in [33], where a sufficient delay-dependent condition for the asymptotic stability was derived, and H_2/H_∞-mixed performance was investigated. Since some parameters, such as absolute temperature and population, in practical systems, are inherent nonnegative, some of the aforementioned periodic piecewise systems, including electrical circuits and cross-road models, may be regarded as periodic piecewise positive systems. Taking advantage of positivity, linear inequalities, rather than linear matrix inequalities, are used to reduce the computational complexity in determining the stability of systems. The delay-free case has been investigated in [34]. As common phenomena in practical systems, time delays could lead to poor performance or even instability. To our best knowledge, there

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are no existing results on periodic piecewise positive systems with time delay. Motivated by the existing literature, we apply a copositive Lyapunov–Krasovskii functional to analyze the stability and L_1-gain performance of the periodic piecewise positive systems. In this article, some major improvements on analyzing periodic piecewise systems with time-delays will be carried out, including: 1) reduction of the conservatism of the stability criteria; 2) characterization of the unweighted input–output gains of the systems. Based on the ideas in [32] and [34], both continuous and discontinuous Lyapunov–Krasovskii functionals are applied to characterize the stability condition and the input–output gain. By introducing more degrees of freedom in the functional, the conservativeness of the systems is reduced.

The article can be briefly outlined as follows. The periodic piecewise positive systems with time delay are characterized in Section II. In Section III, the stability and L_1-gain analysis of the systems are presented. Example illustrating the effectiveness of our theorems is given in Section IV, and Section V concludes the article.

Notation: $\mathbb{R}^n$ denotes the n-dimensional real vector space, $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices, A^{T} denotes the transpose of matrix A, 1_n denotes an n-dimensional column vector with each entry equals to 1, $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\mathbb{R}_+ = \{x \mid x > 0,\ x \in \mathbb{R}\}$, and $\mathbb{R}_{0,+} = \{x \mid x \geq 0,\ x \in \mathbb{R}\}$. $v_{[i]}$ denotes the ith element of the vector v. The product of n matrices $M_{j_1}, M_{j_2}, \ldots, M_{j_n}$ is denoted by $\prod_{j=j_1}^{j_n} M_j = M_{j_n} M_{j_{n-1}} \cdots M_{j_1}$. In addition, $\|v\|_1 = \sum_{i=1}^{n} |v_{[i]}|$ stands for the 1-norm of a vector v, $\|\omega\|_{L_1} = \int_0^\infty \|\omega(t)\|_1 dt$ stands for the L_1-norm of a function ω. We say $\omega \in L_1$, if $\|\omega\|_{L_1} < \infty$. $\|v\|_\infty = \max_{i\in\{1,2,\ldots,n\}} |v_{[i]}|$ stands for the ∞-norm of a vector v, $\|\omega\|_{L_\infty} = \sup_{t\geq 0} \|\omega(t)\|_\infty$ stands for the L_∞-norm of a function ω. Some notations, which are commonly used in positive systems [35] are given as follows. $v \succeq (\succ) 0$ or $v \in \mathbb{R}^n_{0,+} (\mathbb{R}^n_+)$ means a real vector v is a nonnegative (positive) vector whose entries are all nonnegative (positive). $A \succeq (\succ) 0$ or $A \in \mathbb{R}^{m\times n}_{0,+} (\mathbb{R}^{m\times n}_+)$ means a real matrix. $A \in \mathbb{R}^{m\times n}$ is a nonnegative (positive) matrix. For two matrices A and $B \in \mathbb{R}^{m\times n}_{0,+} (\mathbb{R}^{m\times n}_+)$, $A \succeq (\succ) B$ means $A - B$ is a nonnegative (positive) matrix.

II. Problem Formulation

Consider a linear continuous-time periodic piecewise system with time delay given as follows:

$$
\begin{aligned}
\dot{x}(t) &= A(t)x(t) + A_d(t)x(t-d) + B_\omega(t)\omega(t) \\
z(t) &= C(t)x(t) + C_d(t)x(t-d) + D_\omega(t)\omega(t) \\
x(t) &= \phi(t), \quad t \in [-d, 0]
\end{aligned} \tag{1}
$$

where $x(t) \in \mathbb{R}^{n_x}$, $\omega(t) \in \mathbb{R}^{n_\omega}$, and $z(t) \in \mathbb{R}^{n_z}$ are the state, disturbance, and output, respectively. $A(t) = A(t+T)$, $A_d(t) = A_d(t+T)$, $B_\omega(t) = B_\omega(t+T)$, $C(t) = C(t+T)$, $C_d(t) = C_d(t+T)$, and $D_\omega(t) = D_\omega(t+T)$ with T to be the fundamental period. Moreover, the above matrix functions satisfy

$$
\begin{aligned}
A(t) &= A_{\sigma(i)}, \quad A_d(t) = A_{d,\sigma(i)}, \quad B_\omega(t) = B_{\omega,\sigma(i)} \\
C(t) &= C_{\sigma(i)}, \quad C_d(t) = C_{d,\sigma(i)}, \quad D_\omega(t) = D_{\omega,\sigma(i)}
\end{aligned}
$$

when $t \in [t_{i-1,\sigma(i)-1}, t_{i,\sigma(i)})$, where $i \in \{1, 2, \ldots, m\}$, $(\sigma(1), \sigma(2), \ldots, \sigma(m))$ is the cyclic permutation of $(1, 2, \ldots, m)$ with $t_{0,\sigma(1)-1} = 0$ and $t_{m,\sigma(m)} = T$. $x_t = x(t+\theta)$, $\theta \in [-d, 0]$. The time interval for each subsystem is defined as $T_{\sigma(i)} = t_{i,\sigma(i)} - t_{i-1,\sigma(i)-1}$. The definitions of positivity, λ-stability, and L_1-gain of system (1) are given as follows.

Definition 1 (Positivity): Periodic piecewise system (1) with time delay is said to be positive if for any initial condition $x_0 \succeq 0$, disturbance $\omega(t) \succeq 0$, and cyclic permutation $(\sigma(1), \sigma(2), \ldots, \sigma(m))$, one has state $x(t)$ and output $z(t)$ are in the nonnegative orthant for all $t \in \mathbb{R}_{0,+}$.

Lemma 1 (Positivity condition): Periodic piecewise system (1) is positive if and only if matrix A_i is Metzler, matrices $A_{d,i}$, $B_{\omega,i}$, C_i, $C_{d,i}$, and $D_{\omega,i}$ are nonnegative for all $i \in \{1, 2, \ldots, m\}$.

Proof: Periodic piecewise positive systems (1) with constant time delay can be seen as a special kind of switched positive systems with time-varying delays. According to [25, Proposition], the equivalent positivity condition for a switched positive systems with time-varying delays is that matrix A_i is Metzler and matrices $A_{d,i}$, $B_{\omega,i}$, C_i, $C_{d,i}$, $D_{\omega,i}$ are nonnegative for all $i \in \{1, 2, \ldots, m\}$. Thus, when the conditions for system matrices hold, system (1) is positive, and sufficiency is proved. For the necessity part, one can first regard the term $x(t-d)$ as the disturbance when $t \in [0, d)$. According to [36, Th. II.2], for system (1) not satisfying the conditions of system matrices, one can always find a nonnegative initial condition $\phi(t)$, and a nonnegative disturbance $\omega(t)$ such that the state $x(t)$ leaves the nonnegative orthant within a time interval $(0, \varepsilon]$, where $\varepsilon < d$. The necessity part is proved by a contrapositive argument.

Definition 2 (λ-exponential stability): Periodic piecewise positive system (1) with time delay is said to be λ-exponentially stable, if $x(t)$ satisfies

$$
\|x(t)\|_\infty \leq \kappa e^{-\lambda t}\phi_0 \quad \forall t \geq 0 \tag{2}
$$

where $\phi_0 = \sup_{-d\leq t\leq 0} \|\phi(t)\|_\infty$, for constants $\kappa \geq 1$, $\lambda > 0$.

Definition 3 (L_1-gain): For an asymptotically stable periodic piecewise system with time delay (1), under zero initial conditions, the L_1-gain is defined as the smallest $\gamma > 0$ such that

$$
\int_0^\infty \|z(t)\|_1 \, dt \leq \gamma \int_0^\infty \|\omega(t)\|_1 \, dt \tag{3}
$$

holds for all $\omega \in L_1$.

The definitions of λ-exponential stability and L_1-gain performance have been given. In the following sections, the conditions that characterize the decay rate and the L_1-gain of system (1) are discussed.

III. Stability and L_1-Gain Analysis

In this section, the stability and L_1-gain characterization of the periodic piecewise positive system (1) with time delay will be discussed. A continuous copositive Lyapunov–Krasovskii functional is given first. Sufficient time-dependent λ-exponential stability condition and unweighted upper bound of the L_1-gain are given and characterized by linear inequalities. When applying discontinuous Lyapunov–Krasovskii functional, we can reduce the conservativeness of the inequalities to avoid the weighted characterization of the upper bound of input–output gain [33]. It is shown that for both continuous and discontinuous copositive Lyapunov–Krasovskii functionals, unweighted upper bounds of the L_1-gain can be obtained.

Theorem 1 (λ-exponential stability condition): Suppose $\omega(t) \equiv 0$. Periodic piecewise positive system (1) with time delay is λ-exponentially stable if there exist a set of vectors $p_i \succ 0$, $q_i \succeq 0$, where $i = 0, 1, \ldots, m-1$, and $r \succeq 0$ satisfying

$$
\frac{p_i - p_{i-1}}{T_i} + A_i^{\mathrm{T}} p_{i-1} + e^{\lambda d}(q_{i-1} + dr) \prec -\lambda p_{i-1} \tag{4}
$$

$$
\frac{p_i - p_{i-1}}{T_i} + A_i^{\mathrm{T}} p_i + e^{\lambda d}(q_i + dr) \prec -\lambda p_i \tag{5}
$$

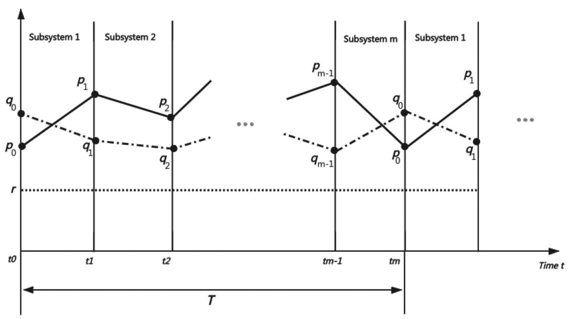

Fig. 1. Continuous vector functions $p(t)$, $q(t)$, and r.

$$A_{d,i}^{\mathrm{T}} p_{i-1} - q_{i-1} \preceq 0 \tag{6}$$

$$A_{d,i}^{\mathrm{T}} p_{i} - q_{i} \preceq 0 \tag{7}$$

$$\frac{q_i - q_{i-1}}{T_i} - r \preceq 0 \tag{8}$$

$$p_0 = p_m,\ q_0 = q_m \tag{9}$$

for all $i \in \{1, 2, \ldots, m\}$.

Proof: First of all, let $\sigma(i) = i$ for all $i \in \{1, 2, \ldots, m\}$. The copositive Lyapunov–Krasovskii functional candidate is given as follows:

$$V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t) \tag{10}$$

where

$$V_1(x_t) = e^{\lambda t} p^{\mathrm{T}}(t) x(t),\ V_2(x_t) = \int_{t-d}^{t} e^{\lambda(\tau+d)} q^{\mathrm{T}}(t) x(\tau) d\tau$$

$$V_3(x_t) = \int_{-d}^{0} \int_{t+\tau}^{t} e^{\lambda(\theta+d)} r^{\mathrm{T}} x(\theta) d\theta d\tau$$

$p(t)$ and $q(t)$ are periodic vector functions with T as the fundamental period, $p(t) = \alpha_i(t) p_{i-1} + (1-\alpha_i(t)) p_i$, $q(t) = \alpha_i(t) q_{i-1} + (1-\alpha_i(t)) q_i$, and $\alpha_i(t) = \frac{kT_p + t_i - t}{T_i}$, where $t \in [kT_p + t_{i-1}, kT_p + t_i)$. When $t \in [kT_p + t_{i-1}, kT_p + t_i)$, the derivatives of $V_1(x_t)$, $V_2(x_t)$, and $V_3(x_t)$ are given as follows:

$$\begin{aligned}\dot{V}_1(x_t) &= \lambda V_1(x_t) + e^{\lambda t} \frac{p_i^{\mathrm{T}} - p_{i-1}^{\mathrm{T}}}{T_i} x(t) \\ &\quad + e^{\lambda t} p^{\mathrm{T}}(t) \left[A_i x(t) + A_{d,i} x(t-d)\right] \end{aligned} \tag{11}$$

$$\begin{aligned}\dot{V}_2(x_t) &= e^{\lambda(t+d)} q^{\mathrm{T}}(t) x(t) - e^{\lambda t} q^{\mathrm{T}}(t) x(t-d) \\ &\quad + \int_{t-d}^{t} e^{\lambda(\tau+d)} \left(\frac{q_i^{\mathrm{T}} - q_{i-1}^{\mathrm{T}}}{T_i}\right) x(\tau) d\tau \end{aligned} \tag{12}$$

$$\begin{aligned}\dot{V}_3(x_t) &= \int_{-d}^{0} \left[e^{\lambda(t+d)} r^{\mathrm{T}} x(t) - e^{\lambda(t+d+\tau)} r^{\mathrm{T}} x(t+\tau)\right] d\tau \\ &= d e^{\lambda(t+d)} r^{\mathrm{T}} x(t) - \int_{t-d}^{t} e^{\lambda(\tau+d)} r^{\mathrm{T}} x(\tau) d\tau. \end{aligned} \tag{13}$$

The vector functions of $p(t)$, $q(t)$, and r for the construction of $V(x_t)$ are illustrated in Fig. 1.

Based on (11)–(13), the derivative of $V(x_t)$ is given as

$$\begin{aligned}\dot{V}(x_t) = e^{\lambda t} \Bigg\{ & \alpha_i(t) \left[\frac{p_i^{\mathrm{T}} - p_{i-1}^{\mathrm{T}}}{T_i} x(t) + p_{i-1}^{\mathrm{T}} \left(A_i x(t) + A_{d,i} x(t-d)\right)\right. \\ & + \lambda p_{i-1}^{\mathrm{T}} x(t) + e^{\lambda d} \left(q_{i-1}^{\mathrm{T}} + d r^{\mathrm{T}}\right) x(t) - q_{i-1}^{\mathrm{T}} x(t-d) \\ & \left. + \left(\frac{q_i^{\mathrm{T}} - q_{i-1}^{\mathrm{T}}}{T_i} - r^{\mathrm{T}}\right) \int_{t-d}^{t} e^{\lambda(\tau+d-t)} x(\tau) d\tau \right] \\ & + (1-\alpha_i(t)) \left[\frac{p_i^{\mathrm{T}} - p_{i-1}^{\mathrm{T}}}{T_i} x(t) + p_i^{\mathrm{T}} \left(A_i x(t) + A_{d,i} x(t-d)\right)\right. \\ & + \lambda p_i^{\mathrm{T}} x(t) + e^{\lambda d} \left(q_i^{\mathrm{T}} + d r^{\mathrm{T}}\right) x(t) - q_i^{\mathrm{T}} x(t-d) \\ & \left. + \left(\frac{q_i^{\mathrm{T}} - q_{i-1}^{\mathrm{T}}}{T_i} - r^{\mathrm{T}}\right) \int_{t-d}^{t} e^{\lambda(\tau+d-t)} x(\tau) d\tau \right] \Bigg\}. \end{aligned} \tag{14}$$

Combining (14) with inequalities (4)–(8), $\dot{V}(x_t) \leq 0$ for all $t \in [kT + t_{i-1}, kT + t_i)$, and $i \in \{1, 2, \ldots, m\}$. Since the Lypapunov–Krasovskii functional (10) is continuous, one has $V(x_t) \leq V(x_0)$ for all $t \in \mathbb{R}_{0,+}$. Since $V_1(x_t)$, $V_2(x_t)$, and $V_3(x_t)$ are nonnegative for all $t \in \mathbb{R}_{0,+}$, one has

$$e^{\lambda t} p^{\mathrm{T}}(t) x(t) \leq V(x_t) \leq V(x_0). \tag{15}$$

When $t = 0$, one has

$$V_1(x_0) = p_0^{\mathrm{T}} x(0) \leq n_x \|p_0\|_\infty \|x(0)\|_\infty$$

$$V_2(x_0) = \int_{-d}^{0} e^{\lambda(\tau+d)} q_0^{\mathrm{T}} \phi(\tau) d\tau \leq n_x d e^{\lambda d} \|q_0\|_\infty \phi_0$$

$$V_3(x_0) = \int_{-d}^{0} \int_{\tau}^{0} e^{\lambda(\theta+d)} r^{\mathrm{T}} x(\theta) d\theta d\tau \leq \frac{n_x d^2}{2} e^{\lambda d} \|r\|_\infty \phi_0.$$

Then, when $t = 0$, the copositive Lyapunov–Krasovskii functional $V(x_0)$ satisfies

$$V(x_0) \leq n_x \left[\|p_0\|_\infty + d e^{\lambda d} \|q_0\|_\infty + \frac{d^2}{2} e^{\lambda d} \|r\|_\infty\right] \phi_0. \tag{16}$$

Since $p^{\mathrm{T}}(t) x(t) \geq \inf_{0 \leq t \leq T_p} (\min_{i \in \{1,2,\ldots,n_x\}} p_{[i]}(t)) \|x(t)\|_\infty$, and $p(t) \succ 0$ for all $t \in [kT, (k+1)T)$, inequality (16) implies that

$$\|x(t)\|_\infty \leq \kappa e^{-\lambda t} \phi_0 \tag{17}$$

where $\kappa = \frac{n_x (\|p_0\|_\infty + d e^{\lambda d} \|q_0\|_\infty + \frac{d^2}{2} e^{\lambda d} \|r\|_\infty)}{\inf_{0 \leq t \leq T} (\min_{i \in \{1,2,\ldots,n_x\}} p_{[i]}(t))}$, and system (1) is λ-exponentially stable when $\sigma(i) = i$ for all $i \in \{1, 2, \ldots, m\}$. When $\sigma(i) \neq i$, without loss of generality, assume $\sigma(i') = 1$. One can let $\phi^*(t) = x(t + t_{i'-1,\sigma(i')-1})$, $t \in [-d, 0]$. Then, a new system starting from first subsystem with initial condition $\phi^*(\cdot)$ is constructed and the above proof of λ-exponential stability can be applied. Therefore, system (1) is λ-exponentially stable for any cyclic permutation $(\sigma(1), \sigma(2), \ldots, \sigma(m))$. □

Remark 1: For the characterization of the decay rate in Theorem 1, the ideas in [37] are applied. When applying the copositive Lyapunov–Krasovskii functional

$$\begin{aligned} V(x_t) &= p^{\mathrm{T}}(t) x(t) + \int_{t-d}^{t} e^{-\lambda(t-\tau)} e^{\lambda d} q^{\mathrm{T}}(t) x(\tau) d\tau \\ &\quad + \int_{-d}^{0} \int_{t+\tau}^{t} e^{-\lambda(t-\theta)} \left(e^{\lambda d} r^{\mathrm{T}}\right) x(\theta) d\theta d\tau \end{aligned} \tag{18}$$

where $t \in [kT + t_{i-1}, kT + t_i)$, conditions (4)–(9) can also be obtained.

Remark 2: When the time-delay d of the system (1) equals to 0, the condition that there exists a scalar $\lambda > 0$ such that (4)–(9) hold is equivalent to the sufficient asymptotic stability condition of [34, Th. 2].

Remark 3 (Time-varying delay): The approach in Theorem 1 can also deal with time-varying delay. For time-varying delay $d(t)$ satisfying $d(t) \in [0, \bar{d}]$ and $\dot{d}(t) \leq h < 1$, a sufficient λ-exponential stability condition could be derived by using functional (10) and replacing $V_2(x_t)$ in (10) with $V_2(x_t) = \int_{t-d(t)}^{t} e^{\lambda(\tau+\overline{d})} q^{\mathrm{T}}(t)x(\tau)d\tau$. The λ-exponential stability could be characterized by linear inequalities (4)–(5), (8)–(9), and

$$A_{d,i}^{\mathrm{T}} p_{i-1} - (1-h)q_{i-1} \preceq 0$$
$$A_{d,i}^{\mathrm{T}} p_{i} - (1-h)q_{i} \preceq 0.$$

By applying the copositive Lyapunov–Krasovskii functional (18) in Remark 1, an upper bound of the L_1-gain of system (1) is given in Theorem 2.

Theorem 2 (L_1-gain characterization): Suppose $\omega \in L_1$. Periodic piecewise positive system with time delay (1) is asymptotically stable and, under zero initial conditions, an upper bound of the L_1-gain is γ, if there exist scalar $\gamma > 0$, and a set of vectors $p_i \succ 0$, $q_i \succeq 0$, where $i = 0, 1, \ldots, m$, and $r \succeq 0$ satisfying

$$\frac{p_i - p_{i-1}}{T_i} + A_i^{\mathrm{T}} p_{i-1} + q_{i-1} + dr + C_i^{\mathrm{T}} 1_{n_z} \prec 0 \qquad (19)$$

$$\frac{p_i - p_{i-1}}{T_i} + A_i^{\mathrm{T}} p_{i} + q_{i} + dr + C_i^{\mathrm{T}} 1_{n_z} \prec 0 \qquad (20)$$

$$A_{d,i}^{\mathrm{T}} p_{i-1} - q_{i-1} + C_{d,i}^{\mathrm{T}} 1_{n_z} \preceq 0 \qquad (21)$$

$$A_{d,i}^{\mathrm{T}} p_{i} - q_{i} + C_{d,i}^{\mathrm{T}} 1_{n_z} \preceq 0 \qquad (22)$$

$$\frac{q_i - q_{i-1}}{T_i} - r \preceq 0 \qquad (23)$$

$$B_{\omega,i}^{\mathrm{T}} p_{i-1} + D_{\omega,i}^{\mathrm{T}} 1_{n_z} \prec \gamma 1_{n_\omega} \qquad (24)$$

$$B_{\omega,i}^{\mathrm{T}} p_{i} + D_{\omega,i}^{\mathrm{T}} 1_{n_z} \prec \gamma 1_{n_\omega} \qquad (25)$$

$$p_0 = p_m,\ q_0 = q_m \qquad (26)$$

for all $i \in \{1, 2, \ldots, m\}$.

Proof: First, the asymptotic stability of system (1) is proved. Since matrices C_i and $C_{d,i}$ are nonnegative matrices for all $i \in \{1, 2, \ldots, m\}$ and (19) and (20) are strict inequalities, according to (19)–(20), one can always find a positive scalar ε such that

$$\frac{p_i - p_{i-1}}{T_i} + A_i^{\mathrm{T}} p_{i-1} + e^{\varepsilon d}\left(q_{i-1} + dr\right) \preceq -\varepsilon p_{i-1} \qquad (27)$$

$$\frac{p_i - p_{i-1}}{T_i} + A_i^{\mathrm{T}} p_{i} + e^{\varepsilon d}\left(q_{i} + dr\right) \preceq -\varepsilon p_{i}. \qquad (28)$$

Combining (27)–(28) with (21)–(23), system (1) is ε-exponentially stable. According to Definition 2, when $t \to \infty$, $\|x(t)\|_\infty \to 0$, and system (1) is asymptotically stable. Then, an upper bound of L_1-gain of the system is investigated. By applying the copositive Lyapunov–Krasovskii functional in (18) with λ replaced by ε in Theorem 1, one has

$$\begin{aligned}&\dot{V}(x_t) + 1_{n_z}^{\mathrm{T}} z(t) - \gamma 1_{n_\omega}^{\mathrm{T}} \omega(t)\\ &= \dot{V}(x_t) + 1_{n_z}^{\mathrm{T}}\left(C_i x(t) + C_{d,i}x(t-d) + D_{\omega,i}\omega(t)\right) - \gamma 1_{n_\omega}^{\mathrm{T}} \omega(t)\\ &= \alpha_i(t)\Phi_{i,1}X(t) + (1-\alpha_i(t))\,\Phi_{i,2}X(t) - \varepsilon V(x_t) \qquad (29)\end{aligned}$$

where

$$X(t) = \begin{bmatrix} x^{\mathrm{T}}(t) & x^{\mathrm{T}}(t-d) & \int_{t-d}^{t} e^{\varepsilon(\tau+d-t)} x^{\mathrm{T}}(\tau)d\tau & \omega^{\mathrm{T}}(t)\end{bmatrix}^{\mathrm{T}}$$

$$\Phi_{i,1} = \begin{bmatrix} \frac{p_i - p_{i-1}}{T_i} + A_i^{\mathrm{T}} p_{i-1} + e^{\varepsilon d}\left(q_{i-1} + dr\right) + C_i^{\mathrm{T}} 1_{n_z} + \varepsilon p_{i-1} \\ A_{d,i}^{\mathrm{T}} p_{i-1} - q_{i-1} + C_{d,i}^{\mathrm{T}} 1_{n_z} \\ \frac{q_i - q_{i-1}}{T_i} - r \\ B_{\omega,i}^{\mathrm{T}} p_{i-1} + D_{\omega,i}^{\mathrm{T}} 1_{n_z} - \gamma 1_{n_\omega}\end{bmatrix}^{\mathrm{T}}$$

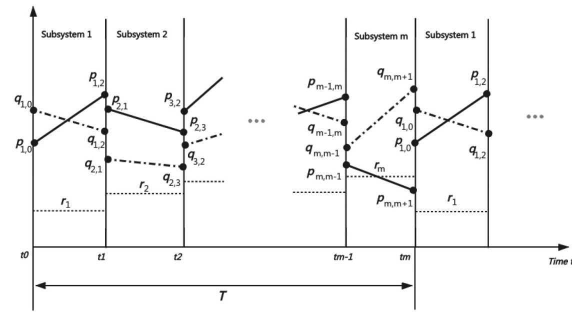

Fig. 2. Discontinuous vector functions $p(t)$, $q(t)$ and r_i.

$$\Phi_{i,2} = \begin{bmatrix} \frac{p_i - p_{i-1}}{T_i} + A_i^{\mathrm{T}} p_{i} + e^{\varepsilon d}\left(q_{i} + dr\right) + C_i^{\mathrm{T}} 1_{n_z} + \varepsilon p_{i} \\ A_{d,i}^{\mathrm{T}} p_{i} - q_{i} + C_{d,i}^{\mathrm{T}} 1_{n_z} \\ \frac{q_i - q_{i-1}}{T_i} - r \\ B_{\omega,i}^{\mathrm{T}} p_{i} + D_{\omega,i}^{\mathrm{T}} 1_{n_z} - \gamma 1_{n_\omega}\end{bmatrix}^{\mathrm{T}}$$

and $t \in [kT + t_{i-1}, kT + t_i)$. Since $V(x_t) \succeq 0$ for all $t \in \mathbb{R}_{0,+}$, according to (19)–(25)

$$\dot{V}(x_t) + 1_{n_z}^{\mathrm{T}} z(t) - \gamma 1_{n_\omega}^{\mathrm{T}} \omega(t) \leq 0 \qquad (30)$$

holds for all $t \in [kT + t_{i-1}, kT + t_i)$. Since $p(t)$ and $q_1(t)$ are continuous for all $t \in \mathbb{R}_{0,+}$, by integrating both sides of the inequality (30) from 0 to ∞, one has

$$\gamma \int_0^\infty 1_{n_\omega}\omega(\tau)d\tau - \int_0^\infty 1_{n_z} z(\tau)d\tau \geq V(x_\infty) \geq 0 \qquad (31)$$

which holds for all $t \in \mathbb{R}_{0,+}$, and an upper bound of the L_1-gain of system (1) is given by γ. □

Remark 4: One can always find a scalar $\lambda > 0$ such that the system (1) is λ-exponentially stable and inequalities

$$\frac{p_i - p_{i-1}}{T_i} + A_i^{\mathrm{T}} p_{i-1} + e^{\lambda d}\left(q_{i-1} + dr\right) + C_i^{\mathrm{T}} 1_{n_z} \prec -\lambda p_{i-1} \quad (32)$$

$$\frac{p_i - p_{i-1}}{T_i} + A_i^{\mathrm{T}} p_{i} + e^{\lambda d}\left(q_{i} + dr\right) + C_i^{\mathrm{T}} 1_{n_z} \prec -\lambda p_{i} \qquad (33)$$

and (21)–(26) hold for all $i \in \{1, 2, \ldots, m\}$. The upper-bound of the L_1 derived from (32)–(33) and (21)–(26) equals to the one derived from Theorem 2. However, when introducing the scalar λ, the couplings between $e^{\lambda d}$ and vectors q_{i-1} and r are introduced and the computational complexity for characterizing the upper-bound of the L_1-gain increases. Therefore, the λ-exponential stability is not used in Theorem 1.

By choosing different copositive Lyapunov–Krasovskii functionals, the conservativeness of the λ-exponential stability conditions and L_1-gain characterization will in general be different. The time-varying functions $p(t)$ and $q(t)$ could be discontinuous at the switching instants and vector r_i is used to replace r in the ith subsystem. The discontinuous vector functions $p(t)$, $q(t)$, and r_i are illustrated in Fig. 2. By applying such kind of functions, the λ-exponential stability condition and characterization of upper bound of L_1-gain are given in Theorem 3.

Theorem 3 (Discontinuous Lyapunov–Krasovskii Functional): Suppose $\omega \in L_1$. Periodic piecewise positive system with time delay (1) is λ^*-exponentially stable and, under zero initial conditions, if there exist scalars $\lambda > 0$, $\gamma \geq 0$, $\mu_i \geq 1$, and a set of vectors $p_{i,i+1} \succ 0$, $p_{i,i-1} \succ 0$, $q_{i,i-1} \succeq 0$, $q_{i,i+1} \succeq 0$ and $r_i \succeq 0$ satisfying

$$\Xi_i + A_i^{\mathrm{T}} p_{i,i-1} + e^{\lambda d}\left(q_{i,i-1} + dr_i\right) + C_i^{\mathrm{T}} 1_{n_z} \prec -\lambda p_{i,i-1} \qquad (34)$$

$$\Xi_i + A_i^{\mathrm{T}} p_{i,i+1} + e^{\lambda d}\left(q_{i,i+1} + dr_i\right) + C_i^{\mathrm{T}} 1_{n_z} \prec -\lambda p_{i,i+1} \qquad (35)$$

$$A_{d,i}^{\mathrm{T}} p_{i,i-1} - q_{i,i-1} + C_{d,i}^{\mathrm{T}} 1_{n_z} \preceq 0 \tag{36}$$

$$A_{d,i}^{\mathrm{T}} p_{i,i+1} - q_{i,i+1} + C_{d,i}^{\mathrm{T}} 1_{n_z} \preceq 0 \tag{37}$$

$$\frac{q_{i,i+1} - q_{i,i-1}}{T_i} - r_i \preceq 0 \tag{38}$$

$$B_{\omega,i}^{\mathrm{T}} p_{i,i-1} + D_{\omega,i}^{\mathrm{T}} 1_{n_z} \prec \gamma 1_{n_\omega} \tag{39}$$

$$B_{\omega,i}^{\mathrm{T}} p_{i,i+1} + D_{\omega,i}^{\mathrm{T}} 1_{n_z} \prec \gamma 1_{n_\omega} \tag{40}$$

$$p_{i+1,i} \preceq \mu_i p_{i,i+1} \tag{41}$$

$$q_{i+1,i} \preceq \mu_i q_{i,i+1} \tag{42}$$

$$r_{i+1} \preceq \mu_i r_i \tag{43}$$

$$\sum_{i=1}^{m} \ln(\mu_i) \leq \lambda T \tag{44}$$

where $\Xi_i \triangleq \frac{p_{i,i+1} - p_{i,i-1}}{T_i}$, $p_{m+1,m} = p_{1,0}$, $q_{m+1,m} = q_{1,0}$ and $r_{m+1} = r_1$, for all $i \in \{1, 2, \ldots, m\}$, then an upper bound of the L_1-gain is $\frac{\lambda \mu_{max} \gamma e^{\lambda T}}{\lambda^*}$, where $\mu_{max} = \max\{\mu_1, \mu_2, \ldots, \mu_m\}$, $\lambda^* = \frac{\lambda T - \sum_{i=1}^{m} \ln(\mu_i)}{T}$.

Proof: A discontinuous copositive Lyapunov–Krasovskii functional is given as

$$V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t) \tag{45}$$

where

$$V_1(x_t) = p^{\mathrm{T}}(t) x(t),\ V_2(x_t) = \int_{t-d}^{t} e^{-\lambda(t-\tau-d)} q^{\mathrm{T}}(t) x(\tau) d\tau$$

$$V_3(x_t) = \int_{-d}^{0} \int_{t+\tau}^{t} e^{-\lambda(t-\theta-d)} r_i^{\mathrm{T}} x(\theta) d\theta d\tau$$

$p(t)$ and $q(t)$ are periodic vector functions with T_p to be the fundamental period, $p(t) = \alpha_i(t) p_{i,i-1} + (1 - \alpha_i(t)) p_{i,i+1}$, $q(t) = \alpha_i(t) q_{i,i-1} + (1 - \alpha_i(t)) q_{i,i+1}$, and $\alpha_i(t) = \frac{kT + t_i - t}{T_i}$, where $t \in [kT + t_{i-1}, kT + t_i)$. Because matrices C_i and $C_{d,i}$ are nonnegative, inequalities (34)–(37) indicate

$$\Xi_i + A_i^{\mathrm{T}} p_{i,i-1} + e^{\lambda d} (q_{i,i-1} + d r_i) \prec 0 \tag{46}$$

$$\Xi_i + A_i^{\mathrm{T}} p_{i,i+1} + e^{\lambda d} (q_{i,i+1} + d r_i) \prec 0 \tag{47}$$

$$A_{d,i}^{\mathrm{T}} p_{i,i-1} - q_{i,i-1} \preceq 0 \tag{48}$$

$$A_{d,i}^{\mathrm{T}} p_{i,i+1} - q_{i,i+1} \preceq 0. \tag{49}$$

Similar to the proof of λ-exponential stability in Theorem 1 proposed by us and [34, Th. 1], system (1) is λ^*-exponentially stable when (46)–(49), (38), and (41)–(44) hold. According to system (1), when $\omega(t) \neq 0$, the derivative of functional (45) is

$$\begin{aligned} \dot{V}(x_t) &= \dot{p}^{\mathrm{T}}(t) x(t) + p^{\mathrm{T}}(t) \left(A_i x(t) + A_{d,i} x(t-d) + B_{\omega,i} \omega(t) \right) \\ &\quad - \lambda V_2(x_t) + e^{\lambda d} q^{\mathrm{T}}(t) x(t) - q^{\mathrm{T}}(t) x(t-d) \\ &\quad + \int_{t-d}^{t} e^{-\lambda(t-\tau-d)} \dot{q}^{\mathrm{T}}(t) x(\tau) d\tau + d e^{\lambda d} r_i^{\mathrm{T}} x(t) \\ &\quad - e^{\lambda d} \int_{t-d}^{t} e^{-\lambda(t-\tau)} r_i^{\mathrm{T}} x(\tau) d\tau - \lambda V_3(x_t). \end{aligned} \tag{50}$$

When $t \in [kT + t_{i-1}, kT + t_i)$, by applying (34)–(40) to (50), one has

$$\dot{V}(x_t) \leq -\lambda V(x_t) + \gamma 1_{n_\omega}^{\mathrm{T}} \omega(t) - 1_{n_z}^{\mathrm{T}} z(t). \tag{51}$$

By integrating (51) from $kT + t_{i-1}$ to $kT + t_i^-$, inequality

$$\begin{aligned} V\left(x_{kT+t_i^-}\right) &\leq e^{-\lambda T_i} V\left(x_{kT+t_{i-1}}\right) + \int_{kT+t_{i-1}}^{kT+t_i} e^{-\lambda(kT+t_i-\tau)} \\ &\quad \times \left(\gamma 1_{n_\omega}^{\mathrm{T}} \omega(\tau) - 1_{n_z}^{\mathrm{T}} z(\tau) \right) d\tau \end{aligned} \tag{52}$$

holds. By applying (41)–(43), inequality $V(x_{kT+t_i}) \leq \mu_i V(x_{kT+t_i^-})$ holds for all i, and the relation between $V(x_{t_i})$ and $V(x_{t_{i-1}})$ satisfies

$$\begin{aligned} V\left(x_{kT+t_i}\right) &\leq \mu_i e^{-\lambda T_i} V\left(x_{kT+t_{i-1}}\right) + \mu_i \int_{kT+t_{i-1}}^{kT+t_i} e^{-\lambda(kT+t_i-\tau)} \\ &\quad \times \left(\gamma 1_{n_\omega}^{\mathrm{T}} \omega(\tau) - 1_{n_z}^{\mathrm{T}} z(\tau) \right) d\tau \end{aligned} \tag{53}$$

for all $i \in \{1, 2, \ldots, m\}$ and $k \in \mathbb{N}$. Let $F(\tau) = \gamma 1_{n_\omega}^{\mathrm{T}} \omega(\tau) - 1_{n_z}^{\mathrm{T}} z(\tau)$. Based on inequalities (51) and (53), when $t \in [kT + t_{i-1}, kT + t_i)$, one has

$$\begin{aligned} V(x_t) &\leq e^{-\lambda(t-kT-t_{i-1})} V\left(x_{kT+t_{i-1}}\right) + \int_{kT+t_{i-1}}^{t} e^{-\lambda(t-\tau)} F(\tau) d\tau \\ &\leq e^{-\lambda(t-kT-t_{i-1})} \left\{ \left(\prod_{j=1}^{km+i-1} \mu_j e^{-\lambda T_j} \right) V(x_0) \right. \\ &\quad + \sum_{j=1}^{km+i-2} \left[\mu_j \left(\prod_{l=j+1}^{km+i-1} \mu_l e^{-\lambda T_l} \right) \int_{t_{j-1}}^{t_j} e^{-\lambda(t_j-\tau)} F(\tau) d\tau \right] \\ &\quad \left. + \mu_{i-1} \int_{kT+t_{i-2}}^{kT+t_{i-1}} e^{-\lambda(kT+t_{i-1}-\tau)} F(\tau) d\tau \right\} \\ &\quad + \int_{kT+t_{i-1}}^{t} e^{-\lambda(t-\tau)} F(\tau) d\tau \end{aligned} \tag{54}$$

where t_j equals $aT + t_b$ when $j = am + b$ for all $a \in \mathbb{N}$ and $b \in \{0, 1, \ldots, m\}$, $\mu_{a+m} = \mu_a$ and $T_{a+m} = T_a$ for all $a \in \mathbb{N}$. Since $V(x_t) \geq 0$ for all $t \in \mathbb{R}_{0,+}$, under zero initial conditions, inequality (54) gives

$$\begin{aligned} &\sum_{j=1}^{km+i-2} \left[\left(\prod_{l=j}^{km+i-1} \mu_l \right) \int_{t_{j-1}}^{t_j} e^{-\lambda(t-\tau)} 1_{n_z}^{\mathrm{T}} z(\tau) d\tau \right] \\ &+ \mu_{i-1} \int_{kT+t_{i-2}}^{kT+t_{i-1}} e^{-\lambda(t-\tau)} 1_{n_z}^{\mathrm{T}} z(\tau) d\tau \\ &+ \int_{kT+t_{i-1}}^{t} e^{-\lambda(t-\tau)} 1_{n_z}^{\mathrm{T}} z(\tau) d\tau \leq \sum_{j=1}^{km+i-2} \left[\left(\prod_{l=j}^{km+i-1} \mu_l \right) \right. \\ &\left. \times \int_{t_{j-1}}^{t_j} e^{-\lambda(t-\tau)} \gamma 1_{n_\omega}^{\mathrm{T}} \omega(\tau) d\tau \right] + \mu_{i-1} \int_{kT+t_{i-2}}^{kT+t_{i-1}} e^{-\lambda(t-\tau)} \\ &\times \gamma 1_{n_\omega}^{\mathrm{T}} \omega(\tau) d\tau + \int_{kT+t_{i-1}}^{t} e^{-\lambda(t-\tau)} \gamma 1_{n_\omega}^{\mathrm{T}} \omega(\tau) d\tau. \end{aligned} \tag{55}$$

Since $\mu_i \geq 1$ for all $i \in \{1, 2, \ldots, m\}$, the left-hand side of (55) is greater than or equivalent to

$$\sum_{j=1}^{km+i-2} \left[\int_{t_{j-1}}^{t_j} e^{-\lambda(t-\tau)} 1_{n_z}^{\mathrm{T}} z(\tau) d\tau \right] + \int_{kT+t_{i-2}}^{kT+t_{i-1}} e^{-\lambda(t-\tau)} 1_{n_z}^{\mathrm{T}} z(\tau) d\tau$$
$$+ \int_{kT+t_{i-1}}^{t} e^{-\lambda(t-\tau)} 1_{n_z}^{\mathrm{T}} z(\tau) d\tau. \quad (56)$$

Since $\prod_{i=1}^{m} \mu_i e^{-\lambda T} \leq 1$, which is indicated by (44), the right-hand side of (55) is less than or equals to

$$\gamma e^{\lambda T} \mu_{max} \times \left(\sum_{j=1}^{km+i-2} \left[\int_{t_{j-1}}^{t_j} e^{-\lambda^*(t-\tau)} \gamma 1_{n_\omega}^{\mathrm{T}} \omega(\tau) d\tau \right] \right.$$
$$+ \int_{kT+t_{i-2}}^{kT+t_{i-1}} e^{-\lambda^*(t-\tau)} 1_{n_\omega}^{\mathrm{T}} \omega(\tau) d\tau$$
$$\left. + \int_{kT+t_{i-1}}^{t} e^{-\lambda^*(t-\tau)} 1_{n_\omega}^{\mathrm{T}} \omega(\tau) d\tau \right) \quad (57)$$

where $\mu_{max} = \max\{\mu_1, \mu_2, \ldots, \mu_m\}$, and $\lambda^* = \frac{\lambda T - \sum_{i=1}^{m} \ln(\mu_i)}{T}$. Replacing the left-hand side and right-hand side of (55) by (56) and (57), respectively, one has

$$\int_0^t e^{-\lambda(t-\tau)} 1_{n_z}^{\mathrm{T}} z(\tau) d\tau \leq \mu_{max} \gamma e^{\lambda T} \int_0^t e^{-\lambda^*(t-\tau)} 1_{n_\omega}^{\mathrm{T}} \omega(\tau) d\tau.$$

By integrating the above inequality from 0 to ∞, we have

$$\frac{1}{\lambda} \int_0^\infty 1_{n_z}^{\mathrm{T}} z(\tau) d\tau \leq \frac{\mu_{max} \gamma e^{\lambda T}}{\lambda^*} \int_0^\infty 1_{n_\omega}^{\mathrm{T}} \omega(\tau) d\tau. \quad (58)$$

Therefore, an upper bound of the L_1-gain of the system (1) is $\frac{\lambda \mu_{max} \gamma e^{\lambda T}}{\lambda^*}$. □

Remark 5: For periodic piecewise positive systems without time delay, λ may take a different value in each subsystem. However, for systems with time delay, it is hard to compare the values of the Lyapunov–Krasovskii functionals at time $kT + t_i^-$ and $kT + t_i$, when applying different values of λ for different subsystems. Therefore, a fixed λ is used for all the subsystems in Theorems 1 and 3.

Remark 6 (Extension to switched positive systems with average dwell time (ADT) constraint): Theorem 3 states an average dwell-time-like result. By replacing $p(t)$ and $q(t)$ in (45) with piecewise constant functions, one can obtain a sufficient stability condition, which has a similar constraint as (44). For a delay-free switched positive system with ADT constraint, one can refer to [28].

Remark 7: The conditions in Theorem 3 contain bilinear terms, which cannot be directly solved by convex optimization. In order to find a local minimum upper bound for a given λ, we propose an iterative algorithm. First, by fixing μ_i, we solve an optimization problem to minimize γ and obtain a set of vectors $p_{i,i+1}$, $p_{i,i-1}$, $q_{i,i+1}$, $q_{i,i-1}$, and r_i. Then for the fixed vectors, we minimize the value of μ_i. Using this iteration scheme, one can obtain an upper bound of the L_1-gain for each λ. The calculated upper bound will finally converge to a local minimum.

IV. Illustrative Example

An illustrative example is given as follows:

$$\dot{x}(t) = A(t)x(t) + A_d(t)x(t-d) + B_\omega(t)\omega(t)$$
$$z(t) = C(t)x(t) + C_d(t)x(t-d) + D_\omega(t)\omega(t)$$
$$x(t) = \phi(t), \quad t \in [-d, 0] \quad (59)$$

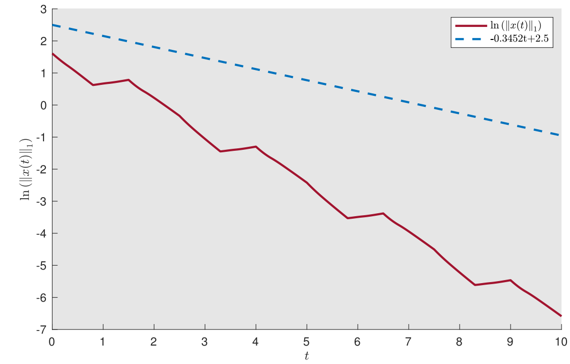

Fig. 3. Trajectory of output $z(t)$ of system (59).

TABLE I
Decay Rate of System (59) With Different Lyapunov–Krasovaskii Functionals

	μ_1	μ_2	μ_3	λ	Decay Rate
Theorem 1	N/A	N/A	N/A	0.3452	0.3452
Theorem 3	1	1	1	0.3625	0.3625
	1.001	1	1	0.3625	0.3629
	1.01	1	1	0.3618	0.3658
	1.1	1	1	0.3544	0.3925
	1	1.01	1	0.3616	0.3656

where $d = 0.3$, $\sigma(i) = i$, $m = 3$, $T_1 = 0.8$, $T_2 = 0.7$, $T_3 = 1$, and

$$A_1 = \begin{bmatrix} -3.5 & 1.2 \\ 0.4 & -2.9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -4.1 & 2.3 \\ 1.1 & -1.3 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} -2.5 & 1.3 \\ 0.2 & -3.7 \end{bmatrix}, \quad A_{d,1} = \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.3 \end{bmatrix}$$
$$A_{d,2} = \begin{bmatrix} 0.4 & 0.4 \\ 0.5 & 0.6 \end{bmatrix}, \quad A_{d,3} = \begin{bmatrix} 0.4 & 0.8 \\ 0.3 & 0.4 \end{bmatrix}$$
$$B_{\omega,1} = \begin{bmatrix} 0.9 \\ 1.2 \end{bmatrix}, \quad B_{\omega,2} = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}, \quad B_{\omega,3} = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix}$$
$$C_1 = \begin{bmatrix} 0.82 \\ 1 \end{bmatrix}^{\mathrm{T}}, \quad C_2 = \begin{bmatrix} 0.85 \\ 1.1 \end{bmatrix}^{\mathrm{T}}, \quad C_3 = \begin{bmatrix} 0.9 \\ 1 \end{bmatrix}^{\mathrm{T}}$$
$$C_{d,1} = \begin{bmatrix} 0.15 \\ 0.1 \end{bmatrix}^{\mathrm{T}}, \quad C_{d,2} = \begin{bmatrix} 0.12 \\ 0.15 \end{bmatrix}^{\mathrm{T}}, \quad C_{d,3} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}^{\mathrm{T}}$$

$D_{\omega,1} = 0.2$, $D_{\omega,2} = 0.1$, $D_{\omega,3} = 0.3$. When $\omega(t) \equiv 0$, according to Theorem 1, the system is λ-exponentially stable with $\lambda = 0.3452$. Fig. 3 depicts variation of the output $z(t)$ in system (59) with $\phi(\cdot) = \begin{bmatrix} 2 & 3 \end{bmatrix}^{\mathrm{T}}$. It shows that the value of $\ln(z(t))$ is always less than $-0.3452t + 2.5$ for all $t \in \mathbb{R}_{0,+}$, and the decay rate is less than 0.3452. According to Theorem 3, for different μ_i, the values of λ can be different. In order to show the effect of varying μ_i to scalar λ and decay rate λ^*, Table I is given. It can be seen that with the increase of μ_1, the value of λ increases and the estimated decay rate varies accordingly. Furthermore, we compare the effect of different μ_i on the estimated decay rate.

When considering the disturbance $\omega \in L_1$, the L_1-gain of the systems is investigated. For single-input single-output periodic piecewise positive systems under zero initial conditions, the L_1-gain of system (59) reaches its maximum when giving an impulse disturbance within

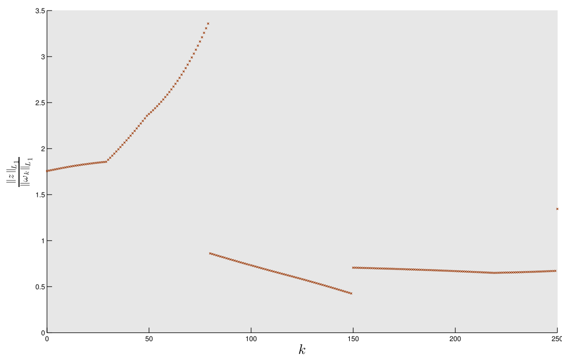

Fig. 4. Trajectory of $\frac{\int_0^\infty \|z(t)\|_1 dt}{\int_0^\infty \|\omega_k(t)\|_1 dt}$ of system (59) with disturbance $\omega_k(t)$.

TABLE II
UPPER BOUND OF L_1-GAIN OF SYSTEM (59) WITH DIFFERENT PARAMETERS

	λ	μ_1	μ_2	μ_3	γ	Upper bound
Theorem 2	N/A	N/A	N/A	N/A	8.300	8.300
Theorem 3	0.001	1	1	1	7.500	7.5192
	0.01	1	1	1	7.6380	7.8309
	0.01	1.01	1	1	7.4852	12.8764

the time interval $[0, T)$. To characterize the exact L_1-gain of system (59) and illustrate the conservativeness of Theorems 2 and 3, a set of disturbances $\omega_k(t)$ is introduced. By giving impulse disturbance $\omega_k(t) = \delta(t - 0.01k)$, where $\delta(t)$ is Dirac delta function and $k \in \{0, 1, \ldots, 250\}$, the ratio of $\frac{\int_0^\infty \|z(t)\|_1 dt}{\int_0^\infty \|\omega_k(t)\|_1 dt}$ is shown in Fig. 4. According to Theorems 2 and 3, for different kinds of Lyapunov–Krasovskii functional and different parameters, the calculated upper bounds are different. To further discuss the effect of the parameters, we fix λ and change the value of μ (or fix μ and change the value of λ). The calculated γ and upper bound of L_1-gain are shown in Table II. It can be observed that with the increase of λ, the value of γ increases. Furthermore, since the value of μ_i affects λ^* and γ, it has a significant influence on the calculated upper bound. One can also observe that for the same value of μ_i, the upper bound obtained by continuous Lyapunov–Krasovskii functional (Theorem 2) is greater than those obtained by discontinuous Lyapunov–Krasovskii functionals (Theorem 3). Therefore, the discontinuous functional can provide a less conservative results of the upper bound of L_1-gain.

V. CONCLUSION

In this article, the λ-exponential stability condition and L_1-gain characterization of periodic piecewise positive systems with time delay have been investigated. By applying a copositive Lyapunov–Krasovskii functional, the decay rate of the system has been characterized, and a delay-dependent λ-exponential stability condition has been given with linear inequalities. Furthermore, a Lyapunov–Krasovskii functional is applied to give an unweighted upper-bound of the L_1-gain of the systems. A numerical example has been given to illustrate the obtained results.

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