Unified Formulation of Multiagent Coordination With Relative Measurements

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***Abstract*—This article provides a unified solution to a general coordination problem of multiagent systems with relative measurements, including uncertain transformations in translation, rotation, reflection, and scale. First, we introduce relative measurements described by a matrix group expressing such transformations in a unified way. Then, we derive a necessary, and sufficient condition to achieve a coordination task by relative, distributed control according to a network topology, and a class of measurement information. Especially, a strict class of all realizable coordination tasks is characterized with an orbit associated with the transformation matrix group on measurement. Next, we show that the network topology required to coordination is the clique rigidity. Then, the clique rigidity for concrete coordination tasks is associated with conventional connectivity, e.g., connectedness and rigidity. Moreover, an intuitive condition is derived as a connectivity condition of the intersection graph of the maximal cliques (i.e., complete subgraphs). Finally, the new method is applied to formation control with unknown, heterogeneous scale factors and its effectiveness is demonstrated through simulations for both 2-D and 3-D spaces.**

***Index Terms*—Distributed control, formation control, multiagent systems, relative measurement.**

I. Introduction

MULTIAGENT systems have attracted a lot of attention in the field of control engineering for decades [1]. A multiagent system consists of a large number of components, called agents, which interact with each other through communication and/or sensing. For multiagent systems, distributed control [2], [3], based on local information of neighboring agents, is important due to its scalability (i.e., applicability regardless of the scales of systems). Actually, many types of distributed controllers have been designed for various coordination tasks, e.g., consensus [4], [5], coverage [6], flocking [7], [8], pursuit [9], [10] attitude synchronization [11], [12], assignment [13], and formation [14]–[21]. Formation is one of the most fundamental coordination tasks since it is applicable to various practical missions, including surveillance with multiple unmanned aerial vehicles [22] and ocean sampling with multiple autonomous underwater vehicles [23].

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TABLE I
Examples of Coordination Tasks (T), Network Topologies (N), Measurement Information (M), and Their Trends

	T	N	M
(i)	Position	No requirements	Absolute position
(ii)	Displacement	Connectedness	Relative positions & Common direction
(iii)	Distance	Rigidity	Relative positions
(iv)	Bearing	Bearing-rigidity	Relative bearings & Common direction
(v)	Angle	Angle-rigidity	Relative bearings
Trends	More inflexible ↓ More flexible	Sparser ↓ Denser	Need more ↓ Need less

The coordination tasks of multiagent systems are aimed at achieving a configuration prescribed by constraints on positional relations between agents, e.g, distances and angles. For this purpose, we have to design a distributed controller that provides a stable equilibrium set on which these constraints are satisfied [21]. Whether such a distributed controller exists or not mainly depends on the two factors: the network topology describing interactions between agents, and measurable information on neighbors and environments. In this respect, coordination problems of multiagent systems can be formulated as follows: for a given coordination task (T), find a required network topology (N), and measurement information (M) with which the coordination task (T) is realizable by distributed control.

According to this formulation, multiagent coordination problems in existing results can be classified as Table I, referring to [21], [24], and [25]. Displacement-based coordination (ii) imposes constraints on displacements, i.e., relative positions between agents (T), and it requires a connected network (N) with the information on relative positions of neighbors and a common direction (M) [5], [14]. Distance-based coordination (iii) prescribes distances between agents (T), requiring a rigid network (N), and relative positions of neighbors (M) [17], [18]. Bearing-based coordination (iv) imposes bearing constraints (T), and a bearing-rigid network (N) is necessary with relative bearings of neighbors and a common direction (M) [26], [27]. A feature of bearing-based coordination is that the scale freedom yields flexibility in coordination. Some research include the scale freedom in different ways [28], [29], and others combine several constraints [30]–[32]. Note that not all papers necessarily obey Table I. For example, for angle-based coordination (v) prescribing angle constraints (T), relative bearings of neighbors

TABLE II
TYPICAL TRANSFORMATION MATRIX SETS AND INVOLVED TRANSFORMATIONS

$\mathcal{M}$	Rotation	Reflection	Scale
$\{I_d\}$	No	No	No
$\mathrm{scaled}(\{I_d\})$	No	No	Yes
$\{I_d, R_w\}$	No	Yes	No
$\mathrm{scaled}(\{I_d, R_w\})$	No	Yes	Yes
$\mathrm{SO}(d)$	Yes	No	No
$\mathrm{scaled}(\mathrm{SO}(d))$	Yes	No	Yes
$\mathrm{O}(d)$	Yes	Yes	No
$\mathrm{scaled}(\mathrm{O}(d))$	Yes	Yes	Yes

(M) are essentially required as [33], but relative positions (M) are used in [34] and [35].

Let us compare the displacement- and distance-based coordination (ii), (iii) in Table I. With regard to the tasks (T), the distance-based coordination (iii) is more flexible than the displacement-based one (ii) in the sense that the former allows both the freedoms of rotation and translation for the coordination, while the latter does not allow the rotational freedom. As for network topologies (N), the distance-based coordination (iii) requires a denser network topology, say rigidity, than the displacement-based one (ii) requires, say connectedness. As for measurements (M), the distance-based coordination (iii) does not need a common direction while the displacement-based one (ii) does. Generally, the triplet (T, N, M) in multiagent coordination problems follow these trends. Namely, to achieve a more flexible coordination task (T), a denser network (N) is required, but less measurement information (M) is needed, as described in Table I.

Although various types of coordination tasks are found in the literature, existing papers have dealt with specific tasks individually, and there is no research that unifies these results. Therefore, once a new coordination task (T) is given, a distributed controller has to be designed over again, and a required network topology and measurement information (N, M) are specified to the designed controller. Moreover, no research has addressed the inverse problem: for a given network topology and measurement information (N, M), find a coordination task (T) realizable by distributed control. This is because a strict class of realizable coordination tasks (T) has not been revealed yet; thus, the class might be larger than expected from Table I. These issues are essential for control problems of multiagent systems, but are open so far.

To address the issues, this article provides a unified solution to a general coordination problem of multiagent systems with relative measurements, including uncertain transformations in translation, rotation, reflection, and scale. For this purpose, we mainly consider the relative measurements described as $x_j^{[i]} = M_i^{-1}(x_j - x_i) \in \mathbb{R}^d$, which denotes the measurement value of agent j at a position $x_j \in \mathbb{R}^d$ from the viewpoint of agent i at $x_i \in \mathbb{R}^d$. The matrix $M_i \in \mathcal{M}$, unknown to anyone, represents a type of such linear transformations, where a transformation matrix set $\mathcal{M} \subset \mathbb{R}^{d\times d}$ is determined by a type of measurement information. For example, if there is no common direction, $\mathcal{M} = \mathrm{SO}(d)$, the special orthogonal group, is employed for uncertain rotation transformations. See Table II for other examples. Through $\mathcal{M}$, the types of measurement information (M) can be expressed in a unified way.

In this setting, this article provides a comprehensive answer to the multiagent coordination problems based on the gradient-flow approach. First, we characterize a strict class of coordination tasks (T) realizable with the measurement information (M) by using an orbit of $\mathcal{M}$, which is a group-theoretical term referring to a collection of elements transformed by $\mathcal{M}$. Next, we show that the network topology (N) required to achieve a coordination task (T) is the clique rigidity, which is a generalized concept of rigidity for cliques (i.e., complete subgraphs). Then, a more intuitive condition on the network topology is derived such that an intersection graph of the maximal cliques is connected. Here, the number of the required intersections is determined by a free dimension of $\mathcal{M}$, which is a new concept associated with freeness, a term in group theory. Finally, to demonstrate the effectiveness of this method, it is applied to formation control with unknown, heterogeneous scale factors, and simulation results for both 2-D and 3-D spaces are shown.

This unified approach brings the following advantages. Systematic methods to specify a network topology and measurement information (N, M) and to design distributed controllers are provided, which are applicable to a wide range of tasks because the realizable coordination task (T) is represented in a general form. All the achievable coordination tasks (T) are specified from a network topology and a type of measurement information (N, M), which indicates that a class of achievable coordination tasks (T) is actually larger than expected in Table I. The trends of the relations among the triplet (T, N, M), shown in Table I, are explained in quantitative manners through the volume and the free dimension of $\mathcal{M}$.

This article is based on the author's conference proceeding paper [36]. Significant differences from [36] are as follows. We newly derive a necessary and sufficient condition on the network topology as clique rigidity, and associate it with conventional graph conditions for concrete coordination tasks, and provide a more intuitive condition with intersection graphs. All the achievable coordination tasks are characterized by an orbit. A simulation for a 3-D space is added to show the effectiveness of the method regardless of the dimension of space. The proofs are complete.

The rest of this article is organized as follows. Section II provides mathematical preliminaries on group and graph theories. Section III formulates the problem tackled in this article. Section IV gives a solution to the target problem and clarifies required network conditions. In Section V, the new method is applied to formation control with unknown, heterogeneous scale factors, and its effectiveness is demonstrated through simulations. Section VI provides the proofs of theorems. Section VII concludes this article.

Notation: Let $\mathbb{R}$ be the set of the real numbers. The d-dimensional identity matrix is denoted as $I_d \in \mathbb{R}^{d\times d}$. The notations $e_{dj} \in \mathbb{R}^d$ and $\mathbf{1}_d \in \mathbb{R}^d$ represent the d-dimensional unit vector with the jth entry one and the vector of all one's. Let $\mathrm{tr}(\cdot)$ and $\det(\cdot)$ denote the trace and the determinant of a square matrix, respectively. Let $\langle X, Y\rangle = \mathrm{tr}(X^\top Y)$ be the inner product of matrices $X, Y \in \mathbb{R}^{d\times n}$, and the Frobenius norm of a matrix is defined as $\|X\| = \sqrt{\langle X, X\rangle}$. For a matrix $X \in \mathbb{R}^{d\times n}$ and a set $\mathcal{T} \subset \mathbb{R}^{d\times n}$, their distance is defined as

$$\mathrm{dist}(X, \mathcal{T}) = \inf_{T\in\mathcal{T}} \|X - T\|. \tag{1}$$

For vectors $x_1, x_2, \ldots, x_n \in \mathbb{R}^d$ and a set $\mathcal{C} \subset \mathcal{N}$ of natural numbers, where $\mathcal{N} = \{1, 2, \ldots, n\}$, let $x_\mathcal{C} \in \mathbb{R}^{d\times|\mathcal{C}|}$ be the matrix consisting of the columns x_i for $i \in \mathcal{C}$ as

$$x_\mathcal{C} = [x_{i_1}\, x_{i_2}\, \cdots\, x_{i_{|\mathcal{C}|}}]$$

where $|\mathcal{C}|$ is the number of the elements of $\mathcal{C}$, and the elements $i_1, i_2, \ldots, i_{|\mathcal{C}|} \in \mathcal{C}$ satisfy $1 \le i_1 < i_2 < \cdots < i_{|\mathcal{C}|} \le n$. Let $\mathrm{col}_m(\cdot)$, $\mathrm{ave}(\cdot)$, and $\mathrm{cen}(\cdot)$ be the mth element, the average, and the center of a collection of vectors, respectively, defined as

$$\mathrm{col}_m(x_\mathcal{N}) = x_m, \qquad \mathrm{ave}(x_\mathcal{N}) = \frac{1}{n}\sum_{i\in\mathcal{N}} x_i$$
$$\mathrm{cen}(x_\mathcal{N}) = x_\mathcal{N} - \mathrm{ave}(x_\mathcal{N})\mathbf{1}_n^\top.$$

For a set $\mathcal{C} \subset \mathcal{N}$, let $\mathrm{proj}_\mathcal{C}(\cdot)$ be the projection of a set onto the $x_\mathcal{C}$-space, defined for a set $\mathcal{T} \subset \mathbb{R}^{d\times n}$ as

$$\mathrm{proj}_\mathcal{C}(\mathcal{T}) = \{x_\mathcal{C} \in \mathbb{R}^{d\times|\mathcal{C}|} : \exists x_1, x_2, \ldots, x_n \in \mathbb{R}^d \text{ s.t. } x_\mathcal{N} \in \mathcal{T}\}. \quad (2)$$

II. Mathematical Preliminaries

A. Group-Theoretical Concepts

This section provides group-theoretical concepts essential for multiagent coordination, e.g., *group, subgroup, scaled set, group action, semidirect product, freeness, orbit, invariant subset, and invariant functions*. The terminology is based on [37]–[39].

1) Group and Subgroup: A set $\mathcal{H}$ is called a *group* (with respect to multiplication) if $H_1H_2 \in \mathcal{H}$ is defined for any $H_1, H_2 \in \mathcal{H}$, satisfying the following:

- $(H_1H_2)H_3 = H_1(H_2H_3)$ for any $H_1, H_2, H_3 \in \mathcal{H}$;
- $I_\mathcal{H} \in \mathcal{H}$, where $I_\mathcal{H}$ is the identity element of $\mathcal{H}$;
- $H^{-1} \in \mathcal{H}$ for any $H \in \mathcal{H}$, where H^{-1} is the inverse element of H.

A group with respect to addition is defined with the zero and minus elements for the identity and inverse elements, respectively. A subset $\check{\mathcal{H}} \subset \mathcal{H}$ is called a *subgroup* of $\mathcal{H}$ if $\check{\mathcal{H}}$ itself is a group. Typical groups are illustrated as follows.

Example 1: The following matrix sets are groups with respect to multiplication:

i) the set of the orthogonal matrices, the orthogonal group $\mathrm{O}(d) \subset \mathbb{R}^{d\times d}$;

ii) the set of the orthogonal matrices with determinant 1, the special orthogonal group $\mathrm{SO}(d)$;

iii) the set $\{I_d\}$, consisting of only the identity matrix;

iv) the set $\{I_d, R_w\}$ with a reflection matrix $R_w \in \mathbb{R}^{d\times d}$, defined as $R_w = I_d - 2ww^\top$ with a unit vector $w \in \mathbb{R}^d$. All $\mathrm{SO}(d)$, $\{I_d\}$, and $\{I_d, R_w\}$ are subgroups of $\mathrm{O}(d)$. ■

Example 2: The sets $\mathbb{R}^d$ and $\{0\} \subset \mathbb{R}^d$ are groups with respect to addition, and $\{0\}$ is a subgroup of $\mathbb{R}^d$. ■

2) Scaled Set: For a group $\mathcal{H}$, a *scaled set* of $\mathcal{H}$, denoted as $\mathrm{scaled}(\mathcal{H})$, is a group consisting of the positive scalar multiples of the elements in $\mathcal{H}$, i.e.,

$$\mathrm{scaled}(\mathcal{H}) = \{sH : s > 0, H \in \mathcal{H}\} \quad (3)$$

with the product $(s_1H_1)(s_2H_2) = (s_1s_2)(H_1H_2)$ for $s_1, s_2 > 0$ and $H_1, H_2 \in \mathcal{H}$. In (3), the multiplier s is called a *scale*. Note that $\mathcal{H}$ is a subgroup of $\mathrm{scaled}(\mathcal{H})$, and that if $\check{\mathcal{H}}$ is a subgroup of $\mathcal{H}$, $\mathrm{scaled}(\check{\mathcal{H}})$ is a subgroup of $\mathrm{scaled}(\mathcal{H})$.

Example 3: From the definition (3) of the scaled set, a matrix $M \in \mathrm{scaled}(\mathrm{SO}(2))$ is parameterized with $s > 0$ and $\theta \in [-\pi, \pi)$ as $M = sR(\theta)$ for the rotation matrix

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \in \mathrm{SO}(2). \quad (4)$$

■

Note that $\mathrm{scaled}(\mathrm{O}(d))$ is described as

$$\mathrm{scaled}(\mathrm{O}(d)) = \{M \in \mathbb{R}^{d\times d} : M^\top M = (\det(M))^{\frac{2}{d}} I_d\} \quad (5)$$

indicating that the scale is given by $s = |\det(M)|^{\frac{1}{d}}$ from (3).

3) Group Action: For a group $\mathcal{H}$ and a set $\mathcal{X}$, we say that $\mathcal{H}$ *acts* on $\mathcal{X}$ if $Hx \in \mathcal{X}$ is defined for any $H \in \mathcal{H}$ and $x \in \mathcal{X}$, satisfying the following:

- $(H_1H_2)x = H_1(H_2x)$ for any $H_1, H_2 \in \mathcal{H}, x \in \mathcal{X}$;
- $I_\mathcal{H}x = x$ for any $x \in \mathcal{X}$.

For a set $\mathcal{X} \subset \mathbb{R}^d$ and a group $\mathcal{H}$ acting on $\mathcal{X}$, $\mathcal{H}$ acts on $\mathcal{X}^n \subset (\mathbb{R}^d)^n = \mathbb{R}^{d\times n}$ to vectors $x_1, x_2, \ldots, x_n \in \mathcal{X}$ as

$$H[x_1\, x_2\, \cdots\, x_n] = [Hx_1\, Hx_2\, \cdots\, Hx_n] \in \mathcal{X}^n. \quad (6)$$

4) Semidirect Product: For groups $\mathcal{M}$ and $\mathcal{B}$ with respect to multiplication and addition, respectively, such that $\mathcal{M}$ acts on $\mathcal{B}$, the *semidirect product* of $\mathcal{M}$ and $\mathcal{B}$, denoted by $\mathcal{M} \ltimes \mathcal{B}$, is a group consisting of the pairs (M, b) of $M \in \mathcal{M}$ and $b \in \mathcal{B}$ with the product

$$(M_1, b_1)(M_2, b_2) = (M_1M_2, b_1 + M_1b_2) \in \mathcal{M} \ltimes \mathcal{B}$$

for $(M_1, b_1), (M_2, b_2) \in \mathcal{M} \ltimes \mathcal{B}$. Accordingly, the identity element of $\mathcal{M} \ltimes \mathcal{B}$ is $(I_\mathcal{M}, 0)$ and the inverse element of $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$ is

$$(M, b)^{-1} = (M^{-1}, -M^{-1}b). \quad (7)$$

For a set $\mathcal{X}$ such that $\mathcal{M}$ and $\mathcal{B}$ act on $\mathcal{X}$ with respect to multiplication and addition, respectively, the semidirect product $\mathcal{M} \ltimes \mathcal{B}$ acts on $\mathcal{X}$ as

$$(M, b)x = Mx + b \in \mathcal{X} \quad (8)$$

for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$ and $x \in \mathcal{X}$.

Example 4: From Example 3, the semidirect product $\mathrm{scaled}(\mathrm{SO}(2)) \ltimes \mathbb{R}^2$ consists of the pairs $(sR(\theta), b)$ for $s > 0$, $\theta \in [-\pi, \pi)$, and $b \in \mathbb{R}^2$. From the action on multiple vectors (6) and the action of a semidirect product (8), $\mathrm{scaled}(\mathrm{SO}(2)) \ltimes \mathbb{R}^2$ acts on $\mathbb{R}^{2\times 2}$ as

$$(sR(\theta), b)[x_1\, x_2] = [sR(\theta)x_1 + b\;\, sR(\theta)x_2 + b] \quad (9)$$

for $x_1, x_2 \in \mathbb{R}^2$. Equation (9) means that the vectors $x_1, x_2 \in \mathbb{R}^d$ are scaled, rotated, and translated with a scale s, angle θ, and vector b, respectively. ■

5) Freeness: For a group $\mathcal{H}$ and a set $\mathcal{X}$ such that $\mathcal{H}$ acts on $\mathcal{X}$, we say that $\mathcal{H}$ is *free* to $\mathcal{X}$ if for every $x \in \mathcal{X}$

$$H_1x = H_2x, H_1, H_2 \in \mathcal{H} \Rightarrow H_1 = H_2. \quad (10)$$

For a group $\mathcal{H}$ acting on $\mathbb{R}^d$, a *free dimension* of $\mathcal{H}$ is defined as

$$\mathrm{fdim}(\mathcal{H}) = \min\{m \in \{0, 1, \ldots\} : \mathcal{H} \text{ is free to } \mathbb{R}^{d\times m}\backslash\mathcal{S}_m\} \quad (11)$$

where $\mathcal{S}_m \subset \mathbb{R}^{d\times m}$ is a set of measure zero such that $\mathcal{H}$ acts on $\mathbb{R}^{d\times m}\backslash\mathcal{S}_m$. See Table III for the free dimensions of typical semidirect products $\mathcal{M} \ltimes \mathcal{B}$. How to derive the free dimension is illustrated as follows.

TABLE III
FREE DIMENSIONS OF TYPICAL SEMIDIRECT PRODUCTS $\mathcal{M} \ltimes \mathcal{B}$

$\mathcal{M}$ \ $\mathcal{B}$	$\{0\}$	$\mathbb{R}^d$
$\{I_d\}$	0	1
$\mathrm{scaled}(\{I_d\})$, $\{I_d, R_w\}$, $\mathrm{scaled}(\{I_d, R_w\})$	1	2
$\mathrm{SO}(d)$, $\mathrm{scaled}(\mathrm{SO}(d))$	$d-1$	d
$\mathrm{O}(d)$, $\mathrm{scaled}(\mathrm{O}(d))$	d	$d+1$

Example 5: We obtain $\mathrm{fdim}(\mathrm{scaled}(\mathrm{SO}(2)) \ltimes \mathbb{R}^2) = 2$ as follows. For two elements $(s_{\mathrm{a}} R(\theta_{\mathrm{a}}), b_{\mathrm{a}}), (s_{\mathrm{b}} R(\theta_{\mathrm{b}}), b_{\mathrm{b}}) \in \mathrm{scaled}(\mathrm{SO}(2)) \ltimes \mathbb{R}^2$ with $s_{\mathrm{a}}, s_{\mathrm{b}} > 0$, $\theta_{\mathrm{a}}, \theta_{\mathrm{b}} \in [-\pi, \pi)$, b_{a}, $b_{\mathrm{b}} \in \mathbb{R}^2$, and two vectors $x_1, x_2 \in \mathbb{R}^2$, $x_1 \neq x_2$, the assumption part in the definition (10) of freeness is reduced to

$$\begin{aligned}(s_{\mathrm{a}} R(\theta_{\mathrm{a}}), b_{\mathrm{a}})[x_1\, x_2] &= [s_{\mathrm{a}} R(\theta_{\mathrm{a}}) x_1 + b_{\mathrm{a}}\; s_{\mathrm{a}} R(\theta_{\mathrm{a}}) x_2 + b_{\mathrm{a}}] \\ = (s_{\mathrm{b}} R(\theta_{\mathrm{b}}), b_{\mathrm{b}})[x_1\, x_2] &= [s_{\mathrm{b}} R(\theta_{\mathrm{b}}) x_1 + b_{\mathrm{b}}\; s_{\mathrm{b}} R(\theta_{\mathrm{b}}) x_2 + b_{\mathrm{b}}]\end{aligned} \tag{12}$$

from (9). By multiplying (12) by $[1\ -1]^\top$ from the right

$$s_{\mathrm{a}} R(\theta_{\mathrm{a}})(x_1 - x_2) = s_{\mathrm{b}} R(\theta_{\mathrm{b}})(x_1 - x_2) \tag{13}$$

is derived. Because $x_1 - x_2$ is not zero, taking the norms of the both sides in (13) yields $s_{\mathrm{a}} = s_{\mathrm{b}}$. Then, $R(\theta_{\mathrm{a}}) = R(\theta_{\mathrm{b}})$ and $b_{\mathrm{a}} = b_{\mathrm{b}}$ are derived from (13) and (12) in order. Hence, the conclusion part of (10) is achieved, and thus, $\mathrm{scaled}(\mathrm{SO}(2)) \ltimes \mathbb{R}^2$ is free to $\mathbb{R}^{2\times 2} \backslash \mathcal{S}_2$ with $\mathcal{S}_2 \subset \mathbb{R}^{2\times 2}$ consisting of the matrices that have equivalent columns. This discussion does not hold for less than two vectors, and thus, the free dimension of $\mathrm{scaled}(\mathrm{SO}(2)) \ltimes \mathbb{R}^2$ is two according to (11). ∎

6) Orbit: For a set $\mathcal{X}$, a group $\mathcal{H}$ acting on $\mathcal{X}$, and a subset $\mathcal{X}^* \subset \mathcal{X}$, an *$\mathcal{H}$-orbit* of $\mathcal{X}^*$ is defined as

$$\mathrm{orb}_{\mathcal{H}}(\mathcal{X}^*) = \{Hx \in \mathcal{X} : H \in \mathcal{H}, x \in \mathcal{X}^*\}. \tag{14}$$

For a singleton $\mathcal{X}^* = \{x^*\}$ of $x^* \in \mathcal{X}$, the $\mathcal{H}$-orbit of $\{x^*\}$ is denoted just as $\mathrm{orb}_{\mathcal{H}}(x^*)$.

Example 6: For vectors $x_1^*, x_2^*, x_3^* \in \mathbb{R}^2$, from (9) and (14), the $(\mathrm{scaled}(\mathrm{SO}(2)) \ltimes \mathbb{R}^2)$-orbit of $\{[x_1^*\, x_2^*\, x_3^*]\}$ is given as

$$\begin{aligned}&\mathrm{orb}_{\mathrm{scaled}(\mathrm{SO}(2))\ltimes\mathbb{R}^2}([x_1^* x_2^* x_3^*]) \\ &= \{[sR(\theta)x_1^* + b\; sR(\theta)x_2^* + b\; sR(\theta)x_3^* + b] : \\ &\quad s > 0, \theta \in [-\pi, \pi), b \in \mathbb{R}^2\}\end{aligned}$$

which is the set of the triangles in a plane similar to the original triangle with the apexes at x_1^*, x_2^*, x_3^*. ∎

7) Invariant Subset: For a set $\mathcal{X}$ and a group $\mathcal{H}$ acting on $\mathcal{X}$, a subset $\mathcal{T} \subset \mathcal{X}$ is said to be *$\mathcal{H}$-invariant* if $\mathcal{T}$ does not change under the action of $\mathcal{H}$, namely, $Hx \in \mathcal{T}$ holds for any $H \in \mathcal{H}$ and $x \in \mathcal{T}$. A typical example of an $\mathcal{H}$-invariant subset is an $\mathcal{H}$-orbit. Actually, any $\mathcal{H}$-invariant subset is characterized by an $\mathcal{H}$-orbit as follows.

Lemma 1: For a set $\mathcal{X}$ and a group $\mathcal{H}$ acting on $\mathcal{X}$, a subset $\mathcal{T} \subset \mathcal{X}$ is $\mathcal{H}$-invariant if and only if $\mathcal{T}$ is of the following form with a subset $\mathcal{X}^* \subset \mathcal{X}$:

$$\mathcal{T} = \mathrm{orb}_{\mathcal{H}}(\mathcal{X}^*). \tag{15}$$

Proof: For necessity, let $\bar{\mathcal{T}} = \mathrm{orb}_{\mathcal{H}}(\mathcal{T})$ for an $\mathcal{H}$-invariant subset $\mathcal{T}$. From the definition (14) of the orbit, for $x \in \bar{\mathcal{T}}$, $x = Hx^* \in \mathcal{T}$ holds with some $H \in \mathcal{H}$ and $x^* \in \mathcal{T}$. Hence, $\bar{\mathcal{T}} \subset \mathcal{T}$ is satisfied. The converse inclusion is obvious, and $\mathcal{T} = \bar{\mathcal{T}}$ holds. Hence, $\mathcal{T}$ is of the form (15) with $\mathcal{X}^* = \mathcal{T}$. The sufficiency is obvious. ∎

Two properties of the projection defined in (2) are given: the commutativity with the orbit operation and the maintenance of the invariance.

Lemma 2: For a group $\mathcal{H}$ acting on $\mathbb{R}^d$, a subset $\mathcal{X}^* \subset \mathbb{R}^{d\times n}$, and $\mathcal{C} \subset \mathcal{N} = \{1, 2, \ldots, n\}$, (i) the following holds:

$$\mathrm{proj}_{\mathcal{C}}(\mathrm{orb}_{\mathcal{H}}(\mathcal{X}^*)) = \mathrm{orb}_{\mathcal{H}}(\mathrm{proj}_{\mathcal{C}}(\mathcal{X}^*)). \tag{16}$$

(ii) If $\mathcal{T}$ is an $\mathcal{H}$-invariant subset of $\mathbb{R}^{d\times n}$, $\mathrm{proj}_{\mathcal{C}}(\mathcal{T})$ is an $\mathcal{H}$-invariant subset of $\mathbb{R}^{d\times|\mathcal{C}|}$.

Proof: (i) For vectors $x_1, x_2, \ldots, x_n \in \mathbb{R}^d$, from the definitions of the projection and the orbit, (2) and (14)

$$\begin{aligned}&\mathrm{proj}_{\mathcal{C}}(\mathrm{orb}_{\mathcal{H}}(\mathcal{X}^*)) = \mathrm{proj}_{\mathcal{C}}(\{Hx_{\mathcal{N}} : H \in \mathcal{H}, x_{\mathcal{N}} \in \mathcal{X}^*\}) \\ &= \{Hx_{\mathcal{C}} : H \in \mathcal{H}, x_{\mathcal{N}} \in \mathcal{X}^*\} \\ &= \{Hy : H \in \mathcal{H}, y \in \mathrm{proj}_{\mathcal{C}}(\mathcal{X}^*)\} = \mathrm{orb}_{\mathcal{H}}(\mathrm{proj}_{\mathcal{C}}(\mathcal{X}^*))\end{aligned}$$

holds, and (16) is achieved.

(ii) From the assumption, Lemma 1 guarantees that $\mathcal{T} = \mathrm{orb}_{\mathcal{H}}(\mathcal{X}^*)$ holds with some $\mathcal{X}^* \subset \mathbb{R}^{d\times n}$. From (i), $\mathrm{proj}_{\mathcal{C}}(\mathcal{T}) = \mathrm{proj}_{\mathcal{C}}(\mathrm{orb}_{\mathcal{H}}(\mathcal{X}^*)) = \mathrm{orb}_{\mathcal{H}}(\mathrm{proj}_{\mathcal{C}}(\mathcal{X}^*))$ holds, and thus $\mathrm{proj}_{\mathcal{C}}(\mathcal{T})$ is $\mathcal{H}$-invariant from Lemma 1. ∎

8) Invariant Functions: For a group $\mathcal{H}$ acting on a set $\mathcal{X}$, a function $v : \mathcal{X} \to \mathbb{R}$ is said to be *$\mathcal{H}$-invariant* if $v(Hx) = v(x)$ holds for any $H \in \mathcal{H}$ and $x \in \mathcal{X}$. A function $v(x)$ is said to be *relatively $\mathcal{H}$-invariant of weight* $\mu : \mathcal{H} \to \mathbb{R}$ if

$$v(Hx) = \mu(H)v(x)\, \forall H \in \mathcal{H}, x \in \mathcal{X}. \tag{17}$$

The distance function in (1) for an invariant subset $\mathcal{T} \subset \mathbb{R}^{d\times n}$ is relatively invariant for a semidirect product as follows.

Lemma 3: Consider subgroups $\mathcal{M}$ and $\mathcal{B}$ of $\mathrm{scaled}(\mathrm{O}(d))$ and $\mathbb{R}^d$, respectively, such that $\mathcal{M}$ acts on $\mathcal{B}$. If a subset $\mathcal{T} \subset \mathbb{R}^{d\times n}$ is $(\mathcal{M} \ltimes \mathcal{B})$-invariant, the distance function $v(X) = \mathrm{dist}(X, \mathcal{T})$ of $X \in \mathbb{R}^{d\times n}$ is relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $|\det(M)|^{\frac{1}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$.

Proof: For $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$, from the property of $\mathrm{scaled}(\mathrm{O}(d))$ in (5), the action on multiple vectors (6), and the action of a semidirect product (8)

$$\begin{aligned}v((M,b)X) &= \mathrm{dist}((M,b)X, \mathcal{T}) = \inf_{T\in\mathcal{T}} \|(M,b)X - T\| \\ &= \inf_{\bar{T}\in\mathcal{T}} \|(M,b)X - (M,b)\bar{T}\| \\ &= \inf_{\bar{T}\in\mathcal{T}} \sqrt{\mathrm{tr}((X - \bar{T})^\top M^\top M (X - \bar{T}))} \\ &= |\det(M)|^{\frac{1}{d}} \inf_{\bar{T}\in\mathcal{T}} \|X - \bar{T}\| = |\det(M)|^{\frac{1}{d}} v(X)\end{aligned}$$

is obtained, where $\bar{T} = (M, b)^{-1} T \in \mathcal{T}$ holds because of the $(\mathcal{M} \ltimes \mathcal{B})$-invariance of $\mathcal{T}$. Hence, from (17), $v(X)$ is relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $|\det(M)|^{\frac{1}{d}}$. ∎

B. Graph-Theoretical Concepts

This section provides graph-theoretical concepts such as *neighbor set, clique, maximal clique, clique rigidity*, and *intersection graph*, based on [32], [40], and [41].

1) Neighbor Set: Consider an undirected graph $G = (\mathcal{N}, \mathcal{E})$ with a node set $\mathcal{N} = \{1, 2, \ldots, n\}$ and an edge set $\mathcal{E}$ consisting of pairs $\{i, j\}$ of nodes $i, j \in \mathcal{N}$. The *neighbor set* of node $i \in \mathcal{N}$

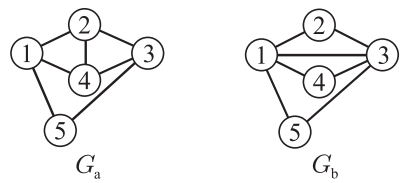


Fig. 1. Examples of graphs.

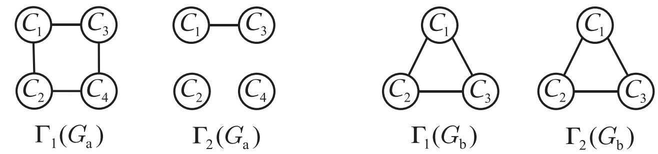


Fig. 2. Examples of one- and two-intersection graphs.

is defined as

$$\mathcal{N}_i = \{j \in \mathcal{N} : \{i,j\} \in \mathcal{E}\} \cup \{i\}. \tag{18}$$

Note that the neighbor set $\mathcal{N}_i$ contains node i itself.

2) Clique and Maximal Clique: For a node subset $\mathcal{C} \subset \mathcal{N}$, a subgraph is said to be *induced* by $\mathcal{C}$ and described as $G|_\mathcal{C} = (\mathcal{C}, \mathcal{E}|_\mathcal{C})$ for the edge subset $\mathcal{E}|_\mathcal{C} \subset \mathcal{E}$ of the pairs of the nodes in $\mathcal{C}$ belonging to $\mathcal{E}$. A node subset $\mathcal{C} \subset \mathcal{N}$ is called a *clique* if the subgraph induced by $\mathcal{C}$ is complete. A clique $\mathcal{C}$ is said to be *maximal* if $\mathcal{C}$ is not contained by each of the other cliques.

Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_q \subset \mathcal{N}$ be the maximal cliques in G, and let $\mathcal{Q} = \{1, 2, \ldots, q\}$ denote the set of the indexes of the maximal cliques. For node $i \in \mathcal{N}$, let $\mathcal{Q}_i \subset \mathcal{Q}$ be the set of the indexes of the maximal cliques that node i belongs to given as follows:

$$\mathcal{Q}_i = \{k \in \mathcal{Q} : i \in \mathcal{C}_k\}. \tag{19}$$

The following lemma indicates that the neighbors of node i can be grouped into cliques that node i belongs to. It will be declared that a distributed controller of multiagent systems can be decomposed into functions depending on the cliques.

Lemma 4: For the neighbor set $\mathcal{N}_i \subset \mathcal{N}$ of node i and the clique index set $\mathcal{Q}_i \subset \mathcal{Q}$ that node i belongs to, defined in (18) and (19), respectively, the following holds:

$$\mathcal{N}_i = \bigcup_{k \in \mathcal{Q}_i} \mathcal{C}_k. \tag{20}$$

Proof: For each $i \in \mathcal{N}$, if $j \in \mathcal{N}_i$, then $i, j \in \mathcal{C}_k$ holds with some $k \in \mathcal{Q}_i$ because any edge is contained by a maximal clique, namely, $i, j \in \mathcal{C}_k$ holds with some $k \in \mathcal{Q}$, and $\mathcal{C}_k$ contains i. The converse relation follows from the definition of the cliques. ■

The following example illustrates Lemma 4.

Example 7: For graph G_a in Fig. 1, the maximal cliques

$$\mathcal{C}_1 = \{1,2,4\}, \mathcal{C}_2 = \{1,5\}, \mathcal{C}_3 = \{2,3,4\}, \mathcal{C}_4 = \{3,5\} \tag{21}$$

are given. The set of the indexes of the maximal cliques is $\mathcal{Q} = \{1,2,3,4\}$, and those of the maximal cliques that each node belongs to are given from the definition (19) as

$$\mathcal{Q}_1 = \{1,2\}, \mathcal{Q}_2 = \mathcal{Q}_4 = \{1,3\}, \mathcal{Q}_3 = \{3,4\}, \mathcal{Q}_5 = \{2,4\}.$$

The neighbor set of node 1 is given as $\mathcal{N}_1 = \{1,2,4,5\}$, and $\mathcal{N}_1 = \sum_{k \in \mathcal{Q}_1} \mathcal{C}_k = \mathcal{C}_1 \cup \mathcal{C}_2$ holds as Lemma 4. ■

3) Intersection Graph: The *r-intersection graph* of the maximal cliques in G is defined as the graph $\Gamma_r(G) = (\mathcal{Q}, \breve{\mathcal{E}}_r)$ with the node set $\mathcal{Q}$ and the edge set

$$\breve{\mathcal{E}}_r = \{\{k, \ell\}, k, \ell \in \mathcal{Q} : |\mathcal{C}_k \cap \mathcal{C}_\ell| \geq r, k \neq \ell\} \tag{22}$$

consisting of the pairs of the maximal cliques such that the number of their intersections is more than or equal to r.

Example 8: The one- and two-intersection graphs of the maximal cliques of G_a in Fig. 1 are illustrated as $\Gamma_1(G_\mathrm{a})$ and $\Gamma_2(G_\mathrm{a})$ in Fig. 2 for the maximal cliques in (21). Those of G_b in Fig. 1 are illustrated as $\Gamma_1(G_\mathrm{b})$ and $\Gamma_2(G_\mathrm{b})$ in Fig. 2 for the maximal cliques $\mathcal{C}_1 = \{1,2,3\}, \mathcal{C}_2 = \{1,3,4\}, \mathcal{C}_3 = \{1,3,5\}$. ■

4) Clique Rigidity: A pair $(G, \mathcal{T})$ of a graph G and a set $\mathcal{T} \subset \mathbb{R}^{d \times n}$ is called a *framework*. The framework $(G, \mathcal{T})$ is said to be *clique rigid* if the following holds for any set of vectors $x_1, x_2, \ldots, x_n \in \mathbb{R}^d$:

$$x_{\mathcal{C}_k} \in \mathrm{proj}_{\mathcal{C}_k}(\mathcal{T}) \, \forall k \in \mathcal{Q} \Rightarrow x_\mathcal{N} \in \mathcal{T}. \tag{23}$$

The framework $(G, \mathcal{T})$ is clique rigid if $\mathcal{T}$ is the only configuration that can be constructed from the configurations $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{T})$, the projections of $\mathcal{T}$ by the maximal cliques $\mathcal{C}_k$.

The clique rigidity is equivalent to conventional graph conditions for specific $\mathcal{T}$ as follows.

Proposition 1: For $\mathcal{T} = \mathrm{orb}_{\{I_d\} \ltimes \mathbb{R}^d}(x^*_\mathcal{N})$ with $x^*_1, x^*_2, \ldots, x^*_n \in \mathbb{R}^d$, framework $(G, \mathcal{T})$ is clique rigid if and only if G is connected.

Proof: From the definition (14) of the orbit

$$\mathcal{T} = \mathrm{orb}_{\{I_d\} \ltimes \mathbb{R}^d}(x^*_\mathcal{N}) = \{(I_d, b) x^*_\mathcal{N} : b \in \mathbb{R}^d\} \tag{24}$$

is obtained. From the action on multiple vectors in (6) and the action of a semidirect product in (8), the conclusion part of the clique rigidity (23) for $\mathcal{T}$ in (24) is equivalent to

$$\exists b \in \mathbb{R}^d \, \mathrm{s.\,t.} \, x_i = x^*_i + b \, \forall i \in \mathcal{N}. \tag{25}$$

Additionally, by using Lemma 2 (i)

$$\begin{aligned} \mathrm{proj}_{\mathcal{C}_k}(\mathcal{T}) &= \mathrm{orb}_{\{I_d\} \ltimes \mathbb{R}^d}(\mathrm{proj}_{\mathcal{C}_k}(x^*_\mathcal{N})) \\ &= \{(I_d, b_k) x^*_{\mathcal{C}_k} : b_k \in \mathbb{R}^d\} \end{aligned}$$

is obtained, which reduces the assumption part of (23) to

$$\forall k \in \mathcal{Q}, \, \exists b_k \in \mathbb{R}^d \, \mathrm{s.\,t.} \, x_i = x^*_i + b_k \, \forall i \in \mathcal{C}_k. \tag{26}$$

For clique rigidity, we just have to show that (25) holds if (26) is satisfied.

For sufficiency, assume that G is connected and that (26) is satisfied. For a pair $\mathcal{C}_k, \mathcal{C}_\ell$, $k, \ell \in \mathcal{Q}$, $k \neq \ell$ of maximal cliques, let $\hat{k} \in \mathcal{C}_k$ and $\hat{\ell} \in \mathcal{C}_\ell$ be contained nodes. From the assumption, between nodes $\hat{k}$ and $\hat{\ell}$, there is a path, $i_1 (= \hat{k}), i_2, \ldots, i_m (= \hat{\ell}) \in \mathcal{N}$, satisfying $\{i_h, i_{h+1}\} \in \mathcal{E}$ for $h \in \{1, 2, \ldots, m-1\}$. Each edge belongs to a maximal clique, and thus, there exists $k_h \in \mathcal{Q}$ satisfying $i_h, i_{h+1} \in \mathcal{C}_{k_h}$. Then, $i_h \in \mathcal{C}_{k_{h-1}} \cap \mathcal{C}_{k_h}$ holds for $h \in \{2, 3, \ldots, m-1\}$, and from (26), $x_{i_h} = x^*_{i_h} + b_{k_{h-1}} = x^*_{i_h} + b_{k_h}$, namely, $b_{k_{h-1}} = b_{k_h}$ is obtained. Iteratively, we obtain $b_{k_1} = b_{k_{m-1}}$, namely, $b_k = b_\ell$. In this way, b_k agrees for all $k \in \mathcal{Q}$. Let b be the agreement, and (26) leads to (25). Hence, the framework is clique rigid. ■

The necessity part is obvious because if G is not connected, the framework is not clique rigid.

Next, the global rigidity is associated with the clique rigidity as follows, where we say that framework $(G, x^*_{\mathcal{N}})$ is globally rigid for a collection of vectors $x^*_1, x^*_2, \ldots, x^*_n \in \mathbb{R}^d$, if the following holds for any vectors $x_1, x_2, \ldots, x_n \in \mathbb{R}^d$:

$$\begin{aligned} &\|x_i - x_j\| = \|x^*_i - x^*_j\| \,\forall i, j \in \mathcal{N} \,\text{s. t.}\, \{i, j\} \in \mathcal{E} \\ &\quad \Rightarrow \|x_i - x_j\| = \|x^*_i - x^*_j\| \,\forall i, j \in \mathcal{N}. \end{aligned} \tag{27}$$

Proposition 2: For $\mathcal{T} = \mathrm{orb}_{\mathrm{O}(d) \ltimes \mathbb{R}^d}(x^*_{\mathcal{N}})$ with $x^*_1, x^*_2, \ldots, x^*_n \in \mathbb{R}^d$, framework $(G, \mathcal{T})$ is clique rigid if and only if framework $(G, x^*_{\mathcal{N}})$ is globally rigid.

Proof: In the same way as (26), for this $\mathcal{T}$, the assumption part of the clique rigidity in (23) is equivalent to

$$\forall k \in \mathcal{Q},\ \exists (M_k, b_k) \in \mathrm{O}(d) \ltimes \mathbb{R}^d \,\text{s. t.}\, x_i = M_k x^*_i + b_k \,\forall i \in \mathcal{C}_k.$$

From [42], this is equivalent to the assumption part of the global rigidity in (27). The conclusion parts of (23) and (27) are shown to be equivalent in the same way. Hence, the statements (23) and (27) are equivalent. ■

III. Problem Setting

A. Agent Models and Implementable Controllers

We consider a multiagent system consisting of n agents. Let $\mathcal{N} = \{1, 2, \ldots, n\}$ be the set of the agent indexes, and let $x_i(t) \in \mathbb{R}^d$ be the position coordinate of agent $i \in \mathcal{N}$ in a d-dimensional space. We have two types of frame: the *global frame* Σ and the *local frame* $\Sigma_i(t)$ of agent $i \in \mathcal{N}$. The origin of $\Sigma_i(t)$, the local origin, is given by a time-varying vector $b_i(t) \in \mathbb{R}^d$ in Σ, and a transformation of $\Sigma_i(t)$ from Σ is determined by a time-varying matrix $M_i(t) \in \mathbb{R}^{d\times d}$. Then, a local coordinate $p^{[i]}(t) \in \mathbb{R}^d$ in $\Sigma_i(t)$ is transformed into a global coordinate $p(t) \in \mathbb{R}^d$ in Σ as

$$p(t) = M_i(t)p^{[i]}(t) + b_i(t). \tag{28}$$

We assume that the transformation matrix $M_i(t)$ belongs to a set $\mathcal{M} \subset \mathbb{R}^{d\times d}$ for any $i \in \mathcal{N}$ at each time t. The set $\mathcal{M}$, called a *transformation matrix set*, is determined by a type of measurements. See Appendix A for details about how to determine $\mathcal{M}$ from measurements. As shown in Table II, various transformations of the frames including rotation, reflection, and scale can be expressed through $\mathcal{M}$. Also, we assume that the local origin $b_i(t)$ belongs to a set $\mathcal{B} \subset \mathbb{R}^d$. The set $\mathcal{B}$, called a *translation vector set*, represents the transformation of the frames in translation. Typically, the agent position is assigned to the local origin as $b_i(t) = x_i(t)$, yielding $\mathcal{B} = \mathbb{R}^d$. In contrast, if there is a landmark (a common observable point to all the agents), its position can be assigned to the global and local origins as $b_i(t) = 0$ to exclude the transformation in translation, and $\mathcal{B} = \{0\}$ is obtained.

Example 9: Suppose that cameras are available with unknown, heterogeneous scale factors, and neither common direction nor landmark is available. Then, $\mathcal{M} = \mathrm{scaled}(\mathrm{SO}(d))$ is adopted from Example 13 in Appendix A with $\mathcal{B} = \mathbb{R}^d$. From Example 3, for $d = 2$, the relation between global and local coordinates $p(t), p^{[i]}(t) \in \mathbb{R}^2$ is expressed as (28) for $M_i(t) = s_i(t)R(\theta_i(t)) \in \mathrm{scaled}(\mathrm{SO}(2))$ and $b_i(t) = x_i(t)$, where $s_i(t) > 0$ and $\theta_i(t) \in [-\pi, \pi)$ represent the scale and the rotation angle of $\Sigma_i(t)$ from Σ, as shown in Fig. 3. ■

Note that $M_i(t) \in \mathcal{M}$ and $b_i(t) \in \mathcal{B}$ are generally heterogeneous and cannot be specified by anyone including agent i itself. We assume that $\mathcal{M}$ and $\mathcal{B}$ are subgroups of $\mathrm{scaled}(\mathrm{O}(d))$ and $\mathbb{R}^d$ with respect to multiplication and addition, respectively, such that $\mathcal{M}$ acts on $\mathcal{B}$. This condition is satisfied by the typical $\mathcal{M}$ and $\mathcal{B}$ listed in Table II. According to the action of the semidirect product $\mathcal{M} \ltimes \mathcal{B}$ in (8), (28) is rewritten by $(M_i(t), b_i(t)) \in \mathcal{M} \ltimes \mathcal{B}$ as

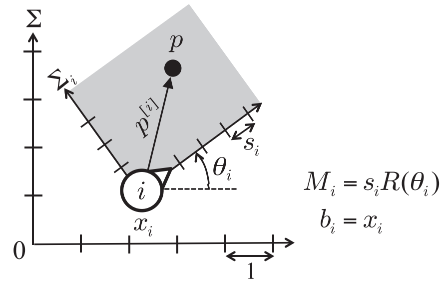


Fig. 3. Relation between global and local coordinates $p, p^{[i]} \in \mathbb{R}^2$ for $\mathcal{M} = \mathrm{scaled}(\mathrm{SO}(2))$ and $\mathcal{B} = \mathbb{R}^2$.

$$p(t) = (M_i(t), b_i(t))p^{[i]}(t). \tag{29}$$

Let $u_i(t) \in \mathbb{R}^d$ be the velocity driving agent i over $\Sigma_i(t)$, and assume that it can be directly controlled. Then, from the transformation (28), for the state $x_i(t)$ and the control input $u_i(t)$, the kinematic model of agent i is represented as

$$\dot{x}_i(t) = M_i(t)u_i(t). \tag{30}$$

When $\mathcal{M} = \mathrm{SO}(d)$, (30) is a common kinematic model of a rigid body, except that the rotation matrix $M_i(t) \in \mathcal{M}$ is not a state but is an uncertain parameter. Hence, even though $u_i(t)$ is determined by agent i, in which direction of Σ the agent moves is uncertain.

Assume that each agent can bilaterally obtain the relative measurements on its neighboring agents by sensing in the local frame. Let $\mathcal{N}_i \subset \mathcal{N}$ be the set of the neighbors of agent i, and let $\mathcal{E} \subset \mathcal{N}$ be the set of the pairs of the agents observing each other, satisfying the relation (18). Then, the topology of the sensing network is represented by an undirected graph $G = (\mathcal{N}, \mathcal{E})$. Let $x^{[i]}_j(t)$ be the local coordinate of a neighbor $j \in \mathcal{N}_i$ observed from agent i over $\Sigma_i(t)$, which is given from (28) and (29) as

$$\begin{aligned} x^{[i]}_j(t) &= M_i^{-1}(t)(x_j(t) - b_i(t)) \\ &= (M_i(t), b_i(t))^{-1}x_j(t) \end{aligned} \tag{31}$$

from the inverse of a semidirect product in (7). We assume that only the states $x^{[i]}_j(t)$ of the neighbors $j \in \mathcal{N}_i$ are available to agent i as relative measurements, and the control input $u_i(t)$ has to be generated as

$$u_i(t) = f_i(x^{[i]}_{\mathcal{N}_i}(t)) \tag{32}$$

with a function $f_i : \mathbb{R}^{d\times|\mathcal{N}_i|} \to \mathbb{R}^d$. We say that the control input of the form (32) is *relative and distributed*.

Remark 1: The control input (32) is available in practical situations. If the local origin is assigned as $b_i(t) = x_i(t)$, (32) requires the relative positions $x^{[i]}_j(t)$ of neighbors except i because the own position satisfies $x^{[i]}_i(t) = 0$ from (31). If the local origin is assigned as $b_i(t) = 0$ for a landmark, (32) requires the relative positions $x^{[i]}_j(t)$ of neighbors including i from the landmark, obtainable with the measurements $x_j^{[i]}(t) - x_i^{[i]}(t)$ of the relative positions from agent i. ■

B. Control Objective and Gradient-Flow Approach

In this article, we consider the following control objective:

$$\lim_{t\to\infty} \operatorname{dist}(x_{\mathcal{N}}(t), \mathcal{T}) = 0 \tag{33}$$

where $\mathcal{T} \subset \mathbb{R}^{d\times n}$, called a *target configuration set*, represents various coordination tasks in a unified way, including those in Table I. See [43] for typical examples of $\mathcal{T}$.

To achieve the control objective (33), the gradient-flow approach is employed. Let $v : \mathbb{R}^{d\times n} \to \mathbb{R}$ be an objective function of $x_{\mathcal{N}}$ with zero as a global minimum, evaluating the achievement of a given task. Under the gradient flow of $v(x_{\mathcal{N}})$, agent i is controlled according to

$$\dot{x}_i(t) = -\frac{\partial v}{\partial x_i}(x_{\mathcal{N}}(t)). \tag{34}$$

Then, $v(x_{\mathcal{N}}(t))$ is monotonically nonincreasing, and $x_{\mathcal{N}}(t)$ locally converges to the zero set $v^{-1}(0)$ of $v(x_{\mathcal{N}})$. Hence, to achieve (33) by (34), we need to find a function $v(x_{\mathcal{N}})$ taking zero exactly in $\mathcal{T}$, i.e., $v^{-1}(0) = \mathcal{T}$. A nonnegative function $v(x_{\mathcal{N}})$ satisfying this equation is called an *indicator* of $\mathcal{T}$. Let $\mathcal{F}_{\text{ind}}(\mathcal{T}) \subset \mathcal{F}_{\text{con}}$ be the set of the indicators, i.e.,

$$\mathcal{F}_{\text{ind}}(\mathcal{T}) = \{v(x_{\mathcal{N}}) \in \mathcal{F}_{\text{con}} : v^{-1}(0) = \mathcal{T},\, v(x_{\mathcal{N}}) \geq 0\} \tag{35}$$

where $\mathcal{F}_{\text{con}}$ is the set of the continuous scalar functions continuously differentiable almost everywhere.

For the system (30), the gradient flow in (34) of $v(x_{\mathcal{N}})$ is generated by the control input

$$u_i(t) = -M_i^{-1}(t)\frac{\partial v}{\partial x_i}(x_{\mathcal{N}}(t)). \tag{36}$$

For a relative, distributed controller (32) to be of the form (36), $v(x_{\mathcal{N}})$ has to satisfy

$$f_i(x_{\mathcal{N}_i}^{[i]}) = -M_i^{-1}\frac{\partial v}{\partial x_i}(x_{\mathcal{N}}) \tag{37}$$

with a function $f_i(\cdot)$ for any $(M_i, b_i) \in \mathcal{M} \ltimes \mathcal{B}$, where $x_j^{[i]} = (M_i, b_i)^{-1}x_j$ for $j \in \mathcal{N}_i$ according to the local coordinate (31). Whether $v(x_{\mathcal{N}})$ can satisfy (37) or not depends on $\mathcal{M}$, $\mathcal{B}$, and G. A set of the functions $v(x_{\mathcal{N}})$ satisfying (37) is defined as

$$\begin{aligned}\mathcal{F}_{\text{rd}}(\mathcal{M} \ltimes \mathcal{B}, G) = &\Big\{v(x_{\mathcal{N}}) \in \mathcal{F}_{\text{con}} : \forall i \in \mathcal{N},\, \exists f_i : \mathbb{R}^{d\times|\mathcal{N}_i|} \to \mathbb{R}^d \\ &\text{s. t.} - M_i^{-1}\frac{\partial v}{\partial x_i}(x_{\mathcal{N}}) = f_i(x_{\mathcal{N}_i}^{[i]}) \\ &\forall x_{\mathcal{N}} \in \mathbb{R}^{d\times n}\backslash\mathcal{S}_v, (M_i, b_i) \in \mathcal{M} \ltimes \mathcal{B}\Big\}\end{aligned} \tag{38}$$

where $\mathcal{S}_v \subset \mathbb{R}^{d\times n}$ is a set of measure zero.

Remark 2: The gradient of a function $v(x_{\mathcal{N}}) \in \mathcal{F}_{\text{con}}$ continuously differentiable almost everywhere is defined except for a set $\mathcal{S}_v$ of measure zero in (38), and can be discontinuous. The solution of the discontinuous gradient flow (34) is defined as an absolutely continuous function $x_{\mathcal{N}}(t)$ satisfying the differential inclusion

$$\dot{x}_{\mathcal{N}}(t) \in \mathcal{K}\left[-\frac{\partial v}{\partial x_{\mathcal{N}}}\right](x_{\mathcal{N}}(t)).$$

Here, for a function $F : \mathbb{R}^{d\times n} \to \mathbb{R}^{d\times n}$ continuous almost everywhere, the set-valued map $\mathcal{K}[F] : \mathbb{R}^{d\times n} \to \operatorname{pow}(\mathbb{R}^{d\times n})$ is defined as

$$\begin{aligned}\mathcal{K}[F](X) = \overline{\text{co}}\Big\{&Y \in \mathbb{R}^{d\times n} : \exists X_h \in \mathbb{R}^{d\times n}\backslash\mathcal{S},\, h = 1, 2, \ldots \\ &\text{s.t.} \lim_{h\to\infty} X_h = X,\, \lim_{h\to\infty} F(X_h) = Y\Big\}\end{aligned}$$

where $\operatorname{pow}(\cdot)$ is the power set of a set, $\overline{\text{co}}(\cdot)$ is the closure of the convex hull of a set, and $\mathcal{S} \subset \mathbb{R}^{d\times n}$ is a set of measure zero. The convergence of this solution is discussed in [32]. ■

C. Target Problem

To realize the control objective (33) by relative, distributed control, we have to design a function $v(x_{\mathcal{N}})$ belonging to the set $\mathcal{F}_{\text{ind}}(\mathcal{T}) \cap \mathcal{F}_{\text{rd}}(\mathcal{M} \ltimes \mathcal{B}, G)$. However, this set is possibly empty, depending on $\mathcal{T}$, $\mathcal{M}$, $\mathcal{B}$, and G. Our goal is to specify $\mathcal{T}$, $\mathcal{M}$, $\mathcal{B}$, and G for which $\mathcal{F}_{\text{ind}}(\mathcal{T}) \cap \mathcal{F}_{\text{rd}}(\mathcal{M} \ltimes \mathcal{B}, G)$ is nonempty to design a relative, distributed controller. The main problem tackled in this article is summarized as follows.

Problem 1: First, derive strict conditions on $\mathcal{T}$, $\mathcal{M}$, $\mathcal{B}$, and G under which the set $\mathcal{F}_{\text{ind}}(\mathcal{T}) \cap \mathcal{F}_{\text{rd}}(\mathcal{M} \ltimes \mathcal{B}, G)$ is nonempty. Next, characterize all the functions $v(x_{\mathcal{N}})$ belonging to this set. Finally, design a relative, distributed controller $f_i(x_{\mathcal{N}_i}^{[i]})$ by using one of the functions as (37). ■

IV. Main Results

A. Solutions to Problem 1

As a solution to the first part of Problem 1, strict conditions on $\mathcal{T}$, $\mathcal{M}$, $\mathcal{B}$, and G for nonemptiness of $\mathcal{F}_{\text{ind}}(\mathcal{T}) \cap \mathcal{F}_{\text{rd}}(\mathcal{M} \ltimes \mathcal{B}, G)$ are derived as follows.

Theorem 1: For a set $\mathcal{T} \subset \mathbb{R}^{d\times n}$, subgroups $\mathcal{M}$ and $\mathcal{B}$ of $\operatorname{scaled}(\text{O}(d))$ and $\mathbb{R}^d$, respectively, such that $\mathcal{M}$ acts on $\mathcal{B}$, and a graph G, the set $\mathcal{F}_{\text{ind}}(\mathcal{T}) \cap \mathcal{F}_{\text{rd}}(\mathcal{M} \ltimes \mathcal{B}, G)$ is nonempty if and only if the following two conditions are satisfied: (A) $\mathcal{T}$ is of the following form with some $\mathcal{X}^* \subset \mathbb{R}^{d\times n}$:

$$\mathcal{T} = \operatorname{orb}_{\mathcal{M}\ltimes\mathcal{B}}(\mathcal{X}^*). \tag{39}$$

(B) Framework $(G, \mathcal{T})$ is clique rigid.

Proof: The necessity is proved in Section VI-A, while the sufficiency follows from Theorem 2 given as follows. ■

As for condition (A), $\mathcal{T}$ in (39), the $(\mathcal{M} \ltimes \mathcal{B})$-orbit of $\mathcal{X}^*$, specifies all the target configuration sets realizable with the measurements transformed by $\mathcal{M} \ltimes \mathcal{B}$, indicating that such $\mathcal{T}$ contains the freedoms in $\mathcal{M} \ltimes \mathcal{B}$, e.g., translation, rotation, reflection, and scale. The trend between the coordination task and the measurement information in Table I is explained in a quantitative way: as the volume of $\mathcal{M} \ltimes \mathcal{B}$ increases [less measurement information is available (M)], the realizable set $\mathcal{T}$ becomes larger [the task becomes more flexible (T)].

Furthermore, $\mathcal{X}^* \subset \mathbb{R}^{d\times n}$ in (39) is selected by the designer, which broadens the class of realizable coordination tasks from

Table I (because a singleton $\mathcal{X}^* = \{x_{\mathcal{N}}^*\}$ for a desired configuration $x_{\mathcal{N}}^* \in \mathbb{R}^{d\times n}$ is employed). For example, by assigning $\mathcal{X}^* = \cup_p \{x_{\mathcal{N}}^{*p}\}$ of multiple points, a formation selection task [43] is expressed such that agents select and form one of multiple configurations $x_{\mathcal{N}}^{*1}, x_{\mathcal{N}}^{*2}, \ldots \in \mathbb{R}^{d\times n}$.

Next, as a solution to the second part of Problem 1, a condition for $v(x_{\mathcal{N}})$ to belong to the set $\mathcal{F}_{\rm ind}(\mathcal{T}) \cap \mathcal{F}_{\rm rd}(\mathcal{M} \ltimes \mathcal{B}, G)$ is derived. Let $\mathcal{Q} = \{1, 2, \ldots, q\}$ be the set of the indexes of the maximal cliques $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_q \subset \mathcal{N}$ in G, and the following theorem is obtained.

Theorem 2: For a set $\mathcal{T} \subset \mathbb{R}^{d\times n}$, subgroups $\mathcal{M}$ and $\mathcal{B}$ of scaled$(\mathrm{O}(d))$ and $\mathbb{R}^d$, respectively, such that $\mathcal{M}$ acts on $\mathcal{B}$, and a graph G, assume that conditions (A) and (B) in Theorem 1 hold. Then, a function $v : \mathbb{R}^{d\times n} \to \mathbb{R}$ belongs to $\mathcal{F}_{\rm ind}(\mathcal{T}) \cap \mathcal{F}_{\rm rd}(\mathcal{M} \ltimes \mathcal{B}, G)$ if and only if it is of the form

$$v(x_{\mathcal{N}}) = \sum_{k\in\mathcal{Q}} v_k(x_{\mathcal{C}_k}) \tag{40}$$

with some indicators $v_k : \mathbb{R}^{d\times|\mathcal{C}_k|} \to \mathbb{R}$ of $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{T})$, relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $(\det(M))^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$ for $k \in \mathcal{Q}$.

Proof: See Section VI-B. ■

Theorem 2 characterizes the function $v(x_{\mathcal{N}})$ with the parameters $v_k(x_{\mathcal{C}_k})$ of indicators of $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{T})$, relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $(\det(M))^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$. An example of such $v_k(x_{\mathcal{C}_k})$ is given as follows.

Lemma 5: Assume that condition (A) in Theorem 1 holds. Then, the following function is an indicator of $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{T})$, relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $(\det(M))^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$:

$$v_k(x_{\mathcal{C}_k}) = \frac{1}{2}(\mathrm{dist}(x_{\mathcal{C}_k}, \mathrm{proj}_{\mathcal{C}_k}(\mathcal{T})))^2. \tag{41}$$

Proof: The part of the indicator follows from the distance function (1). From (A), $\mathcal{T}$ is an $(\mathcal{M} \ltimes \mathcal{B})$-orbit as (39). Then, Lemmas 1 and 2 (ii) guarantee that $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{T})$ is an $(\mathcal{M} \ltimes \mathcal{B})$-invariant subset of $\mathbb{R}^{d\times|\mathcal{C}_k|}$. Hence, Lemma 3 leads to the relative $(\mathcal{M} \ltimes \mathcal{B})$-invariance of (41) of weight $(\det(M))^{\frac{2}{d}}$. ■

A meaning of the objective function (40) with (41) is explained as follows. From (1), achieving the control objective (33) is equivalent to solving the optimization problem

$$\mathrm{dist}(x_{\mathcal{N}}, \mathcal{T}) = \min_{T\in\mathcal{T}} \|x_{\mathcal{N}} - T\|. \tag{42}$$

Note that (42) depends on all the states $x_1, x_2, \ldots, x_n$, and can be solved in a centralized way. To overcome this issue, (42) is divided into cliquewise problems as

$$\mathrm{dist}(x_{\mathcal{C}_k}, \mathrm{proj}_{\mathcal{C}_k}(\mathcal{T})) = \min_{T_k\in\mathrm{proj}_{\mathcal{C}_k}(\mathcal{T})} \|x_{\mathcal{C}_k} - T_k\| \tag{43}$$

by projecting $x_{\mathcal{N}}$ and $\mathcal{T}$ onto the state space of $x_{\mathcal{C}_k}$ for each maximal clique $\mathcal{C}_k$. Because (43) depends only on the states of the agents belonging to a clique $\mathcal{C}_k$, it can be solved in a distributed way. The cliquewise problem (43) corresponds to (41), and combining it for all the maximal cliques yields (40). The solution of the combined problem is equal to the solution to (42) under conditions (A) and (B) in Theorem 1.

Finally, a solution to the last part of Problem 1 is given by partially differentiating $v(x_{\mathcal{N}})$ in (40) with (41). Let $\mathcal{Q}_i \subset \mathcal{Q}$ be the set of the indexes of the maximal cliques that agent i belongs to, defined as (19), and the following is obtained.

Theorem 3: For a set $\mathcal{T} \subset \mathbb{R}^{d\times n}$, subgroups $\mathcal{M}$ and $\mathcal{B}$ of scaled$(\mathrm{O}(d))$ and $\mathbb{R}^d$, respectively, such that $\mathcal{M}$ acts on $\mathcal{B}$, and a graph G, assume that condition (A) in Theorem 1 holds. Then, the gradient-based controller (36) of $v(x_{\mathcal{N}})$ in (40) with (41) is reduced to a relative, distributed controller (32) with

$$f_i(x_{\mathcal{N}_i}^{[i]}) = -\sum_{k\in\mathcal{Q}_i} (x_i^{[i]} - \mathrm{col}_{n_{ki}}(\hat{T}_{ki}(x_{\mathcal{C}_k}^{[i]}))) \tag{44}$$

where $n_{ki} \in \{1, 2, \ldots, |\mathcal{C}_k|\}$ is the place of $i \in \mathcal{N}$ in $\mathcal{C}_k$ such that $x_{\mathcal{C}_k} = [\cdots \overset{n_{ki}}{x_i} \cdots]$, and

$$\hat{T}_{ki}(x_{\mathcal{C}_k}^{[i]}) = (\hat{M}_{ki}(x_{\mathcal{C}_k}^{[i]}), \hat{b}_{ki}(x_{\mathcal{C}_k}^{[i]}))\hat{\Xi}_{ki}(x_{\mathcal{C}_k}^{[i]}) \tag{45}$$

for

$$((\hat{M}_{ki}(x_{\mathcal{C}_k}^{[i]}), \hat{b}_{ki}(x_{\mathcal{C}_k}^{[i]})), \hat{\Xi}_{ki}(x_{\mathcal{C}_k}^{[i]})) \in \mathop{\mathrm{argmin}}_{((M_{ki},b_{ki}),\Xi_{ki})\in(\mathcal{M}\ltimes\mathcal{B})\times\mathrm{proj}_{\mathcal{C}_k}(\mathcal{X}^*)} \|x_{\mathcal{C}_k}^{[i]} - (M_{ki}, b_{ki})\Xi_{ki}\|. \tag{46}$$

Proof: See Section VI-C. ■

The function in (44) depends only on the states $x_j^{[i]}$ of the agents $j \in \mathcal{C}_k$ for $k \in \mathcal{Q}_i$. Lemma 4 guarantees that these agents are only the neighbors $j \in \mathcal{N}_i$ and, thus, (44) is relative and distributed from (32). To implement this controller, agents need to solve the optimization problem (46). See Section V for how to solve a specific case. Note that (46) corresponds to the optimization problem (43) for the projection of $\mathcal{T}$ in (39), the $(\mathcal{M} \ltimes \mathcal{B})$-orbit of $\mathcal{X}^*$, regarding a maximal clique $\mathcal{C}_k$.

Remark 3: Theorem 3 does not require condition (B) in Theorem 1, namely, clique rigidity. Even when it is not satisfied, the relative, distributed controller (44) is guaranteed to provide the best performance in terms of minimizing the difference between the target configuration set $\mathcal{T}$ and the equilibrium set derived by the controller [43]. Hence, we can use this controller regardless of the graph topology. ■

B. Graph Conditions

From condition (A) in Theorem 1, the target configuration set $\mathcal{T}$ has to be an $(\mathcal{M} \ltimes \mathcal{B})$-orbit of $\mathcal{X}^*$ as (39). Here, we assume that $\mathcal{X}^* = \{x_{\mathcal{N}}^*\}$ is a singleton for $x_1^*, x_2^*, \ldots, x_n^* \in \mathbb{R}^d$. Then, for a specific $\mathcal{M} \ltimes \mathcal{B}$, the control objective (33) of this $\mathcal{T}$ is associated with a conventional task, and a familiar graph condition is derived from condition (B).

First, for the $(\mathcal{M} \ltimes \mathcal{B})$-orbit $\mathcal{T}$ in (39) with $\mathcal{M} \ltimes \mathcal{B} = \{I_d\} \ltimes \mathbb{R}^d$, the control objective (33) is equivalent to the displacement-based coordination in Table I (ii). Actually, from the definition (14) of the orbit, (33) is reduced to

$$\exists b \in \mathbb{R}^d \text{ s.t. } \lim_{t\to\infty} \|x_i(t) - (x_i^* + b)\| = 0\, \forall i \in \mathcal{N}$$

where b provides the freedom of translation. From Proposition 1, the clique rigidity is equivalent to connectedness.

Next, for $\mathcal{M} \ltimes \mathcal{B} = \mathrm{O}(d) \ltimes \mathbb{R}^d$, we can show that the control objective (33) is equivalent to the distance-based coordination in Table I (iii), in the same way as the displacement-based one. In this case, Proposition 2 guarantees that framework $(G, \mathcal{T})$ is clique rigid if and only if $(G, x_{\mathcal{N}}^*)$ is globally rigid.

As for $\mathcal{M} \ltimes \mathcal{B} = \mathrm{SO}(d) \ltimes \mathbb{R}^d$, the control objective (33) is the distance-based coordination without reflection. In this case, a clique rigid framework is rigid, but the converse relation does not necessarily hold. Actually, we have no equivalent, conventional graph condition as discussed in [32].

Generally, the clique rigidity is difficult to check in a direct way. A more intuitive, easily checkable condition is provided by using the intersection graph as follows.

Theorem 4: For a set $\mathcal{T} \subset \mathbb{R}^{d\times n}$, subgroups $\mathcal{M}$ and $\mathcal{B}$ of $\mathrm{scaled}(\mathrm{O}(d))$ and $\mathbb{R}^d$, respectively, such that $\mathcal{M}$ acts on $\mathcal{B}$, and a graph G, assume that condition (A) in Theorem 1 holds for $\mathcal{X}^* = \{x^*_{\mathcal{N}}\}$ with reference vectors $x^*_1, x^*_2, \ldots, x^*_n \in \mathbb{R}^d$. Then, if the $\mathrm{fdim}(\mathcal{M} \ltimes \mathcal{B})$-intersection graph of the maximal cliques, say $\Gamma_{\mathrm{fdim}(\mathcal{M}\ltimes\mathcal{B})}(G)$, is connected, framework $(G, \mathcal{T})$ is clique rigid for almost every $x^*_{\mathcal{N}} \in \mathbb{R}^{d\times n}$.

Proof: See Section VI-D. ■

Example 10: Consider graphs G_{a}, G_{b} in Fig. 1, and the space of dimension $d = 2$. For $\mathcal{M} = \mathrm{scaled}(\{I_2\})$, $\mathrm{SO}(2)$, and $\mathrm{scaled}(\mathrm{SO}(2))$, $\mathrm{fdim}(\mathcal{M} \ltimes \mathbb{R}^2) = 2$ is derived from Table III. The two-intersection graphs of the maximal cliques in G_{a} and G_{b} are given as $\Gamma_2(G_{\mathrm{a}})$ and $\Gamma_2(G_{\mathrm{b}})$, respectively, in Fig. 2. The intersection graph $\Gamma_2(G_{\mathrm{b}})$ is connected, and, thus, framework $(G_{\mathrm{b}}, \mathcal{T})$ is clique rigid for $\mathcal{T}$ in (39) from Theorem 4. ■

Theorem 4 characterizes a topology of G from the connectivity of the maximal cliques, associated with the free dimension $\mathrm{fdim}(\mathcal{M} \ltimes \mathcal{B})$. This result indicates the trend of the relation between the measurement information and the network topology (M, N) in Table I in a quantitative manner. Namely, as the volume of $\mathcal{M} \ltimes \mathcal{B}$ increases [less measurement information is available (M)], the required number $\mathrm{fdim}(\mathcal{M} \ltimes \mathcal{B})$ of the intersections between maximal cliques becomes larger [a required network becomes denser (N)].

The reason why the number $\mathrm{fdim}(\mathcal{M} \ltimes \mathcal{B})$ is required to intersections is explained as follows. Over the r-intersection graph of the maximal cliques, information on r variables can be conveyed between maximal cliques through their r intersections. In contrast, each maximal clique has to determine $\mathrm{fdim}(\mathcal{M} \ltimes \mathcal{B})$ variables, associated with the degree of freedom in $\mathcal{T} = \mathrm{orb}_{\mathcal{M}\ltimes\mathcal{B}}(x^*_{\mathcal{N}})$, to achieve the control objective (33). To agree the variables between maximal cliques, the number $r = \mathrm{fdim}(\mathcal{M} \ltimes \mathcal{B})$ of intersections is required.

Remark 4: Theorem 4 guarantees that framework $(G, \mathcal{T})$ is clique rigid for almost every $x^*_{\mathcal{N}} \in \mathbb{R}^{d\times n}$. This means that the vectors $x^*_1, x^*_2, \ldots, x^*_n$ should be scattered enough to determine the degree $\mathrm{fdim}(\mathcal{M} \ltimes \mathcal{B})$ of the freedoms in $\mathcal{T} = \mathrm{orb}_{\mathcal{M}\ltimes\mathcal{B}}(x^*_{\mathcal{N}})$ by avoiding the set $\mathcal{S}_m$ of measure zero in the definition of the free dimension (11). For example, the pairs (resp. trios) of the vectors associated over G cannot be the same (resp. collinear) for $\mathrm{fdim}(\mathcal{M} \ltimes \mathcal{B}) \geq 2$ (resp. 3). See Section VI-D for more details. ■

Remark 5: Constructing the intersection graph of the maximal cliques and verifying its connectedness can be done with polynomial running time with respect to the number n of the agents if the numbers $|\mathcal{N}_i|$ of neighbors are limited. Let $N_{\max} = \max_{i\in\mathcal{N}} |\mathcal{N}_i|$. First, list the maximal cliques with running time $O(n3^{N_{\max}})$ [13], [44]. Next, construct the r-intersection graph by checking the intersections of the maximal cliques, which takes running time $O(n^2 N_{\max}^2)$ because for each of the $q(q-1)/2$ pairs of the maximal cliques, at most $(\max_{k\in\mathcal{Q}} |\mathcal{C}_k|)^2$-times matching is required, $|\mathcal{C}_k| \leq N_{\max}$ holds from (4), and $q \leq n$. Finally, check its connectedness as a normal graph with running time $O(q|\breve{\mathcal{E}}_r|)$ [45], less than or equal to $O(n^3)$ from $|\breve{\mathcal{E}}_r| \leq q(q-1)/2$ and $q \leq n$. ■

V. Numerical Examples

A. Controller Design

A way of applying this method is illustrated to the situation of Example 9, i.e., cameras are available with unknown, heterogeneous scale factors, and neither common direction nor landmark is available. Then, $\mathcal{M} \ltimes \mathcal{B} = \mathrm{scaled}(\mathrm{SO}(d)) \ltimes \mathbb{R}^d$ is adopted. Following conditions (A) and (B) in Theorem 1, let us consider a target configuration set $\mathcal{T}$ in (39), namely, an $(\mathcal{M} \ltimes \mathcal{B})$-orbit of $\mathcal{X}^* = \{x^*_{\mathcal{N}}\}$ for reference vectors $x^*_1, x^*_2, \ldots, x^*_n \in \mathbb{R}^d$, and assume that $(G, \mathcal{T})$ is clique rigid. Then, Theorem 2 guarantees that $v(X)$ in (40) with (41) belongs to $\mathcal{F}_{\mathrm{ind}}(\mathcal{T}) \cap \mathcal{F}_{\mathrm{rd}}(\mathcal{M} \ltimes \mathcal{B}, G)$, and Theorem 3 guarantees that its gradient-based controller consists of (44)–(46), which is relative and distributed.

The optimization problem (46) is a Procrustes problem [46] and is analytically solvable as follows.

Proposition 3: For $\mathcal{M} \ltimes \mathcal{B} = \mathrm{scaled}(\mathrm{SO}(d)) \ltimes \mathbb{R}^d$, a solution to the optimization problem (46) with $\mathcal{X}_* = \{x^*_{\mathcal{N}}\}$ is given by $\hat{\Xi}_{ki} = x^*_{\mathcal{C}_k}$ and

$$(\hat{M}_{ki}, \hat{b}_{ki}) = (\hat{s}_{ki}\hat{U}_{ki}\hat{D}_{ki}\hat{V}_{ki}^\top, \mathrm{ave}(x^{[i]}_{\mathcal{C}_k} - \hat{M}_{ki}\hat{\Xi}_{ki})) \quad (47)$$

where the variables $x^{[i]}_{\mathcal{C}_k}$ are dropped in the functions. Here

$$\hat{s}_{ki} = \frac{\langle \hat{S}_{ki}, \hat{D}_{ki}\rangle}{\|\mathrm{cen}(x^*_{\mathcal{C}_k})\|^2}$$

$$\hat{D}_{ki} = \mathrm{diag}(1, 1, \ldots, 1, \det(\hat{U}_{ki}\hat{V}_{ki}^\top)) \in \mathbb{R}^{d\times d}$$

with $\hat{S}_{ki} = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_d)$ $(\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_d \geq 0)$ and $\hat{U}_{ki}, \hat{V}_{ki} \in \mathrm{O}(d)$ satisfying

$$\hat{U}_{ki}\hat{S}_{ki}\hat{V}_{ki}^\top = \mathrm{cen}(x^{[i]}_{\mathcal{C}_k})\mathrm{cen}(x^*_{\mathcal{C}_k})^\top. \quad (48)$$

Proof: See [46]. ■

Equation (48) indicates that the singular values in $\hat{S}_{ki}$ evaluate the correlation between $\mathrm{cen}(x^{[i]}_{\mathcal{C}_k})$ and $\mathrm{cen}(x^*_{\mathcal{C}_k})$, the centers of the current relative positions and desired ones of the agents belonging to a maximal clique $\mathcal{C}_k$. The resultant matrix $\hat{M}_{ki}$ in (47) consists of the scale $\hat{s}_{ki} > 0$ and the matrix $\hat{U}_{ki}\hat{D}_{ki}\hat{V}_{ki}^\top \in \mathrm{SO}(d)$. Thus, $\hat{M}_{ki} \in \mathrm{scaled}(\mathrm{SO}(d))$ holds.

B. Simulation Results

Simulations are carried out for multiagent systems in $d = 2$, 3-D spaces for the kinematic model (30) with the local coordinates (31) for $b_i(t) = x_i(t) \in \mathbb{R}^d$ and $M_i(t) \in \mathrm{scaled}(\mathrm{SO}(d))$ randomly chosen for each agent. The relative, distributed control input (32) with the controller designed in the previous section is employed with gain 5.

First, in $d = 2$-D space, we consider $n = 11$ agents with the reference vectors $x^*_1, x^*_2, \ldots, x^*_{11} \in \mathbb{R}^2$ and the edges of G given by the squares and lines in Fig. 4(a), respectively. The two-intersection graph $\Gamma_2(G)$ of the maximal cliques is connected, and $\mathrm{fdim}(\mathrm{scaled}(\mathrm{SO}(2)) \ltimes \mathbb{R}^2) = 2$ is obtained from Table III. Therefore, Theorem 4 guarantees that $(G, \mathcal{T})$ is clique rigid for the $(\mathcal{M} \ltimes \mathcal{B})$-orbit $\mathcal{T}$ of $\mathcal{X}^* = \{x^*_{\mathcal{N}}\}$. Fig. 4(b)–(d) shows simulation results from different initial positions, where the circles and squares with numbers describe the agent positions at $t = 0$ and $t = 10$, respectively. Notice that the scales of

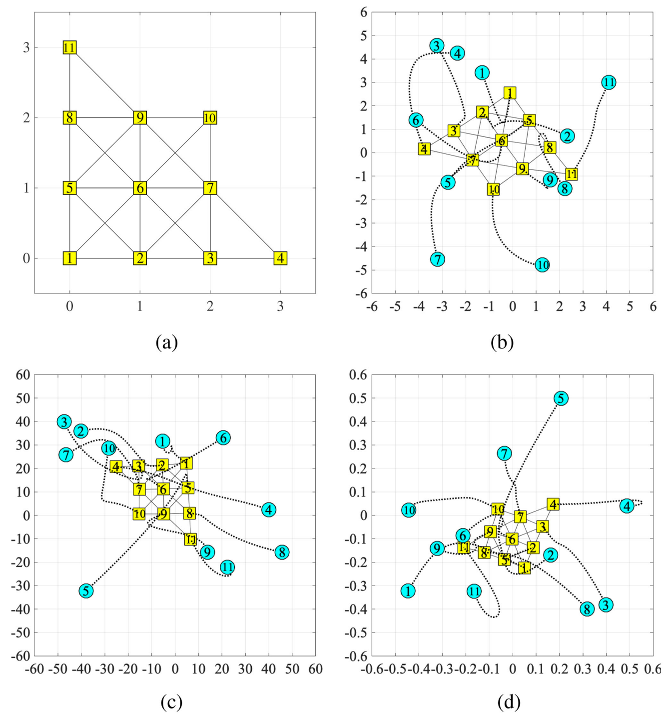


Fig. 4. Simulation results from different initial positions: the circles and squares with numbers describe the agent positions at $t = 0$ and $t = 10$, respectively. (a) Desired configuration and edges. (b) Simulation result 1. (c) Simulation result 2. (d) Simulation result 3.

Fig. 4(a)–(d) are different, and we can see that the desired configuration in Fig. 4(a) is obtained in every case with different translation, rotation, and scale. This is because the target configuration set $\mathcal{T}$ in (39) has the corresponding freedoms in $\mathcal{M} \ltimes \mathcal{B} = \mathrm{scaled}(\mathrm{SO}(2)) \ltimes \mathbb{R}^2$.

Next, in $d = 3$-D space, $n = 22$ agents are considered with the reference vectors and the edges of G given in Fig. 4(a). Because the three-intersection graph of the maximal cliques $\Gamma_3(G)$, is connected and $\mathrm{fdim}(\mathrm{scaled}(\mathrm{SO}(3)) \ltimes \mathbb{R}^3) = 3$ holds from Table III, framework $(G, \mathcal{T})$ is clique rigid for $\mathcal{T}$ in (39) from Theorem 4. Fig. 5 depicts a simulation result, where the circles and squares with numbers describe the positions at $t = 0$ and $t = 10$, respectively. We can see that the desired configuration is achieved at the terminal time.

These results demonstrate the effectiveness of the method regardless of the dimension of space.

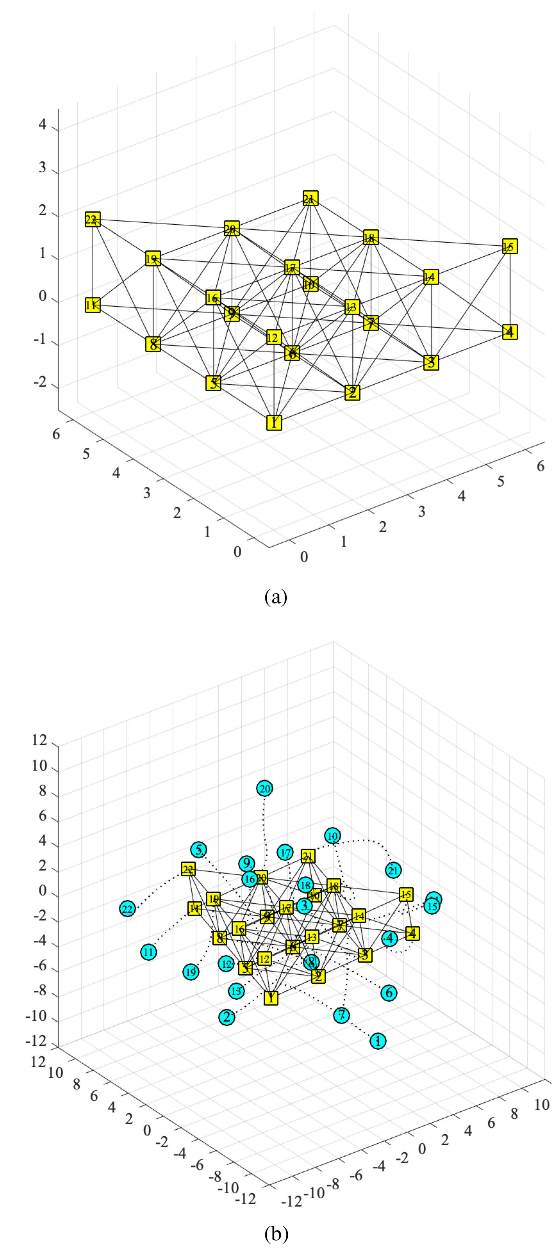


Fig. 5. Simulation result in 3-D space: the circles and squares with numbers describe the positions at $t = 0$ and $t = 10$, respectively. (a) Desired configuration and edges. (b) Simulation result.

VI. Proofs of Theorems

A. Proof of the Necessity of Theorem 1

We investigate the structure of $\mathcal{F}_{\mathrm{rd}}(\mathcal{M} \ltimes \mathcal{B}, G)$ defined in (38) via the two requirements to gradients: (i) relativeness with respect to $\mathcal{M} \ltimes \mathcal{B}$, (ii) distributedness over G, characterized by the sets $\mathcal{F}_{\mathrm{r}}(\mathcal{M} \ltimes \mathcal{B})$ and $\mathcal{F}_{\mathrm{d}}(G)$, respectively, defined as

$$\begin{aligned}&\mathcal{F}_{\mathrm{r}}(\mathcal{M} \ltimes \mathcal{B})\\&= \left\{ v(x_{\mathcal{N}}) \in \mathcal{F}_{\mathrm{con}} : \frac{\partial v}{\partial x_i}((M_i, b_i) x_{\mathcal{N}}) = M_i \frac{\partial v}{\partial x_i}(x_{\mathcal{N}}) \right.\\&\qquad \left. \forall x_{\mathcal{N}} \in \mathbb{R}^{d \times n} \backslash \mathcal{S}_v,\, (M_i, b_i) \in \mathcal{M} \ltimes \mathcal{B},\, i \in \mathcal{N} \right\} \end{aligned} \tag{49}$$

$$\begin{aligned}&\mathcal{F}_{\mathrm{d}}(G)\\&= \left\{ v(x_{\mathcal{N}}) \in \mathcal{F}_{\mathrm{con}} : \forall i \in \mathcal{N},\, \exists f_i : \mathbb{R}^{d \times |\mathcal{N}_i|} \to \mathbb{R}^d \right.\\&\qquad \left. \mathrm{s.\,t.}\ \frac{\partial v}{\partial x_i}(x_{\mathcal{N}}) = -f_i(x_{\mathcal{N}_i})\, \forall x_{\mathcal{N}} \in \mathbb{R}^{d \times n} \backslash \mathcal{S}_v \right\} \end{aligned} \tag{50}$$

with sets $\mathcal{S}_v \subset \mathbb{R}^{d \times n}$ of measure zero. See Table IV for the summary of the sets of functions. Actually, the following relation holds:

$$\mathcal{F}_{\mathrm{rd}}(\mathcal{M} \ltimes \mathcal{B}, G) = \mathcal{F}_{\mathrm{r}}(\mathcal{M} \ltimes \mathcal{B}) \cap \mathcal{F}_{\mathrm{d}}(G) \tag{51}$$

TABLE IV
SUMMARY OF SETS OF FUNCTIONS

Sets	Elements	Def.
$\mathcal{F}_{\mathrm{con}}$	Scalar continuous functions differentiable a.e.	–
$\mathcal{F}_{\mathrm{ind}}(\mathcal{T})$	Non-negative functions taking zero exactly in $\mathcal{T}$	(35)
$\mathcal{F}_{\mathrm{rd}}(\mathcal{M} \ltimes \mathcal{B}, G)$	Functions deriving relative, distributed gradients	(38)
$\mathcal{F}_{\mathrm{r}}(\mathcal{M} \ltimes \mathcal{B})$	Functions deriving relative gradients	(49)
$\mathcal{F}_{\mathrm{d}}(G)$	Functions deriving distributed gradients	(50)

which is verified by substituting $\partial v/\partial x_i(x_\mathcal{N})$ as (50) and x_j with $x_j^{[i]} = (M_i, b_i)^{-1}x_j$ in (49) to yield $\mathcal{F}_{\mathrm{rd}}(\mathcal{M} \ltimes \mathcal{B}, G)$ defined in (38), and vice versa.

For preliminaries, the functions belonging to $\mathcal{F}_{\mathrm{r}}(\mathcal{M} \ltimes \mathcal{B})$ and $\mathcal{F}_{\mathrm{d}}(G)$ are individually characterized as follows.

Lemma 6: For subgroups $\mathcal{M}$ and $\mathcal{B}$ of $\mathrm{scaled}(\mathrm{O}(d))$ and $\mathbb{R}^d$, respectively, such that $\mathcal{M}$ acts on $\mathcal{B}$, a nonnegative function $v : \mathbb{R}^{d\times n} \to \mathbb{R}$ with zero as a global minimum belongs to $\mathcal{F}_{\mathrm{r}}(\mathcal{M} \ltimes \mathcal{B})$ if and only if it is relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $(\det(M))^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$.

Proof: From the chain rule, the action on multiple vectors (6), and the action of a semidirect product (8), the following holds for any $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$ with $y_1, y_2, \dots, y_n \in \mathbb{R}^d$:

$$\begin{aligned}\frac{\partial v((M,b)x_\mathcal{N})}{\partial x_i} &= \frac{\partial ((M,b)x_i)^\top}{\partial x_i} \left.\frac{\partial v(y_\mathcal{N})}{\partial y_i}\right|_{y_\mathcal{N}=(M,b)x_\mathcal{N}} \\ &= M^\top \frac{\partial v}{\partial x_i}((M,b)x_\mathcal{N}). \end{aligned} \tag{52}$$

For sufficiency, assume that $v(x_\mathcal{N})$ is relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $(\det(M))^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$. By partially differentiating the definition (17) of the relative invariance with respect to x_i, we obtain

$$\frac{\partial v(Hx_\mathcal{N})}{\partial x_i} = \frac{\partial \mu(H) v(x_\mathcal{N})}{\partial x_i} = \mu(H)\frac{\partial v(x_\mathcal{N})}{\partial x_i}.$$

Apply $H = (M_i, b_i) \in \mathcal{M} \ltimes \mathcal{B}$ and $\mu(H) = (\det(M_i))^{\frac{2}{d}}$ to this equation, and from the property of $\mathrm{scaled}(\mathrm{O}(d))$ in (5)

$$\begin{aligned}&(\det(M_i))^{\frac{2}{d}} M_i \frac{\partial v(x_\mathcal{N})}{\partial x_i} = M_i \frac{\partial v((M_i,b_i)x_\mathcal{N})}{\partial x_i} \\ &= M_i M_i^\top \frac{\partial v}{\partial x_i}((M_i,b_i)x_\mathcal{N}) = (\det(M_i))^{\frac{2}{d}} \frac{\partial v}{\partial x_i}((M_i,b_i)x_\mathcal{N}) \end{aligned} \tag{53}$$

is obtained with (52). From (53) and $\det(M_i) \neq 0$, $v(x_\mathcal{N}) \in \mathcal{F}_{\mathrm{r}}(\mathcal{M} \ltimes \mathcal{B})$, defined in (49), holds.

For necessity, assume that $v(x_\mathcal{N}) \in \mathcal{F}_{\mathrm{r}}(\mathcal{M} \ltimes \mathcal{B})$ is nonnegative with zero as a global minimum. From the property of $\mathrm{scaled}(\mathrm{O}(d))$ in (5), (49), and (52)

$$\begin{aligned}\frac{\partial v((M,b)x_\mathcal{N})}{\partial x_i} &= M^\top \frac{\partial v}{\partial x_i}((M,b)x_\mathcal{N}) = M^\top M \frac{\partial v}{\partial x_i}(x_\mathcal{N}) \\ &= (\det(M))^{\frac{2}{d}} \frac{\partial v(x_\mathcal{N})}{\partial x_i}\end{aligned}$$

holds for any $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$, which leads to

$$\frac{\partial (v((M,b)x_\mathcal{N}) - (\det(M))^{\frac{2}{d}} v(x_\mathcal{N}))}{\partial x_\mathcal{N}} = 0. \tag{54}$$

From the gradient theorem, integrating (54) with respect to $x_\mathcal{N}$ yields

$$v((M,b)x_\mathcal{N}) = (\det(M))^{\frac{2}{d}} v(x_\mathcal{N}) + \gamma(M,b) \tag{55}$$

with a function $\gamma : \mathcal{M} \ltimes \mathcal{B} \to \mathbb{R}$ independent of $x_\mathcal{N}$. Let $\tilde{x}_\mathcal{N} \in \mathbb{R}^{d\times n}$ be a global minimum point of $v(x_\mathcal{N})$. Then, from (55)

$$\begin{aligned} v((M,b)\tilde{x}_\mathcal{N}) &= (\det(M))^{\frac{2}{d}} v(\tilde{x}_\mathcal{N}) + \gamma(M,b) \\ &= \gamma(M,b) \end{aligned} \tag{56}$$

$$\begin{aligned} 0 = v(\tilde{x}_\mathcal{N}) &= v((M,b)(M,b)^{-1}\tilde{x}_\mathcal{N}) \\ &= (\det(M))^{\frac{2}{d}} v((M,b)^{-1}\tilde{x}_\mathcal{N}) + \gamma(M,b) \end{aligned} \tag{57}$$

are obtained. From (56) and (57)

$$\gamma(M,b) = v((M,b)\tilde{x}_\mathcal{N}) = -(\det(M))^{\frac{2}{d}} v((M,b)^{-1}\tilde{x}_\mathcal{N})$$

is obtained, which leads to $\gamma(M, b) = 0$ for any $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$ because of the non-negativeness of $v(x_\mathcal{N})$. Hence, from (55), $v(x_\mathcal{N})$ satisfies the definition (17) of the relative invariance with weight $(\det(M))^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$. ■

Lemma 7: For a graph G, a nonnegative function $v : \mathbb{R}^{d\times n} \to \mathbb{R}$ belongs to $\mathcal{F}_{\mathrm{d}}(G)$ if and only if it is decomposable as (40) with non-negative functions $v_k : \mathbb{R}^{d\times|\mathcal{C}_k|} \to \mathbb{R}$ for $k \in \mathcal{Q}$.

Proof: See [43]. ■

To prove the necessity of Theorem 1, assume that $\mathcal{F}_{\mathrm{ind}}(\mathcal{T}) \cap \mathcal{F}_{\mathrm{rd}}(\mathcal{M} \ltimes \mathcal{B}, G)$ is nonempty, which implies that both $\mathcal{F}_{\mathrm{ind}}(\mathcal{T}) \cap \mathcal{F}_{\mathrm{r}}(\mathcal{M} \ltimes \mathcal{B})$ and $\mathcal{F}_{\mathrm{ind}}(\mathcal{T}) \cap \mathcal{F}_{\mathrm{d}}(G)$ are nonempty from the relation of these sets in (51). The nonemptiness of the two sets leads to conditions (A) and (B) as follows.

First, from the nonemptiness of $\mathcal{F}_{\mathrm{ind}}(\mathcal{T}) \cap \mathcal{F}_{\mathrm{r}}(\mathcal{M} \ltimes \mathcal{B})$, we prove condition (A), equivalent to the $(\mathcal{M} \ltimes \mathcal{B})$-invariance of $\mathcal{T}$ from Lemma 1. To do so, consider $\tilde{x}_\mathcal{N} \in \mathcal{T}$ and $(\tilde{M}, \tilde{b}) \in \mathcal{M} \ltimes \mathcal{B}$, and we show $(\tilde{M}, \tilde{b})\tilde{x}_\mathcal{N} \in \mathcal{T}$. From the assumption, there exists a function $v(x_\mathcal{N}) \in \mathcal{F}_{\mathrm{ind}}(\mathcal{T}) \cap \mathcal{F}_{\mathrm{r}}(\mathcal{M} \ltimes \mathcal{B})$, and from the definition of $\mathcal{F}_{\mathrm{ind}}(\mathcal{T})$ in (35), $v(\tilde{x}_\mathcal{N}) = 0$ holds. Because $v(x_\mathcal{N}) \in \mathcal{F}_{\mathrm{ind}}(\mathcal{T})$ is nonnegative and takes zero as a global minimum, from Lemma 6, $v(x_\mathcal{N}) \in \mathcal{F}_{\mathrm{r}}(\mathcal{M} \ltimes \mathcal{B})$ is relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $(\det(M))^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$. Thus, from the definition (17) of the relative invariance, $v((\tilde{M}, \tilde{b})\tilde{x}_\mathcal{N}) = (\det(\tilde{M}))^{\frac{2}{d}} v(\tilde{x}_\mathcal{N}) = 0$ holds, and $(\tilde{M}, \tilde{b})\tilde{x}_\mathcal{N} \in v^{-1}(0) = \mathcal{T}$ is obtained from (35).

Second, from the nonemptiness of $\mathcal{F}_{\mathrm{ind}}(\mathcal{T}) \cap \mathcal{F}_{\mathrm{d}}(G)$, we derive condition (B), the clique rigidity of $(G, \mathcal{T})$. From the assumption, there exists a function $v(x_\mathcal{N}) \in \mathcal{F}_{\mathrm{ind}}(\mathcal{T}) \cap \mathcal{F}_{\mathrm{d}}(G)$. Then, from the definition of $\mathcal{F}_{\mathrm{ind}}(\mathcal{T})$ in (35), $v^{-1}(0) = \mathcal{T}$ holds, and from Lemma 7, $v(x_\mathcal{N})$ is decomposable as (40) with nonnegative functions $v_k(x_{\mathcal{C}_k})$, which lead to

$$\mathcal{T} = v^{-1}(0) = \bigcap_{k\in\mathcal{Q}} \{x_\mathcal{N} \in \mathbb{R}^{d\times n} : x_{\mathcal{C}_k} \in v_k^{-1}(0)\}. \tag{58}$$

From (58), for each $k \in \mathcal{Q}$

$$\begin{aligned}\mathrm{proj}_{\mathcal{C}_k}(\mathcal{T}) &\subset \mathrm{proj}_{\mathcal{C}_k}(\{x_\mathcal{N} \in \mathbb{R}^{d\times n} : x_{\mathcal{C}_k} \in v_k^{-1}(0)\}) \\ &= v_k^{-1}(0)\end{aligned} \tag{59}$$

holds. From (58) and (59), we obtain

$$\begin{aligned} x_{\mathcal{C}_k} \in \mathrm{proj}_{\mathcal{C}_k}(\mathcal{T})\, \forall k \in \mathcal{Q} &\Rightarrow x_{\mathcal{C}_k} \in v_k^{-1}(0)\, \forall k \in \mathcal{Q} \\ &\Rightarrow x_\mathcal{N} \in \mathcal{T} \end{aligned}$$

and the definition (23) of the clique rigidity is derived.

B. Proof of Theorem 2

Assume conditions (A) and (B) in Theorem 1.

For sufficiency, consider a function $v(x_{\mathcal{N}})$ in (40), decomposed with indicators $v_k(x_{\mathcal{C}_k})$ of $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{T})$, relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $(\det(M))^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$. Note that such $v_k(x_{\mathcal{C}_k})$ exists under condition (A) from Lemma 5. To show that $v(X) \in \mathcal{F}_{\mathrm{ind}}(\mathcal{T}) \cap \mathcal{F}_{\mathrm{rd}}(\mathcal{M} \ltimes \mathcal{B}, G)$, we just have to prove that it belongs to each of $\mathcal{F}_{\mathrm{ind}}(\mathcal{T})$, $\mathcal{F}_{\mathrm{r}}(\mathcal{M} \ltimes \mathcal{B})$, and $\mathcal{F}_{\mathrm{d}}(G)$ from (51). First, $v(x_{\mathcal{N}}) \in \mathcal{F}_{\mathrm{ind}}(\mathcal{T})$, defined in (35), holds because $v(x_{\mathcal{N}})$ is nonnegative and $v^{-1}(0) = \mathcal{T}$ holds as follows: from condition (B), the definition (23) of the clique rigidity, we obtain the relation

$$\begin{aligned} v(x_{\mathcal{N}}) = 0 &\Rightarrow v_k(x_{\mathcal{C}_k}) = 0\, \forall k \in \mathcal{Q} \\ &\Rightarrow x_{\mathcal{C}_k} \in \mathrm{proj}_{\mathcal{C}_k}(\mathcal{T})\, \forall k \in \mathcal{Q} \Rightarrow x_{\mathcal{N}} \in \mathcal{T} \end{aligned}$$

for indicators $v_k(x_{\mathcal{C}_k})$ of $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{T})$, and the converse relation is obvious. Next, $v(x_{\mathcal{N}}) \in \mathcal{F}_{\mathrm{r}}(\mathcal{M} \ltimes \mathcal{B})$ holds from Lemma 6 because $v(x_{\mathcal{N}}) \in \mathcal{F}_{\mathrm{ind}}(\mathcal{T})$ is nonnegative and takes zero as a global minimum, $v(x_{\mathcal{N}})$ in (40) satisfies the definition of the relative invariance (17) with weight $(\det(M))^{\frac{2}{d}}$ as

$$\begin{aligned} v((M,b)x_{\mathcal{N}}) &= \sum_{k \in \mathcal{Q}} v_k((M,b)x_{\mathcal{C}_k}) = \sum_{k \in \mathcal{Q}} (\det(M))^{\frac{2}{d}} v_k(x_{\mathcal{C}_k}) \\ &= (\det(M))^{\frac{2}{d}} v(x_{\mathcal{N}}) \end{aligned}$$

where the relative $(\mathcal{M} \ltimes \mathcal{B})$-invariance of $v_k(x_{\mathcal{C}_k})$ is used. Finally, Lemma 7 guarantees $v(x_{\mathcal{N}}) \in \mathcal{F}_{\mathrm{d}}(G)$.

For necessity, consider a function $v(x_{\mathcal{N}}) \in \mathcal{F}_{\mathrm{ind}}(\mathcal{T}) \cap \mathcal{F}_{\mathrm{rd}}(\mathcal{M} \ltimes \mathcal{B}, G)$. Lemma 7 guarantees that the nonnegative function $v(x_{\mathcal{N}}) \in \mathcal{F}_{\mathrm{d}}(G)$ is decomposable as (40) with nonnegative $v_k(x_{\mathcal{C}_k})$ for $k \in \mathcal{Q}$. Additionally, we have to show that each $v_k(x_{\mathcal{C}_k})$ is an indicator of $\mathrm{proj}_{\mathcal{C}_k}(\mathcal{T})$ and is relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $(\det(M))^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$. This can be done by using the following lemma.

Lemma 8: If a pair $i, j \in \mathcal{N}$ satisfies $\{i, j\} \notin \mathcal{E}$, $i \neq j$, $v(x_{\mathcal{N}}) \in \mathcal{F}_{\mathrm{ind}}(\mathcal{T}) \cap \mathcal{F}_{\mathrm{rd}}(\mathcal{M} \ltimes \mathcal{B}, G)$ is decomposable as

$$v(x_{\mathcal{N}}) = w_1(x_{\mathcal{N}\setminus\{i\}}) + w_2(x_{\mathcal{N}\setminus\{j\}}) \tag{60}$$

with some indicators $w_1(x_{\mathcal{N}\setminus\{i\}})$ and $w_2(x_{\mathcal{N}\setminus\{j\}})$ of $\mathrm{proj}_{\mathcal{N}\setminus\{i\}}(\mathcal{T})$ and $\mathrm{proj}_{\mathcal{N}\setminus\{j\}}(\mathcal{T})$, respectively, relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $(\det(M))^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$.

Proof: See Appendix B. ■

According to Lemma 8, any function in $v(x_{\mathcal{N}})$ can be decomposed into two relatively invariant indicators as long as it depends on the states x_i, x_j satisfying $\{i, j\} \notin \mathcal{E}$. Repeating this operation results in the sum of relatively invariant indicators $v_k(x_{\mathcal{C}_k})$ depending only on the states $x_{\mathcal{C}_k}$ for maximal cliques $\mathcal{C}_k$ to obtain the decomposed form (40).

C. Proof of Theorem 3

We show that (37) holds for $v(x_{\mathcal{N}})$ in (40) with (39), (41) and for $f_i(x^{[i]}_{\mathcal{N}_i})$ in (44) with (45), (46). To do so, we just have to show the following for each $k \in \mathcal{Q}_i$:

$$M_i^{-1} \frac{\partial v_k}{\partial x_i}(x_{\mathcal{C}_k}) = x_i^{[i]} - \mathrm{col}_{n_{ki}}(\hat{T}_{ki}(x^{[i]}_{\mathcal{C}_k})). \tag{61}$$

For a preliminary, the following lemma gives the gradient of the distance function in (1) without assumptions on $\mathcal{T}$.

Lemma 9: For a set $\mathcal{T} \subset \mathbb{R}^{d \times n}$ and vectors $x_1, x_2, \ldots, x_n \in \mathbb{R}^d$

$$\frac{1}{2} \frac{\partial (\mathrm{dist}(x_{\mathcal{N}}, \mathcal{T}))^2}{\partial x_i} = x_i - \mathrm{col}_i(\hat{T}(x_{\mathcal{N}})) \tag{62}$$

holds for a matrix-valued function $\hat{T} : \mathbb{R}^{d \times n} \to \mathcal{T}$ satisfying

$$\hat{T}(x_{\mathcal{N}}) \in \operatorname*{argmin}_{T \in \mathcal{T}} \|x_{\mathcal{N}} - T\|. \tag{63}$$

Proof: See Appendix C. ■

From condition (A) in Theorem 1, Lemma 5 guarantees that $v_k(x_{\mathcal{C}_k})$ in (41) is relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $(\det(M))^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$. This function is nonnegative and takes zero as a global minimum. Thus, from Lemma 6, $v_k(x_{\mathcal{C}_k}) \in \mathcal{F}_{\mathrm{r}}(\mathcal{M} \ltimes \mathcal{B})$ holds. From the inverse of a semidirect product in (7), the definition (49) of $\mathcal{F}_{\mathrm{r}}(\mathcal{M} \ltimes \mathcal{B})$, and Lemma 9

$$\begin{aligned} M_i^{-1} \frac{\partial v_k}{\partial x_i}(x_{\mathcal{C}_k}) &= \frac{\partial v_k}{\partial x_i}((M_i, b_i)^{-1} x_{\mathcal{C}_k}) = \frac{\partial v_k}{\partial x_i}(x^{[i]}_{\mathcal{C}_k}) \\ &= x_i^{[i]} - \mathrm{col}_{n_{ki}}(\hat{T}_{ki}(x^{[i]}_{\mathcal{C}_k})) \end{aligned} \tag{64}$$

is obtained, where $x_j^{[i]} = (M_i, b_i)^{-1} x_j$ according to the local coordinate (31) and $\hat{T}_{ki} : \mathbb{R}^{d \times |\mathcal{C}_k|} \to \mathrm{proj}_{\mathcal{C}_k}(\mathcal{T})$ satisfies

$$\hat{T}_{ki}(x^{[i]}_{\mathcal{C}_k}) \in \operatorname*{argmin}_{T_{ki} \in \mathrm{proj}_{\mathcal{C}_k}(\mathcal{T})} \|x^{[i]}_{\mathcal{C}_k} - T_{ki}\|. \tag{65}$$

From the definitions of projection and orbit in (2) and (14), and Lemma 2 (i), $T_{ki} \in \mathrm{proj}_{\mathcal{C}_k}(\mathcal{T})$ holds for the orbit $\mathcal{T}$ in (39) if and only if $T_{ki} = (M_{ki}, b_{ki})\Xi_{ki}$ holds for some $((M_{ki}, b_{ki}), \Xi_{ki}) \in (\mathcal{M} \ltimes \mathcal{B}) \times \mathrm{proj}_{\mathcal{C}_k}(\mathcal{X}^*)$. Hence, (65) is equivalent to (45) with (46). Then, from (64), (61) is satisfied.

D. Proof of Theorem 4

Assume that condition (A) in Theorem 1 holds and that the $\mathrm{fdim}(\mathcal{M} \ltimes \mathcal{B})$-intersection graph of the maximal cliques is connected. Let $x_1, x_2, \ldots, x_n \in \mathbb{R}^d$ be vectors satisfying the assumption part of the definition (23) of the clique rigidity, and consider $k, \ell \in \mathcal{Q}$, $k \neq \ell$, satisfying $\{k, \ell\} \in \check{\mathcal{E}}_{\mathrm{fdim}(\mathcal{M} \ltimes \mathcal{B})}$. From the definition of $\check{\mathcal{E}}_r$ in (22)

$$|\mathcal{C}_k \cap \mathcal{C}_\ell| \geq \mathrm{fdim}(\mathcal{M} \ltimes \mathcal{B}) \tag{66}$$

holds. From the assumption part of (23), $x_{\mathcal{C}_k} \in \mathrm{proj}_{\mathcal{C}_k}(\mathcal{T})$ and $x_{\mathcal{C}_\ell} \in \mathrm{proj}_{\mathcal{C}_\ell}(\mathcal{T})$ hold, which leads to

$$x_{\mathcal{C}_k} = (M_k, b_k) x^*_{\mathcal{C}_k},\ x_{\mathcal{C}_\ell} = (M_\ell, b_\ell) x^*_{\mathcal{C}_\ell} \tag{67}$$

with some $(M_k, b_k), (M_\ell, b_\ell) \in \mathcal{M} \ltimes \mathcal{B}$ from the definitions of projection and orbit in (2) and (14), respectively, and Lemma 2 (i) for the $(\mathcal{M} \ltimes \mathcal{B})$-orbit $\mathcal{T}$ in (39). From (67)

$$x_{\mathcal{C}_k \cap \mathcal{C}_\ell} = (M_k, b_k) x^*_{\mathcal{C}_k \cap \mathcal{C}_\ell} = (M_\ell, b_\ell) x^*_{\mathcal{C}_k \cap \mathcal{C}_\ell} \tag{68}$$

is obtained. From (66), $\mathcal{M} \ltimes \mathcal{B}$ is free to $\mathbb{R}^{d \times |\mathcal{C}_k \cap \mathcal{C}_\ell|} \setminus \mathcal{S}_{k\ell}$ with a set $\mathcal{S}_{k\ell}$ of measure zero, and from (68), $(M_k, b_k) = (M_\ell, b_\ell)$ is obtained for $x^*_{\mathcal{C}_k \cap \mathcal{C}_\ell} \in \mathbb{R}^{d \times |\mathcal{C}_k \cap \mathcal{C}_\ell|} \setminus \mathcal{S}_{k\ell}$, according to the definitions of the free dimension in (11) and the freeness in (10) in order. Due to the assumption of the connectedness of $\Gamma_{\mathrm{fdim}(\mathcal{M} \ltimes \mathcal{B})}(G)$, (M_k, b_k) agree for all $k \in \mathcal{Q}$ as long as

$x^*_{\mathcal{C}_k\cap\mathcal{C}_\ell} \in \mathbb{R}^{d\times|\mathcal{C}_k\cap\mathcal{C}_\ell|}\backslash\mathcal{S}_{k\ell}$ hold for any such $k, \ell \in \mathcal{Q}$. Let (M, b) be the agreement. Because every node belongs to a maximal clique, $x_{\mathcal{N}} = (M, b)x^*_{\mathcal{N}}$ is obtained from (67), which belongs to $\mathcal{T}$ in (39) with $\mathcal{X}^* = \{x^*_{\mathcal{N}}\}$ from the definition (14) of the orbit. Hence, the conclusion part of (23) is fulfilled, and $(G, \mathcal{T})$ is clique rigid.

VII. Conclusion

This article provided a unified solution to a general multiagent coordination problem with relative measurements, including transformations in translation, rotation, reflection, and scale. First, we described relative measurements with transformation matrix and translation vector sets, defined by subgroups $\mathcal{M}$ and $\mathcal{B}$, respectively. This formulation enabled us to describe various types of measurement information in a unified way. Next, as the main result, we derived a necessary and sufficient condition for a network topology and measurement information with which a given coordination task is achievable by relative, distributed control. By employing the semidirect product $\mathcal{M} \ltimes \mathcal{B}$ to describe relative measurements, sophisticated tools of group theory could be utilized. Actually, the class of the realizable coordination tasks was specified with the target configuration sets expressed by $(\mathcal{M} \ltimes \mathcal{B})$-orbits. Then, the gradient-based controllers to achieve these tasks were characterized by relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant functions. Moreover, we showed that a required network topology is clique rigidity and derived a more intuitive condition such that the intersection graph of the maximal cliques is connected, where the required number of the intersections is more than or equal to the free dimension of $\mathcal{M} \ltimes \mathcal{B}$. Finally, the new method was applied to formation control with unknown, heterogeneous scale factors, and its effectiveness was demonstrated through simulations for both 2-D and 3-D spaces.

These results indicate the trends of the relations between coordination tasks, required network topologies, and measurement information (T, N, M) shown in Table I in quantitative ways as follows: a realizable set $\mathcal{T}$ is larger [a coordination task is more flexible (T)] for larger volume of $\mathcal{M} \ltimes \mathcal{B}$ [less measurement information (M)], and then the number of intersections required to G increases according to $\mathrm{fdim}(\mathcal{M} \ltimes \mathcal{B})$ [a denser network topology is necessary (N)]. This article is the first that strictly shows these trends. Moreover, this method is applicable to a wide range of coordination tasks due to its unified approach. Actually, this approach covers coordination tasks (i), (ii), and (iii) in Table I by employing $\mathcal{M} \ltimes \mathcal{B} = \{I_d\} \ltimes \{0\}, \{I_d\} \ltimes \mathbb{R}^d$, and $\mathrm{O}(d) \ltimes \mathbb{R}^d$, respectively. Additionally, various other coordination tasks can be attained by employing $\mathcal{M}$ and $\mathcal{B}$ from Table II and others. For example, formation control with unknown scale factors is achieved with $\mathcal{M} \ltimes \mathcal{B} = \mathrm{scaled}(\mathrm{SO}(d)) \ltimes \mathbb{R}^d$, and target enclosure is with $\mathcal{M} \ltimes \mathcal{B} = \mathrm{SO}(d) \ltimes \{0\}$. Nevertheless, coordination tasks (iv) and (v) in Table I are not covered because measurements in terms of only bearings cannot be expressed as (31). Future work includes more comprehensive generalization including all these tasks.

Appendix A
Examples of Transformation Matrix Sets

We derive the four transformation matrix sets $\mathcal{M} = \mathrm{SO}(d)$, $\{I_d\}$, $\mathrm{scaled}(\mathrm{SO}(d))$, and $\mathrm{O}(d)$ in Table II according to types of measurement information with the translation vector set $\mathcal{B} = \mathbb{R}^d$ for $d = 2$.

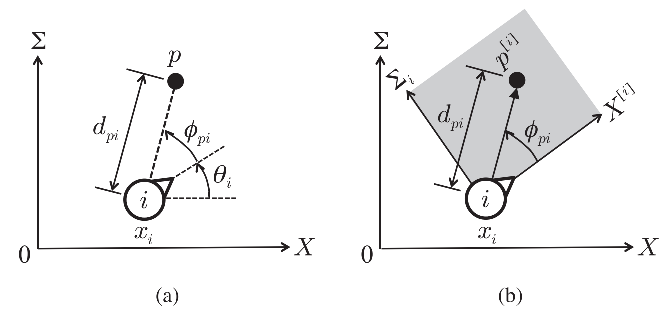


Fig. 6. Global and local coordinates for $\mathcal{M} = \mathrm{SO}(2)$. (a) Global coordinate p. (b) Corresponding local coordinate $p^{[i]}$.

Example 11: Consider a situation where each agent is equipped with a camera or laser-range-finder to measure the distance and relative bearing of an object. Then, $\mathcal{M} = \mathrm{SO}(2)$ is adopted, which is explained from Fig. 6 as follows. Suppose that agent i is at a position coordinate $x_i \in \mathbb{R}^2$ with facing to the direction of an angle $\theta_i \in [-\pi, \pi)$ from the X-axis of the global frame Σ, and that an object is set at a position coordinate $p \in \mathbb{R}^2$. Let $d_{pi} \geq 0$ and $\phi_{pi} \in [-\pi, \pi)$ be the distance and relative bearing of the object from agent i, and only d_{pi} and ϕ_{pi} are assumed to be measurable. Then, from Fig. 6(a), the global coordinate p satisfies

$$p = x_i + d_{pi} \begin{bmatrix} \cos(\theta_i + \phi_{pi}) \\ \sin(\theta_i + \phi_{pi}) \end{bmatrix}. \quad (69)$$

Suppose that the $X^{[i]}$-axis of the local frame Σ_i is set to the face of agent i, as illustrated in Fig. 6(b). Then, the local coordinate $p^{[i]} \in \mathbb{R}^2$ of the object satisfies

$$p^{[i]} = d_{pi} \begin{bmatrix} \cos\phi_{pi} \\ \sin\phi_{pi} \end{bmatrix}. \quad (70)$$

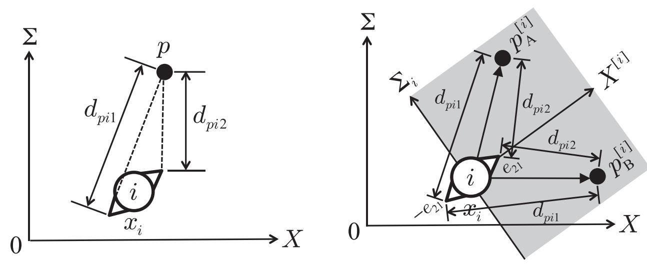
From (69) and (70), the relation between global and local coordinates (28) holds with $M_i = R(\theta_i) \in \mathrm{SO}(2)$ and $b_i = x_i$ for the rotation matrix $R(\cdot)$ defined in (4). Hence, $\mathcal{M} = \mathrm{SO}(2)$ is obtained. ■

Example 12: When each agent carries a compass in addition to the equipment in Example 11, a common direction is available. Then, $\mathcal{M} = \{I_2\}$ is adopted. Actually, set the X-axis of Σ to this direction, and the angle θ_i of the face of agent i from the X-axis is obtainable as Fig. 6(a). Then, the local coordinate $p^{[i]}$ can be transformed from (70) to

$$p^{[i]} = d_{pi} \begin{bmatrix} \cos(\theta_i + \phi_{pi}) \\ \sin(\theta_i + \phi_{pi}) \end{bmatrix}. \quad (71)$$

From (69) and (71), (28) holds with $M_i = I_2$ and $b_i = x_i$. Hence, $\mathcal{M} = \{I_2\}$ is achieved. ■

Example 13: Suppose that the distance from the object is measured with an unknown scale factor. Then, $\mathcal{M} = \mathrm{scaled}(\mathrm{SO}(2))$ is adopted as explained as follows. Let $\hat{d}_{pi} \geq 0$ be the value obtained by the measurement of the distance, and



(a) (b)

Fig. 7. Global and local coordinates for $\mathcal{M} = \mathrm{O}(2)$. (a) Global coordinate p. (b) Corresponding local coordinate $p^{[i]}$.

the relative coordinate $p^{[i]}$ of the object is described as

$$p^{[i]} = \hat{d}_{pi} \begin{bmatrix} \cos\phi_{pi} \\ \sin\phi_{pi} \end{bmatrix} \tag{72}$$

instead of (70). Let $s_i > 0$ be the unknown scale factor and $\hat{d}_{pi} = s_i d_i$ holds. Then, from (69) and (72), (28) holds with $M_i = s_i R(\theta_i) \in \mathrm{scaled}(\mathrm{SO}(2))$ and $b_i = x_i$, as illustrated in Fig. 3. Thus, $\mathcal{M} = \mathrm{scaled}(\mathrm{SO}(2))$ is achieved. ■

Example 14: Suppose that each agent detects a beacon on the object by two receivers. Then, $\mathcal{M} = \mathrm{O}(2)$ is adopted. In this case, as shown in Fig. 7(a), the distances $d_{pi1}, d_{pi2} \geq 0$ of the beacon from the receivers are measurable. Without the loss of generality, let $[1\,0]^\top$ and $-[1\,0]^\top$ be the local coordinates of the receivers. Then, the relative coordinate $p^{[i]}$ of the object satisfies

$$\|p^{[i]} - [1\,0]^\top\| = d_{pi1}, \; \|p^{[i]} + [1\,0]^\top\| = d_{pi2}. \tag{73}$$

As shown in Fig. 7(b), (73) allows the two possible coordinates $p_{\mathrm{A}}^{[i]}, p_{\mathrm{B}}^{[i]} \in \mathbb{R}^2$ for $p^{[i]}$, where

$$p_{\mathrm{A}}^{[i]} = d_{pi} \begin{bmatrix} \cos\hat{\phi}_{pi} \\ \sin\hat{\phi}_{pi} \end{bmatrix}, \; p_{\mathrm{B}}^{[i]} = d_{pi} \begin{bmatrix} \cos\hat{\phi}_{pi} \\ -\sin\hat{\phi}_{pi} \end{bmatrix} \tag{74}$$

with

$$d_{pi} = \sqrt{\frac{d_{pi1}^2 + d_{pi2}^2}{2} - 1}, \; \hat{\phi}_{pi} = \cos^{-1}\frac{d_{pi2}^2 - d_{pi1}^2}{4d_{pi}} \in [0, \pi].$$

Because the relative bearing $\phi_{pi} \in (-\pi, \pi]$ of the object is equivalent to either $\hat{\phi}_{pi}$ or $-\hat{\phi}_{pi}$, from (74)

$$p^{[i]} \in \{p_{\mathrm{A}}^{[i]}, p_{\mathrm{B}}^{[i]}\} = \left\{ d_{pi} \begin{bmatrix} \cos\phi_{pi} \\ \sin\phi_{pi} \end{bmatrix}, d_{pi} R_w \begin{bmatrix} \cos\phi_{pi} \\ \sin\phi_{pi} \end{bmatrix} \right\} \tag{75}$$

is obtained with the reflection matrix $R_w \in \mathbb{R}^{2\times 2}$ for $w = [0\,1]^\top$. From (69) and (75), (28) holds for $M_i = R(\theta_i)$ or $R(\theta_i)R_w$, and $b_i = x_i$. The duality of the possible positions is called a flip ambiguity in distance-based localization [47]. We can match the ambiguity to either $M_i = R(\theta_i)$ or $R(\theta_i)R_w$ for all neighbors by using a landmark beacon. Then

$$\begin{aligned} \mathcal{M} &= \{R(\theta) : \theta \in [-\pi, \pi)\} \cup \{R(\theta)R_w : \theta \in [-\pi, \pi)\} \\ &= \mathrm{O}(2) \end{aligned}$$

is obtained. ■

Appendix B
Proof of Lemma 8

From (51), we can assume that $v(x_\mathcal{N})$ belongs to $\mathcal{F}_{\mathrm{ind}}(\mathcal{T})$, $\mathcal{F}_{\mathrm{r}}(\mathcal{M} \ltimes \mathcal{B})$, and $\mathcal{F}_{\mathrm{d}}(G)$. For functions $\eta_i, \eta_j : \mathbb{R}^{d\times(n-2)} \to \mathbb{R}^d$ defined later, we let:

$$\begin{aligned} w_1(x_{\mathcal{N}\backslash\{i\}}) &= v(x_\mathcal{N})|_{x_i = \eta_i(x_{\mathcal{N}\backslash\{i,j\}})} \\ &\quad - \frac{1}{2} v(x_\mathcal{N})|_{x_{ij} = \eta_{ij}(x_{\mathcal{N}\backslash\{i,j\}})} \end{aligned} \tag{76}$$

$$\begin{aligned} w_2(x_{\mathcal{N}\backslash\{j\}}) &= v(x_\mathcal{N})|_{x_j = \eta_j(x_{\mathcal{N}\backslash\{i,j\}})} \\ &\quad - \frac{1}{2} v(x_\mathcal{N})|_{x_{ij} = \eta_{ij}(x_{\mathcal{N}\backslash\{i,j\}})} \end{aligned} \tag{77}$$

where $v(x_\mathcal{N})|_{x_i=\eta_i} = v([x_1 \cdots x_{i-1}\, \eta_i\, x_{i+1} \cdots x_n])$ and $v(x_\mathcal{N})|_{x_{ij}=\eta_{ij}} = v(x_\mathcal{N})|_{x_i=\eta_i}|_{x_j=\eta_j}$ for $x_{ij} = [x_i\, x_j]$ and $\eta_{ij} = [\eta_i\, \eta_j]$. We just have to show that (i) the decomposed form (60) is obtained, that (ii) $w_1(x_{\mathcal{N}\backslash\{i\}})$ is an indicator of $\mathrm{proj}_{\mathcal{N}\backslash\{i\}}$, and that (iii) it is relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $(\det(M))^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$. The properties of $w_2(x_{\mathcal{N}\backslash\{j\}})$ can be shown in the same way.

(i) For $v(x_\mathcal{N}) \in \mathcal{F}_{\mathrm{d}}(G)$, defined in (50), and a vector $y \in \mathbb{R}^d$, from the gradient theorem

$$\begin{aligned} v(x_\mathcal{N}) - v(x_\mathcal{N})|_{x_i=y} &= \int_y^{x_i} \frac{\partial v}{\partial x_i}(x_\mathcal{N}) \mathrm{d}x_i \\ &= -\int_y^{x_i} f_i(x_{\mathcal{N}_i}) \mathrm{d}x_i = \tilde{v}_i(x_{\mathcal{N}_i}, y) \end{aligned} \tag{78}$$

is obtained with a function $\tilde{v}_i : \mathbb{R}^{d\times|\mathcal{N}_i|} \times \mathbb{R}^d \to \mathbb{R}$. Let $y = \eta_i(x_{\mathcal{N}\backslash\{i,j\}})$ in (78), and

$$v(x_\mathcal{N}) - v(x_\mathcal{N})|_{x_i=\eta_i(x_{\mathcal{N}\backslash\{i,j\}})} = \tilde{v}_i(x_{\mathcal{N}_i}, \eta_i(x_{\mathcal{N}\backslash\{i,j\}})) \tag{79}$$

is obtained. From the assumption $\{i, j\} \notin \mathcal{E}$, $j \notin \mathcal{N}_i$ holds, and thus, $\tilde{v}_i(\cdot)$ in (79) does not depend on x_j, and

$$\begin{aligned} & v(x_\mathcal{N}) - v(x_\mathcal{N})|_{x_i=\eta_i(x_{\mathcal{N}\backslash\{i,j\}})} \\ &= (v(x_\mathcal{N}) - v(x_\mathcal{N})|_{x_i=\eta_i(x_{\mathcal{N}\backslash\{i,j\}})})|_{x_j=\eta_j(x_{\mathcal{N}\backslash\{i,j\}})} \end{aligned}$$

is obtained, which leads to the decomposed form (60) with $w_1(x_{\mathcal{N}\backslash\{i\}})$ and $w_2(x_{\mathcal{N}\backslash\{j\}})$ in (76) and (77).

(ii) We choose $\eta_{ij}(x_{\mathcal{N}\backslash\{i,j\}})$ as a function satisfying

$$\eta_{ij}(x_{\mathcal{N}\backslash\{i,j\}}) \in \operatorname*{argmin}_{x_{ij}\in\mathbb{R}^{d\times 2}} v(x_\mathcal{N}) \tag{80}$$

which is well-defined because $v(x_\mathcal{N}) \in \mathcal{F}_{\mathrm{ind}}(\mathcal{T})$ has a global minimum. From (80)

$$v(x_\mathcal{N})|_{x_i=\eta_i(x_{\mathcal{N}\backslash\{i,j\}})} \geq v(x_\mathcal{N})|_{x_{ij}=\eta_{ij}(x_{\mathcal{N}\backslash\{i,j\}})} \tag{81}$$

holds. From (81) and the nonnegativeness of $v(x_\mathcal{N}) \in \mathcal{F}_{\mathrm{ind}}(\mathcal{T})$, $w_1(x_{\mathcal{N}\backslash\{i\}})$ in (76) is nonnegative. Also, $w_2(x_{\mathcal{N}\backslash\{i\}})$ in (77) is shown to be nonnegative. Let $\bar{x}_{\mathcal{N}\backslash\{i\}} \in \mathrm{proj}_{\mathcal{N}\backslash\{i\}}(\mathcal{T})$, and from the definition (2) of the projection, $\bar{x}_\mathcal{N} \in \mathcal{T}$ holds for some $\bar{x}_i \in \mathbb{R}^d$. From $v(x_\mathcal{N}) \in \mathcal{F}_{\mathrm{ind}}(\mathcal{T})$, defined in (35), and the decomposed form (60), $0 = w_1(\bar{x}_{\mathcal{N}\backslash\{i\}}) + w_2(\bar{x}_{\mathcal{N}\backslash\{j\}})$ is obtained. Therefore, $w_1(\bar{x}_{\mathcal{N}\backslash\{i\}}) = w_2(\bar{x}_{\mathcal{N}\backslash\{j\}}) = 0$ holds from their nonnegativeness. Let $\tilde{x}_{\mathcal{N}\backslash\{i\}} \notin \mathrm{proj}_{\mathcal{N}\backslash\{i\}}(\mathcal{T})$, and from (2), $\tilde{x}_\mathcal{N} \notin \mathcal{T}$ holds for any $\tilde{x}_i \in \mathbb{R}^d$. Let $\tilde{x}_i = \eta_i(\tilde{x}_{\mathcal{N}\backslash\{i,j\}})$, and from

(76), (77), and (81), $w_1(\tilde{x}_{\mathcal{N}\setminus\{j\}}) \geq w_2(\tilde{x}_{\mathcal{N}\setminus\{j\}})$ holds. From $v(x_\mathcal{N}) \in \mathcal{F}_{\rm ind}(\mathcal{T})$, $0 < v(\tilde{x}_\mathcal{N}) = w_1(\tilde{x}_{\mathcal{N}\setminus\{i\}}) + w_2(\tilde{x}_{\mathcal{N}\setminus\{j\}})$ is obtained, and thus $w_1(\tilde{x}_{\mathcal{N}\setminus\{j\}}) > 0$ holds. Hence, $w_1(x_{\mathcal{N}\setminus\{i\}})$ is an indicator of $\mathrm{proj}_{\mathcal{N}\setminus\{i\}}(\mathcal{T})$.

(iii) From Lemma 6, $v(x_\mathcal{N}) \in \mathcal{F}_{\rm r}(\mathcal{M} \ltimes \mathcal{B})$ is relatively $(\mathcal{M} \ltimes \mathcal{B})$-invariant of weight $(\det(M))^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$. Hence, from (80) and the definition (17) of relative invariance

$$\begin{aligned}\eta_{ij}((M,b)x_{\mathcal{N}\setminus\{i,j\}}) &\in (M,b) \operatorname*{argmin}_{x_{ij}\in\mathbb{R}^{d\times 2}} v((M,b)x_\mathcal{N}) \\ &= (M,b) \operatorname*{argmin}_{x_{ij}\in\mathbb{R}^{d\times 2}} (\det(M))^{\frac{2}{d}} v(x_\mathcal{N}) \\ &= (M,b) \operatorname*{argmin}_{x_{ij}\in\mathbb{R}^{d\times 2}} v(x_\mathcal{N}) \end{aligned} \tag{82}$$

holds, where $(M, b)\operatorname{argmin}(\cdot)$ is the set of the elements in $\operatorname{argmin}(\cdot)$ multiplied by (M, b). From (80) and (82)

$$\eta_{ij}((M,b)x_{\mathcal{N}\setminus\{i,j\}}) = (M,b)\eta_{ij}(x_{\mathcal{N}\setminus\{i,j\}}) \tag{83}$$

is obtained. From (76), (83), and the relative $(\mathcal{M} \ltimes \mathcal{B})$-invariance of $v(x_\mathcal{N})$

$$\begin{aligned}&w_1((M,b)x_{\mathcal{N}\setminus\{i\}}) \\ &\quad = v((M,b)x_\mathcal{N})|_{x_i=(M,b)^{-1}\eta_i((M,b)x_{\mathcal{N}\setminus\{i,j\}})} \\ &\qquad - \frac{1}{2}v((M,b)x_\mathcal{N})|_{x_{ij}=(M,b)^{-1}\eta_{ij}((M,b)x_{\mathcal{N}\setminus\{i,j\}})} \\ &\quad = (\det(M))^{\frac{2}{d}} v(x_\mathcal{N})|_{x_i=\eta_i(x_{\mathcal{N}\setminus\{i,j\}})} \\ &\qquad - \frac{1}{2}(\det(M))^{\frac{2}{d}} v(x_\mathcal{N})|_{x_{ij}=\eta_{ij}(x_{\mathcal{N}\setminus\{i,j\}})} \\ &\quad = (\det(M))^{\frac{2}{d}} w_1(x_{\mathcal{N}\setminus\{i\}})\end{aligned}$$

holds. Thus, $w_1(x_{\mathcal{N}\setminus\{i\}})$ satisfies the definition (17) of relative invariance with weight $(\det(M))^{\frac{2}{d}}$ for $(M, b) \in \mathcal{M} \ltimes \mathcal{B}$.

Appendix C
Proof of Lemma 9

For a preliminary, the directional derivative of a matrix-valued function $F : \mathbb{R}^{d\times n} \to \mathbb{R}^{d\times n}$ at $x_\mathcal{N} \in \mathbb{R}^{d\times n}$ in the direction $W \in \mathbb{R}^{d\times n}$ is defined as

$$\nabla_W F(x_\mathcal{N}) = \lim_{h\to 0} \frac{1}{h}(F(x_\mathcal{N} + hW) - F(x_\mathcal{N})). \tag{84}$$

Then, we obtain the following lemma.

Lemma 10: For a set $\mathcal{T} \subset \mathbb{R}^{d\times n}$, let $\hat{T} : \mathbb{R}^{d\times n} \to \mathcal{T}$ be a solution to the optimization problem (63). Then, for any $x_\mathcal{N}, W \in \mathbb{R}^{d\times n}$, the following holds:

$$\langle \nabla_W \hat{T}(x_\mathcal{N}), x_\mathcal{N} - \hat{T}(x_\mathcal{N})\rangle = 0. \tag{85}$$

Proof: From (63)

$$\|x_\mathcal{N} - \hat{T}(x_\mathcal{N})\| \leq \|x_\mathcal{N} - \tilde{T}\| \, \forall \tilde{T} \in \mathcal{T}$$

holds, and by applying $\tilde{T} = \hat{T}(x_\mathcal{N} + hW)$, we obtain

$$\begin{aligned}&\langle \hat{T}(x_\mathcal{N} + hW) - \hat{T}(x_\mathcal{N}), x_\mathcal{N} - \hat{T}(x_\mathcal{N})\rangle \\ &= \langle \hat{T}(x_\mathcal{N} + hW) - x_\mathcal{N}, x_\mathcal{N} - \hat{T}(x_\mathcal{N})\rangle + \|x_\mathcal{N} - \hat{T}(x_\mathcal{N})\|^2 \\ &\leq \langle \hat{T}(x_\mathcal{N} + hW) - x_\mathcal{N}, x_\mathcal{N} - \hat{T}(x_\mathcal{N})\rangle \\ &\quad + \frac{1}{2}\|x_\mathcal{N} - \hat{T}(x_\mathcal{N})\|^2 + \frac{1}{2}\|\hat{T}(x_\mathcal{N} + hW) - x_\mathcal{N}\|^2 \\ &= \frac{1}{2}\|\hat{T}(x_\mathcal{N} + hW) - x_\mathcal{N} + (x_\mathcal{N} - \hat{T}(x_\mathcal{N}))\|^2 \\ &= \frac{1}{2}\|\hat{T}(x_\mathcal{N} + hW) - \hat{T}(x_\mathcal{N}))\|^2. \end{aligned} \tag{86}$$

From (86), for $h > 0$, we obtain

$$\begin{aligned}&\frac{1}{h}\langle \hat{T}(x_\mathcal{N} + hW) - \hat{T}(x_\mathcal{N}), x_\mathcal{N} - \hat{T}(x_\mathcal{N})\rangle \\ &\qquad \leq \frac{h}{2}\left\|\frac{1}{h}(\hat{T}(x_\mathcal{N} + hW) - \hat{T}(x_\mathcal{N}))\right\|^2. \end{aligned} \tag{87}$$

From (84), as $h \to 0+$, the left-hand side of (87) converges as

$$\begin{aligned}&\lim_{h\to 0+}\left\langle \frac{1}{h}(\hat{T}(x_\mathcal{N} + hW) - \hat{T}(x_\mathcal{N})), x_\mathcal{N} - \hat{T}(x_\mathcal{N})\right\rangle \\ &\qquad = \langle \nabla_W \hat{T}(x_\mathcal{N}), x_\mathcal{N} - \hat{T}(x_\mathcal{N})\rangle\end{aligned}$$

while the right-hand side of (87) converges to 0. Hence

$$\langle \nabla_W \hat{T}(x_\mathcal{N}), x_\mathcal{N} - \hat{T}(x_\mathcal{N})\rangle \leq 0$$

holds. By considering $h < 0$, the converse inequality is obtained. Thus, (85) is achieved. ■

From (1) and (63), $(\mathrm{dist}(x_\mathcal{N}, \mathcal{T}))^2 = \|x_\mathcal{N} - \hat{T}(x_\mathcal{N})\|^2$ holds. Partially differentiate both the sides of this equation with respect to x_{ji}, the jth entry of x_i, and we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\partial(\mathrm{dist}(x_\mathcal{N}, \mathcal{T}))^2}{\partial x_{ji}} = \frac{1}{2}\frac{\partial}{\partial x_{ji}}\langle x_\mathcal{N} - \hat{T}(x_\mathcal{N}), x_\mathcal{N} - \hat{T}(x_\mathcal{N})\rangle \\ &= \left\langle e_{dj}e_{ni}^\top - \frac{\partial \hat{T}}{\partial x_{ji}}(x_\mathcal{N}), x_\mathcal{N} - \hat{T}(x_\mathcal{N})\right\rangle \\ &= \left\langle e_{dj}e_{ni}^\top - \nabla_{e_{dj}e_{ni}^\top}\hat{T}(x_\mathcal{N}), x_\mathcal{N} - \hat{T}(x_\mathcal{N})\right\rangle \\ &= \langle e_{dj}e_{ni}^\top, x_\mathcal{N} - \hat{T}(x_\mathcal{N})\rangle = \mathrm{tr}(e_{ni}e_{dj}^\top(x_\mathcal{N} - \hat{T}(x_\mathcal{N}))) \\ &= e_{dj}^\top(x_\mathcal{N} - \hat{T}(x_\mathcal{N}))e_{ni} = e_{dj}^\top(x_i - \mathrm{col}_i(\hat{T}(x_\mathcal{N})))\end{aligned} \tag{88}$$

from (84) and Lemma 10 for $W = e_{dj}e_{ni}^\top$. By collecting (88) for all $j \in \{1, 2, \ldots, d\}$, (62) is achieved.

References

[1] J. Shamma, Ed., *Cooperative Control of Distributed Multi-Agent Systems*. New York, NY, USA: Wiley, 2008.

[2] S. Martínez, J. Cortés, and F. Bullo, "Motion coordination with distributed information," *IEEE Control Syst. Mag.*, vol. 27, no. 4, pp. 75–88, Jul. 2007.

[3] Y. Cao, W. Yu, W. Ren, and G. Chen, "An overview of recent progress in the study of distributed multi-agent coordination," *IEEE Trans. Ind. Informat.*, vol. 9, no. 1, pp. 427–438, Feb. 2012.

[4] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[5] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol. 95, no. 1, pp. 215–233, Jan. 2007.

[6] J. Cortés, S. Martinez, T. Karatas, and F. Bullo, "Coverage control for mobile sensing networks," *IEEE Trans. Robot. Autom.*, vol. 20, no. 2, pp. 243–255, Apr. 2004.
[7] R. Olfati-Saber, "Flocking for multi-agent dynamic systems: Algorithms and theory," *IEEE Trans. Autom. Control*, vol. 51, no. 3, pp. 401–420, Mar. 2006.
[8] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, "Flocking in fixed and switching networks," *IEEE Trans. Autom. Control*, vol. 52, no. 5, pp. 863–868, May 2007.
[9] J. A. Marshall, M. E. Broucke, and B. A. Francis, "Formations of vehicles in cyclic pursuit," *IEEE Trans. Autom. Control*, vol. 49, no. 11, pp. 1963–1974, Nov. 2004.
[10] T.-H. Kim, S. Hara, and Y. Hori, "Cooperative control of multi-agent dynamical systems in target-enclosing operations using cyclic pursuit strategy," *Int. J. Control*, vol. 83, no. 10, pp. 2040–2052, 2010.
[11] Y. Igarashi, T. Hatanaka, M. Fujita, and M. W. Spong, "Passivity-based attitude synchronization in SE(3)," *IEEE Trans. Control Syst. Technol.*, vol. 17, no. 5, pp. 1119–1134, Sep. 2009.
[12] W. Ren, "Distributed cooperative attitude synchronization and tracking for multiple rigid bodies," *IEEE Trans. Control Syst. Technol.*, vol. 18, no. 2, pp. 383–392, Mar. 2010.
[13] K. Sakurama and H. Ahn, "Multi-agent coordination over local indexes via clique-based distributed assignment," *Automatica*, vol. 112, 2020, Art. no. 108670.
[14] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1465–1476, Sep. 2004.
[15] Z. Lin, B. Francis, and M. Maggiore, "Necessary and sufficient graphical conditions for formation control of unicycles," *IEEE Trans. Autom. Control*, vol. 50, no. 1, pp. 121–127, Jan. 2005.
[16] W. Ren and R. Beard, *Distributed Consensus in Multi-Vehicle Cooperative Control Theory and Applications*. New York, NY, USA: Springer, 2008.
[17] B. D. O. Anderson, C. Yu, B. Fidan, and J. M. Hendrickx, "Rigid graph control architectures for autonomous formations," *IEEE Control Syst. Mag.*, vol. 28, no. 6, pp. 48–63, Dec. 2008.
[18] L. Krick, M. E. Broucke, and B. A. Francis, "Stabilisation of infinitesimally rigid formations of multi-robot networks," *Int. J. Control*, vol. 82, no. 3, pp. 423–439, Mar. 2009.
[19] F. Dörfler and B. Francis, "Geometric analysis of the formation problem for autonomous robots," *IEEE Trans. Autom. Control*, vol. 55, no. 10, pp. 2379–2384, Oct. 2010.
[20] P. Lin and Y. Jia, "Distributed rotating formation control of multi-agent systems," *Syst. Control Lett.*, vol. 59, no. 10, pp. 587–595, 2010.
[21] K. Oh, M. Park, and H. Ahn, "A survey of multi-agent formation control," *Automatica*, vol. 53, no. 3, pp. 424–440, Mar. 2015.
[22] I. Maza, F. Caballero, J. Capitán, J. R. M. de Dios, and A. Ollero, "Experimental results in multi-UAV coordination for disaster management and civil security applications," *J. Intell. Robot. Syst.*, vol. 61, no. 1, pp. 563–585, Jan. 2011.
[23] E. Fiorelli, N. E. Leonard, P. Bhatta, D. A. Paley, R. Bachmayer, and D. M. Fratantoni, "Multi-AUV control and adaptive sampling in Monterey bay," *IEEE J. Ocean. Eng.*, vol. 31, no. 4, pp. 935–948, Oct. 2006.
[24] R. Tron, J. Thomas, G. Loianno, K. Daniilidis, and V. Kumar, "A distributed optimization framework for localization and formation control," *IEEE Control Syst. Mag.*, vol. 36, no. 4, pp. 22–44, Aug. 2016.
[25] H. Ahn, *Formation Control*. Cham, Switzerland: Springer, 2020.
[26] S. Zhao and D. Zelazo, "Bearing rigidity and almost global bearing-only formation stabilization," *IEEE Trans. Autom. Control*, vol. 61, no. 5, pp. 1255–1268, May 2016.
[27] S. Zhao and D. Zelazo, "Bearing rigidity theory and its applications for control and estimation of network systems," *IEEE Control Syst. Mag.*, vol. 39, no. 2, pp. 66–83, Apr. 2019.
[28] Z. Han, L. Wang, Z. Lin, and R. Zheng, "Formation control with size scaling via a complex Laplacian-based approach," *IEEE Trans. Cybern.*, vol. 46, no. 10, pp. 2348–2359, Oct. 2016.
[29] K. Sakurama, S. Azuma, and T. Sugie, "Multi-agent coordination to high-dimensional target subspaces," *IEEE Trans. Control Netw. Syst.*, vol. 5, no. 1, pp. 345–357, Mar. 2018.
[30] B. D. O. Anderson, Z. Sun, T. Sugie, S. Azuma, and K. Sakurama, "Formation shape control with distance and area constraints," *IFAC J. Syst. Control*, vol. 1, pp. 2–12, 2017.
[31] Z. Sun, M.-C. Park, B. D. Anderson, and H. Ahn, "Distributed stabilization control of rigid formations with prescribed orientation," *Automatica*, vol. 78, pp. 250–257, 2017.
[32] K. Sakurama, S. Azuma, and T. Sugie, "Multi-agent coordination via distributed pattern matching," *IEEE Trans. Autom. Control*, vol. 64, no. 8, pp. 3210–3225, Aug. 2019.
[33] L. Chen, M. Cao, and C. Li, "Angle rigidity and its usage to stabilize multi-agent formations in 2D," *IEEE Trans. Autom. Control*, to be published, doi: 10.1109/TAC.2020.3025539.
[34] G. Jing, G. Zhang, H. W. J. Lee, and L. Wang, "Angle-based shape determination theory of planar graphs with application to formation stabilization," *Automatica*, vol. 105, pp. 117–129, 2019.
[35] G. Jing and L. Wang, "Multi-agent flocking with angle-based formation shape control," *IEEE Trans. Autom. Control*, vol. 65, no. 2, pp. 817–823, Feb. 2019.
[36] K. Sakurama, "Formation control of multi-agent systems over generalized relative measurement," in *Proc. 57th IEEE Conf. Decis. Control*, 2018, pp. 2878–2882.
[37] P. J. Olver, *Equivalence, Invariants, and Symmetry*. New York, NY, USA: Cambridge Univ. Press, 1995.
[38] P. J. Olver, *Classical Invariant Theory*. Cambridge, U.K.: Cambridge Univ. Press, 1999.
[39] F. Bullo and A. D. Lewis, *Geometric Control of Mechanical Systems: Modeling, Analysis, and Design for Simple Mechanical Control Systems*. New York, NY, USA: Springer Science and Business Media, 2004.
[40] B. Bolloás, *Modern Graph Theory*. New York, NY, USA: Springer Science and Business Media, 1998.
[41] T. A. McKee and F. R. McMorris, *Topics in Intersection Graph Theory*. Philadelphia PA, USA: SIAM, 1999.
[42] M. Boutin and G. Kemper, "On reconstructing n-point configurations from the distribution of distances or areas," *Adv. Appl. Math.*, vol. 32, no. 4, pp. 709–735, 2004.
[43] K. Sakurama, S. Azuma, and T. Sugie, "Distributed controllers for multi-agent coordination via gradient-flow approach," *IEEE Trans. Autom. Control*, vol. 60, no. 6, pp. 1471–1485, Jun. 2015.
[44] E. Tomita, A. Tanaka, and H. Takahashi, "The worst-case time complexity for generating all maximal cliques and computational experiments," *Theor. Comput. Sci.*, vol. 363, pp. 28–42, 2006.
[45] S. Even and Y. Shiloach, "An on-line edge-deletion problem," *J. ACM*, vol. 28, pp. 1–4, 1981.
[46] J. C. Gower and G. B. Dijksterhuis, *Procrustes Problems*. New York, NY, USA: Oxford Univ. Press, 2004.
[47] A. A. Kannan, B. Fidan, G. Mao, and B. D. Anderson, "Analysis of flip ambiguities in distributed network localization," in *Proc. Inf., Decis. Control*, 2007, pp. 193–198.

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