Model Reference DSMC With a Relative Degree Two Switching Variable

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***Abstract*—Reaching-law-based discrete sliding mode controllers are well known to be easy to tune and robust with respect to disturbance. In recent years, it has been demonstrated that their robustness can be further enhanced with the use of arbitrary relative degree sliding variables. However, reaching laws using such variables only ensure a good sliding mode performance of the system when the perturbations affecting the plant are matched, which is a very restrictive assumption. To address this issue and to further improve robustness of the plant, in this article, we propose a new model reference approach for strategies with relative degree two sliding variables. In the proposed approach, the reaching law is first used to control the evolution of a disturbance-free model of the plant, and then, the original system state is driven toward that of the model with a secondary controller. It will be shown that the proposed approach ensures better system robustness compared to the conventional reaching law approach and that it does not require the assumption about matched uncertainties.**

***Index Terms*—Control theory, sliding mode control, robust control, discrete-time systems.**

I. Introduction

Rejection of disturbance and model uncertainties has been a topic of extensive research in the field of the control theory. Since unpredictable perturbations can significantly degrade the performance of the controlled plant and even negatively affect its stability, development of robust control schemes proved to be necessary. One of the most significant developments in robust control was the introduction of continuous-time sliding modes [1]–[3]. Sliding mode control strategies provide complete insensitivity to a class of disturbance and model uncertainties that satisfy the so-called matching conditions [4]. Furthermore, since most modern control processes are applied digitally, sliding mode methodology has been further applied to discrete-time systems [5], [6]. The field of sliding mode control has been further developed by various researches, both in the area of continuous-time systems [7]–[11] and discrete-time ones [12]–[16].

A major development in the sliding mode control theory was the introduction of the so-called reaching law approach for both continuous-time [17] and discrete-time systems [18]. Rather than analyzing the stability of the sliding motion using Lyapunov's theorem, this approach allows one to *a priori* specify the evolution of the system representative point, and then, synthesize the control signal that enforces this evolution. This approach has gained significant popularity in discrete-time sliding mode control (DSMC) and various authors have introduced new reaching laws with the aim of ensuring favorable properties of quasi-sliding motion [19]–[28].

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Since the size of the quasi-sliding mode band to which the system representative point is ultimately driven reflects robustness of the plant with respect to uncertainties, it is vital for the control engineer to ensure the smallest possible width of this band. In recent years, various researchers have used discrete-time sliding variables with arbitrary relative degree to improve the disturbance rejection capabilities of reaching-law-based strategies [29]–[32]. Since such variables are not affected by the control signal and matched perturbations from a number of time instants ago smaller than their relative degree, it is possible to obtain precise information about their future evolution up to $r-1$ steps in advance, where r is the relative degree of the variable. This information can then be used in the sliding mode controller design procedure with the aim of improving robustness of the plant. However, even though many recent works on discrete-time sliding mode control have abandoned the classic assumption about matched uncertainties [23], [24], this restrictive assumption is still necessary for strategies using higher than one relative degree sliding variables.

Discrete-time reaching laws rely on a recursive function to specify future values of the sliding variable. However, this creates a significant and often neglected problem in practical implementation of such strategies. Since value of the sliding variable is determined on a step-by-step basis, it is affected by disturbance and model uncertainties from all previous steps. As a result, the desired trajectory of the system can become distorted at any stage of the control process. To remedy this issue, a recent work [33] proposed the use of a disturbance-free reference model of the plant. The new approach uses the reference model of the plant to obtain a desired trajectory of its representative point, and then, drives the state of the original plant alongside that trajectory, thus eliminating the residual effect of past perturbations on the motion of the system.

In this article, we introduce a new discrete-time reaching-law-based sliding mode control strategy using a reference model of the plant and apply it to the system subject to unmatched uncertainties. Contrary to the approach from [33] where Gao's classic reaching law [18] has been applied to the model, in our article, a more sophisticated reaching law is considered. In particular, we use generalization of Gao's reaching law for sliding variables with relative degree two [32]. However, even though this reaching law has been shown to provide better dynamical properties of the system than its relative degree one equivalent, its application is usually not feasible when matching conditions are not satisfied. Since the effect of unmatched uncertainties on a relative degree two sliding variable is significantly amplified, one risks deteriorating the sliding mode performance of the system. In order to overcome this challenge, in our article, we have applied the relative degree two reaching law to the reference model of the plant, and then, designed a secondary, relative degree one controller for the actual plant with the aim of driving its state alongside that of the model. As a result, favorable properties ensured by the relative degree two strategy [32] are obtained without the restrictive assumption about matched uncertainties, which was not previously possible. It has been further demonstrated that the proposed strategy confines the system representative point to a narrower vicinity of the sliding hyperplane than the previously proposed model reference approach [33] and the reaching law [32] applied individually.

This is a significant property, since (as shown in this article) achieving a narrower quasi-sliding mode band directly reduces the error of all state variables.

II. Considered Class of Systems

In this article, we consider a class of discrete-time single-input and single-output systems subject to perturbations that do not satisfy the matching conditions. State equation of the considered plants has the following general form:

$$\boldsymbol{x}_p(k+1) = \boldsymbol{A}\boldsymbol{x}_p(k) + \boldsymbol{b}u_p(k) + \boldsymbol{p}\tilde{d}(k) \tag{1}$$

where $\boldsymbol{x}_p \in \boldsymbol{R}^n$ is the system state, $u_p \in \boldsymbol{R}$ is the control signal, $\tilde{d} \in \boldsymbol{R}$ collectively represents model uncertainties and external disturbance, and $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{p}$ are of appropriate dimensions. For all k uncertainties affecting the plant are assumed to be bounded by constants

$$d^{\min} \leq \tilde{d}(k) \leq d^{\max}. \tag{2}$$

Mean effect of these perturbations on the system and its maximum admissible deviation from the mean are defined as

$$d^{\text{avg}} = 0.5(d^{\max} + d^{\min}), \quad d^{\delta} = 0.5(d^{\max} - d^{\min}) \tag{3}$$

which implies that the following inequalities hold for all k:

$$|\tilde{d}(k) - d^{\text{avg}}| \leq d^{\delta}, \quad |\tilde{d}(k)| \leq |d^{\text{avg}}| + d^{\delta}. \tag{4}$$

Since vectors $\boldsymbol{b}$ and $\boldsymbol{p}$ are not necessarily collinear, perturbations affecting the plant (1) are not matched. Initial conditions of the plant are known and equal to $\boldsymbol{x}_p(0)$ and the goal of the control process is to drive the state of the plant to zero. This objective will be achieved with a new sliding mode control strategy using a reference model of the plant. State equation of this model has the following form:

$$\boldsymbol{x}_m(k+1) = \boldsymbol{A}\boldsymbol{x}_m(k) + \boldsymbol{b}u_m(k) \tag{5}$$

where the state matrix $\boldsymbol{A}$ and input distribution vector $\boldsymbol{b}$ are the same as in (1). Furthermore, $\boldsymbol{x}_m(0) = \boldsymbol{x}_p(0)$ and the target state of the model is also $\boldsymbol{0}$. In order to design a discrete-time sliding mode controller, one must first define an appropriate sliding variable and its corresponding switching hyperplane. In particular, in this article, discrete-time sliding variables with relative degrees equal to one and two will be considered. In order to make the reaching law with a relative degree two sliding variable applicable to the plant (1), a new approach using the reference model (5) will be proposed. In this approach, two control signals are designed. First, a controller based on the reaching law with a relative degree two variable will be applied to the reference model to obtain favorable properties of its sliding motion. Then, a dead-beat controller with a relative degree one sliding variable will be applied to the original plant with the aim of driving its trajectory alongside that of the model in the presence of nonmatched uncertainties. It will be demonstrated that this approach effectively eliminates the effect of perturbations on the original plant's sliding motion, save for the single most recent disturbance term. This allows one to obtain smaller state error compared to the conventional reaching-law-based strategy and allows one to use sliding variables with relative degree higher than one without the need to satisfy matching conditions.

A. Relative Degree One and Two Sliding Variables

In this section, we will specify the appropriate sliding hyperplanes for both the original plant and its reference model. For the plant (1), we define the conventional relative degree one sliding variable and its corresponding switching hyperplane in the following way:

$$s_p(k) = \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_p(k) = 0 \tag{6}$$

where the "1" subscript signifies the relative degree and $\boldsymbol{c}_1$ is selected to guarantee that $\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b} \neq 0$. Although it is sufficient for the stability of the plant that all eigenvalues of the closed-loop system state matrix are inside the unit circle, in this article, the eigenvalues will be placed in zero to ensure the smallest possible error of all state variables in the sliding phase. To that end, elements of $\boldsymbol{c}_1$ must satisfy the following equation [27]:

$$\det\left[\lambda\boldsymbol{I}_{n\times n} - \boldsymbol{A} + \boldsymbol{b}(\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b})^{-1}\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{A}\right] = \lambda^n. \tag{7}$$

DSMC strategies using a relative degree one sliding variable do not require the assumption about matched disturbance, which makes them applicable to the considered class of systems.

In a similar way, one can design a sliding variable with relative degree higher than one. Although strategies using such variables require the assumption about matched disturbance and cannot be used for the system (1), they can successfully be applied to the disturbance-free model (5) in order to ensure its good dynamical properties. For the reference model, we define the following relative degree two sliding variable and its corresponding switching hyperplane:

$$s_m(k) = \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_m(k) = 0 \tag{8}$$

where $\boldsymbol{c}_2^{\mathrm{T}}$ is a vector selected so that $\boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{b} = 0$ and $\boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{A}\boldsymbol{b} \neq 0$. In practice, this vector is commonly chosen to ensure $\boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{A}\boldsymbol{b} = 1$ in order to streamline the analysis of system sliding motion. Just like for the conventional sliding variable (6), vector $\boldsymbol{c}_2$ is selected to guarantee a finite-time performance of the closed-loop system, which means it must satisfy [27]

$$\det\left[\lambda\boldsymbol{I}_{n\times n} - \boldsymbol{A} + \boldsymbol{b}(\boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{A}\boldsymbol{b})^{-1}\boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{A}^2\right] = \lambda^n. \tag{9}$$

Sliding variables with a relative degree higher than one are only affected by the control signal (and matched uncertainties) from a number of time instants ago equal to or greater than their relative degree. This is a vital property as it allows one to calculate values of such variable several steps in advance and use this additional information when designing a sliding mode controller. Indeed, since $\boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{b} = 0$, substitution of the model state equation (5) into relation (8) yields

$$\begin{aligned} s_m(k) &= \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}_m(k-1) \\ &= \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{A}^2\boldsymbol{x}_m(k-2) + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{A}\boldsymbol{b}u(k-2) \end{aligned} \tag{10}$$

which implies that a relative degree two sliding variable is not affected by the control signal from the previous time instant. This property is always ensured for the considered reference model since it is free of uncertainties. In the next two sections of this article, a new model reference sliding mode control strategy will be proposed and its design will be divided in two stages. First, a reaching law using the relative degree two sliding variable (8) will be applied to the reference model (5), and then, a new controller using the sliding variable (6) will be proposed for the original plant.

Remark 1: In formerly published literature [27], it has been demonstrated that for vectors $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ selected according to (7) and (9), respectively, one obtains

$$\boldsymbol{c}_1^{\mathrm{T}} = \gamma\boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{A} \tag{11}$$

where $\gamma \neq 0$ is an arbitrary constant. For the purpose of fair comparison between different strategies, further in this article, vectors $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ will be chosen so that $\gamma = 1$.

III. Reaching Law for the Disturbance-Free Model

In this section, the desired evolution of the disturbance-free model (5) will be obtained with the use of a reaching-law-based control strategy. In particular, a generalization of Gao's seminal reaching law for the case of relative degree two sliding variables will be used since such a strategy has been shown to provide better dynamical properties of the system than its relative degree one equivalent [32]. The considered reaching law is expressed in the following way:

$$s_m(k+2) = q^2 s_m(k) - q\varepsilon \mathrm{sgn}[s_m(k)] - \varepsilon \mathrm{sgn}[s_m(k+1)] \tag{12}$$

where $s_m(k)$ is the relative degree two sliding variable (8) and $\varepsilon > 0$, $1 > q > 0$ are the design parameters. The objective of this strategy is to drive the system representative point to a narrow vicinity of the sliding hyperplane and to ensure that the hyperplane is crossed in each step. The control signal that satisfies these properties can be obtained by substituting (10) into the left-hand side of the reaching law (12) and solving the obtained equation for $u_m(k)$. Then, the control signal has the following form:

$$u_m(k) = (\boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{A}\boldsymbol{b})^{-1}\{q^2 s_m(k) - q\varepsilon\mathrm{sgn}[s_m(k)] - \varepsilon\mathrm{sgn}[s_m(k+1)] - \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{A}^2\boldsymbol{x}_m(k)\}. \tag{13}$$

In formerly published literature [32], several advantageous properties of the reaching law (12) have been proven. Two of those properties will now be quoted.

Property 1: For any initial conditions of the system, its representative point will cross the sliding hyperplane in finite time and cross this hyperplane again in each subsequent step.

Property 2: If $\varepsilon > |\boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{A}\boldsymbol{b}|d^\delta/(1-q)$ (where d^δ is the maximum deviation of the matched disturbance from the mean) and the representative point enters the quasi-sliding mode band

$$\left\{\boldsymbol{x}_m : |\boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_m| \le \frac{\varepsilon}{1+q} + \frac{|\boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{A}\boldsymbol{b}|d^\delta}{1-q^2}\right\} \tag{14}$$

it will remain inside that band for all future time instants.

The first property directly applies to the model presented in this article. However, since the disturbance-free case is considered in this section, a stronger version of the second property can be proven for the model (5). This extension of Property 2 will now be formulated in the following theorem.

Theorem 1: If the control signal for the model (5) is defined according to the relation (13), then for any initial conditions of the system, its representative point will at least asymptotically approach the quasi-sliding mode band

$$\left\{\boldsymbol{x}_m : |\boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}_m| \le \frac{\varepsilon}{1+q}\right\}. \tag{15}$$

Furthermore, if the representative point enters this band, it will remain inside it for all future time instants.

Proof: Let $\boldsymbol{x}_m(k)$ be the first such a state that $\mathrm{sgn}[s_m(k+1)] = -\mathrm{sgn}[s_m(k)]$. The existence of this state is ensured by the Property 1. First, it will be shown that if $\boldsymbol{x}_m(k)$ is out of the band (15), this band will be approached at least asymptotically. Indeed, the relation (12) gives

$$\begin{aligned} s_m(k+2) &= q^2 s_m(k) + (1-q)\varepsilon\mathrm{sgn}[s_m(k)] \\ &= s_m(k) - (1-q^2)s_m(k) + (1-q)\varepsilon\mathrm{sgn}[s_m(k)] \\ &= s_m(k) - (1-q)\left\{(1+q)s_m(k) - \varepsilon\mathrm{sgn}[s_m(k)]\right\}. \end{aligned} \tag{16}$$

Since $0 < q < 1$ and $|s_m(k)| > \varepsilon/(1+q)$, then for positive $s_m(k)$, one gets

$$s_m(k+2) < s_m(k) - (1-q)\left(\frac{1+q}{1+q}\varepsilon - \varepsilon\right) = s_m(k). \tag{17}$$

Therefore, if $s_m(k) > \varepsilon/(1+q)$, then the variable will always decrease in the next step. Thus, s_m will either become smaller than $\varepsilon/(1+q)$ in finite time or asymptotically converge to a certain positive value. Suppose that $s_m(k)$ tends to $s_+ > 0$ for $k \to \infty$. Then, $s_m(k+2)$ is also convergent to s_+ and

$$s_+ = \lim_{k\to\infty} s_m(k+2) \le q^2 s_+ + (1-q)\varepsilon. \tag{18}$$

Solving (18) for s_+, one gets

$$s_+ \le \frac{1-q}{1-q^2}\varepsilon = \frac{\varepsilon}{1+q}. \tag{19}$$

Thus, if the sliding variable does not become smaller than $\varepsilon/(1+q)$ in finite time, it must asymptotically converge to this exact value. Repeating derivations (17)–(19) for negative sliding variables, one concludes that s_m will either become bigger than $-\varepsilon/(1+q)$ in finite time or approach this value asymptotically. Therefore, it has been proven that the representative point of the model (5) will always approach the quasi-sliding mode band (1) at least asymptotically.

The proof that the system representative point, after entering the quasi-sliding mode band, will remain inside it in the next step follows directly from Lemma 2 presented in [32] for $r = 2$ and $d^\delta = 0$. Since the proof can be found in the existing literature [32], it will not be repeated in this article. ■

In this section, a reaching-law-based control strategy using a relative degree two sliding variable has been applied to the disturbance-free reference model of the plant. It has been shown that the application of this strategy ensures a switching-type quasi-sliding motion of the system and confines its representative point to a narrow vicinity of the switching hyperplane. In the next section, desirable system dynamics obtained from the disturbance-free model with the use of the reaching law approach will be applied to design a control strategy for the original plant (1).

IV. New Model Reference DSMC Strategy

The objective of the control strategy for the considered plants is to drive its output alongside the trajectory specified by the reaching law (12) in the presence of uncertainties that do not satisfy matching conditions. To that end, a new sliding mode control strategy will be designed using values of s_m obtained from the disturbance-free model (5). Since the plant (1) is subject to disturbance and parameter uncertainties that do not satisfy matching conditions, relative degree one sliding variable (6) will be used in order to make the proposed strategy applicable. The reaching law proposed for the plant is expressed as

$$s_p(k+1) = s_m(k+2) + \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}\tilde{d}(k) - \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}d^{\mathrm{avg}} \tag{20}$$

where $s_p(k)$ is the relative degree one sliding variable (6), $s_m(k)$ are values obtained from the reference model (5) with the use of the reaching law (12) and $\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}\tilde{d}(k)$ represents the total effect of disturbance and model uncertainties on the sliding variable. It is important to highlight that even though the function (20) requires values of s_m from the time instant $k+2$, it can be calculated at time k. Indeed, as shown in relation (10), values of s_m can be obtained two time instants in advance, which means it is possible to use $s_m(k+2)$ in the reaching law (20). Just like in the previous section, the proposed reaching law is applied to design the control signal. To that end, the left-hand side of the state equation (1) is first substituted into (20), giving

$$\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}_p(k) + \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b}u_p(k) + \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}\tilde{d}(k) = s_m(k+2) + \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}\tilde{d}(k) - \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}d^{\mathrm{avg}}. \tag{21}$$

Then, solving (21) for u_p, we obtain the control signal

$$u_p(k) = (\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b})^{-1}[s_m(k+2) - \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}d^{\mathrm{avg}} - \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}_p(k)]. \tag{22}$$

It will now be demonstrated that the proposed control scheme using the reference model of the plant maintains favorable properties of sliding motion ensured by the reaching law (12), and at the same time, provides better robustness of the system with respect to disturbance. In particular, in the following theorem, it will be shown that the new strategy ensures a switching-type quasi-sliding motion exactly as defined by Gao *et al.* [18].

Theorem 2: If the control signal for the system (1) is defined by (22) where values of s_m are obtained from the reference model (5) and

$$\varepsilon > \frac{|\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}|d^{\delta}}{(1-q)(1+q^2)} \tag{23}$$

then the system representative point will cross the sliding hyperplane in finite time and cross it again for all future steps.

Proof: To prove the theorem, it will be shown that $\mathrm{sgn}[s_p(k+1)] = \mathrm{sgn}[s_m(k)]$ for all k after a finite number of initial steps. Let k_0 be the first time instant for which the model (5) subject to control (13) operates in the sliding mode. In other words, $\boldsymbol{x}(k_0)$ is the first state for which $\mathrm{sgn}[s_m(k_0-2)] = -\mathrm{sgn}[s_m(k_0-1)] = \mathrm{sgn}[s_m(k_0)]$. Then, the Property 1 described in Section III of this article implies that variable s_m will change its sign in each subsequent step. Thus, for any $k \geq k_0$, substitution of (12) into the right-hand side of (20) yields

$$s_p(k+1) = q^2 s_m(k) + (1-q)\varepsilon\mathrm{sgn}[s_m(k)] + \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}\tilde{d}(k) - \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}d^{\mathrm{avg}}. \tag{24}$$

Further substituting (12) into the right-hand side of (24), we obtain

$$s_p(k+1) = q^4 s_m(k-2) + q^2(1-q)\varepsilon\mathrm{sgn}[s_m(k-2)] + (1-q)\varepsilon\mathrm{sgn}[s_m(k)] + \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}\tilde{d}(k) - \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}d^{\mathrm{avg}}. \tag{25}$$

Since $\mathrm{sgn}[s_m(k-2)] = \mathrm{sgn}[s_m(k)]$, then for a positive $s_m(k-2)$ relations (4) and (25) give

$$s_p(k+1) \geq 0 + (1+q^2)(1-q)\varepsilon + \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}[\tilde{d}(k) - d^{\mathrm{avg}}] > \frac{(1+q^2)(1-q)}{(1-q)(1+q^2)}|\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}|d^{\delta} - |\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}|d^{\delta} = 0. \tag{26}$$

Likewise, for negative $s_m(k)$, relations (4) and (25) yield

$$s_p(k+1) \leq 0 - (1+q^2)(1-q)\varepsilon + \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}[\tilde{d}(k) - d^{\mathrm{avg}}] < -\frac{(1+q^2)(1-q)}{(1-q)(1+q^2)}|\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}|d^{\delta} + |\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}| = 0. \tag{27}$$

In conclusion, after a finite number of initial time instants, the sign of $s_p(k+1)$ is always equal to that of $s_m(k)$. Thus, since variable s_m is guaranteed to change its sign in finite time and change it again in each subsequent step, the same is true for variable s_p. ■

It has been demonstrated that with the right choice of parameters ε and q, the proposed control scheme ensures a switching-type motion of the system representative point, similarly to the reaching law (12) applied to the disturbance-free model. In the next theorem, it will be shown that the strategy (22) drives the representative point of the system (1) to a specified, narrow vicinity of the sliding hyperplane.

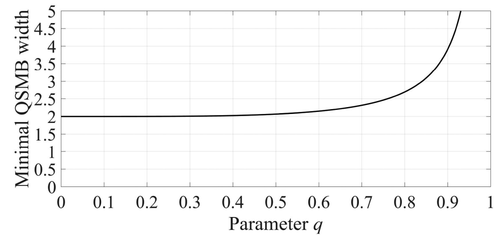

Fig. 1. Smallest possible quasi-sliding mode band width.

Theorem 3: If the control signal for the system (1) is defined by (22) where values of s_m are obtained from the reference model (5), then the system representative point approaches the following quasi-sliding mode band:

$$\left\{\boldsymbol{x}_p : |\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_p| \leq B = \frac{\varepsilon}{1+q} + |\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}|d^{\delta}\right\} \tag{28}$$

around the sliding hyperplane at least asymptotically.

Proof: It will be demonstrated that the upper bound of $|s_p(k)|$ converges to B as k tends to infinity. From the reaching law (20), one obtains

$$|s_p(k+1)| = |s_m(k+2) + \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}\tilde{d}(k) - \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}d^{\mathrm{avg}}| \leq |s_m(k+2)| + |\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}| \cdot |\tilde{d}(k) - d^{\mathrm{avg}}|. \tag{29}$$

Theorem 1 states that the upper bound of $|s_m(k)|$ approaches $\varepsilon/(1+q)$ as k tends to infinity. Naturally, the same is true for $|s_m(k+2)|$. Consequently, relations (4) and (29) give

$$\limsup_{k\to\infty}|s_p(k+1)| \leq \limsup_{k\to\infty}|s_m(k+2)| + |\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}|d^{\delta} \leq \frac{\varepsilon}{1+q} + |\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}|d^{\delta}. \tag{30}$$

Relation (30) implies that the representative point of the system (1) will always approach the quasi-sliding mode band (28) at least asymptotically. ■

It has been proven that the proposed control scheme using the disturbance-free reference model of the plant drives the representative point of the original system to a specified vicinity of the sliding hyperplane. This is an important property, since the ability to confine the system representative point to a small area in the state space illustrates robustness of the system with respect to disturbance. Furthermore, as one can see from relations (23) and (28), choice of parameters q and ε allows one to determine the convergence rate of the sliding variable to zero as well as the quasi-sliding mode band width. The relationship between parameter q and the smallest possible quasi-sliding mode band width has been illustrated in Fig. 1. This width is achieved when ε tends to its lower bound expressed by the inequality (23), assuming that $|\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}|d^{\delta} = 1$.

It can be seen from Fig. 1 that the quasi-sliding mode band width increases sharply for q greater than $\sim$0.8, which means selecting large values of this parameter is not recommended. On the other hand, for any values of q smaller than $\sim$0.8, the width of the band does not change dramatically. Thus, one can easily adjust q and ε to obtain the desired convergence rate of the sliding variable to zero without significant impact on the system robustness. The robustness property of the plant will be further elaborated upon in Section V of this article.

Remark 2: It is important to notice that the quasi-sliding mode band (28) originating from the newly proposed strategy is strictly narrower than the band (15) obtained from the reaching law [32] itself when design parameters for both cases are selected to ensure similar properties of the system in the reaching phase. Furthermore, the obtained band is also narrower than the one described in [33], where Gao's classic reaching law has been used in conjunction with a reference model approach.

V. State Boundedness in Sliding Mode

In the previous section, it has been demonstrated that the representative point of the system (1) is confined to a specified quasi-sliding mode band. However, this property alone does not give us sufficient information about evolution of individual state variables. It will now be shown that the proposed strategy ensures limited error of all state variables in the sliding phase, which illustrates robustness of the whole system with respect to disturbance. Furthermore, it will be demonstrated that this error is proportional to the quasi-sliding mode band width, which implies that a smaller band directly results in improved performance of the system. The property describing the state error of all state variables will be formulated in the following theorem.

Theorem 4: If the control signal for the system (1) is defined by (22) and the system representative point belongs to the band (29) for n consecutive time instants, then there exists k_0 such that for all $k \geq k_0$ and for every $i = 1, 2, \ldots, n$, the ith state variable of the plant

$$|x_{p,i}(k)| \leq B \cdot \left|\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b}\right|^{-1} \sum_{j=0}^{n-1} \left|\boldsymbol{q}_i \boldsymbol{A}_{\mathrm{cl}}^j \boldsymbol{b}\right| + \left(|d^{\mathrm{avg}}| + d^{\delta}\right) \sum_{j=0}^{n-1} \left|\boldsymbol{q}_i \boldsymbol{A}_{\mathrm{cl}}^j \left(\boldsymbol{p} - \boldsymbol{b}\frac{\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}}{\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b}}\right)\right| \tag{31}$$

where B is the width of the band (28), vector

$$\boldsymbol{q}_i = [\underbrace{0 \ \ldots \ 0}_{i-1} \ \ 1 \ \ \underbrace{0 \ \ldots \ 0}_{n-i}] \tag{32}$$

and the closed-loop system state matrix

$$\boldsymbol{A}_{\mathrm{cl}} = \left[\boldsymbol{A} - \boldsymbol{b}(\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b})^{-1}\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{A}\right]. \tag{33}$$

On the other hand, if the system representative point approaches the quasi-sliding mode band asymptotically, absolute value of each state variable will also asymptotically converge to its respective value on the right-hand side of relation (31).

Proof: It will be shown that if the system representative point belongs to the band (28), then inequality (31) is satisfied for all i. To that end, the state vector will first be expressed as a function of the sliding variable. We substitute control signal (22) into the state equation (1) and obtain

$$\begin{aligned}\boldsymbol{x}_p(k+1) &= \boldsymbol{A}\boldsymbol{x}_p(k) + \boldsymbol{b}(\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b})^{-1}[s_m(k+2) \\ &\quad - \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}d^{\mathrm{avg}} - \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}_p(k)] + \boldsymbol{p}\tilde{d}(k) \\ &= \left[\boldsymbol{A} - \boldsymbol{b}(\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b})^{-1}\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{A}\right]\boldsymbol{x}_p(k) + \boldsymbol{p}\tilde{d}(k) \\ &\quad + \boldsymbol{b}(\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b})^{-1}[s_m(k+2) - \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}d^{\mathrm{avg}}].\end{aligned} \tag{34}$$

Taking into account relations (20) and (33), (34) can be rewritten in the following way:

$$\begin{aligned}\boldsymbol{x}_p(k+1) &= \boldsymbol{A}_{\mathrm{cl}}\boldsymbol{x}_p(k) + \boldsymbol{b}(\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b})^{-1}[s_m(k+2) - \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}d^{\mathrm{avg}}] \\ &\quad + \left(\boldsymbol{p} - \boldsymbol{b}\frac{\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}}{\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b}} + \boldsymbol{b}\frac{\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}}{\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b}}\right)\tilde{d}(k) \\ &= \boldsymbol{A}_{\mathrm{cl}}\boldsymbol{x}_p(k) + \boldsymbol{b}(\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b})^{-1}s_p(k+1) \\ &\quad + \left(\boldsymbol{p} - \boldsymbol{b}\frac{\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}}{\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b}}\right)\tilde{d}(k).\end{aligned} \tag{35}$$

Repeated substitution of the left-hand side of the relation (35) into its right-hand side yields

$$\boldsymbol{x}_p(k) = \boldsymbol{A}_{\mathrm{cl}}^n\boldsymbol{x}_p(k-n) + \sum_{j=0}^{n-1}\boldsymbol{A}_{\mathrm{cl}}^j\boldsymbol{b}(\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b})^{-1}s_p(k-j) + \sum_{j=0}^{n-1}\boldsymbol{A}_{\mathrm{cl}}^j\left(\boldsymbol{p} - \boldsymbol{b}\frac{\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}}{\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b}}\right)\tilde{d}(k-j-1). \tag{36}$$

Since vector $\boldsymbol{c}_1$ selected according to the relation (7) ensures a dead-beat response of the closed-loop system, then the matrix $\boldsymbol{A}_{\mathrm{cl}}$ specified by (33) satisfies $\boldsymbol{A}_{\mathrm{cl}}^n = \boldsymbol{0}$. Furthermore, since the relation (32) gives $x_{p,i}(k) = \boldsymbol{q}_i\boldsymbol{x}_p(k)$ for all $i = 1, 2, \ldots, n$, the absolute value of ith state variable is expressed as

$$|x_{p,i}(k)| = \left|0 + \sum_{j=0}^{n-1}\boldsymbol{q}_i\boldsymbol{A}_{\mathrm{cl}}^j\boldsymbol{b}(\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b})^{-1}s_p(k-j) + \sum_{j=0}^{n-1}\boldsymbol{q}_i\boldsymbol{A}_{\mathrm{cl}}^j\left(\boldsymbol{p} - \boldsymbol{b}\frac{\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}}{\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b}}\right)\tilde{d}(k-j-1)\right|. \tag{37}$$

Suppose now that $k \geq k_0 + n$, where k_0 is the first time instant for which the system representative point has entered the band (28). Then, Theorem 3 implies that $|s_p(k-j)| \leq B$ for all $j = 0, 1, \ldots, n-1$. Consequently, the relation (37) yields

$$\begin{aligned}|x_{p,i}(k)| &\leq \sum_{j=0}^{n-1}\left|\boldsymbol{q}_i\boldsymbol{A}_{\mathrm{cl}}^j\boldsymbol{b}\right| \cdot \left|\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b}\right|^{-1} \cdot |s_p(k-j)| \\ &\quad + \sum_{j=0}^{n-1}\left|\boldsymbol{q}_i\boldsymbol{A}_{\mathrm{cl}}^j\left(\boldsymbol{p} - \boldsymbol{b}\frac{\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}}{\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b}}\right)\right| \cdot \left|\tilde{d}(k-j-1)\right| \\ &\leq B \cdot \left|\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b}\right|^{-1}\sum_{j=0}^{n-1}\left|\boldsymbol{q}_i\boldsymbol{A}_{\mathrm{cl}}^j\boldsymbol{b}\right| \\ &\quad + \left(|d^{\mathrm{avg}}| + d^{\delta}\right)\sum_{j=0}^{n-1}\left|\boldsymbol{q}_i\boldsymbol{A}_{\mathrm{cl}}^j\left(\boldsymbol{p} - \boldsymbol{b}\frac{\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}}{\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b}}\right)\right|\end{aligned} \tag{38}$$

which means that the ith state variable satisfies the inequality (31). On the other hand, if the absolute value of $s_p(k)$ asymptotically approaches B as k tends to infinity, the relation (38) gives

$$\limsup_{k\to\infty}|x_{p,i}(k)| \leq B \cdot \left|\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b}\right|^{-1}\sum_{j=0}^{n-1}\left|\boldsymbol{q}_i\boldsymbol{A}_{\mathrm{cl}}^j\boldsymbol{b}\right| + \left(|d^{\mathrm{avg}}| + d^{\delta}\right)\sum_{j=0}^{n-1}\left|\boldsymbol{q}_i\boldsymbol{A}_{\mathrm{cl}}^j\left(\boldsymbol{p} - \boldsymbol{b}\frac{\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}}{\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b}}\right)\right|. \tag{39}$$

In other words, if the system representative point approaches the quasi-sliding mode band asymptotically, each state variable converges to the interval specified by (31). ■

Although Theorem 4 is true for all systems, it proves to be too conservative in many practical cases. Indeed, a better estimation of state variables can often be obtained due to the nature of the switching-type quasi-sliding motion, or in systems subject to matched disturbance. These cases will be discussed in the following two remarks.

Remark 3: From Theorem 2, it is known that variable s_p changes its sign in each step in the sliding phase. Thus, if assumptions of Theorem 4 are satisfied, and for a given i, elements $\boldsymbol{q}_i\boldsymbol{A}_{\mathrm{cl}}^j\boldsymbol{b}$ have the same sign for $j = 0, 1, \ldots, n-1$, then the absolute value of state variable $x_{p,i}$ has a smaller upper bound than the one shown in the relation (31). Indeed, in this case, relation (37) implies

$$|x_{p,i}(k)| \le B \cdot \left|\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b}\right|^{-1} \max\{\alpha_{\text{even},i}, \alpha_{\text{odd},i}\}$$
$$+ (|d^{\text{avg}}| + d^{\delta}) \sum_{j=0}^{n-1} \left|\boldsymbol{q}_i \boldsymbol{A}_{\text{cl}}^{j} \left(\boldsymbol{p} - \boldsymbol{b}\frac{\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{p}}{\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b}}\right)\right| \quad (40)$$

where

$$\alpha_{\text{even},i} = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \left|\boldsymbol{q}_i \boldsymbol{A}_{\text{cl}}^{2j} \boldsymbol{b}\right|$$
$$\alpha_{\text{odd},i} = \sum_{j=1}^{\lceil (n-1)/2-1 \rceil} \left|\boldsymbol{q}_i \boldsymbol{A}_{\text{cl}}^{2j+1} \boldsymbol{b}\right| \quad (41)$$

and $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are the floor and ceiling function, respectively.

Remark 4: The formula describing the upper bound of each state variable can be greatly simplified if the disturbance affecting the system is matched. Indeed, if matching conditions for the system (1) are satisfied, then $\boldsymbol{b} = \kappa \boldsymbol{p}$ for a certain constant $\kappa \neq 0$. As a result, the second line of the relation (31) is reduced to zero and for all $i = 1, 2, \ldots, n$

$$|x_{p,i}(k)| \le B \cdot \left|\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{b}\right|^{-1} \sum_{j=0}^{n-1} \left|\boldsymbol{q}_i \boldsymbol{A}_{\text{cl}}^{j} \boldsymbol{b}\right|. \quad (42)$$

Furthermore, it is worth noticing that if the disturbance affecting the plant can be divided into a matched part $d_m(k)$ and an unmatched one $d_u(k)$, only the latter will appear in the second line of the relation (31). One can conclude that the unmatched disturbance has a significantly larger effect on the state error than uncertainties that satisfy the matching conditions. The relation (40) presented in Remark 3 can be simplified in the same way if the disturbance affecting the system is matched.

It has been demonstrated that the proposed control scheme limits the error of all state variables in the sliding mode. Furthermore, since the obtained upper bounds of all state variables are proportional to the quasi-sliding mode band width B, Remark 2 implies that the new strategy ensures smaller state error than either the relative degree two reaching law (12) or the reference model scheme using Gao's conventional reaching law [33]. The advantage of the new approach will be further highlighted in the simulation example in the next section.

VI. Simulation Examples

The effectiveness of the proposed method will now be verified by means of a simulation example. In particular, we will conduct comparison of the proposed model reference scheme using the reaching law (12) with the one introduced in [33] where Gao's classic reaching law is used instead. It will be shown that the new strategy can successfully be applied to a plant with unmatched disturbance and that it ensures better system robustness than its relative degree one equivalent. The considered control strategies will be applied to a fourth-order discrete time system (1), where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0.5 & 0.167 \\ 0 & 1 & 1 & 0.5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} 0.04167 \\ 0.167 \\ 0.5 \\ 1 \end{bmatrix}, \ \boldsymbol{p} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (43)$$

the initial state is $\boldsymbol{x}_p(0) = [40 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}$ and total perturbations affecting the plant are expressed as

$$\tilde{d}(k) = (-1)^{\lfloor k/20 \rfloor} \quad (44)$$

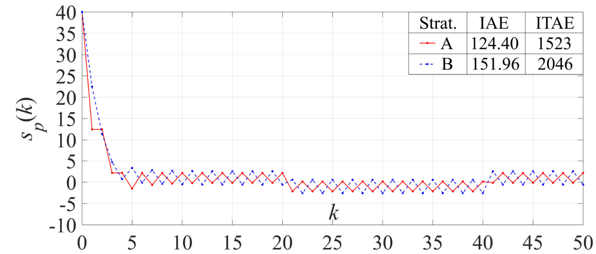

Fig. 2. Sliding variable.

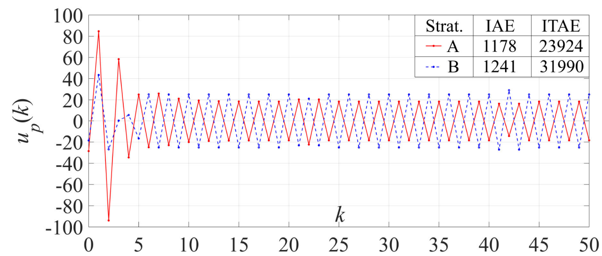

Fig. 3. Control signal.

with $d^{\max} = -d^{\min} = 1$, which further implies $d^{\text{avg}} = 0$ and $d^{\delta} = 1$. Furthermore, since $\boldsymbol{b} \neq \boldsymbol{p}$, this disturbance does not satisfy the matching conditions. According to relations (7) and (9), vectors $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ for the considered system are selected as

$$\boldsymbol{c}_1 = [1 \quad 1.5 \quad 0.9167 \quad 0.25]^{\mathrm{T}}$$
$$\boldsymbol{c}_2 = [1 \quad 0.5 \quad -0.0833 \quad -0.0833]^{\mathrm{T}}. \quad (45)$$

With this in mind, sliding variables with relative degree one and two are defined as in relations (6) and (8), respectively. The following two control strategies are applied to such a system.

a) Our model reference strategy, where the reaching law (12) for the disturbance-free model has parameters $q = 0.6$ and $\varepsilon = 1.84$.
b) Model reference strategy proposed in [33], using Gao's classic reaching law with $q = 0.6$ and $\varepsilon = 2.51$.

Results of this comparison are illustrated by the following three figures depicting the sliding variable, control signal, and the first state variable. Additionally, a numerical comparison of these three variables for both strategies has been conducted using the following two criteria: integral absolute error (IAE) and integral time-weighted absolute error (ITAE). Results of the comparison are shown in the figures.

Fig. 2 demonstrates that our approach drives the sliding variable closer to zero than the strategy B in the sliding phase while ensuring a faster convergence rate to the vicinity of zero in the reaching phase. In particular, the width of the quasi-sliding mode band equals 2.15 for the strategy A and 2.64 for the strategy B. Fig. 3 shows that, despite generating greater values of the control signal in a few initial steps, our strategy requires significantly less control effort in the sliding phase. Finally, we can see from Fig. 4 that both strategies result in a similar error of the first state variable, but our method guarantees faster convergence of this variable to zero. Our strategy drives the variable below 1.5 in finite time, while the strategy B makes it approach this value asymptotically. In conclusion, the approach proposed in this article ensures similar dynamical properties to the method presented in [33], but requires significantly less control effort in the sliding mode.

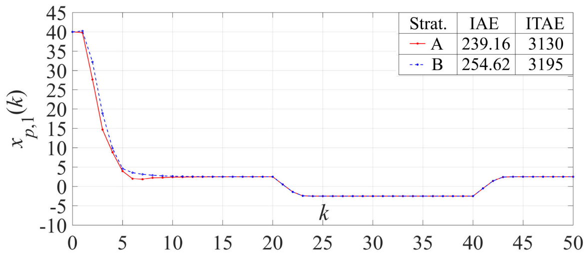

Fig. 4. First state variable.

VII. Conclusion

In this article, we have proposed a new discrete-time sliding mode control strategy using a reference model of the plant. In order to mitigate the effect of past perturbations on the system, a reaching law has first been applied to a disturbance-free model of the plant, and then, a secondary control signal has been used to drive the state of the original system toward that of the model. The reaching law applied to the model uses a relative degree two sliding variable, which ordinarily would make it difficult to apply when matching conditions are not satisfied. However, the proposed model reference scheme allows one to keep the favorable properties of this reaching law without the need for matched perturbations. It has been proven that the proposed approach ensures a switching type quasi-sliding motion of the system and limits the error of all state variables in the sliding phase. Furthermore, simulation examples have shown that the proposed method ensures better dynamical properties of the system than a formerly published model reference strategy using Gao's classic reaching law.

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