Asymptotic Stability Analysis of Discrete-Time Switched Cascade Nonlinear Systems With Delays

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***Abstract*—This paper addresses the stability issue of a class of delayed switched cascade nonlinear systems consisting of separate subsystems and coupling terms between them. Some global and local asymptotic stability sufficient conditions are proposed, drawing stability conclusion of the overall cascade system from those of separate systems. These results essentially rely on the following observation: For a general delayed switched nonlinear system being asymptotically stable, the trajectories of the perturbed system asymptotically approach zero if so does the perturbation. This observation is one of the main results in this paper.**

***Index Terms*—Asymptotic stability, cascade systems, delays, exponential stability, switched systems.**

I. Introduction

A cascade system consists of some separate subsystems and coupling terms between them. It is known that cascade systems have broad applications in various fields and possess many important particular properties [1]–[4]. In recent years, switched cascade systems, a class of more complicated cascade systems, have gained much attention from communities of researchers and engineers [5]–[7]. Since delays and nonlinearities are frequently encountered in diverse real systems and may lead to very complex dynamics [8]–[11], this paper will study switched cascade nonlinear systems (SCNSs) with delays.

To facilitate subsequent analysis, assume that a cascade system consists of two separate systems and that the state of separate system 1 is not affected by that of separate system 2 but does affect that of separate system 2 via a coupling term.

The well-known Lyapunov theory is widely employed when stability of cascade systems is studied [12]–[14]. Due to particular structures of cascade systems, the used Lyapunov functions or functionals are usually of particular forms, say, the composite Lyapunov function [1, Appendix C]. The key idea to construct such a function (functional) is by constructing a separate function (functional) for each separate system. A more general problem naturally arises: Can we draw a stability conclusion for SCNSs with delays if some stability properties of separate systems are known?

The above-mentioned problem has been partially solved. For a discrete-time SCNS, with assumptions that separate system 2 is exponentially stable and that the coupling term has a linear growth bound [15, p. 340], it was shown that the overall cascade system is exponentially stable if and only if so is separate system 1, and is asymptotically stable if so is separate system 1 [16]. Continuous-time SCNSs with delays were discussed in [17], and parallel results have been established. Note that the approach used in [16] and [17] is the so-called covering method rather that Lyapunov one, and the underlying reason is that a converse Lyapunov theorem has not been established in corresponding contexts. The covering method will also be employed in this paper. Note that the main results in [16] essentially rely on the following fact: For a nominal delayed switched nonlinear system being exponentially stable, the trajectories of the corresponding perturbed system behave as the perturbation, i.e., the perturbed system decays at an exponential rate if the perturbation decays at an exponential rate and asymptotically approaches zero if so does the perturbation.

The main motivations of this paper lie in the following three aspects.

1) There exists a wide class of dynamical systems being asymptotically rather than exponentially stable. Note that the key assumption in [16] and [17] is separate system 2 being exponentially stable. We need to consider the case where the separate system is asymptotically stable.
2) The core of a covering method is to construct a covering function. In the situation of separate system 2 being exponentially stable [16], it is relatively easy to construct such a function, since the information of exponentially decaying rate can be used. However, if separate system 2 is just asymptotically stable, constructing such a function would be more challenging.
3) The basic idea employed here is to view the coupling term of a cascade system as a perturbation of separate system 2. In order to establish our main results, we have to study the convergence property of delayed switched nonlinear systems subject to perturbations, and such a topic is also of importance [18], [19].

On this ground, asymptotic stability of delayed SCNSs is studied in this paper. The main contribution of the paper lies in the following aspects: 1) It is shown that, with the assumption that the nominal delayed switched nonlinear system is asymptotically stable, trajectories of the perturbed system asymptotically approach zero if so does the perturbation, both global and local cases are investigated. 2) These findings on convergence are applied to SCNSs with delays, and some sufficient asymptotic stability conditions are presented. According to these results, stability of SCNSs can be analyzed in a decomposition (and, therefore, a simpler) manner.

The rest of this paper is organized as follows. Preliminaries are presented in Section II. Convergence properties of delayed switched nonlinear systems subject to perturbations are explored in Section III,

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cascade systems are studied in Section IV, and a numerical example is provided in Section V. Finally, Section VI concludes this paper.

Notation: A^{T} is the transpose of matrix A. $\mathbb{R}^{n\times m}$ denotes the set of all real matrices of $n\times m$ dimension and $\mathbb{R}^n=\mathbb{R}^{n\times 1}$. $\mathbb{R}_+=(0,\infty)$. $\mathbb{N}_0$ stands for the set of nonnegative integers and $\mathbb{N}=\mathbb{N}_0\backslash\{0\}$. $\mathbb{N}_q=\{q,q+1,\ldots\}$ for $q\in\mathbb{N}_0$ and $\underline{m}=\{1,\ldots,m\}$ for $m\in\mathbb{N}$. $|a|$ means the absolute value of real number a. $\|\boldsymbol{x}\|_\infty=\max\{|x_1|,\ldots,|x_n|\}$ is the l_∞ norm of vector $\boldsymbol{x}\in\mathbb{R}^n$, and for simplicity, it is denoted by $\|\boldsymbol{x}\|$. If $\boldsymbol{x}(s)$ is defined on the set $\{-d,\ldots,a\}$ with $a\in\mathbb{N}_0$, then for any $k\in\{0,\ldots,a\}$, $\boldsymbol{x}_k(\theta)=\boldsymbol{x}(k+\theta)$ for $\theta\in\{-d,\ldots,0\}$, $\|\boldsymbol{x}_k\|=\max_{s\in\{k-d,\ldots,k\}}\{\|\boldsymbol{x}(s)\|\}$, and $\|\boldsymbol{\varphi}\|=\max_{s\in\{k_0-d,\ldots,k_0\}}\{\|\boldsymbol{\varphi}(s)\|\}$. $\mathbf{0}$ is the zero vector of dimension n. $\mathrm{col}(\boldsymbol{x},\boldsymbol{y})=[\boldsymbol{x}^{\mathrm{T}}\,\boldsymbol{y}^{\mathrm{T}}]^{\mathrm{T}}$, with $\boldsymbol{x}$ and $\boldsymbol{y}$ being vectors of arbitrary dimensions. $\mathcal{B}_a=\{\boldsymbol{x}\in\mathbb{R}^n:\|\boldsymbol{x}\|<a\}$. Throughout this paper, the dimensions of matrices and vectors will not be explicitly mentioned if clear from context.

II. Preliminaries

Consider the following switched nonlinear system:

$$\begin{aligned}\boldsymbol{x}(k+1)=&\ \boldsymbol{f}_{\sigma(k)}\left(k,\boldsymbol{x}(k),\boldsymbol{x}\left(k-d_{\sigma(k)}(k)\right)\right),k\geq k_0\\ \boldsymbol{x}(k)=&\ \boldsymbol{\varphi}(k),\ k\in\{k_0-d,\ldots,k_0\}\end{aligned}\tag{1}$$

where $k_0\in\mathbb{N}_0,\boldsymbol{x}(k)\in\mathbb{R}^n$ is the state, $\boldsymbol{\varphi}$ is an initial vector-valued function, and $\sigma:\mathbb{N}_{k_0}\to\underline{m}$ is a switching signal with m being the number of subsystems. It is assumed that σ is with switching instant sequence $\{k_i\}_{i=0}^\infty$ satisfying $k_i\in\mathbb{N}_{k_0},k_i>k_{i-1}(i\in\mathbb{N})$. For each $l\in\underline{m}$, delay $d_l(k)\in\{\underline{d}_l,\ldots,\bar{d}_l\}$ with $\underline{d}_l,\bar{d}_l\in\mathbb{N}_0$ and $d=\max_{l\in\underline{m}}\{\bar{d}_l\}$. $\boldsymbol{f}_l,l\in\underline{m}$, are mappings from $\mathbb{N}_{k_0}\times\mathbb{R}^n\times\mathbb{R}^n$ into $\mathbb{R}^n$ and satisfy the following assumption.

Assumption 1 ($\boldsymbol{f}_l(\cdot,\mathbf{0},\mathbf{0})=\mathbf{0}$): There exist two positive scalars L and δ such that $\|\boldsymbol{f}_l(\cdot,\boldsymbol{x}_1,\boldsymbol{y}_1)-\boldsymbol{f}_l(\cdot,\boldsymbol{x}_2,\boldsymbol{y}_2)\|\leq L\|\mathrm{col}(\boldsymbol{x}_1-\boldsymbol{x}_2,\boldsymbol{y}_1-\boldsymbol{y}_2)\|$ $\forall\boldsymbol{x}_i,\boldsymbol{y}_i\in\mathcal{B}_\delta,i\in\{1,2\}$, $l\in\underline{m}$. That is, $\boldsymbol{f}_l$ is locally Lipschitz at origin in the second and third arguments, uniformly in the first one. If $\delta=+\infty$, then $\boldsymbol{f}_l$ is globally Lipschitz.

There are different kinds of switching signals some of which are defined as follows.

Definition 1 (See [20]): For switching signal σ and any $T>k\geq k_0$, let $N_\sigma(T,k)$ be the switching numbers of σ on the open interval (k,T). σ is said to has an average dwell time τ_a and "chatter-bound" N_0 if there exist two positive numbers N_0 and τ_a such that $N_\sigma(T,k)\leq N_0+\frac{T-k}{\tau_a}$. A switching signal σ is said to be periodic if there exists an integer $\kappa>1$ such that $\sigma(k+\kappa)=\sigma(k)$ holds for any $k\in\mathbb{N}_{k_0}$, and such a minimal κ is called the period of σ. σ is constant if $\kappa=1$.

Listed here are four classes of switching signals, which are frequently encountered in the literature: $\mathbb{S}_1=\{\sigma:\sigma\text{ is an arbitrary switching signal}\}$; $\mathbb{S}_2(N_0,\tau_a)=\{\sigma:\sigma\text{ has an average dwell time }\tau_a\text{ and chatter bound }N_0\}$; $\mathbb{S}_3(\kappa)=\{\sigma:\sigma\text{ has a period }\kappa\}$; $\mathbb{S}_4(\tau_d)=\{\sigma:k_{i+1}-k_i\geq\tau_d\geq 2\ \forall i\in\mathbb{N}_0\}$. For $\mathbb{S}_4(\tau_d)$, if $\tau_d=1$, then $\mathbb{S}_4(1)$ is actually the set of arbitrary switching signals, i.e., $\mathbb{S}_4(1)=\mathbb{S}_1$. Hereafter, it is always assumed that $\mathbb{S}$ is arbitrarily chosen from $\{\mathbb{S}_1,\mathbb{S}_2(N_0,\tau_a),\mathbb{S}_3(\kappa),\mathbb{S}_4(\tau_d)\}$.

A system has a certain property over a given set $\mathbb{S}$ of switching signals if the property holds for all switching signals in $\mathbb{S}$. In order to avoid any ambiguity, we will explicitly point out a system has a property " over $\mathbb{S}''$ when the underlying set $\mathbb{S}$ needs to be mentioned, otherwise "over $\mathbb{S}$" will be omitted.

Definition 2 (See [15]): A continuous function $\alpha:[0,a)\to[0,\infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0)=0$, where a is any positive real number or ∞, and it is said to belong to class $\mathcal{K}_\infty$ if it belongs to class $\mathcal{K}$ and $\alpha(r)\to\infty$ as $r\to\infty$. A continuous function $\beta:[0,a)\times[0,\infty)\to[0,\infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r,s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s)\to 0$ as $s\to\infty$.

Definition 3 (See [21]): Consider system (1) and fix $\mathbb{S}$. Denote $\boldsymbol{x}(k;k_0,\boldsymbol{\varphi})$ the solution to (1) with a starting time k_0 and an initial function $\boldsymbol{\varphi}$. System (1) is locally uniformly exponentially stable (LUES) if there exist three scalars $\alpha>0,\gamma>1$, and $\delta>0$ such that $\|\boldsymbol{x}(k;k_0,\boldsymbol{\varphi})\|\leq\alpha\gamma^{-(k-k_0)}\|\boldsymbol{\varphi}\|\ \forall k_0\in\mathbb{N}_0,k\geq k_0,\|\boldsymbol{\varphi}\|\leq\delta,\sigma\in\mathbb{S}$; if δ can be arbitrarily large, then system (1) is globally uniformly exponentially stable (GUES). System (1) is locally uniformly asymptotically stable (LUAS) if there exist $\delta>0$ and a function $\beta\in\mathcal{KL}$ such that $\|\boldsymbol{x}(k;k_0,\boldsymbol{\varphi})\|\leq\beta(\|\boldsymbol{\varphi}\|,k-k_0)\ \forall k_0\in\mathbb{N}_0,k\geq k_0,\|\boldsymbol{\varphi}\|\leq\delta,\sigma\in\mathbb{S}$; if δ can be arbitrarily large, then system (1) is globally uniformly asymptotically stable (GUAS). It is seen that $\beta(a,0)\geq a$ holds for $0<a\leq\delta(0<a<\infty)$ if the system is LUAS (GUAS). Note that when the considered system is locally exponentially or asymptotically stable, the scalar δ is called domain of attraction.

The following lemma is key to establish main results.

Lemma 1: Assume that (1) is GUAS, i.e., there exists $\beta\in\mathcal{KL}$ such that $\|\boldsymbol{x}(k;k_0,\boldsymbol{\varphi})\|\leq\beta(\|\boldsymbol{\varphi}\|,k-k_0)$. Fix positive scalars δ_1,δ_2, and ν with $\delta_1\leq\delta_2$. There exists a scalar $\tau\in\mathbb{N}$ such that $\|\boldsymbol{x}(k;k_0,\boldsymbol{\varphi})\|\leq\beta(\|\boldsymbol{\varphi}\|,k-k_0)\leq\frac{\|\boldsymbol{\varphi}\|}{\nu}\ \forall k\geq\tau+k_0,\delta_1\leq\|\boldsymbol{\varphi}\|\leq\delta_2$.

Proof: Suppose that the lemma does not hold. That is, for any $\tilde{\tau}\in\mathbb{N}$, there exists a pair $(\boldsymbol{\varphi},\tau)$ with $\delta_1\leq\|\boldsymbol{\varphi}\|\leq\delta_2$ and $\tau\geq\tilde{\tau}$ such that $\beta(\|\boldsymbol{\varphi}\|,\tau)>\frac{\|\boldsymbol{\varphi}\|}{\nu}$. Particularly, take a sequence $\{\tilde{\tau}_i\}_{i=1}^\infty$ with $\tilde{\tau}_i\in\mathbb{N},\tilde{\tau}_i<\tilde{\tau}_{i+1}$, and there correspondingly exists a sequence of pairs $\{(\boldsymbol{\varphi}_i,\tau_i)\}_{i=1}^\infty$ satisfying $\delta_1\leq\|\boldsymbol{\varphi}_i\|\leq\delta_2,\tilde{\tau}_i\leq\tau_i$, and $\beta(\|\boldsymbol{\varphi}_i\|,\tau_i)>\frac{\|\boldsymbol{\varphi}_i\|}{\nu}$. Note that $\|\boldsymbol{\varphi}_i\|\leq\delta_2$, and one has $\lim_{i\to\infty}\beta(\delta_2,\tau_i)\geq\lim_{i\to\infty}\beta(\|\boldsymbol{\varphi}_i\|,\tau_i)>\frac{\|\boldsymbol{\varphi}_i\|}{\nu}\geq\frac{\delta_1}{\nu}>0$, contradicting the fact $\lim_{i\to\infty}\beta(\delta_2,\tau_i)=0$. ■

The following corollary immediately follows from the previous lemma.

Corollary 1: Assume that (1) is LUAS with domain of attraction δ. Lemma 1 holds with $\delta_1\leq\delta_2\leq\delta$.

III. Convergence of Switched Systems With Perturbations

Consider the perturbed system of (1) as

$$\begin{aligned}\boldsymbol{y}(k+1)=&\boldsymbol{f}_{\sigma(k)}\left(k,\boldsymbol{y}(k),\boldsymbol{y}\left(k-d_{\sigma(k)}(k)\right)\right)+\boldsymbol{u}(k),k\geq k_0\\ \boldsymbol{y}(k)=&\boldsymbol{\varphi}(k),\quad k\in\{k_0-d,\ldots,k_0\}\end{aligned}\tag{2}$$

where $\boldsymbol{y}(k)$ and $\boldsymbol{u}(k)\in\mathbb{R}^n$ are the state of system (2) and the perturbation, respectively.

The following lemma is a simple extension of [16, Prop. 1].

Lemma 2: Let L be as in Assumption 1 and $\mu(s)=\frac{1-L^s}{1-L}$ if $L\neq 1$ and $\mu(s)=s$ if $L=1$. Fix $\imath\in\mathbb{N}_{k_0},\sigma$ and $\boldsymbol{\phi}(k)$ on the set $\{\imath-d,\ldots,\imath\}$. Denote $\boldsymbol{x}(k)$ and $\boldsymbol{y}(k)$ be solutions to (1) and (2) with $k_0=\imath$ and $\boldsymbol{\varphi}(\cdot)=\boldsymbol{\phi}(\cdot)$, respectively. Let $\boldsymbol{e}(s;\imath)=\boldsymbol{y}(\imath+s)-\boldsymbol{x}(\imath+s),s\in\mathbb{N}_0$. It holds that $\|\boldsymbol{e}(0;\imath)\|=0$ and

$$\|\boldsymbol{e}(s;\imath)\|\leq\mu(s)\max_{\jmath\in\{0,\ldots,s-1\}}\{\|\boldsymbol{u}(\imath+\jmath)\|\}\quad\forall s\in\mathbb{N}.\tag{3}$$

For system (2), the following assumption is always imposed on $\boldsymbol{u}(k)$.

Assumption 2: There exist scalars $\alpha>0,\gamma>1$ such that $\|\boldsymbol{u}(k)\|\leq\alpha\gamma^{-(k-k_0)}\ \forall k\geq k_0$.

By Assumption 2 and (3), one has that

$$\|\boldsymbol{e}(k-\imath;\imath)\|\leq\mu(k-\imath)\alpha\gamma^{-(\imath-k_0)},\ k\geq\imath.\tag{4}$$

The identity $\boldsymbol{y}(k;k_0,\boldsymbol{\varphi})=\boldsymbol{y}(k;\imath,\boldsymbol{y}_\imath)$ clearly holds for any $\imath\geq k_0,k\geq\imath$, where $\boldsymbol{y}_\imath$ is the solution to system (2) on set $\{\imath-$

$d, \ldots, \imath\}$. Moreover, if (1) is GUAS, then there exists a $\beta \in \mathcal{KL}$ such that $\|\boldsymbol{x}(k; \imath, \boldsymbol{y}_\imath)\| \leq \beta(\|\boldsymbol{y}_\imath\|, k-\imath)$. Then, it follows from the definition of $\boldsymbol{e}$ that $\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| = \|\boldsymbol{y}(k; \imath, \boldsymbol{y}_\imath)\| \leq \|\boldsymbol{x}(k; \imath, \boldsymbol{y}_\imath)\| + \|\boldsymbol{e}(k-\imath; \imath)\| \leq \beta(\|\boldsymbol{y}_\imath\|, k-\imath) + \|\boldsymbol{e}(k-\imath; \imath)\| \, \forall k \geq \imath$. This inequality, together with (4), implies that

$$\|\boldsymbol{y}\,(k; k_0, \boldsymbol{\varphi})\| \leq \beta\,(\|\boldsymbol{y}_\imath\|\,, k-\imath) + \mu(k-\imath)\alpha\gamma^{-(\imath-k_0)}, k \geq \imath \geq k_0 \tag{5}$$

$$\|\boldsymbol{y}\,(k; k_0, \boldsymbol{\varphi})\| \leq \beta\,(\|\boldsymbol{y}_\imath\|\,, 0) + \mu(k-\imath)\alpha\gamma^{-(\imath-k_0)}, k \geq \imath \geq k_0. \tag{6}$$

Lemma 3: Pick $\mathbb{S}$ and suppose that system (1) is GUAS. The solution to system (2) is bounded, i.e., for any given $\boldsymbol{\varphi}$, there exists a scalar $\bar{\delta} > 0$ such that $\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| < \bar{\delta} \, \forall k \geq k_0$.

Proof: According to [20, pp. 58–59], $\mathbb{S}_4(\tau_d) = \mathbb{S}_2(1, \tau_a)$ if $\tau_d = \tau_a$. Hereafter, we just discuss the general case $\mathbb{S}_2(N_0, \tau_a)$ without considering $\mathbb{S}_4(\tau_d)$ additionally. Pick $\mathbb{S}$ from $\{\mathbb{S}_1, \mathbb{S}_2(N_0, \tau_a), \mathbb{S}_3(\kappa)\}$. If switching signal $\sigma : \mathbb{N}_{k_0} \to m$ belongs to $\mathbb{S}$, then $\sigma : \mathbb{N}_{k_0+b} \to m$ also belongs to $\mathbb{S}$ for any $b \in \mathbb{N}$. Note that this property of $\mathbb{S}$ is very important to the later reasoning. Indeed, the assumption that system (1) is uniformly asymptotically stable over $\mathbb{S}$ implies the following fact: For any fixed k_0, there exists a function $\beta \in \mathcal{KL}$ such that $\|\boldsymbol{x}(k; k_0 + b, \boldsymbol{\varphi})\| \leq \beta(\|\boldsymbol{\varphi}\|, k - k_0 - b) \, \forall b \in \mathbb{N}, k \geq k_0 + b$ holds over $\mathbb{S}$ with $\boldsymbol{\varphi}$ defined on $\{k_0 + b - d, \ldots, k_0 + b\}$. This fact will be repeatedly employed without explicit statement in the sequel.

Suppose for contradiction that the conclusion of this lemma is false, which implies that there exist some $\boldsymbol{\varphi}, \boldsymbol{u}$ satisfying $\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \to \infty$ as $k \to \infty$. Therefore, it is always possible to pick two increasing sequences $\{r_i\}_{i=1}^\infty$ and $\{\boldsymbol{y}_{r_i}\}_{i=1}^\infty$ satisfying $\|\boldsymbol{y}_{r_i}\| \geq \sup_{k_0 - d \leq k < r_i}\{\|\boldsymbol{y}(k)\|\}$, $\lim_{i\to\infty} r_i = \infty$, and $\lim_{i\to\infty} \|\boldsymbol{y}_{r_i}\| = \infty$. Let $\epsilon_i = 0.5\|\boldsymbol{y}_{r_i}\|$. By Lemma 1 (with $\delta_1 = \delta_2 = \|\boldsymbol{y}_{r_i}\|$) and the fact system (1) being GUAS over $\mathbb{S}$, there exists a sequence $\{\tau_i\}_{i=1}^\infty$ such that $\|\boldsymbol{x}(k; r_i, \boldsymbol{y}_{r_i})\| \leq \epsilon_i$ for $k \geq r_i + \tau_i$.

Let $r_{i,j} = r_i + j(\tau_i + d)$ for $j \in \mathbb{N}_0$ and $\mathbb{I}_{i,j} = \{r_{i,j-1}, \ldots, r_{i,j}\}$, $\bar{\mathbb{I}}_{i,j} = \{r_{i,j-1} + \tau_i, \ldots, r_{i,j}\}$, $\eta_{i,j} = \alpha \sum_{k \in \mathbb{I}_{i,j}} \gamma^{-(k-k_0)}$ for $j \in \mathbb{N}$. For any finite set $\mathbb{I} = \{\imath, \ldots, \imath + \jmath\} \subset \mathbb{N}_{k_0}$ with $\jmath \in \mathbb{N}_0$, it holds that $\max_{k\in\mathbb{I}} \|\boldsymbol{e}(k-\imath; \imath)\| \to 0$ as $\sum_{k\in\mathbb{I}} \|\boldsymbol{u}(k)\| \to 0$ due to continuous dependence of $\boldsymbol{e}$ on perturbation $\boldsymbol{u}$. Since $r_i \to \infty$, $\|\boldsymbol{y}_{r_i}\| \to \infty$, and $\eta_{i,1} \to 0$ as $i \to \infty$, there necessarily exists some l satisfying

$$\left\|\boldsymbol{x}\left(k; r_l, \boldsymbol{y}_{r_l}\right)\right\| \leq \epsilon_l \quad \forall k \geq r_l + \tau_l \tag{7}$$

$$\max_{k\in\mathbb{I}_{l,1}} \|\boldsymbol{e}(k - r_l; r_l)\| < \epsilon_l. \tag{8}$$

It follows from (6) and (8) that $\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \leq \beta(\|\boldsymbol{y}_{r_l}\|, 0) + \epsilon_l$ for $k \in \mathbb{I}_{l,1}$, and follows from (7) and (8) that $\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \leq \|\boldsymbol{x}(k; r_l, \boldsymbol{y}_{r_l})\| + \|\boldsymbol{e}(k - r_l; r_l)\| \leq \|\boldsymbol{y}_{r_l}\|$ for $k \in \bar{\mathbb{I}}_{l,1}$, that is, $\|\boldsymbol{y}_{r_{l,1}}\| \leq \|\boldsymbol{y}_{r_l}\|$.

The fact $\|\boldsymbol{y}_{r_{l,1}}\| \leq \|\boldsymbol{y}_{r_l}\|$ implies that $\|\boldsymbol{x}(k; r_{l,1}, \boldsymbol{y}_{r_{l,1}})\| \leq \epsilon_l \, \forall k \geq r_{l,1} + \tau_l$. Moreover, since $\eta_{l,2} < \eta_{l,1}$, one has that $\max_{k\in\mathbb{I}_{l,2}} \|\boldsymbol{e}(k - r_{l,1}; r_{l,1})\| < \epsilon_l$. These facts mean that $\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \leq \|\boldsymbol{y}_{r_l}\|$ for $k \in \bar{\mathbb{I}}_{l,2}$ and that $\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \leq \beta(\|\boldsymbol{y}_{r_l}\|, 0) + \epsilon_l$ for $k \in \mathbb{I}_{l,2}$. Analogously, one may show inductively that $\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \leq \|\boldsymbol{y}_{r_l}\|$ for $k \in \bar{\mathbb{I}}_{l,j}$ and $\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \leq \beta(\|\boldsymbol{y}_{r_{l,j}}\|, 0) + \epsilon_l$ for $k \in \mathbb{I}_{l,j}$ hold for all $j \in \mathbb{N}$, which means that the solution to system (2) is bounded. ■

Corollary 2: Pick $\mathbb{S}$ and suppose that system (1) is LUAS. The solution to system (2) is locally bounded, i.e., there exist two positive scalars $\hat{\delta}$ and $\bar{\delta}$ such that $\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| < \bar{\delta} \, \forall k \geq k_0, \|\boldsymbol{\varphi}\| \leq \hat{\delta}$, $\alpha \leq \hat{\delta}$.

Proof: Since system (1) is LUAS, there exist $\delta > 0$ and a function $\beta \in \mathcal{KL}$ such that $\|\boldsymbol{x}(k; k_0, \boldsymbol{\varphi})\| \leq \beta(\|\boldsymbol{\varphi}\|, k - k_0) \, \forall k_0 \in \mathbb{N}_0, k \geq k_0, \|\boldsymbol{\varphi}\| \leq \delta, \sigma \in \mathbb{S}$. Applying Corollary 1 with $\delta_1 = \delta_2 = \delta$, one claims that there exist positive scalars τ and r satisfying the following properties:

$$\beta\,(\delta, k - r) \leq 0.5\delta, \;\forall k \geq r + \tau \tag{9}$$

$$\|\boldsymbol{e}(s; r)\| \leq \mu(\tau + d)\alpha\gamma^{-(r-k_0)} \leq 0.5\delta \;\; \forall 0 \leq s \leq \tau + d \tag{10}$$

with small α. Since $\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})$ continuously depends on $\boldsymbol{\varphi}$ and α on any finite set, there exists a $\hat{\delta} > 0$ such that

$$\|\boldsymbol{y}\,(k; k_0, \boldsymbol{\varphi})\| \leq \delta, k_0 \leq k \leq r, \|\boldsymbol{\varphi}\| \leq \hat{\delta}, \alpha \leq \hat{\delta}. \tag{11}$$

Let $r_0 = r, r_j = r + j(\tau + d), \mathbb{I}_j = \{r_{j-1}, \ldots, r_j\}$, and $\bar{\mathbb{I}}_j = \{r_{j-1} + \tau, \ldots, r_j\}, j \in \mathbb{N}$. It follows from (9)–(11) that $\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \leq \beta(\|\boldsymbol{y}_r\|, k - r) + 0.5\delta \, \forall k \in \mathbb{I}_1$, which clearly implies that $\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \leq \beta(\delta, 0) + 0.5\delta \, \forall k \in \mathbb{I}_1$ and $\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \leq \delta \, \forall k \in \bar{\mathbb{I}}_1$.

Repeating a similar process and using mathematical induction principle, we can show that, for every $j \in \mathbb{N}$, the inequalities $\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \leq \beta(\delta, 0) + 0.5\delta \, \forall k \in \mathbb{I}_j$ and $\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \leq \delta \, \forall k \in \bar{\mathbb{I}}_j$ hold, which, combining (11), means that solution to system (2) is locally bounded with $\bar{\delta} = \beta(\delta, 0) + 0.5\delta$. ■

The following theorem shows that, if system (1) is GUAS, then solution to system (2) asymptotically converges to zero provided that Assumption 2 holds.

Theorem 1: Consider system (2) and pick $\mathbb{S}$. Suppose that system (1) is GUAS and that Assumption 1 holds with $\delta = +\infty$. Then, there exists a function $\tilde{\beta} \in \mathcal{KL}$ satisfying

$$\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \leq \tilde{\beta}\,(\theta, k - k_0) \, \forall k \geq k_0, \; k_0 \geq 0$$

where $\theta = \max\{\|\boldsymbol{\varphi}\|, \alpha\}$.

Proof: As in the proof of Lemma 3, here we only consider the case $\mathbb{S} \in \{\mathbb{S}_1, \mathbb{S}_2(N_0, \tau_a), \mathbb{S}_3(\kappa)\}$.

Since system (1) is GUAS, there exists a $\beta \in \mathcal{KL}$ such that

$$\|\boldsymbol{x}\,(k; k_0, \boldsymbol{\varphi})\| \leq \beta\,(\|\boldsymbol{\varphi}\|\,, k - k_0) \quad \forall k_0 \in \mathbb{N}_0, k \geq k_0$$

By Lemma 3, $\bar{\delta} = \sup_{\|\boldsymbol{\varphi}\| \leq \theta, \alpha \leq \theta, k \geq k_0} \|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\|$ exists for any given θ. Arbitrarily fix $\boldsymbol{\varphi}$. In view of the definitions of θ and $\bar{\delta}$, the next inequality naturally holds

$$\|\boldsymbol{y}\,(k; k_0, \boldsymbol{\varphi})\| \leq \bar{\delta} \quad \forall k \geq k_0. \tag{12}$$

Pick $0 < q < 1$ satisfying $\beta(q\bar{\delta}, 0) + \frac{q^2}{2}\bar{\delta} \leq \bar{\delta}$. Such a q does exist since $\bar{\delta} > 0$ and $\beta(q\bar{\delta}, 0) + \frac{q^2}{2}\bar{\delta}$ approaches zero as so does q.

By Lemma 1, there exists a sequence $\{\tau_i\}_{i=1}^\infty$ with $\tau_i \in \mathbb{N}$ such that

$$\beta\,(\varsigma, k - k_0) \leq \frac{q}{2}\varsigma \quad \forall k_0 \geq 0, \; k \geq k_0 + \tau_i, \; q^i\bar{\delta} \leq \varsigma \leq q^{i-1}\bar{\delta}. \tag{13}$$

Denote $\mu_i = \mu(\tau_i + d)$, with μ being defined in Lemma 2. It is not difficult to construct two sequences $\{h_i\}_{i=1}^\infty$ and $\{n_i\}_{i=1}^\infty$ with the following properties:

$$h_i \in \mathbb{N}, \; n_i \in \mathbb{N}, \; \mu_i\theta\gamma^{-h_i} \leq \frac{q^i}{2}\bar{\delta}$$
$$h_{i+1} = h_i + n_i\,(\tau_i + d)\,, i \in \mathbb{N}. \tag{14}$$

Indeed, h_1 satisfying $\mu_1\theta\gamma^{-h_1} \leq \frac{q}{2}\bar{\delta}$ exists. Then, one can fix an n_1 satisfying $\mu_2\theta\gamma^{-h_2} \leq \frac{q^2}{2}\bar{\delta}$ with $h_2 = h_1 + n_1(\tau_1 + d)$. The construction is completed by repeating a similar way for any other

$i \in \mathbb{N} \setminus \{1, 2\}$. Let h_i and n_i be the minimal integers satisfying (14). Note that h_i and n_i are independent of k_0.

The following symbols will be used repeatedly:

$$h_{i,j} = h_i + j(\tau_i + d), \; s_{i,j} = k_0 + h_{i,j}, \; s_i = k_0 + h_i \quad j \in \{0, 1, \ldots, n_i\}$$
$$\mathbb{I}_{i,j} = \{s_{i,j-1}, \ldots, s_{i,j}\}, \bar{\mathbb{I}}_{i,j} = \{s_{i,j-1} + \tau_i, \ldots, s_{i,j}\}$$
$$\underline{\mathbb{I}}_{i,j} = \{s_{i,j-1}, \ldots, s_{i,j} - d\}, j \in \{1, \ldots, n_i\}. \tag{15}$$

Clearly, $h_{i,0} = h_{i-1,n_{i-1}}$ for $i \in \mathbb{N}\setminus\{1\}$ and all $h_{i,j}$'s are independent of k_0.

Note that $\mu(s)$ is increasing in s and $\boldsymbol{u}(k)$ satisfies Assumption 2. It follows from (4) and (14) that

$$\|\boldsymbol{e}(k - s_{i,j-1}; s_{i,j-1})\| \le \frac{q^i}{2}\bar{\delta} \quad \forall k \in \mathbb{I}_{i,j}, \; i \in \mathbb{N}, j \in \{1, \ldots, n_i\} \tag{16}$$

Now, prove that

$$\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \le q\bar{\delta}, k \in \bar{\mathbb{I}}_{1,n_1}. \tag{17}$$

By the definition of τ_1 and the fact $\|\boldsymbol{y}_{s_1}\| \le \bar{\delta}$, it follows from (16) (with $i = j = 1$) and (5) that

$$\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \le \beta(\|\boldsymbol{y}_{s_1}\|, k - s_1) + \frac{q}{2}\bar{\delta} \le \frac{q}{2}\|\boldsymbol{y}_{s_1}\| + \frac{q}{2}\bar{\delta} \le q\bar{\delta}, k \in \bar{\mathbb{I}}_{1,1}.$$

Then, (17) holds if $n_1 = 1$; otherwise suppose that $\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \le q\bar{\delta}(k \in \bar{\mathbb{I}}_{1,j})$ with $j \in \{1, \ldots, n_1 - 1\}$. (5), together with (16), shows that $\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \le \beta(\|\boldsymbol{y}_{s_{1,j}}\|, k - s_{1,j}) + \frac{q}{2}\bar{\delta} \le q\bar{\delta}, k \in \bar{\mathbb{I}}_{1,j+1}$, that is, $\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \le q\bar{\delta}, k \in \bar{\mathbb{I}}_{1,j+1}$. By induction, $\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \le q\bar{\delta}, k \in \bar{\mathbb{I}}_{1,j}$ holds for any $j \in \{1, \ldots, n_1\}$. Therefore, (17) is true.

Next, we will show that for $i \in \mathbb{N}\setminus\{1\}$, it holds that

$$\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \le \beta(q^{i-1}\bar{\delta}, 0) + \frac{q^i}{2}\bar{\delta}, \; s_i < k \le s_{i+1} \tag{18}$$
$$\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \le q^i\bar{\delta}, \; k \in \bar{\mathbb{I}}_{i,n_i}. \tag{19}$$

Prove (18) and (19) for $i = 2$ first. By condition (17) and the definition of s_2, we have

$$\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \le \beta(\|\boldsymbol{y}_{s_2}\|, k - s_2) + \frac{q^2}{2}\bar{\delta}, k \in \mathbb{I}_{2,1} \tag{20}$$

which, by the definition of τ_2, further implies that

$$\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \le \frac{q^2}{2}\bar{\delta} + \frac{q^2}{2}\bar{\delta} = q^2\bar{\delta}, k \in \bar{\mathbb{I}}_{2,1}. \tag{21}$$

Following a reasoning similar to the process from (20) to (21), and using the mathematical induction principle, it yields that

$$\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \le \beta\left(\left\|\boldsymbol{y}_{s_{2,j}}\right\|, k - s_{2,j}\right) + \frac{q^2}{2}\bar{\delta} \quad k \in \mathbb{I}_{2,j}, \; j \in \{1, \ldots n_2\} \tag{22}$$
$$\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \le q^2\bar{\delta}, \; k \in \bar{\mathbb{I}}_{2,j}, \; j \in \{1, \ldots n_2\}. \tag{23}$$

Since β is decreasing in the second argument, (22) and (23), respectively, imply that (18) and (19) hold for $i = 2$. Moreover, following a very similar manner proving the case $i = 2$, it is straightforward to verify that (18) and (19) hold for $i + 1$ provided that they hold for some $i \ge 2$. Therefore, by the mathematical induction principle, (18) and (19) hold for any $i \in \mathbb{N}\setminus\{1\}$.

Inequality (12), together with (18) and the fact $\beta(q\bar{\delta}, 0) + \frac{q^2}{2}\bar{\delta} \le \bar{\delta}$, implies that

$$\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \le \bar{\delta}, k_0 \le k \le s_2$$
$$\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \le \beta(q^{i-1}\bar{\delta}, 0) + \frac{q^i}{2}\bar{\delta} \quad s_i < k \le s_{i+1}, i \in \mathbb{N}\setminus\{1\}. \tag{24}$$

Pick a small positive scalar ε and introduce

$$\hat{\beta}(\theta, k) = \begin{cases} \bar{\delta} + \varepsilon\theta, & 0 \le k \le h_2 \\ \hat{\beta}_i(\theta, k), & h_i < k \le h_{i+1}, i \in \mathbb{N}\setminus\{1\} \end{cases} \tag{25}$$

where $\hat{\beta}_i(\theta, k) = \beta(q^{i-1}\bar{\delta}, 0) + \frac{q^i}{2}(\bar{\delta} + \varepsilon\theta)$.

It is observed from (25) that, for any fixed θ, $\hat{\beta}(\theta, k)$ monotonically decreases in k, and approaches zero as k approaches infinity, and for fixed k, $\hat{\beta}(\theta, k)$ strictly increases in θ since $\bar{\delta}$ is nondecreasing in θ. Moreover, $\hat{\beta}$ is independent of k_0.

For any $k \ge 0$, define $\tilde{\beta}(\theta, k) = \sup_{0<r\le\theta} \hat{\beta}(r, k)$. Then, one can check that $\tilde{\beta}(\theta, k)$ strictly increases in θ for any fixed $k \ge 0$, and for any fixed θ, decreases in k and approaches zero as k approaches infinity. That is, $\tilde{\beta} \in \mathcal{KL}$. $\tilde{\beta}$ is independent of k_0 as so is $\hat{\beta}$.

Equation (24) indicates that $\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \le \hat{\beta}(\theta, k - k_0)$. It is clear that $\hat{\beta}(\theta, k - k_0) \le \tilde{\beta}(\theta, k - k_0)\ \forall\theta \ge 0, k \ge k_0$. Therefore, it holds that $\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \le \tilde{\beta}(\theta, k - k_0)$. ■

Theorem 1 has the following local version.

Corollary 3: Consider system (2) and fix $\mathbb{S}$. Suppose that system (1) is LUAS and that Assumption 1 holds. Let $\theta = \max\{\|\boldsymbol{\varphi}\|, \alpha\}$. There exists a function $\tilde{\beta} \in \mathcal{KL}$ and a scalar $\upsilon > 0$ such that

$$\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| \le \tilde{\beta}(\theta, k - k_0) \quad \forall k \ge k_0, \; k_0 \ge 0$$

for any $\theta < \upsilon$.

Proof: By Corollary 2, there exist two positive scalars $\hat{\delta}, \tilde{\delta}$ such that $\|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\| < \tilde{\delta}\ \forall k \ge k_0, k_0 \ge 0, \|\boldsymbol{\varphi}\| \le \hat{\delta}, \alpha \le \hat{\delta}$, which further means that there exists a positive scalar $\hat{\theta} < \hat{\delta}$ such that the supremum $\bar{\delta} = \sup_{\|\boldsymbol{\varphi}\|\le\hat{\theta}, \alpha\le\hat{\theta}, k\ge k_0} \|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\|$ exists and satisfies $\bar{\delta} \le \tilde{\delta}$.

Clearly, there exists some positive scalar $\delta < \hat{\theta}$ such that Assumption 1 holds and that, since system (1) is LUAS, there exists $\beta \in \mathcal{KL}$ satisfying

$$\|\boldsymbol{x}(k; k_0, \boldsymbol{\varphi})\| \le \beta(\|\boldsymbol{\varphi}\|, k - k_0)\ \forall k \ge k_0, \; k_0 \ge 0, \|\boldsymbol{\varphi}\| \le \delta$$

Fix $\boldsymbol{\varphi}, \alpha$ with $\|\boldsymbol{\varphi}\| < \delta, \alpha < \delta$, and $\theta = \max\{\|\boldsymbol{\varphi}\|, \alpha\} < \delta$. Thus, $\bar{\delta} = \sup_{\|\boldsymbol{\varphi}\|\le\theta, \alpha\le\theta, k\ge k_0} \|\boldsymbol{y}(k; k_0, \boldsymbol{\varphi})\|$ exists and satisfies $\bar{\delta} \le \tilde{\delta}$.

Pick $0 < q < 1$ satisfying $0 < \beta(q\bar{\delta}, 0) + \frac{q^2}{2}\bar{\delta} \le \bar{\delta}$ and then determine, by applying Lemma 1, a sequence $\{\tau_i\}_{i=1}^{\infty}$ satisfying (13). Let $\mu_i = \mu(\tau_i + d)$ and then construct two sequences $\{h_i\}_{i=1}^{\infty}, \{n_i\}_{i=1}^{\infty}$ satisfying condition (14). Finally, define all notations in (15). Now, following an analogous process from (16) to the end of the proof of Theorem 1 produces the required conclusion. ■

Remark 1: All the results obtained previously can be easily extended to systems with multiple bounded delays in a similar way.

We are in a position to discuss Assumption 2. One may say that the constraint $\|\boldsymbol{u}(k)\| \le \alpha\gamma^{-(k-k_0)}$ is too restrictive. In our context, if $\boldsymbol{u}(k)$ is upper bounded by a function asymptotically, rather than exponentially, decaying to zero, then the perturbed system may diverge, see Example 1 for details.

Example 1: Consider the following scalar system:

$$x(k+1) = -x(k) + 0.5x(k - d(k)) + u(k), k \in \mathbb{N}_0. \tag{26}$$

Define $\Lambda_0 = \{0\}$ and associate each $i \in \mathbb{N}$ with a finite set $\Lambda_i = \{2^i - 1, \ldots, 2^{i+1} - 2\}$. Let $d(0) = 0$. For $k \in \Lambda_i (i \in \mathbb{N})$, $d(k)$ is

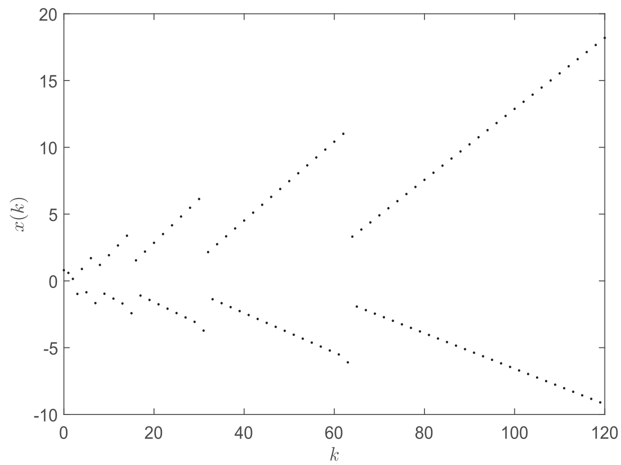

Fig. 1. Trajectory of (26) with $x(0) = 0.8, u_0 = 1$.

defined in this way: $d(2^i - 1) = d(2^i) = 0$, $d(k) = 0$ for even $k \in \Lambda_i \backslash \{2^i - 1, 2^i\}$, and $d(k) = 1$ for odd $k \in \Lambda_i \backslash \{2^i - 1, 2^i\}$. It is shown that (26) with $u(k) = 0$ is asymptotically stable, but not exponentially stable [22].

Now define a sequence of perturbation $\{u(k)\}_{k=0}^{\infty}$ with the property: $u(0) = u_0$ being given, $u(1) = \frac{0.9}{2} u_0$ and $u(2) = -0.9 u_0$. For $k \geq 3$, $u(k)$ is defined in the following manner: $u(k) = \frac{0.9^i}{2} u_0$ for odd $k \in \Lambda_i \backslash \{2^{i+1} - 2\}$, $u(k) = -\frac{0.9^i}{2} u_0$ for even $k \in \Lambda_i \backslash \{2^{i+1} - 2\}$, and $u(2^{i+1} - 2) = -0.9^i u_0$. With this asymptotically decaying perturbation, the perturbed system diverges, as shown in Fig. 1.

IV. Stability of Delayed SCNSs

In this section, we will apply the proposed results in the previous section to SCNSs with delays.

Consider the following delayed SCNS:

$$\hat{\boldsymbol{x}}(k+1) = \hat{\boldsymbol{f}}_{\sigma(k)}\left(k, \hat{\boldsymbol{x}}(k), \hat{\boldsymbol{x}}\left(k - d_{1\sigma(k)}(k)\right)\right), k \geq k_0 \quad (27a)$$

$$\bar{\boldsymbol{x}}(k+1) = \bar{\boldsymbol{f}}_{\sigma(k)}\left(k, \bar{\boldsymbol{x}}(k), \bar{\boldsymbol{x}}\left(k - d_{2\sigma(k)}(k)\right)\right) + \boldsymbol{g}_{\sigma(k)}\left(k, \hat{\boldsymbol{x}}(k), \hat{\boldsymbol{x}}\left(k - d_{3\sigma(k)}(k)\right)\right), k \geq k_0$$

$$\boldsymbol{x}(k) = \boldsymbol{\varphi}(k), k \in \{k_0 - d, \ldots, k_0\} \quad (27b)$$

where $\boldsymbol{x}(k) = \mathrm{col}(\hat{\boldsymbol{x}}(k), \bar{\boldsymbol{x}}(k))$ with $\hat{\boldsymbol{x}}(k) \in \mathbb{R}^{n_1}, \bar{\boldsymbol{x}}(k) \in \mathbb{R}^{n_2}$, $\sigma : \mathbb{N}_{k_0} \to m$, $0 \leq d_{1l}(k), d_{2l}(k), d_{3l}(k) \leq d$, and $\boldsymbol{\varphi} = \mathrm{col}(\hat{\boldsymbol{\varphi}}, \bar{\boldsymbol{\varphi}})$, which means that for each $k \in \{k_0 - d, \ldots, k_0\}$, $\boldsymbol{\varphi}(k) = \mathrm{col}(\hat{\boldsymbol{\varphi}}(k), \bar{\boldsymbol{\varphi}}(k))$ with $\hat{\boldsymbol{\varphi}}(k) \in \mathbb{R}^{n_1}, \bar{\boldsymbol{\varphi}}(k) \in \mathbb{R}^{n_2}$. The separate systems of (27) are (27a) and the following equation:

$$\tilde{\boldsymbol{x}}(k+1) = \bar{\boldsymbol{f}}_{\sigma(k)}\left(k, \tilde{\boldsymbol{x}}(k), \tilde{\boldsymbol{x}}\left(k - d_{2\sigma(k)}(k)\right)\right), k \geq k_0$$

$$\tilde{\boldsymbol{x}}(k) = \tilde{\boldsymbol{\varphi}}(k), k \in \{k_0 - d, \ldots, k_0\}. \quad (28)$$

Deleting $\boldsymbol{g}_{\sigma(k)}(k, \hat{\boldsymbol{x}}(k), \hat{\boldsymbol{x}}(k - d_{3\sigma(k)}(k)))$ from (27b), one gets $\bar{\boldsymbol{x}}(k+1) = \bar{\boldsymbol{f}}_{\sigma(k)}(k, \bar{\boldsymbol{x}}(k), \bar{\boldsymbol{x}}(k - d_{2\sigma(k)}(k)))$, which is in fact the same as (28) except with different notations for states in order to distinguish the separate and overall systems, respectively.

Assumption 3: $\boldsymbol{g}_l, l \in m$, have a linear growth bound, that is, there exist two positive scalars L and δ such that

$$\|\boldsymbol{g}_l(\cdot, \boldsymbol{x}, \boldsymbol{y})\| \leq L \|\mathrm{col}(\boldsymbol{x}, \boldsymbol{y})\| \ \forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{B}_\delta, l \in m. \quad (29)$$

Theorem 2: Consider system (27) and fix $\mathbb{S}$. Suppose that Assumption 3 holds with $\delta = \infty$ and that $\hat{\boldsymbol{f}}_l, \bar{\boldsymbol{f}}_l$ are globally Lipschitz, $\hat{\boldsymbol{f}}_l(\cdot, \mathbf{0}, \mathbf{0}) = \mathbf{0}, \bar{\boldsymbol{f}}_l(\cdot, \mathbf{0}, \mathbf{0}) = \mathbf{0}, l \in m$. System (27) is GUAS if system (27a) is GUES and system (28) is GUAS.

Proof: Fix $\boldsymbol{\varphi} = \mathrm{col}(\hat{\boldsymbol{\varphi}}, \bar{\boldsymbol{\varphi}})$. Since (27a) is GUES, there exist $\alpha > 0, \gamma > 1$ such that $\|\hat{\boldsymbol{x}}(k)\| \leq \alpha\gamma^{-(k-k_0)}\|\hat{\boldsymbol{\varphi}}\| \ \forall k \geq k_0$. Clearly, $\|\hat{\boldsymbol{x}}(k - d_{1\sigma(k)}(k))\| \leq \max\{\alpha, 1\}\gamma^{d-(k-k_0)}\|\hat{\boldsymbol{\varphi}}\| \ \forall k \geq k_0$.

Let $\boldsymbol{u}(k) = \boldsymbol{g}_{\sigma(k)}(k, \hat{\boldsymbol{x}}(k), \hat{\boldsymbol{x}}(k - d_{3\sigma(k)}(k)))$. By Assumption 3, we have

$$\begin{aligned} &\|\boldsymbol{u}(k)\| \\ &\leq L \max\left\{\|\hat{\boldsymbol{x}}(k)\|, \left\|\hat{\boldsymbol{x}}(k - d_{3\sigma(k)}(k))\right\|\right\} \\ &\leq L \max\left\{\alpha\gamma^{-(k-k_0)}\|\hat{\boldsymbol{\varphi}}\|, \max\{\alpha, 1\}\gamma^{d-(k-k_0)}\|\hat{\boldsymbol{\varphi}}\|\right\} \\ &\leq L \max\{\alpha, 1\}\gamma^{d-(k-k_0)}\|\hat{\boldsymbol{\varphi}}\| \\ &= \hat{\alpha}\gamma^{-(k-k_0)}\|\hat{\boldsymbol{\varphi}}\| \end{aligned}$$

with $\hat{\alpha} = L\max\{\alpha, 1\}\gamma^d$.

Note that (28) is GUAS. Let $\tilde{\boldsymbol{\varphi}} = \bar{\boldsymbol{\varphi}}$. By Theorem 1, there exists a function $\beta \in \mathcal{KL}$ satisfying

$$\|\bar{\boldsymbol{x}}(k; k_0, \bar{\boldsymbol{\varphi}})\| \leq \beta\left(\max\left\{\|\bar{\boldsymbol{\varphi}}\|, \hat{\alpha}\|\hat{\boldsymbol{\varphi}}\|\right\}, k - k_0\right), \forall k \geq k_0$$

where $\bar{\boldsymbol{x}}(k; k_0, \bar{\boldsymbol{\varphi}})$ is the solution to (27b). Hence,

$$\begin{aligned} &\|\boldsymbol{x}(k; k_0, \boldsymbol{\varphi})\| \\ &\leq \|\hat{\boldsymbol{x}}(k; k_0, \hat{\boldsymbol{\varphi}})\| + \|\bar{\boldsymbol{x}}(k; k_0, \bar{\boldsymbol{\varphi}})\| \\ &\leq \alpha\gamma^{-(k-k_0)}\|\hat{\boldsymbol{\varphi}}\| + \beta\left(\max\left\{\|\bar{\boldsymbol{\varphi}}\|, \hat{\alpha}\|\hat{\boldsymbol{\varphi}}\|\right\}, k - k_0\right) \\ &\leq \alpha\gamma^{-(k-k_0)}\|\boldsymbol{\varphi}\| + \beta\left(\max\{1, \hat{\alpha}\}\|\boldsymbol{\varphi}\|, k - k_0\right). \end{aligned}$$

The proof is completed. ■

Theorem 2 proposes a global stability result for system (27). The next local version also holds, whose proof is based on Corollary 3 and is similar to that of Theorem 2.

Corollary 4: Consider system (27) and fix $\mathbb{S}$. Suppose that Assumption 3 holds with $\delta \in \mathbb{R}_+$ and that $\hat{\boldsymbol{f}}_l, \bar{\boldsymbol{f}}_l$ are local Lipschitz, $\hat{\boldsymbol{f}}_l(\cdot, \mathbf{0}, \mathbf{0}) = \mathbf{0}, \bar{\boldsymbol{f}}_l(\cdot, \mathbf{0}, \mathbf{0}) = \mathbf{0}, l \in m$. System (27) is LUES if (27a) is LUES and (28) is LUAS.

Remark 2: Both Theorem 2 and Corollary 4 assume that system (28) is asymptotically stable. Note that (28) serves as the nominal system (1) in Theorem 1 and Corollary 3, and $\boldsymbol{g}_{\sigma(k)}(k, \hat{\boldsymbol{x}}(k), \hat{\boldsymbol{x}}(k - d_{3\sigma(k)}(k)))$ in system (27) can be viewed as the perturbation $\boldsymbol{u}(k)$ in (2). This is the key idea to establish Theorem 2 and Corollary 4.

Remark 3: This section proposes several results drawing stability of the cascade system from that of separate systems. Since separate systems have lower dimensions and can be handled more flexibly, these results provide an effective way for analyzing stability of the overall cascade system.

V. Example

Consider the following SCNS:

$$\hat{\boldsymbol{x}}(k+1) = \hat{\boldsymbol{f}}_{\sigma(k)}\left(\hat{\boldsymbol{x}}(k), \hat{\boldsymbol{x}}(k - d_1(k))\right), k \geq 0$$

$$\bar{\boldsymbol{x}}(k+1) = \bar{\boldsymbol{f}}_{\sigma(k)}\left(\bar{\boldsymbol{x}}(k), \bar{\boldsymbol{x}}(k - d_2(k))\right) + \boldsymbol{g}\left(\hat{\boldsymbol{x}}(k), \hat{\boldsymbol{x}}(k - d_3(k))\right), k \geq 0$$

$$\boldsymbol{x}(k) = \boldsymbol{\varphi}(k), k \in \{-d, \ldots, 0\} \quad (30)$$

where $\hat{\boldsymbol{x}} = \mathrm{col}(\hat{x}_1, \hat{x}_2) \in \mathbb{R}^2, \bar{\boldsymbol{x}} = \mathrm{col}(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2, \boldsymbol{x} = \mathrm{col}(\hat{\boldsymbol{x}}, \bar{\boldsymbol{x}})$, and

$$\hat{\boldsymbol{f}}_1(\boldsymbol{x}, \boldsymbol{y}) = \begin{bmatrix} 0.75 & 0.24 \\ 0.78 & 0.21 \end{bmatrix} \begin{bmatrix} \sqrt{x_1 x_2} \\ \sqrt{y_1 y_2} \end{bmatrix}$$

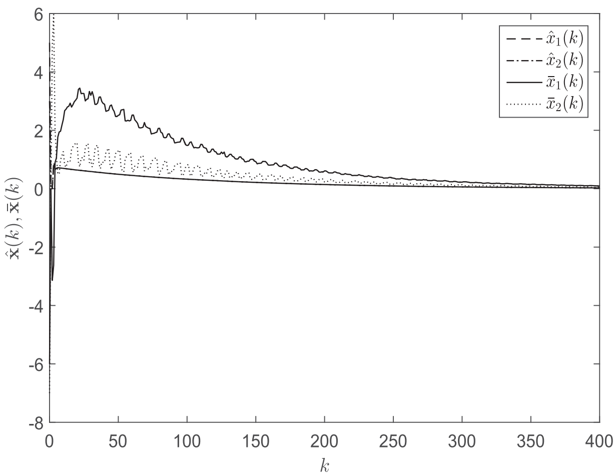

Fig. 2. Trajectory of (30) with $\boldsymbol{\varphi}(\cdot) = \mathrm{col}(5, 3, -6, -7)$.

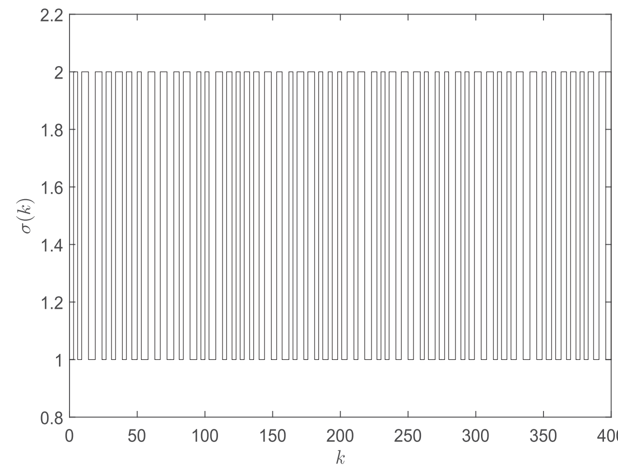

Fig. 3. Switching signal.

$$\hat{\boldsymbol{f}}_2(\boldsymbol{x}, \boldsymbol{y}) = \begin{bmatrix} 0.8 & 0.19 \\ 0.81 & 0.18 \end{bmatrix} \begin{bmatrix} \sqrt{x_1 x_2} \\ \sqrt{y_1 y_2} \end{bmatrix}$$

$$\bar{\boldsymbol{f}}_1(\boldsymbol{x}, \boldsymbol{y}) = \begin{bmatrix} 0.33 & 0.38 - 0.02\cos y_2 \\ 0.1 & 0.28 + 0.02\cos x_2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0.36 - 0.04\sin x_2 & 0.14 \\ 0.08\sin y_2 & 0.06 \end{bmatrix} \boldsymbol{y}$$

$$\bar{\boldsymbol{f}}_2(\boldsymbol{x}, \boldsymbol{y}) = \begin{bmatrix} 0.3 + 0.02\cos x_2 & 0.13 \\ 0.1 - 0.08\cos x_2 & 0.04 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0.41 & 0.38 + 0.01\sin y_2 \\ 0.15 & -0.28\sin y_2 \end{bmatrix} \boldsymbol{y}$$

$$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) = \begin{bmatrix} y_2 \sin x_1 \\ y_1 \cos x_2 \end{bmatrix}$$

$$\boldsymbol{x} = \mathrm{col}(x_1, x_2) \in \mathbb{R}^2, \boldsymbol{y} = \mathrm{col}(y_1, y_2) \in \mathbb{R}^2.$$

Using [23, Th. 4], it is not difficult to show that the system

$$\hat{\boldsymbol{x}}(k+1) = \hat{\boldsymbol{f}}_{\sigma(k)}(\hat{\boldsymbol{x}}(k), \hat{\boldsymbol{x}}(k - d_1(k)))$$

is exponentially stable under the switching law with the dwell time greater than or equal to 3 for any delay $d_1(k) \le 2$.

Observe that the system

$$\tilde{\boldsymbol{x}}(k+1) = A_{\sigma(k)}\tilde{\boldsymbol{x}}(k) + B_{\sigma(k)}\tilde{\boldsymbol{x}}(k - d_2(k)) \tag{31}$$

with $A_1 = \left[\begin{smallmatrix} 0.33 & 0.4 \\ 0.1 & 0.3 \end{smallmatrix}\right]$, $B_1 = \left[\begin{smallmatrix} 0.4 & 0.14 \\ 0.08 & 0.06 \end{smallmatrix}\right]$, $A_2 = \left[\begin{smallmatrix} 0.32 & 0.13 \\ 0.18 & 0.04 \end{smallmatrix}\right]$, and $B_2 = \left[\begin{smallmatrix} 0.41 & 0.39 \\ 0.15 & 0.28 \end{smallmatrix}\right]$. By [24, Th. 1], system (31) is asymptotically stable for an arbitrary switching signal and any finite delay $d_2(k)$ since there exists a vector $\lambda = \mathrm{col}(139.68, 69.2)$ such that $(A_i + B_i - I)\lambda, i = 1, 2$, are strictly negative vectors. Hence, system (31) is asymptotically stable under the switching signal with the dwell time greater than or equal to 3 and any finite delay $d_2(k)$. Applying [25, Th. 3] with domain of attraction being infinity, one can claim that $\tilde{\boldsymbol{x}}(k+1) = \bar{\boldsymbol{f}}_{\sigma(k)}(\tilde{\boldsymbol{x}}(k), \tilde{\boldsymbol{x}}(k - d_2(k)))$ is asymptotically stable under the switching signal with the dwell time greater than or equal to 3 and any finite delay $d_2(k)$. According to Theorem 1, system (30) is asymptotically stable under the same switching signal with $d_1(k), d_2(k)$, and $d_3(k) \le 2$, as shown in Fig. 2. Fig. 3 shows the corresponding switching signal.

VI. Conclusion

The convergence properties of switched nonlinear systems with bounded delays and perturbations have been investigated. It was revealed that if the nominal switched system is asymptotically stable and if the perturbation decays exponentially to zero, then trajectories of the perturbed system asymptotically converge to zero. Based on these results, some sufficient asymptotic stability conditions were established for SCNSs with delays. Note that admissible delays in this paper are required to be bounded and that the method in this paper is inapplicable to the case of unbounded delays. Some numerical experiments show that our main results are also true even if the delays are unbounded. Therefore, one of the challenging studies in the future is to investigate the case of unbounded delays.

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