Discrete-Time Sliding-Mode Control With a Desired Switching Variable Generator

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***Abstract*—This paper introduces a new, reference trajectory following, sliding-mode control strategy for disturbed discrete-time systems. The strategy uses an external trajectory generator based on a switching type reaching law. The trajectory following strategy not only ensures all the properties of the quasi-sliding-mode as defined by Gao *et al.*, but with a certain choice of the control parameters it also guarantees a significant reduction of the quasi-sliding-mode band width and the errors of all the state variables. This paper also considers the problem of error calculation in the discrete-time, reaching law based sliding-mode control systems. It is shown that the limitation of the sliding variable in the sliding phase directly results in bounded errors of all state variables. The results are verified with a simulation example.**

***Index Terms*—Discrete-time systems, reaching law, sliding-mode control, trajectory generator.**

I. Introduction

Variable structure control algorithms are classified as a form of nonlinear state feedback control strategies. They have been first introduced in Russia in the 1950s [1] and thoroughly studied ever since [2]–[4]. The idea of adjusting the structure of the controller according to the current value of the state vector of the system allows us to stabilize nominally unstable plants and ensures high robustness with respect to external disturbances and parameter uncertainties [3]. Early studies focused on continuous time systems and among different approaches the sliding-mode control strategy has gained the most interest [5]–[11]. The method assumes restricting the dynamics of the plant to an arbitrarily chosen sliding surface, which guarantees a stable steady-state behavior. Therefore, the idea ensures a great reduction of the dimensions of the problem. The task becomes to control one function of the system's state instead of n state variables.

However, the ending of the 20th century has put another issue into the perspective. The development of electronic measurements and digital control systems triggered the genesis of discrete-time sliding-mode control. The pioneers of the method were Milosavljević [12], Utkin with Drakunov [13], and Furuta [14]. Their concept was later developed by Gao, Wang, and Homaifa, who defined the quasi-sliding motion and introduced a reaching law approach [15]. According to their definition the representative point of the system must move monotonically from any initial position to the sliding hyperplane and cross it in finite time. Once the representative point of the system has crossed the sliding surface for the first time, it recrosses it in each consecutive step in a so-called sliding phase. The sign of the sliding variable in the sliding phase changes in each step and its absolute value does not exceed an *a priori* known constant. Over the years, several authors have elaborated upon the strategy of Gao *et al*. [16]–[20] and multiple new reaching laws have been introduced [21]–[35]. Nowadays, the researchers focus mainly on the restriction of the control input and minimization of the errors of all state variables. Nevertheless, the definition of the quasi-sliding-mode introduced in [15] is now ubiquitous in the discrete-time sliding-mode control literature and it will also be applied in this paper. Furthermore, we will refer to the switching type strategy [17], which guarantees a restriction of the rate of change of the sliding variable and a reduction of the quasi-sliding-mode band width in comparison to the original strategy [15].

In nowadays control engineering, one of the main factors determining the quality of control is the system's robustness. Therefore, the designers over the world are working on reducing the influence of external disturbances on the systems' dynamics. Following this path, in this paper, we propose a new discrete-time sliding-mode control strategy employing a trajectory generator. Our aim is to minimize the impact of disturbances on the system by applying a new trajectory following reaching law for the disturbed plant and using a trajectory generator based on the reaching law [17]. As the desired trajectory is obtained externally it will not hold any disturbance effects. We will prove that with an appropriate choice of the control parameters, our strategy ensures the existence of the quasi-sliding-mode as defined in [15] and provides a reduction of the quasi-sliding-mode band width in comparison to the strategy [17]. Moreover, we will provide a way to calculate all state variable errors directly from the values of the sliding variable, and therefore, prove that with our new control strategy the state errors are reduced as well.

This paper is organized in the following way. Section II-A introduces the problem we consider. Next, in Section II-B, we present a trajectory generator based on the switching type strategy [17] and, in Section II-C, we introduce a new, trajectory following reaching law for the disturbed system. This paper also shows how to choose the control parameters in order to ensure the existence of the quasi-sliding motion. Furthermore, in Section II-D, we present an additional modification of our method for the systems with slowly varying disturbances and we demonstrate the benefits that it brings to their performance. In Section III, our results are verified with a simulation example and Section IV provides final remarks and conclusions.

II. Sliding-Mode Control Strategy With Trajectory Generator

A. Problem Formulation

In this paper, we consider a discrete-time plant subject to bounded external disturbance. The system is described by the following state

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equation:

$$\boldsymbol{\eta}(k+1) = \boldsymbol{\Phi}\boldsymbol{\eta}(k) + \boldsymbol{\Gamma}v(k) + \boldsymbol{\Gamma}d(k) \tag{1}$$

where $\boldsymbol{\eta}(k)$ is an $n \times 1$ state vector of the plant, $\boldsymbol{\Phi}$ is the plant's state matrix, $\boldsymbol{\Gamma}$ is the input distribution vector, $v(k)$ represents a scalar control signal, and $d(k)$ is a scalar disturbance upper and lower bounded by $d_{\max}$ and $d_{\min}$, respectively. The initial conditions are denoted by $\boldsymbol{\eta}_0 = \boldsymbol{\eta}(0)$ and we aim to drive the system to the demand state $\boldsymbol{\eta}_d$. Therefore, we define the state error: $\boldsymbol{e}(k) = \boldsymbol{\eta}_d - \boldsymbol{\eta}(k)$ and introduce the following sliding hyperplane:

$$\sigma(k) = \boldsymbol{c}\boldsymbol{e}(k) = 0 \tag{2}$$

where $\boldsymbol{c}$ is an $1 \times n$ vector selected so that $(\boldsymbol{c}\boldsymbol{\Gamma})^{-1} \neq 0$. We define $D(k) = \boldsymbol{c}\boldsymbol{\Gamma}(k)$ and we denote the mean value of $D(k)$ with D_1 and the maximum deviation of $D(k)$ from the average value with D_2

$$D_1 = \frac{1}{2}(\mathbf{c}\boldsymbol{\Gamma}d_{\max} + \boldsymbol{c}\boldsymbol{\Gamma}d_{\min}), \quad D_2 = \frac{1}{2}(|\boldsymbol{c}\boldsymbol{\Gamma}d_{\max} - \boldsymbol{c}\boldsymbol{\Gamma}d_{\min}|). \tag{3}$$

The nominal system (1) may be controlled according to one of various sliding-mode reaching laws presented in the literature [15], [17], [21]–[31]. In this paper, we will focus on the switching type reaching law introduced in [17]. The idea of [17] is especially attractive in the case of restricted magnitude of control signal and bounded rate of change of the sliding variable. It ensures faster convergence and smaller width of the ultimate band than the fundamental strategy of Gao *et al*. [15]. The switching type reaching law [17] has the following form:

$$\begin{aligned}\sigma(k+1) = &\{1 - q[\sigma(k)]\}\sigma(k) \\ &- (\varepsilon + D_2)\,\mathrm{sgn}[\sigma(k)] - D(k) + D_1\end{aligned} \tag{4}$$

where the q function is defined as: $q[\sigma(k)] = \sigma_0 / (|\sigma(k)| + \sigma_0)$ and ε and σ_0 are real positive constants. According to [17], these constants must satisfy

$$\sigma_0 > 2D_2, \quad \varepsilon > \frac{4D_2^2}{\sigma_0 - 2D_2} \tag{5}$$

to ensure monotonic convergence of the system to the sliding surface and all the properties of the quasi-sliding-mode defined in [15]. The quasi-sliding-mode band width is expressed as

$$|\sigma(k)| \le \varepsilon + 2D_2. \tag{6}$$

In this paper, we modify the strategy [17] in order to minimize the state error and the quasi-sliding-mode band width. For that purpose, we generate a reference sliding variable profile and apply a new reaching law to make the actual sliding variable of the disturbed system (1) follow the profile. To generate the desired trajectory we will use the unperturbed version of the reaching law [17].

B. Desired Switching Variable Generator

We denote the reference sliding variable with $\sigma_g(k)$ and propose to generate its desired evolution externally. The demand trajectory may be determined using the plant's mathematical model. Then, we will use a new reaching law to drive the system's switching variable to its desired value. To guarantee monotonic convergence of the disturbed system to the sliding hyperplane the initial conditions are set the same for both sliding variables $\sigma(k)$ and $\sigma_g(k)$, so

$$\sigma_g(0) = \sigma(0) = \boldsymbol{c}[\boldsymbol{\eta_d} - \boldsymbol{\eta}_0]. \tag{7}$$

The quasi-sliding-mode will exist if the trajectory of $\sigma_g(k)$ converges to the sliding hyperplane $\sigma_g(k) = 0$ and crosses it in finite time. After the sign of the sliding variable changes for the first time it has to change again in each successive step and the absolute value of $\sigma_g(k)$ may not exceed an *a priori* known constant. To ensure the above, in the trajectory generator we will use the switching type reaching law presented in [17]

$$\sigma_g(k+1) = \{1 - q[\sigma_g(k)]\}\sigma_g(k) - \varepsilon\mathrm{sgn}[\sigma_g(k)] \tag{8}$$

where q is defined by the following function:

$$q[\sigma_g(k)] = \frac{\sigma_0}{|\sigma_g(k)| + \sigma_0} \in (0, 1\rangle \tag{9}$$

and σ_0 and ε are real positive values. For the sake of clarity, we define the signum function as follows:

$$\mathrm{sgn}(z) = \begin{cases} 1 & \text{for } z \ge 0 \\ -1 & \text{for } z < 0 \end{cases}. \tag{10}$$

From the practical point of view, we notice that $\sigma_g(k)$ may be obtained in an external trajectory generator by the application of the reaching law (8) to the mathematical model of the system. According to the results of [17] and considering (10) the change of sign of the sliding variable occurs when its value accesses the region described by

$$\sigma_g(k) \in \left\langle -\frac{1}{2}\left(\varepsilon + \sqrt{\varepsilon^2 + 4\varepsilon\sigma_0}\right), \frac{1}{2}\left(\varepsilon + \sqrt{\varepsilon^2 + 4\varepsilon\sigma_0}\right)\right). \tag{11}$$

We define k_{0g} as the first time instant when the generated sliding variable satisfies

$$\mathrm{sgn}[\sigma_g(k_{0g})] = -\mathrm{sgn}[\sigma_g(k_{0g} + 1)] \tag{12}$$

so the sign of σ_g changes for the first time between steps k_{0g} and $k_{0g} + 1$. As ε is a real positive constant, it follows from (8) and (9) that the finite k_{0g} always exists. Furthermore, (8) and (9) imply that when the signs of $\sigma_g(k)$ and $\sigma_g(k+1)$ are opposite the absolute value of the sliding variable is bounded by

$$|\sigma_g(k+1)| \le \varepsilon \tag{13}$$

which describes the quasi-sliding-mode band width. We now conclude that (13) guarantees that, for any $k \ge k_{0g} + 1$, (11) is satisfied and

$$\mathrm{sgn}[\sigma_g(k)] = -\mathrm{sgn}[\sigma_g(k+1)] \tag{14}$$

holds in all the future steps.

Furthermore, the strategy [17] restricts the maximum rate of change of the generated sliding variable in the reaching phase. We can express the change of σ_g as follows:

$$\sigma_g(k) - \sigma_g(k+1) = \frac{\sigma_0\sigma_g(k)}{|\sigma_g(k)| + \sigma_0} + \varepsilon\mathrm{sgn}[\sigma_g(k)]. \tag{15}$$

The maximum absolute rate of change of σ_g satisfies

$$|\Delta\sigma_g| = \left|\varepsilon\mathrm{sgn}[\sigma_g(k)] + \frac{\sigma_0\sigma_g(k)}{|\sigma_g(k)| + \sigma_0}\right| \le \varepsilon + \frac{\sigma_0|\sigma_g(k)|}{|\sigma_g(k)| + \sigma_0}. \tag{16}$$

Considering that the function on the right-hand side of (16) is even, decreasing for any $\sigma_g \in (-\infty, 0)$ and increasing for any $\sigma_g \in (0, \infty)$, in order to obtain its boundary values, we calculate the limits at zero

$$\lim_{\sigma_g(k)\to 0}\left[\varepsilon + \frac{\sigma_0|\sigma_g(k)|}{|\sigma_g(k)| + \sigma_0}\right] = \varepsilon \tag{17}$$

and at plus/minus infinity

$$\lim_{\sigma_g(k)\to\pm\infty} \left[\varepsilon + \sigma_0 \left|\sigma_g(k)\right| / \left(\left|\sigma_g(k)\right| + \sigma_0\right)\right]$$
$$= \lim_{\sigma_g(k)\to\pm\infty} \left\{\varepsilon + \sigma_0 \left|\sigma_g(k)\right| / \left[1 + \frac{\sigma_0}{\left|\sigma_g(k)\right|}\right] \left|\sigma_g(k)\right|\right\} = \varepsilon + \sigma_0. \tag{18}$$

From (17) and (18), we conclude that the rate of change of σ_g in the reaching phase is restricted by

$$\left|\Delta\sigma_g\right| < \varepsilon + \sigma_0. \tag{19}$$

Therefore, the designer may influence the duration of the reaching phase by an appropriate choice of the parameter σ_0 and control the width of the quasi-sliding-mode band with the choice of ε.

C. Reaching Law-Based Control Strategy

In this section, we present a new reaching law for the disturbed system (1). Our strategy aims to drive the system's sliding variable σ in the step $k + 1$ to its desired value $\sigma_g(k + 1)$ and to minimize the impact of perturbations by compensating for the average disturbance D_1. We propose the following reaching law for the plant:

$$\sigma(k+1) = \sigma_g(k+1) - D(k) + D_1. \tag{20}$$

In the nominal strategy [17] the value of $\sigma(k + 1)$ depends on $\sigma(k)$, as visible from (4). Consequently, it is influenced by all the disturbance values from the beginning of the control process. In our reaching law the plant's sliding variable is controlled according to the desired trajectory, which does not depend on disturbances. Therefore, $\sigma(k + 1)$ only bears the impact of the disturbance $D(k)$ from one control step and is not affected by the previous disturbance values $D(0), D(1), \dots, D(k - 1)$. As a result, our reaching law guarantees an improvement of the robustness of the system by reducing the ultimate band width and the state error.

Considering (1), (2), and (20), we get the following control law:

$$v(k) = (\boldsymbol{c\Gamma})^{-1}\left[\boldsymbol{c\eta_d} - \boldsymbol{c\Phi\eta}(k) - \sigma_g(k+1) - D_1\right]. \tag{21}$$

From (21), we notice that the desired trajectory σ_g must be obtained in advance to generate the control signal for the plant. Moreover, as the disturbance may push the plant's sliding variable away from the sliding surface, the control parameters ε and σ_0 must be properly chosen to ensure the existence of the quasi-sliding motion.

Let us define k_0 as the first time instant when

$$\operatorname{sgn}\left[\sigma(k_0)\right] = -\operatorname{sgn}\left[\sigma(k_0+1)\right]. \tag{22}$$

With this notation the first change of sign of σ occurs between steps k_0 and $k_0 + 1$. For the quasi-sliding-mode as defined in [15] to emerge, for any $k \geq k_0$, the following equality must be satisfied:

$$\operatorname{sgn}\left[\sigma(k)\right] = -\operatorname{sgn}\left[\sigma(k+1)\right]. \tag{23}$$

Theorem 1: If, in the reaching law (8), $\sigma_0 > D_2$ and $\varepsilon > \sigma_0 D_2 / (\sigma_0 - D_2)$, then $k_0 \leq k_{0g} + 2$ and (23) is satisfied for any $k \geq k_0$. Furthermore, for any $k \geq k_{0g} + 1$ the absolute value of the plant's sliding variable does not exceed $\varepsilon + D_2$.

Proof: We first present $\sigma(k + 2)$ according to the reaching law (20)

$$\sigma(k+2) = \sigma_g(k+2) - D(k+1) + D_1. \tag{24}$$

Taking into account (8) we obtain

$$\begin{aligned}\sigma&(k+2)\\ &= \left\{1 - q\left[\sigma_g(k+1)\right]\right\}\left\{\left\{1 - q\left[\sigma_g(k)\right]\right\}\sigma_g(k) - \varepsilon\operatorname{sgn}\left[\sigma_g(k)\right]\right\}\\ &\quad - \varepsilon\operatorname{sgn}\left[\sigma_g(k+1)\right] - D(k+1) + D_1.\end{aligned} \tag{25}$$

Considering that σ_g converges to the vicinity of zero in finite time and, for any $k \geq k_{0g}$, the change of its sign occurs, we can transform (25) into

$$\begin{aligned}&\operatorname{sgn}\left[\sigma(k+2)\right]\left|\sigma(k+2)\right|\\ &= \operatorname{sgn}\left[\sigma_g(k)\right]\left\{\left\{1 - q\left[\sigma_g(k+1)\right]\right\}\left\{1 - q\left[\sigma_g(k)\right]\right\}\left|\sigma_g(k)\right|\right.\\ &\quad \left. + \varepsilon q\left[\sigma_g(k+1)\right]\right\} - D(k+1) + D_1.\end{aligned} \tag{26}$$

If the following inequality is satisfied:

$$\begin{aligned}&\left\{1 - q\left[\sigma_g(k+1)\right]\right\}\left\{1 - q\left[\sigma_g(k)\right]\right\}\left|\sigma_g(k)\right|\\ &\quad + \varepsilon q\left[\sigma_g(k+1)\right] > \left|D(k+1) - D_1\right|,\end{aligned} \tag{27}$$

then

$$\operatorname{sgn}\left[\sigma(k+2)\right] = \operatorname{sgn}\left[\sigma_g(k)\right]. \tag{28}$$

One may notice that both terms on the left-hand side of (27) are always nonnegative. Since in the worst case the value of $\sigma_g(k)$ tends to zero and $|D(k + 1) - D_1| \leq D_2$ it follows that if:

$$\varepsilon q\left[\sigma_g(k+1)\right] > D_2, \tag{29}$$

then (28) holds. As the value of $\sigma_g(k + 1)$, for any $k \geq k_{0g}$, satisfies (13), we can write

$$q\left[\sigma_g(k+1)\right] = \frac{\sigma_0}{\left|\sigma_g(k+1)\right| + \sigma_0} > \frac{\sigma_0}{\varepsilon + \sigma_0} \tag{30}$$

and the left-hand side of (29) satisfies

$$\varepsilon q\left[\sigma_g(k+1)\right] > \frac{\varepsilon\sigma_0}{\varepsilon + \sigma_0}. \tag{31}$$

Therefore, (28) is satisfied if

$$\frac{\varepsilon\sigma_0}{\varepsilon + \sigma_0} > D_2. \tag{32}$$

Since $\sigma_g(k)$ changes its sign for any $k \geq k_{0g}$, we conclude that (23) holds for any $k \geq k_{0g} + 2$, if the control parameters are chosen as

$$\sigma_0 > D_2 \tag{33}$$

and

$$\varepsilon > \frac{D_2\sigma_0}{\sigma_0 - D_2}. \tag{34}$$

In the worst case of external disturbance, the sliding variable of the plant converges to the vicinity of zero not later than two steps after the reference sliding variable. The sign of σ changes for the first time between the time instants $k_0 = k_{0g} + 2$ and $k_0 + 1 = k_{0g} + 3$. Moreover, we may use the reaching law (20) to determine the quasi-sliding-mode band width. For any $k \geq k_{0g} + 1$, the reference sliding variable σ_g satisfies (13), so the maximal absolute value of $\sigma(k)$ satisfies

$$\left|\sigma(k)\right| \leq \varepsilon + D_2. \tag{35}$$

■

From the above, we may conclude that our new control strategy ensures the existence of the quasi-sliding motion in a smaller vicinity of zero than the strategy [17]. The combination of reaching laws (8) and (20) allows to determine the maximum rate of change of the plant's sliding variable in the reaching phase. We express the change of the plant's sliding variable as

$$\begin{aligned}\Delta\sigma(k) &= \sigma(k) - \sigma(k+1)\\ &= \sigma_g(k) - D(k-1) + D_1 - \sigma_g(k+1) + D(k) - D_1.\end{aligned} \tag{36}$$

TABLE I
COMPARISON OF THE CONSIDERED CONTROL STRATEGIES

Property	The original control strategy	The trajectory following control strategy
Control parameters	$\sigma_0 > 2D_2,\ \varepsilon > \frac{4D_2^2}{\sigma_0 - 2D_2}$	$\sigma_0 > D_2,\ \varepsilon > \frac{D_2\sigma_0}{\sigma_0 - D_2}$
The quasi-sliding mode band	$\lvert\sigma(k)\rvert \le \varepsilon + 2D_2$	$\lvert\sigma(k)\rvert \le \varepsilon + D_2$
Maximum rate of change of the sliding variable	$\lvert\Delta\sigma\rvert \le \varepsilon + \sigma_0 + 2D_2$	$\lvert\Delta\sigma\rvert \le \varepsilon + \sigma_0 + 2D_2$

From (36), we conclude that the maximum absolute rate of change of σ is limited by

$$|\Delta\sigma| \le |\Delta\sigma_g| + 2D_2. \quad (37)$$

Using (19), we obtain the upper bound of the rate of change of the plant's sliding variable described as

$$|\Delta\sigma| \le \varepsilon + \sigma_0 + 2D_2. \quad (38)$$

From the comparison of (33) and (5), we notice that in our strategy σ_0 may have smaller values than in the original control law. As follows, for the smallest admissible σ_0 the parameter ε in our method is also reduced. Consequently, comparing (35) and (6), we conclude that application of the proposed trajectory following reaching law with an appropriate choice of σ_0 guarantees a restriction of the maximum rate of change of the sliding variable along with a reduction of the ultimate band width by the value of D_2. However, the smaller the value of σ_0, the longer the reaching phase lasts. The price to pay for reducing the width of the quasi-sliding-mode band is slower convergence of the system to the sliding hyperplane than in the original strategy. The comparison of the considered control methods is presented in Table I.

D. Modified Strategy

For the slowly varying disturbances, i.e., when for any $k \ge 0$ $|d(k+1) - d(k)| \le \delta$, the trajectory following reaching law may be modified so that the quasi-sliding-mode band is further reduced. We define $\Delta = \boldsymbol{c}\boldsymbol{\Gamma}\delta$ and assume that

$$|D(k+1) - D(k)| \le \Delta < D_2. \quad (39)$$

Inspired by [21], we propose a new reaching law

$$\sigma(k+1) = \sigma_g(k+1) - D(k) + D_1 - \sum_{l=0}^{k}[\sigma(l) - \sigma_g(l)]. \quad (40)$$

Considering (1), (2), and (40), we obtain the following control signal:

$$v(k) = (\boldsymbol{c}\boldsymbol{\Gamma})^{-1}\Big\{\boldsymbol{c}\boldsymbol{\eta_d} - \boldsymbol{c}\boldsymbol{\Phi}\boldsymbol{\eta}(k) - \sigma_g(k+1) - D_1 + \sum_{l=0}^{k}[\sigma(l) - \sigma_g(l)]\Big\}. \quad (41)$$

Theorem 2: When the control parameters for the trajectory generator satisfy: $\sigma_0 > \Delta$ and $\varepsilon > \sigma_0\Delta\ /\ (\sigma_0 - \Delta)$ and the disturbance D satisfies (39), then $k_0 \le k_{0g} + 2$ and (23) is satisfied for any $k \ge k_0$. Moreover, for any $k \ge \max\{k_{0g} + 1, 2\}$, the absolute value of the plant's sliding variable does not exceed $\varepsilon + \Delta$.

Proof: We begin by expressing $\sigma(k+2)$ according to (40)

$$\sigma(k+2) = \sigma_g(k+2) - D(k+1) + D_1 - \sum_{l=0}^{k+1}[\sigma(l) - \sigma_g(l)]. \quad (42)$$

Taking into account (8) and (14), for any $k \ge k_{0g}$, we obtain

$$\begin{aligned}&\operatorname{sgn}[\sigma(k+2)]\,|\sigma(k+2)|\\ &= \operatorname{sgn}[\sigma_g(k)]\,\{\{1 - q[\sigma_g(k+1)]\}\{1 - q[\sigma_g(k)]\}|\sigma_g(k)|\\ &\quad + \varepsilon q[\sigma_g(k+1)]\} - D(k+1) + D_1 - \sum_{l=0}^{k+1}[\sigma(l) - \sigma_g(l)].\end{aligned} \quad (43)$$

The last element on the right-hand side of (43) can be expressed as

$$\sum_{l=0}^{k+1}[\sigma(l) - \sigma_g(l)] = \sigma(k+1) - \sigma_g(k+1) + \sum_{l=0}^{k}[\sigma(l) - \sigma_g(l)]. \quad (44)$$

From the reaching law (40), we get

$$\sum_{l=0}^{k+1}[\sigma(l) - \sigma_g(l)] = -D(k) + D_1. \quad (45)$$

By substituting (45) into (43), we obtain

$$\begin{aligned}&\operatorname{sgn}[\sigma(k+2)]\,|\sigma(k+2)|\\ &= \operatorname{sgn}[\sigma_g(k)]\,\{\{1 - q[\sigma_g(k+1)]\}\{1 - q[\sigma_g(k)]\}|\sigma_g(k)|\\ &\quad + \varepsilon q[\sigma_g(k+1)]\} - D(k+1) + D(k).\end{aligned} \quad (46)$$

In the worst case $|\sigma_g(k)|$ can be arbitrarily small. Therefore, in order to ensure that (28) is satisfied, the following inequality must hold:

$$\varepsilon q[\sigma_g(k+1)] > |D(k+1) - D(k)|. \quad (47)$$

Taking into account (31) and (39), we notice that this is the case when

$$\frac{\varepsilon\sigma_0}{\varepsilon + \sigma_0} > \Delta. \quad (48)$$

From (48), we conclude that the control parameters

$$\sigma_0 > \Delta \quad (49)$$

$$\varepsilon > \frac{\sigma_0\Delta}{\sigma_0 - \Delta} \quad (50)$$

ensure that for any $k \ge k_{0g} + 2$, the equality (23) holds and the quasi-sliding-mode exists. Furthermore, it follows from (40) and (45) that for any $k \ge 2$:

$$\sigma(k) = \sigma_g(k) - D(k-1) + D(k-2). \quad (51)$$

Consequently, for any $k \ge \max\{k_{0g} + 1, 2\}$, the quasi-sliding-mode band width is reduced to

$$|\sigma(k)| \le \varepsilon + \Delta. \quad (52)$$

■

As Δ satisfies (39), the modified strategy guarantees a further reduction of the ultimate band. Moreover, the rate of change of the sliding variable is also reduced. In order to demonstrate this property, we use the reaching law (40) and we follow the same routine as in Section II-C. We write the change of σ between steps k and $k+1$ as

$$\Delta\sigma(k) = \sigma(k) - \sigma(k+1) = \sigma_g(k) - D(k-1) + D_1 - \sum_{l=0}^{k-1}[\sigma(l) - \sigma_g(l)] - \sigma_g(k+1) + D(k) - D_1 + \sum_{l=0}^{k}[\sigma(l) - \sigma_g(l)]. \quad (53)$$

Taking into account (19) and (45), we conclude that the rate of change of the sliding variable is upper bounded by

$$|\Delta\sigma| \leq \varepsilon + \sigma_0 + 2\Delta. \quad (54)$$

It should be highlighted that the restriction of the rate of change of the sliding variable directly influences a restriction of the rate of change of all state variables of the system. Therefore, the control process is smoother and the control input less aggressive.

E. Regulation Error

In this section, we will present how to calculate each state variable's error based on the values of the sliding variable. We will proceed with our reasoning for a certain choice of the sliding surface. In order to properly choose the elements of vector $\boldsymbol{c}$ we first calculate the new state matrix of the closed-loop system $\boldsymbol{\Phi}_c$

$$\boldsymbol{\Phi_c} = \left[\mathbf{1} - \boldsymbol{\Gamma}(\boldsymbol{c\Gamma})^{-1}\boldsymbol{c}\right]\boldsymbol{\Phi}. \quad (55)$$

The choice of the sliding surface is crucial to ensure stability of the system. In the following calculations, we assume that the vector $\boldsymbol{c}$ is selected so that:

$$M(z) = \det(\mathbf{1}z - \boldsymbol{\Phi_c}) = z^n = 0. \quad (56)$$

Although this choice of vector $\boldsymbol{c}$ places all the poles at the origin of the complex plane, there is no risk of unacceptable increase of the control signal, as it is limited by the reaching law. Moreover, satisfying (56) guarantees that the new state matrix $\boldsymbol{\Phi}_c$ is nilpotent, which results in the fastest possible disturbance rejection.

Theorem 3: The error vector at any time instant $k+j$, where $k \geq 0$ and $j > 0$ may be expressed as

$$\boldsymbol{e}(k+j) = \boldsymbol{\eta_d} - \boldsymbol{\eta}(k+j) = \boldsymbol{\eta_d} - \sum_{l=0}^{j-1}\boldsymbol{\Phi_c^l\Gamma}(\boldsymbol{c\Gamma})^{-1}\boldsymbol{c\eta_d} - \boldsymbol{\Phi_c^j\eta}(k) + \sum_{l=0}^{j-1}\boldsymbol{\Phi_c^l\Gamma}(\boldsymbol{c\Gamma})^{-1}\sigma(k+j-l). \quad (57)$$

Moreover, if the characteristic polynomial of the closed-loop system satisfies (56) and $j \geq n$, where n represents the order of the system, then

$$\boldsymbol{e}(k+j) = \boldsymbol{\eta_d} - \boldsymbol{\eta}(k+j) = \boldsymbol{\eta_d} - \sum_{l=0}^{n-1}\boldsymbol{\Phi_c^l\Gamma}(\boldsymbol{c\Gamma})^{-1}\boldsymbol{c\eta_d} + \sum_{l=0}^{n-1}\boldsymbol{\Phi_c^l\Gamma}(\boldsymbol{c\Gamma})^{-1}\sigma(k+j-l). \quad (58)$$

Proof: To prove the abovementioned equation, we use the principle of mathematical induction. First, let us consider $j = 1$. We may express the error vector at time instant $k+1$ as

$$\boldsymbol{e}(k+1) = \boldsymbol{\eta_d} - \boldsymbol{\eta}(k+1) = \boldsymbol{\eta_d} - \boldsymbol{\Phi\eta}(k) - \boldsymbol{\Gamma}v(k) - \boldsymbol{\Gamma}d(k). \quad (59)$$

Applying the control signal (21), we get

$$\boldsymbol{e}(k+1) = \boldsymbol{\eta_d} - \boldsymbol{\Phi\eta}(k) - \boldsymbol{\Gamma}(\boldsymbol{c\Gamma})^{-1}\boldsymbol{c\eta_d} + \boldsymbol{\Gamma}(\boldsymbol{c\Gamma})^{-1}\boldsymbol{c\Phi\eta}(k) + \boldsymbol{\Gamma}(\boldsymbol{c\Gamma})^{-1}\sigma_g(k+1) + \boldsymbol{\Gamma}(\boldsymbol{c\Gamma})^{-1}D_1 - \boldsymbol{\Gamma}d(k). \quad (60)$$

Next, using the reaching law (20), we obtain

$$\boldsymbol{e}(k+1) = \boldsymbol{\eta_d} - \boldsymbol{\Phi_c^0\Gamma}(\boldsymbol{c\Gamma})^{-1}\boldsymbol{c\eta_d} - \boldsymbol{\Phi_c\eta}(k) + \boldsymbol{\Phi_c^0\Gamma}(\boldsymbol{c\Gamma})^{-1}\sigma(k+1) = \boldsymbol{\eta_d} - \sum_{l=0}^{0}\boldsymbol{\Phi_c^l\Gamma}(\boldsymbol{c\Gamma})^{-1}\boldsymbol{c\eta_d} - \boldsymbol{\Phi_c^1\eta}(k) + \sum_{l=0}^{0}\boldsymbol{\Phi_c^l\Gamma}(\boldsymbol{c\Gamma})^{-1}\sigma(k+1-l) \quad (61)$$

which proves that (57) holds for the time instant $k+1$, where $k \geq 0$.

Next, according to the mathematical induction method, we verify if the fact that (57) holds for $k+j$ implies that it also holds for $k+j+1$. We start by writing the error vector at time instant $k+j+1$ as

$$\boldsymbol{e}(k+j+1) = \boldsymbol{\eta_d} - \boldsymbol{\eta}(k+j+1). \quad (62)$$

We substitute the disturbed plant's state (1)

$$\boldsymbol{e}(k+j+1) = \boldsymbol{\eta_d} - \boldsymbol{\Phi\eta}(k+j) - \boldsymbol{\Gamma}v(k+j) - \boldsymbol{\Gamma}d(k+j). \quad (63)$$

Now, we use the control law (21) to get

$$\boldsymbol{e}(k+j+1) = \boldsymbol{\eta_d} - \boldsymbol{\Gamma}(\boldsymbol{c\Gamma})^{-1}\boldsymbol{c\eta_d} - \boldsymbol{\Phi_c\eta}(k+j) + \boldsymbol{\Gamma}(\boldsymbol{c\Gamma})^{-1}\sigma_g(k+j+1) + \boldsymbol{\Gamma}(\boldsymbol{c\Gamma})^{-1}D_1 - \boldsymbol{\Gamma}d(k+j) \quad (64)$$

and furthermore, from the application of the reaching law (20), we obtain

$$\boldsymbol{e}(k+j+1) = \boldsymbol{\eta_d} - \boldsymbol{\Phi_c^0\Gamma}(\boldsymbol{c\Gamma})^{-1}\boldsymbol{c\eta_d} - \boldsymbol{\Phi_c\eta}(k+j) + \boldsymbol{\Phi_c^0\Gamma}(\boldsymbol{c\Gamma})^{-1}\sigma(k+j+1). \quad (65)$$

Assuming that (57) holds for $k+j$, we can express $\boldsymbol{\eta}(k+j)$ as

$$\boldsymbol{\eta}(k+j) = \sum_{l=0}^{j-1}\boldsymbol{\Phi_c^l\Gamma}(\boldsymbol{c\Gamma})^{-1}\boldsymbol{c\eta_d} + \boldsymbol{\Phi_c^j\eta}(k) - \sum_{l=0}^{j-1}\boldsymbol{\Phi_c^l\Gamma}(\boldsymbol{c\Gamma})^{-1}\sigma(k+j-l). \quad (66)$$

We substitute (66) into (64)

$$\boldsymbol{e}(k+j+1) = \boldsymbol{\eta_d} - \boldsymbol{\Phi_c^0\Gamma}(\boldsymbol{c\Gamma})^{-1}\boldsymbol{c\eta_d} - \boldsymbol{\Phi_c}\left[\sum_{l=0}^{j-1}\boldsymbol{\Phi_c^l\Gamma}(\boldsymbol{c\Gamma})^{-1}\boldsymbol{c\eta_d} + \boldsymbol{\Phi_c^j\eta}(k) - \sum_{l=0}^{j-1}\boldsymbol{\Phi_c^l\Gamma} \times (\boldsymbol{c\Gamma})^{-1}\sigma(k+j-l)\right] + \boldsymbol{\Phi_c^0\Gamma}(\boldsymbol{c\Gamma})^{-1}\sigma(k+j+1). \quad (67)$$

After some further calculations we obtain

$$e(k+j+1) = \eta_d - \sum_{l=0}^{j} \Phi_c^l \Gamma (c\Gamma)^{-1} c\eta_d - \Phi_c^{j+1}\eta(k) + \sum_{l=0}^{j} \Phi_c^l \Gamma (c\Gamma)^{-1} \sigma(k+j+1-l) \quad (68)$$

which proves that (57) holds for $k+j+1$. Therefore, considering the principle of mathematical induction we conclude that (57) holds for any $k \geq 0$ and $j > 0$. If additionally the closed-loop system's state matrix Φ_c satisfies (56) then $\Phi_c{}^n = \Phi_c{}^{n+1} = \cdots = \Phi_c{}^{j+1} = \mathbf{0}$. Consequently, the error vector for any $j+1 \geq n$

$$e(k+j+1) = \eta_d - \sum_{l=0}^{n-1} \Phi_c^l \Gamma (c\Gamma)^{-1} c\eta_d + \sum_{l=0}^{n-1} \Phi_c^l \Gamma (c\Gamma)^{-1} \sigma(k+j+1-l). \quad (69)$$

■

To get the deviation of the ith state variable from its demand value, we multiply both sides of (58) by the following row vector:

$$w_i = \begin{bmatrix} \underbrace{0 \ \dots \ 0}_{i-1} & 1 & \underbrace{0 \ \dots \ 0}_{n-i} \end{bmatrix} \quad (70)$$

and obtain

$$e_i(k+j) = \eta_{di} - w_i \sum_{l=0}^{n-1} \Phi_c^l \Gamma (c\Gamma)^{-1} c\eta_d + w_i \sum_{l=0}^{n-1} \Phi_c^l \Gamma (c\Gamma)^{-1} \sigma(k+j-l). \quad (71)$$

Moreover, we notice that with the reaching law approach we restrict the sliding variable in the sliding phase with known constants $\pm\sigma_{\max}$. This leads to the conclusion that the error values are also bounded. We may state that for any $k \geq k_0 + n$ the state error satisfies

$$e_i(k) \leq e_{i\max} = \left| \eta_{di} - w_i \sum_{l=0}^{n-1} \Phi_c^l \Gamma (c\Gamma)^{-1} c\eta_d + \sigma_{\max} w_i \sum_{l=0}^{n-1} \Phi_c^l \Gamma (c\Gamma)^{-1} \right|. \quad (72)$$

We have come to the conclusion that the bounds of the state errors depend directly on the quasi-sliding-mode band width. Therefore, as the trajectory following reaching law allows the designer to reduce the ultimate band, all the state variable errors will be reduced as well.

III. Simulation Results

In order to present the benefits of the trajectory following control strategy, we consider a first order integrator supplied by three sources with different control allocation coefficients α_i, different delay times T_{di} and subject to bounded disturbance $f(t)$. The model of such a system is presented in Fig. 1.

As continuous-time state measurement and control may not be feasible, we discretize the model, so that each of the delay times is a multiple of the discretization period $T = 1$. We obtain $T_{d1} = 9T$, $T_{d2} = 4T$, and $T_{d3} = 2T$. Moreover, $\alpha_1 = 0.5$, $\alpha_2 = 0.3$, and $\alpha_3 = 0.2$. Therefore,

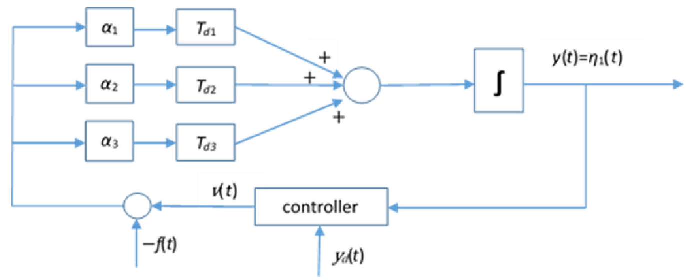

Fig. 1. Model of the continuous time plant.

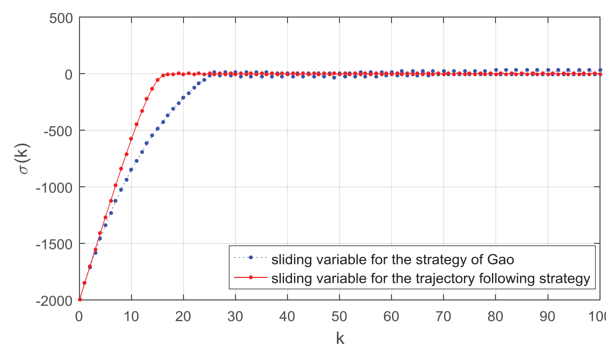

Fig. 2. Evolution of the sliding variable of the plant.

the discrete-time model is represented by

$$\eta(k+1) = \begin{bmatrix} 1 & a_9 & a_8 & \cdots & a_1 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \eta(k) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ T \end{bmatrix} v(k) - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ T \end{bmatrix} d(k) \quad (73)$$

where $a_9 = 0.5$, $a_4 = 0.3$, $a_2 = 0.2$, and the remaining coefficients $a_1 = a_3 = a_5 = a_6 = a_7 = a_8 = 0$. The initial conditions of the system are $\eta_0 = \eta(0) = [2000\ 0\ \dots\ 0]^{\mathrm{T}}$ and we aim to drive the system to the desired state $\eta_d = [0\ \dots\ 0]^{\mathrm{T}}$. We select the sliding surface in the dead-beat manner and set vector $c = [1\ 0.5\ 0.5\ 0.5\ 0.5\ 0.5\ 0.8\ 0.8\ 1\ 1]$, so that (56) is satisfied. The disturbance $d(k)$ has the following values: $d(k) = -1$ for $k \in [1, 50]$ and $d(k) = 1$ for $k \in [51, 100]$. Therefore, $D_1 = 0$ and $D_2 = 1$.

To present the effect of the constraint of the control effort we compare our control method with the strategy presented in the seminal paper of Gao *et al.* [15]. The control signal is required not to exceed the maximum value of 150. To satisfy this constraint we choose the control parameters in our strategy $\sigma_0 = 160$ and $\varepsilon = 1.01$ and in the Gao's strategy $q = 0.058$ and $\varepsilon = 32.5$. The results of our simulations are presented in Figs. 2–8. We marked the trajectories obtained with the original strategy [15] with blue and those of the trajectory following strategy we plot in red.

Figs. 2 and 3 show the evolution of the sliding variable of the plant in the whole control process and in the sliding phase only. Both of the strategies ensure monotonic convergence to the sliding plane and guarantee all the properties of the quasi-sliding-mode as defined in [15]. It is clearly visible that the implementation of the trajectory generator significantly reduced the width of the ultimate band and shortened the duration of the reaching phase. In the Gao's control strategy, the representative point of the system crosses the sliding plane for the first

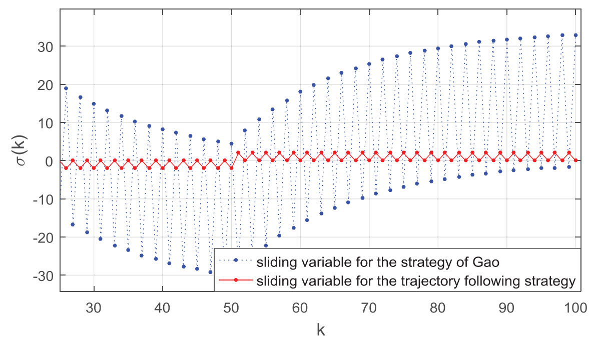

Fig. 3. Comparison of the ultimate bands.

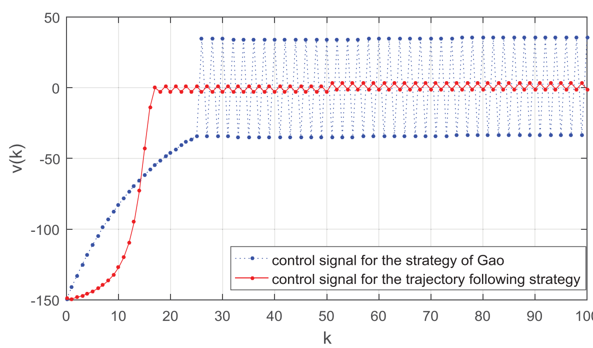

Fig. 4. Control signal for the plant.

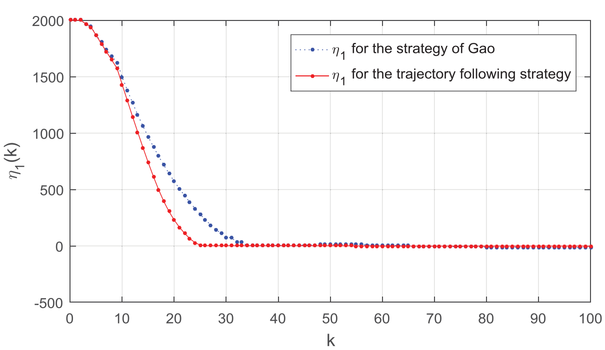

Fig. 5. First state variable η_1 of the plant.

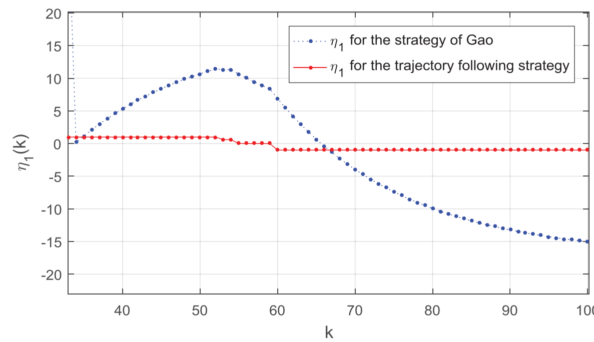

Fig. 6. First state variable η_1 of the plant in the sliding phase.

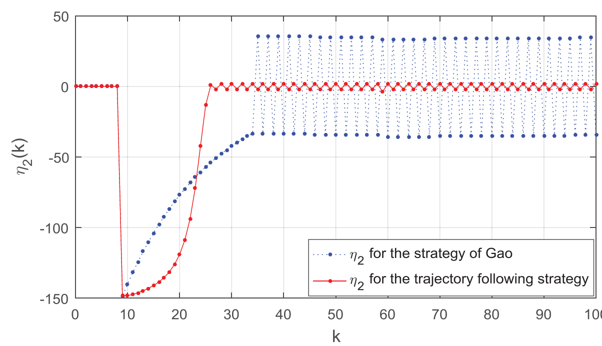

Fig. 7. State variable η_i $(i = 2)$.

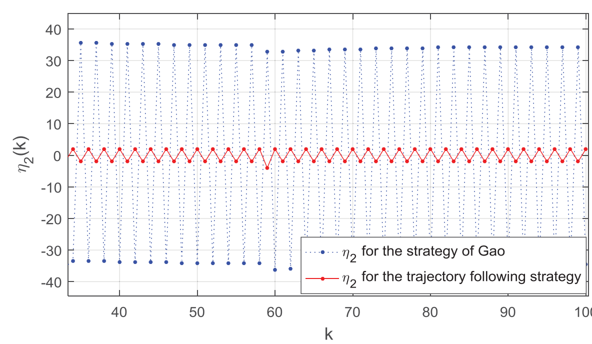

Fig. 8. State variable η_i $(i = 2)$ in the sliding phase.

time between steps $k = 25$ and $k = 26$, while in our control method the first crossing takes place between instants $k = 18$ and $k = 19$.

The quasi-sliding-mode band width obtained with the original strategy equals 34.5, while for the new reference trajectory following strategy it is 2.01. Therefore, the width of the band has been reduced by more than 90%.

Fig. 4 presents the control signal in both strategies. It may be noticed that the constraint is satisfied in both cases. However, in the trajectory following method the control input in the sliding phase has been significantly reduced.

We used (71) to calculate the state variable errors in the quasi-steady state. The errors, for any $k \geq k_0 + n = k_0 + 10$, in our control strategy are limited as follows: $|e_1| \leq 1.0085$ and, for $i = 2, \ldots, 10$, $|e_i| \leq 2.023$, while for Gao's strategy $|e_1| \leq 16.5$ and $|e_i| \leq 36$. We may conclude that the state error has been reduced by over 90%. The first state variable error may be seen from Figs. 5 and 6. Moreover, as for $i = 2, \ldots, 10$, the state variables η_i have the same evolution with appropriate time shifts, their error values may be verified from Figs. 7 and 8.

It has been shown that in comparison to the traditional control method, the application of the trajectory following control strategy ensures faster convergence of the system and a reduction of the quasi-sliding-mode band width while satisfying the same control constraint. Moreover, the deviations of all the state variables from their demand values have been reduced as well. Therefore, the robustness of the system has been significantly improved.

IV. Conclusion

In this paper, we have introduced a new trajectory following, sliding-mode control strategy for discrete-time disturbed systems. First, we proposed an external trajectory generator based on the modified version of the Gao's reaching law presented in [17]. Therefore, the obtained desired trajectory satisfies all the properties of the quasi-sliding-mode as defined by Gao *et al*. Afterward, we introduced a new trajectory following reaching law for the disturbed system. We have presented how to select the control parameters in order to guarantee that the representative point of the system converges to the sliding hyperplane in finite time and then moves along it with a characteristic zigzagging motion. This paper shows that with an appropriate choice of the control parameters the trajectory following strategy ensures a significant reduction of the width of the quasi-sliding-mode band and a bounded rate of change of the sliding variable. Furthermore, we have obtained an equation, which allows us to calculate the errors of all the state variables and their upper limits in the sliding phase. We have proved that the reduction of the ultimate band directly results in the reduction of the deviations of all state variables from their demand values. Finally, we have verified our results with a simple simulation example of a tenth order discrete integrator plant with three input channels and constrained control signal.

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