Input Estimation Over Frequency Region in Presence of Disturbances

Jovan Stefanovski and Đani Juričić

***Abstract*—A new formulation of the unknown input estimation problem in presence of unknown disturbances over a frequency region is presented. Necessary and sufficient conditions for existence of a solution are given and three estimation filters are given. The minimal attenuation is equal to the norm of a rational matrix over the frequency region. As a result, the minimum can be computed by solving linear matrix inequalities. An example is given, and the estimation with the proposed filters is illustrated by a numerical simulation and is compared with some state-of-the-art estimation filters.**

***Index Terms*—Generalized spectral factorization, inertia of Hermitian matrix, input estimation, para-Hermitian rational matrix.**

I. Introduction

The estimation of inputs in dynamical systems is an important task. For instance, if the input is a fault, the fault estimation (FE) is needed in the design of active and passive fault tolerant control (FTC), see [12].

Consider a plant with the following output and inputs: $y(t) \in \mathbb{R}^{n_y}$ is the output, $f(t) \in \mathbb{R}^{n_f}$ the fault input, $d(t) \in \mathbb{R}^{n_d}$ the disturbance input, and $u(t) \in \mathbb{R}^{n_u}$ the control input. Consider also, for some given rational matrices (RMs) $\boldsymbol{G}_u$, $\boldsymbol{G}_f$, and $\boldsymbol{G}_d$, that the following identity between the Laplace transforms of y, u, f, and d holds:

$$\boldsymbol{y} = \boldsymbol{G}_u \boldsymbol{u} + \boldsymbol{G}_f \boldsymbol{f} + \boldsymbol{G}_d \boldsymbol{d}. \tag{1}$$

The input estimation problem is to determine the magnitude of an unknown input [in this note, fault $f(t)$] in presence of other unknown input [in this note, disturbance $d(t)$], using the known $y(t)$ and $u(t)$. There are two general classes of solutions in the literature: online and offline input estimation. The online algorithms, realized as filters, can be applied for FTC, if the estimate can be obtained in a reasonable short time. One of the oldest online input estimation algorithms is the well-known Wiener deconvolution filter, based on minimization of least squares. For $\mathscr{H}_\infty$-type online algorithms, see Problems 1 and 2 and the short review of solutions. A review of offline input estimation algorithms is given in [15]. This class of algorithms could give reliable estimates after a regularization, but cannot be generalized on the case that the spectra of the input signals are bounded.

We use the same notation from "Remarks on the notation" in [17]. Consider that there exist proper stable RMs $\boldsymbol{M}$, $\boldsymbol{N}_u$, $\boldsymbol{N}_f$, and $\boldsymbol{N}_d$, such that $\boldsymbol{M}$ and $[\boldsymbol{N}_u, \boldsymbol{N}_f, \boldsymbol{N}_d]$ are left coprime and

$$[\boldsymbol{G}_u, \boldsymbol{G}_f, \boldsymbol{G}_d] = \boldsymbol{M}^{-1}[\boldsymbol{N}_u, \boldsymbol{N}_f, \boldsymbol{N}_d]. \tag{2}$$

For some unknown $n_f \times n_y$-dimensional FE filter transfer matrix $\boldsymbol{F}(s)$, introduce the following residual vector $r(t) \in \mathbb{R}^{n_f}$ by its Laplace transform:

$$\boldsymbol{r} = \boldsymbol{F}(\boldsymbol{M}\boldsymbol{y} - \boldsymbol{N}_u \boldsymbol{u}). \tag{3}$$

Replacing (1) in (3), we obtain

$$\boldsymbol{r} = \boldsymbol{T}_{rf} \boldsymbol{f} + \boldsymbol{T}_{rd} \boldsymbol{d} \tag{4}$$

where $\boldsymbol{T}_{rf} := \boldsymbol{F}\boldsymbol{N}_f$ and $\boldsymbol{T}_{rd} := \boldsymbol{F}\boldsymbol{N}_d$.

In a perfect situation, a filter $\boldsymbol{F}$ can be constructed such that $\boldsymbol{F}[\boldsymbol{N}_f, \boldsymbol{N}_d] = [I_{n_f}, 0]$ (see [4, Sec. 14.2.1]). In practice, the perfect situation rarely happens; therefore, the role of a filter is to attenuate the effect of the disturbances on the residual in respect to the faults. In formal terms, this can be re-stated as follows:

Problem 1: Find a possibly minimal β with $0 \le \beta < \infty$ and a proper and stable filter transfer matrix $\boldsymbol{F}$ such that $\|\boldsymbol{F}[\boldsymbol{N}_f, \boldsymbol{N}_d] - [I_{n_f}, 0]\|_\infty \le \beta$

Problem 1 minimizes the $\mathscr{H}_\infty$ distance from the perfect FE. It is elaborated in the textbooks (see [1, Sec. 6.5.2] and [2, Sec. 8.3.3]), and it can be solved by the standard $\mathscr{H}_\infty$-filtering theory. As shown in Theorem 17.5 of [20], the filter can be constructed in an observer form.

Another (not equivalent) FE problem is

Problem 2: Find a possibly minimal γ with $0 \le \gamma < \infty$ and a proper and stable filter transfer matrix $\boldsymbol{F}$ such that $\|\boldsymbol{T}_{rd}\|_\infty \le \gamma$, and $\boldsymbol{T}_{rf} = I_{n_f}$.

Problem 2 is elaborated in [10] and [13], by considering the FE filter realization in an observer form, and formulating a matrix inequality that includes the unknown coefficients of the observer. It can also be solved by the algorithm of [18].

Another approach for FE is presented in [5], by considering that f is a state variable of a descriptor system, and by constructing an observer for the state variables of the original system and for f. This approach has been applied in [14] on systems with disturbances d, so that the constructed observer has an $\mathscr{H}_\infty$ bound on the estimation error with respect to the disturbance. In this approach, the estimation performance depends on the structure of the observer. Indeed, it is shown by an example in [14] that the observer of that paper has a superior estimation performance in respect to the so called proportional observer. Also, this approach cannot be applied directly, when the signals f and d have bounded spectra.

Problem formulation, interpretation, and paper's results. We consider that the spectra of the fault and disturbance are located in some frequency regions $\mathbb{B}_f$ and $\mathbb{B}_d$, respectively, which are union of closed intervals in $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, which can include the infinity point.

If we intend to apply a pre- or post-filter to eliminate the part in $\mathbb{B}_d \backslash \mathbb{B}_f$ of the disturbance spectrum, where by $\backslash$ we denote the subtraction of sets, then we define $\mathbb{B} := \mathbb{B}_f \cap \mathbb{B}_d$, otherwise we define $\mathbb{B} := \mathbb{B}_d$. Then we formulate the following problem.

Problem 3:$\mathbb{B}$ Find a possibly minimal γ with $0 < \gamma < \infty$ and a proper and stable filter transfer matrix $\boldsymbol{F}$ such that the poles of $\boldsymbol{T}_{rf}$ (if

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J. Stefanovski is with the Control & Informatics Division, JP "Streževo," Bitola 7000, Republic of Macedonia (e-mail: jovanstef@t.mk).

Đ. Juričić is with the Department of Systems and Control, Jožef Stefan Institute, Jamova cesta 39, Ljubljana SI-1000, Slovenia (e-mail: Dani.Juricic@ijs.si).

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any) are arbitrary assignable, and

$$\boldsymbol{T}_{rd}\boldsymbol{T}_{rd}^{\#} \le \gamma^2 \boldsymbol{T}_{rf}\boldsymbol{T}_{rf}^{\#}, \ \ j\omega \in j\mathbb{B} \tag{5}$$

$$\boldsymbol{T}_{rf}(0) = I_{n_f}, \tag{6}$$

where $\boldsymbol{T}_{rd}^{\#} := \boldsymbol{T}_{rd}(-s)^{\mathrm{T}}$.

We simplify the notation Problem 3:$\overline{\mathbb{R}}$ by writing Problem 3.

A physical interpretation of (5) can be given analogously as in Remark 1 in [17]. The requirement on the assignability of the poles of $\boldsymbol{T}_{rf}$ enables us to choose them in the region $\Re[s] \le -\alpha$, where $\alpha > 0$ is chosen to guaranty a fast decay of the transients. Together with the requirement (6), we have $r(t) \to f(t)$, as $t \to \infty$, very fast, therefore, the residual $r(t)$ can be used as an estimate for the fault $f(t)$.

The filter from [9] that solves the $\mathscr{H}_-/\mathscr{H}_\infty$ problem to find a proper stable RM $\boldsymbol{F}$ such that $\|\boldsymbol{T}_{rd}\|_\infty \le \gamma$ and $\|\boldsymbol{T}_{rf}\|_- \ge 1$, for some minimal γ, could be a solution of Problem 3, if we premultiply it by an orthogonal matrix. Indeed, the RM $\boldsymbol{T}_{rf} = \boldsymbol{F}\boldsymbol{N}_f$, obtained in [9], is inner; therefore, the matrix $\boldsymbol{T}_{rf}(0)$ is orthogonal. The filter $\boldsymbol{T}_{rf}(0)^{-1}\boldsymbol{F}$ is a solution of Problem 3. However, it is not clear how to modify the algorithm of [9] in order to adjust the poles of $\boldsymbol{T}_{rf}$, and how to modify it to work over a frequency region.

The contributions of this note are three solutions to Problems 3 and 3:$\mathbb{B}$, given in Theorems 1, 2, and 3. In particular, in Theorem 1, we present necessary and sufficient conditions for Problem 3 including an FE filter. In Corollary 1, we simplify the FE filter from Theorem 1, so that the FE filter has an observer form. In Theorem 2, we present an FE filter solving Problem 3:$\mathbb{B}$, which can be constructed under a sufficient condition that is close to the necessary one. The order of the filter is slightly greater than the order of the plant. In Theorem 3, we present a necessary and sufficient condition for Problem 3:$\mathbb{B}$, and an FE filter that is independent of γ, such that frequency region $\mathbb{B}$ is maximal with respect to inclusion of subspaces, and $\mathbb{B}$ is specific for given γ. For that reason, we call it universal estimator, in the sense that it works for all feasible frequency regions and γ's. In Corollary 2, we transform the conditions for Problem 3:$\mathbb{B}$ to a solvability of certain LMIs.

The given example illustrates the FE by a comparison of six filters, solving Problems 1, 2, deconvolution problem [15], and the filters from the three theorems of this note.

II. Main Results

Consider a plant given by the following descriptor system:

$$E\dot{x} = Ax + B_u u + B_f f + B_d d, \ \ x \in \mathbb{R}^n$$
$$y = Cx + D_u u + D_f f + D_d d \tag{7}$$

such that matrix pencil $A - sE$ is regular, which satisfies

Assumption 1: The pair $(C, A - sE)$ is finite-mode detectable and impulse observable.

Then in formula (1), the RMs $\boldsymbol{G}_u, \boldsymbol{G}_f$ and $\boldsymbol{G}_d$ are

$$[\boldsymbol{G}_u, \boldsymbol{G}_f, \boldsymbol{G}_d] = \left[\begin{array}{c|ccc} A - sE & B_u & B_f & B_d \\ \hline C & D_u & D_f & D_d \end{array}\right]. \tag{8}$$

Under Assumption 1, there is a matrix $\mathscr{H}$ such that $(A - \mathscr{H}C - sE)^{-1}$ is a proper and stable RM. Then the RMs in (2) can be constructed as

$$[\boldsymbol{M}, \boldsymbol{N}_u, \boldsymbol{N}_f, \boldsymbol{N}_d] = \left[\begin{array}{c|cccc} \mathscr{A} - sE & -\mathscr{H} & \mathscr{B}_u & \mathscr{B}_f & \mathscr{B}_d \\ \hline C & I_{n_y} & D_u & D_f & D_d \end{array}\right] \tag{9}$$

where $\mathscr{A} = A - \mathscr{H}C$, $\mathscr{B}_u = B_u - \mathscr{H}D_u$, $\mathscr{B}_f = B_f - \mathscr{H}D_f$ and $\mathscr{B}_d = B_d - \mathscr{H}D_d$.

Now define a para-Hermitian RM $\boldsymbol{\Pi}_\gamma(s)$, by

$$\boldsymbol{\Pi}_\gamma = \boldsymbol{N}_d\boldsymbol{N}_d^{\#} - \gamma^2\boldsymbol{N}_f\boldsymbol{N}_f^{\#}. \tag{10}$$

Introducing the matrices

$$\Sigma = \begin{bmatrix} \mathscr{B}_d & \mathscr{B}_f \\ D_d & D_f \end{bmatrix} J_\gamma \begin{bmatrix} \mathscr{B}_d & \mathscr{B}_f \\ D_d & D_f \end{bmatrix}^{\mathrm{T}}, \ J_\gamma = \begin{bmatrix} I_{n_d} & 0 \\ 0 & -\gamma^2 I_{n_f} \end{bmatrix}$$

the RM $\boldsymbol{\Pi}_\gamma$ can be rewritten as

$$\boldsymbol{\Pi}_\gamma = [C(sE - \mathscr{A})^{-1}, I]\Sigma \begin{bmatrix} (-sE^{\mathrm{T}} - \mathscr{A}^{\mathrm{T}})^{-1}C^{\mathrm{T}} \\ I \end{bmatrix}. \tag{11}$$

Assumption 2:$\{0\}$ The RM $[\boldsymbol{N}_d, \boldsymbol{N}_f]$ is right-invertible at $s = 0$.

If the RM $[\boldsymbol{N}_d, \boldsymbol{N}_f]$ is right-invertible, we simply write Assumption 2. Assumption 2 is not restrictive, at least in the generic case. Indeed, if the RM $[\boldsymbol{N}_d, \boldsymbol{N}_f]$ is not right-invertible, it is left-invertible in the generic case. Then $\boldsymbol{f} = [0, I_{n_f}][\boldsymbol{N}_d, \boldsymbol{N}_f]^\dagger(\boldsymbol{M}\boldsymbol{y} - \boldsymbol{N}_u\boldsymbol{u})$, where $[\boldsymbol{N}_d, \boldsymbol{N}_f]^\dagger$ is a left-inverse of $[\boldsymbol{N}_d, \boldsymbol{N}_f]$, and therefore, the fault $f(t)$ can be determined without solving Problem 3.

We start by deriving necessary conditions for Problem 3:$\mathbb{B}$. Let $\boldsymbol{F}$ be a solution of Problem 3:$\mathbb{B}$, and denote $\boldsymbol{\Theta} := \boldsymbol{F}\boldsymbol{N}_f$. The matrix $\boldsymbol{N}_f(0)$ has full column rank, as a consequence of the requirement (6). With a possible renumeration of the indices of y, the following partition $[\boldsymbol{N}_d, \boldsymbol{N}_f] = \left[\begin{smallmatrix} \boldsymbol{N}_{d1} & \boldsymbol{N}_{f1} \\ \boldsymbol{N}_{d2} & \boldsymbol{N}_{f2} \end{smallmatrix}\right]$ such that the RM $\boldsymbol{N}_{f1}$ is square and nonsingular, is well defined. Compatibly partition RM $\boldsymbol{F}$ as $\boldsymbol{F} = [\boldsymbol{F}_1, \boldsymbol{F}_2]$.

We have $\boldsymbol{F}_1 = \boldsymbol{\Theta}\boldsymbol{N}_{f1}^{-1} - \boldsymbol{F}_2\boldsymbol{N}_{f2}\boldsymbol{N}_{f1}^{-1}$, and

$$\boldsymbol{F}\boldsymbol{N}_d = \boldsymbol{F}_1\boldsymbol{N}_{d1} + \boldsymbol{F}_2\boldsymbol{N}_{d2} = \boldsymbol{\Theta}\boldsymbol{H}_1 + \boldsymbol{F}_2\boldsymbol{H}_2$$

where $\boldsymbol{H}_1 = \boldsymbol{N}_{f1}^{-1}\boldsymbol{N}_{d1}$ and $\boldsymbol{H}_2 = \boldsymbol{N}_{d2} - \boldsymbol{N}_{f2}\boldsymbol{N}_{f1}^{-1}\boldsymbol{N}_{d1}$.

Next we prove that the RM $\boldsymbol{H}_2$ is right-invertible, under the right-invertibility of $[\boldsymbol{N}_d, \boldsymbol{N}_f]$, which is a consequence of Assumption 2:$\{0\}$. Indeed, let $\boldsymbol{x}\boldsymbol{H}_2 = 0$, for some rational vector $\boldsymbol{x} \neq 0$. Then, $\boldsymbol{x}\boldsymbol{N}_{d2} + \boldsymbol{y}\boldsymbol{N}_{d1} = 0$, where $\boldsymbol{y} = -\boldsymbol{x}\boldsymbol{N}_{f2}\boldsymbol{N}_{f1}^{-1}$. Combining the latter two equations, we obtain $[\boldsymbol{y}, \boldsymbol{x}][\boldsymbol{N}_d, \boldsymbol{N}_f] = 0$, hence $[\boldsymbol{y}, \boldsymbol{x}] = 0$.

The inequality (5) is equivalent with the inequality

$$\boldsymbol{Q} := (\boldsymbol{H}_1 + \boldsymbol{F}_3\boldsymbol{H}_2)(\boldsymbol{H}_1 + \boldsymbol{F}_3\boldsymbol{H}_2)^{\#} \le \gamma^2 I_{n_f} \tag{12}$$

$\widetilde{\text{on}}$ j$\mathbb{B}$,[1] where $\boldsymbol{F}_3 := \boldsymbol{\Theta}^{-1}\boldsymbol{F}_2$. We have

$$\begin{bmatrix} \boldsymbol{N}_{f1}^{-1} & 0 \\ -\boldsymbol{N}_{f2}\boldsymbol{N}_{f1}^{-1} & I_{n_y - n_f} \end{bmatrix} \begin{bmatrix} \boldsymbol{N}_{f1} \\ \boldsymbol{N}_{f2} \end{bmatrix} = \begin{bmatrix} I_{n_f} \\ 0 \end{bmatrix} \tag{13}$$

$$\begin{bmatrix} \boldsymbol{N}_{f1}^{-1} & 0 \\ -\boldsymbol{N}_{f2}\boldsymbol{N}_{f1}^{-1} & I_{n_y - n_f} \end{bmatrix} \begin{bmatrix} \boldsymbol{N}_{d1} \\ \boldsymbol{N}_{d2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{H}_1 \\ \boldsymbol{H}_2 \end{bmatrix}. \tag{14}$$

Using the identities (13) and (14), we obtain

$$\begin{bmatrix} I & \boldsymbol{F}_3 \\ 0 & I \end{bmatrix} \begin{bmatrix} \boldsymbol{N}_{f1}^{-1} & 0 \\ -\boldsymbol{N}_{f2}\boldsymbol{N}_{f1}^{-1} & I \end{bmatrix} \boldsymbol{\Pi}_\gamma \begin{bmatrix} \boldsymbol{N}_{f1}^{-1} & 0 \\ -\boldsymbol{N}_{f2}\boldsymbol{N}_{f1}^{-1} & I \end{bmatrix}^{\#} \begin{bmatrix} I & \boldsymbol{F}_3 \\ 0 & I \end{bmatrix}^{\#}$$

$$= \begin{bmatrix} \boldsymbol{Q} - \gamma^2 I & (\boldsymbol{H}_1 + \boldsymbol{F}_3\boldsymbol{H}_2)\boldsymbol{H}_2^{\#} \\ \boldsymbol{H}_2(\boldsymbol{H}_1 + \boldsymbol{F}_3\boldsymbol{H}_2)^{\#} & \boldsymbol{H}_2\boldsymbol{H}_2^{\#} \end{bmatrix}$$

$$\cong \begin{bmatrix} \boldsymbol{Q} - \gamma^2 I - \boldsymbol{P} & 0 \\ 0 & \boldsymbol{H}_2\boldsymbol{H}_2^{\#} \end{bmatrix} \tag{15}$$

$\widetilde{\text{on}}$ j$\mathbb{B}$, where by $\cong$ we denote the congruence of Hermitian matrices, and where

$$\boldsymbol{P} := (\boldsymbol{H}_1 + \boldsymbol{F}_3\boldsymbol{H}_2)\boldsymbol{H}_2^{\#}(\boldsymbol{H}_2\boldsymbol{H}_2^{\#})^{-1}\boldsymbol{H}_2(\boldsymbol{H}_1 + \boldsymbol{F}_3\boldsymbol{H}_2)^{\#}.$$

The congruence in (15) has been obtained using the Schur complement. Having in mind the inequalities (12) and $\boldsymbol{P} \ge 0$ $\widetilde{\text{on}}$ j$\mathbb{B}$, we obtain that

[1] We say that a RM has a property $\widetilde{\text{on}}$ j$\mathbb{B}$ if it has the same property everywhere on j$\mathbb{B}$, except a finite number of points.

$\boldsymbol{Q} - \gamma^2 I - \boldsymbol{P} \leq 0$, and from (15), that $\mathrm{In}(\boldsymbol{\Pi}_\gamma) = (n_f,\, 0,\, n_y - n_f)$ $\widetilde{\text{on}}$ j$\mathbb{B}$.[2]

We have

$$\boldsymbol{Q} - \boldsymbol{P} = \boldsymbol{H}_1[I - \boldsymbol{H}_2^{\#}(\boldsymbol{H}_2\boldsymbol{H}_2^{\#})^{-1}\boldsymbol{H}_2]\boldsymbol{H}_1^{\#} =: \boldsymbol{Z} \tag{16}$$

because $\boldsymbol{H}_2[I - \boldsymbol{H}_2^{\#}(\boldsymbol{H}_2\boldsymbol{H}_2^{\#})^{-1}\boldsymbol{H}_2] = 0$. In (16), we have introduced the new symbol $\boldsymbol{Z}$, instead of $\boldsymbol{Q} - \boldsymbol{P}$, because the RMs $\boldsymbol{Q}$ and $\boldsymbol{P}$ depend on the unknown filter $\boldsymbol{F}$ (although their difference does not depend on $\boldsymbol{F}$).

Since $I - \boldsymbol{H}_2^{\#}(\boldsymbol{H}_2\boldsymbol{H}_2^{\#})^{-1}\boldsymbol{H}_2 \geq 0$, $\widetilde{\text{on}}$ the EIA, there is a full column normal rank RM $\boldsymbol{\Xi}$ such that

$$\boldsymbol{Z} = \boldsymbol{\Xi} \cdot \boldsymbol{\Xi}^{\#}\,. \tag{17}$$

We see by inspecting the relations (15) that the inequality

$$\boldsymbol{\Xi} \cdot \boldsymbol{\Xi}^{\#} \leq \gamma^2 I \;\; \widetilde{\text{on}} \;\; \mathrm{j}\mathbb{B} \tag{18}$$

is a necessary and sufficient condition for $\mathrm{In}(\boldsymbol{\Pi}_\gamma) = (n_f,\, 0,\, n_y - n_f)$ $\widetilde{\text{on}}$ j$\mathbb{B}$.

Using Lemma 1 in [16], a $2n$th order Hamiltonian realization of the RM $\boldsymbol{Z}(s^{-1})$, which is suitable for the factorization (17), can be derived. Then, $\boldsymbol{\Xi}$ has a realization order n.

Assumption 3: The RM $\boldsymbol{\Pi}_\gamma$ is nonsingular.

Theorem 1: Under Assumptions 1, 2:$\{0\}$ and 3, Problem 3 admits a solution if and only if $\boldsymbol{N}_f(0)$ is a full column rank matrix and the RM $\boldsymbol{\Pi}_\gamma$ has a constant inertia $\widetilde{\text{on}}$ the EIA, equal to $(n_f,\, 0,\, n_y - n_f)$.

Proof: The necessity is proved. To prove the sufficiency, consider that $\mathrm{In}(\boldsymbol{\Pi}_\gamma) = (n_f,\, 0,\, n_y - n_f)$ $\widetilde{\text{on}}$ the EIA. By Theorem 1.1 of [3], there is the following factorization of $\boldsymbol{\Pi}_\gamma$:

$$\boldsymbol{\Pi}_\gamma = \boldsymbol{\Psi} J \boldsymbol{\Psi}^{\#}\,, \quad J = \mathrm{diag}\{I_{n_y - n_f}, -I_{n_f}\} \tag{19}$$

where the RM $\boldsymbol{\Psi}$ is a nonsingular factor, and both $\boldsymbol{\Psi}$ and $\boldsymbol{\Psi}^{-1}$ are possibly improper and possibly unstable, respectively.

Define the RM $\widetilde{\boldsymbol{F}} = [0, I_{n_f}]\boldsymbol{\Psi}^{-1}$, and let $\widetilde{\boldsymbol{F}}$ be given by the descriptor realization $\widetilde{\boldsymbol{F}} = \widetilde{D} + \widetilde{C}(s\widetilde{E} - \widetilde{A})^{-1}\widetilde{B}$ that is finite mode observable, impulse observable, and finite mode controllable at $s = 0$ (the latter property means that $[\widetilde{A},\, \widetilde{B}]$ is a full row rank matrix).

Using the realizations (9) of $\boldsymbol{N}_f$ and $\boldsymbol{N}_d$, we obtain

$$\widetilde{\boldsymbol{F}}\boldsymbol{N}_f = \left[\begin{array}{cc|c} \widetilde{A} - s\widetilde{E} & \widetilde{B}C & \widetilde{B}D_f \\ 0 & \mathscr{A} - sE & \mathscr{B}_f \\ \hline \widetilde{C} & \widetilde{D}C & \widetilde{D}D_f \end{array}\right] =: \left[\begin{array}{c|c} \overline{A} - s\overline{E} & \overline{B}_f \\ \hline \overline{C} & \overline{D}_f \end{array}\right].$$

Using the fact that $\mathscr{A} - sE$ is stable and impulse-free matrix pencil, and that the pair $(\widetilde{C}, \widetilde{A} - s\widetilde{E})$ is finite mode observable and impulse observable, it can be proved that the pair $(\overline{C}, \overline{A} - s\overline{E})$ is finite mode observable and impulse observable. (The proof of impulse observability uses Theorems 1 and 2 of [7].)

By those properties of $(\overline{C}, \overline{A} - s\overline{E})$, there exists a matrix $\overline{K}$ such that matrix pencil $\overline{A} - \overline{K}\,\overline{C} - s\overline{E}$ is regular, stable, impulse-free, and its finite zeros are arbitrary assignable.

Define the RM $\boldsymbol{R} := I + \overline{C}(s\overline{E} - \overline{A})^{-1}\overline{K}$. Its finite zeros are the finite zeros of matrix pencil $\overline{A} - \overline{K}\,\overline{C} - s\overline{E}$, and $\boldsymbol{R}^{-1}$ is proper. Define the RM $\widehat{\boldsymbol{F}} := \boldsymbol{R}^{-1}\widetilde{\boldsymbol{F}}$.

By (19), we have the following identity:

$$\widehat{\boldsymbol{F}}\boldsymbol{N}_d\boldsymbol{N}_d^{\#}\widehat{\boldsymbol{F}}^{\#} - \gamma^2\widehat{\boldsymbol{F}}\boldsymbol{N}_f\boldsymbol{N}_f^{\#}\widehat{\boldsymbol{F}}^{\#} = -\boldsymbol{R}^{-1}\boldsymbol{R}^{-\#}\,. \tag{20}$$

[2]By $\mathrm{In}(H) = (m_-, m_0, m_+)$, we denote the inertia of Hermitian matrix H, where m_-, m_0, and m_+ are the numbers of its negative, zero and positive eigenvalues.

Using the following (nonminimal) realization of $\widetilde{\boldsymbol{F}}$

$$\widetilde{\boldsymbol{F}} = \widetilde{D} + \overline{C}(s\overline{E} - \overline{A})^{-1}\widetilde{B}_1\,, \quad \widetilde{B}_1 = \begin{bmatrix} \widetilde{B} \\ 0 \end{bmatrix}$$

and the algebra of transfer matrices, we obtain

$$\widehat{\boldsymbol{F}} = \boldsymbol{R}^{-1}\widetilde{\boldsymbol{F}} = [0, I_{n_f}]\begin{bmatrix} \overline{A} - s\overline{E} & \overline{K} \\ \overline{C} & I_{n_f} \end{bmatrix}^{-1}\begin{bmatrix} \widetilde{B}_1 \\ \widetilde{D} \end{bmatrix} \tag{1}$$

$$= \left[\begin{array}{cc|c} \widetilde{A} - K_1\widetilde{C} - s\widetilde{E} & (\widetilde{B} - K_1\widetilde{D})C & \widetilde{B} - K_1\widetilde{D} \\ -K_2\widetilde{C} & \mathscr{A} - K_2\widetilde{D}C - sE & -K_2\widetilde{D} \\ \hline \widetilde{C} & \widetilde{D}C & \widetilde{D} \end{array}\right] \tag{21}$$

$$\left[\widehat{\boldsymbol{F}}\boldsymbol{N}_d,\, \widehat{\boldsymbol{F}}\boldsymbol{N}_f,\, \boldsymbol{R}^{-1}\right] =$$

$$[0, I_{n_f}]\begin{bmatrix} \overline{A} - s\overline{E} & \overline{K} \\ \overline{C} & I \end{bmatrix}^{-1}\begin{bmatrix} \overline{B}_f & \overline{B}_d & 0 \\ \overline{D}_f & \overline{D}_d & I_{n_f} \end{bmatrix} \tag{22}$$

where $\overline{B}_d := \left[\begin{smallmatrix} \widetilde{B}D_d \\ \mathscr{B}_d \end{smallmatrix}\right]$, $\overline{D}_d := \widetilde{D}D_d$ and $\overline{K} =: \left[\begin{smallmatrix} K_1 \\ K_2 \end{smallmatrix}\right]$.

In order to construct a solution $\boldsymbol{F}$ to Problem 3, we put

$$\boldsymbol{F} = Y\widehat{\boldsymbol{F}} \tag{23}$$

for some nonsingular matrix Y, chosen to satisfy the requirement (6) of Problem 3. For that purpose, we shall prove that $\widehat{\boldsymbol{F}}(0)\boldsymbol{N}_f(0)$ is a nonsingular matrix. Then, we can take $Y = (\widehat{\boldsymbol{F}}(0)\boldsymbol{N}_f(0))^{-1}$.

Let $x\widehat{\boldsymbol{F}}(0)\boldsymbol{N}_f(0) = 0$, for some vector-row x. By (20), we obtain $x\widehat{\boldsymbol{F}}(0)\boldsymbol{N}_d(0) = 0$ and $x\boldsymbol{R}^{-1}(0) = 0$. By (22) we obtain

$$[z,\, y]\begin{bmatrix} \overline{B}_f & \overline{B}_d & 0 \\ \overline{D}_f & \overline{D}_d & I_{n_f} \end{bmatrix} = 0 \;\; \text{where}$$

$$[z,\, y] = x[0, I_{n_f}]\begin{bmatrix} \overline{A} & \overline{K} \\ \overline{C} & I \end{bmatrix}^{-1}.$$

From these equations we obtain $y = 0$, $z[\overline{A},\, \overline{B}_f,\, \overline{B}_d] = 0$. Putting $z = [z_1, z_2]$, the latter identity becomes

$$[z_1, z_2]\begin{bmatrix} \widetilde{A} & \widetilde{B}C & \widetilde{B}D_f & \widetilde{B}D_d \\ 0 & \mathscr{A} & \mathscr{B}_f & \mathscr{B}_d \end{bmatrix} = 0\,, \text{ i.e.,}$$

$$z_1\widetilde{A} = 0\,, \;\; z_1\widetilde{B}[C, D_f, D_d] + z_2[\mathscr{A}, \mathscr{B}_f, \mathscr{B}_d] = 0\,. \tag{24}$$

Right-multiplying the latter identity by the matrix $\left[\begin{smallmatrix} -\mathscr{A}^{-1}[\mathscr{B}_f,\, \mathscr{B}_d] \\ I \end{smallmatrix}\right]$, we obtain $z_1\widetilde{B}[\boldsymbol{N}_f(0),\, \boldsymbol{N}_d(0)] = 0$, and $z_1 = 0$, as a consequence of the full row ranks of the matrices $[\boldsymbol{N}_f(0),\, \boldsymbol{N}_d(0)]$ and $[\widetilde{A},\, \widetilde{B}]$. From $z_2 = -z_1\widetilde{B}C\mathscr{A}^{-1}$, which is included in the second equation in (24), it follows $z_2 = 0$. ■

The filter realization given by (21) and (23) in the sufficiency part of the proof of Theorem 1 is of a theoretical importance, as it is constructed under a minimal set of assumptions. However, the realization is of order $2n$. In Corollary 1, we reduce this realization to an nth order observer realization, which exists under a slightly stronger set of assumptions. Namely, besides Assumptions 1 and 2:$\{0\}$, we introduce the additional Assumptions 4 and 5 (which imply Assumption 3).

Introduce the matrices $B := [B_d,\, B_f]$ and $D := [D_d,\, D_f]$.

Assumption 4: (I) The matrix $D_dD_d^T - \gamma^2D_fD_f^T = DJ_\gamma D^{\mathrm{T}}$ has inertia $(n_f, 0, n_y - n_f)$.

(II) There is a real matrix X that solves the following generalized algebraic Riccati equation (GARE):

$$EX^T = XE^T\,, \quad AX^T + XA^T + BJ_\gamma B^T$$
$$-\,(XC^T + BJ_\gamma D^T)(DJ_\gamma D^T)^{-1}(CX^T + DJ_\gamma B^T) = 0. \tag{25}$$

Introduce matrix $L = (XC^T + BJ_\gamma D^T)(DJ_\gamma D^T)^{-1}$. Sufficient conditions for existence of a stabilizing[3] solution X are given in [11], Theorem 1. Note that we do not need a stabilizing solution X.

Define the matrix $\mathcal{L} := L - \mathscr{H}$, define the nonsingular matrix $\mathcal{M}$ such that $DJ_\gamma D^T = \mathcal{M}J\mathcal{M}^T$, where $J = \mathrm{diag}\{I_{n_y - n_f}, -I_{n_f}\}$, and define the matrix $\mathcal{N} := [0, I_{n_f}]\mathcal{M}^{-1}$. The matrix $\mathcal{N}D_f$ is nonsingular.

Denote $A_L := \mathscr{A} - \mathcal{L}C = A - LC$, $B_{fL} := \mathscr{B}_f - \mathcal{L}D_f = B_f - LD_f$ and $B_{dL} := \mathscr{B}_d - \mathcal{L}D_d = B_d - LD_d$. With the found matrix X, we can obtain less-order realizations of the RMs, which are defined in the proof of Theorem 1. Namely, the RM $\boldsymbol{\Psi}$ satisfying (19) is given by $\boldsymbol{\Psi} = [I_{n_y} + C(sE - \mathscr{A})^{-1}\mathcal{L}]\mathcal{M}$, and RM $\widetilde{\boldsymbol{F}}$ satisfying $\widetilde{\boldsymbol{F}} = [0, I_{n_f}]\boldsymbol{\Psi}^{-1}$ is given by $\widetilde{\boldsymbol{F}} = \mathcal{N} - \mathcal{N}C(sE - A_L)^{-1}\mathcal{L}$.

Using the algebra of transfer matrices, we obtain that RMs $\widehat{\boldsymbol{F}}$ in (21) and $\widehat{\boldsymbol{F}}\boldsymbol{N}_f$ in (22) reduce to

$$\widehat{\boldsymbol{F}} = \left[\begin{array}{c|c} A_L + K_1\mathcal{N}C - sE & \mathcal{L} - K_1\mathcal{N} \\ \hline -\mathcal{N}C & \mathcal{N} \end{array}\right] \quad (26)$$

$$\widehat{\boldsymbol{F}}\boldsymbol{N}_f = \left[\begin{array}{c|c} A_L + K_1\mathcal{N}C - sE & B_{fL} + K_1\mathcal{N}D_f \\ \hline \mathcal{N}C & \mathcal{N}D_f \end{array}\right]. \quad (27)$$

Assumption 5: The pair $(\mathcal{N}C,\ A_L - sE)$ is finite mode observable and impulse observable.

We see from (27) that, under Assumption 5, the matrix K_1 can be used to assign the finite poles of $\widehat{\boldsymbol{F}}\boldsymbol{N}_f$, as well as to render the RM $\widehat{\boldsymbol{F}}\boldsymbol{N}_f$ proper.

If $\widehat{\boldsymbol{F}}(0)\boldsymbol{N}_f(0)$ is a nonsingular matrix for all choices of matrices K_1, we can define matrix $Y = (\widehat{\boldsymbol{F}}(0)\boldsymbol{N}_f(0))^{-1}$. The filter $\boldsymbol{F} = Y\widehat{\boldsymbol{F}}$ solves Problem 3.

It remains to prove that the matrix $\widehat{\boldsymbol{F}}(0)\boldsymbol{N}_f(0)$ is nonsingular.[4] Let $x\widehat{\boldsymbol{F}}(0)\boldsymbol{N}_f(0) = 0$, for some vector-row x. Using (27), we obtain the identity $[y, z]\left[\begin{smallmatrix} A_L & B_{fL} \\ \mathcal{N}C & \mathcal{N}D_f \end{smallmatrix}\right] = 0$, where $y = -x\mathcal{N}C(A_L + K_1\mathcal{N}C)^{-1}$ and $z = x + yK_1$. The proof of nonsingularity of $\widehat{\boldsymbol{F}}(0)\boldsymbol{N}_f(0)$ is completed by the fact that matrix $\left[\begin{smallmatrix} A_L & B_{fL} \\ -\mathcal{N}C & \mathcal{N}D_f \end{smallmatrix}\right]$ is nonsingular (proof omitted).

It is easy to check that the estimation algorithm (3) takes the following descriptor observer form:

$$\begin{aligned} E\dot{\widehat{x}} &= A\widehat{x} + B_u u + H(y - C\widehat{x} - D_u u) \\ r &= Y\mathcal{N}(y - C\widehat{x} - D_u u) \end{aligned} \quad (28)$$

where $H := L - K_1\mathcal{N}$. Note that observer (28) does not depend on matrix $\mathscr{H}$. The following corollary is proved.

Corollary 1: Under Assumptions 1, 2:$\{0\}$, 4 and 5, Problem 3 is solvable. A solution is given by the observer (28).

Next Theorem 2 is a sufficiency result on Problem 3:$\mathbb{B}$.

Theorem 2: In addition to Assumptions 1, 2:$\{0\}$, and the full column rank of the matrix $\boldsymbol{N}_f(0)$, assume that following is a generalized factorization of $\boldsymbol{\Pi}_\gamma$:

$$\boldsymbol{\Pi}_\gamma = \boldsymbol{\Psi} \cdot \mathrm{diag}\{\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2\} \cdot \boldsymbol{\Psi}^\# \quad (29)$$

where RM $\boldsymbol{\Psi}$ is a nonsingular factor, possibly improper, and with unspecified location of poles and zeros, $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$ are nonsingular para-Hermitian $(n_y - n_f)$- and n_f-dimensional polynomial matrices, respectively, such that $\boldsymbol{\Phi}_1$ is positive semi-definite on j$\mathbb{B}$, $\boldsymbol{\Phi}_2$ is negative semi-definite on j$\mathbb{B}$, and $\det(\boldsymbol{\Phi}_2(0)) \neq 0$. Then, the filter given by (21) and (23) solves Problem 3:$\mathbb{B}$.

[3]A stabilizing solution of the GARE (25) is a solution X such that the matrix pencil $A - LC - sE$ is regular, stable and impulse-free.

[4]The arguments from the proof of Theorem 1 cannot be applied, because $[A_L, \mathcal{L}]$ is not necessarily a full row rank matrix.

Proof: The proof is analogous to the proof of the sufficiency part of Theorem 1, except the identity (20), which is now: $\widehat{\boldsymbol{F}}\boldsymbol{N}_d\boldsymbol{N}_d^\#\widehat{\boldsymbol{F}}^\# - \gamma^2\widehat{\boldsymbol{F}}\boldsymbol{N}_f\boldsymbol{N}_f^\#\widehat{\boldsymbol{F}}^\# = \boldsymbol{R}^{-1}\boldsymbol{\Phi}_2\boldsymbol{\Phi}_2^\#\boldsymbol{R}^{-\#}$. ■

To formulate Theorem 3, we define at first the RMs $\boldsymbol{F}_4 := \boldsymbol{H}_1\boldsymbol{H}_2^\#(\boldsymbol{H}_2\boldsymbol{H}_2^\#)^{-1}$ and

$$\boldsymbol{F}_\ell := [I_{n_f}, -\boldsymbol{F}_4]\begin{bmatrix} \boldsymbol{N}_{f1} & 0 \\ \boldsymbol{N}_{f2} & I \end{bmatrix}^{-1}. \quad (30)$$

Using the identities (13) and (14), we obtain that

$$\begin{aligned} &\boldsymbol{F}_\ell\boldsymbol{N}_f = I_{n_f}, \quad \boldsymbol{F}_\ell\boldsymbol{N}_d\boldsymbol{N}_d^\#\boldsymbol{F}_\ell^\# = \boldsymbol{\Xi}\cdot\boldsymbol{\Xi}^\#, \\ &\boldsymbol{F}_\ell\boldsymbol{N}_d\boldsymbol{N}_d^\#\boldsymbol{F}_\ell^\# - \gamma^2\boldsymbol{F}_\ell\boldsymbol{N}_f\boldsymbol{N}_f^\#\boldsymbol{F}_\ell^\# = \boldsymbol{\Xi}\cdot\boldsymbol{\Xi}^\# - \gamma^2 I. \end{aligned} \quad (31)$$

By (31), if $\|\boldsymbol{\Xi}\| \leq \gamma$ on j$\mathbb{B}$, the RM $\boldsymbol{F}_\ell$ can be a solution of Problem 3:$\mathbb{B}$, if we stabilize it.

Let $(A_\ell, B_\ell, C_\ell, D_\ell, E_\ell)$ be at least a finite mode observable and impulse observable descriptor realization of $\boldsymbol{F}_\ell$. The right- and left-multiplication of the inequality $\boldsymbol{\Xi}\cdot\boldsymbol{\Xi}^\# - \gamma^2 I \leq 0$ on j$\mathbb{B}$ by RMs $\boldsymbol{M}_\ell$ and $\boldsymbol{M}_\ell^\#$, correspondingly, does not change its validity. We choose the RM $\boldsymbol{M}_\ell$ such that

$$[\boldsymbol{M}_\ell, \boldsymbol{M}_\ell\boldsymbol{F}_\ell] = \left[\begin{array}{c|cc} A_\ell - H_\ell C_\ell - sE_\ell & -H_\ell & B_\ell - H_\ell D_\ell \\ \hline C_\ell & I_{n_f} & D_\ell \end{array}\right]$$

where H_ℓ is a matrix such that matrix pencil $A_\ell - H_\ell C_\ell - sE_\ell$ is stable and impulse-free, and its zeros are arbitrary.

Then, we choose the FE filter

$$\boldsymbol{F} = \boldsymbol{M}_\ell(0)^{-1}\boldsymbol{M}_\ell\boldsymbol{F}_\ell. \quad (32)$$

Theorem 3: Under Assumptions 1 and 2, Problem 3:$\mathbb{B}$ admits a solution if and only if $\boldsymbol{N}_f(0)$ is a full column rank matrix and $\|\boldsymbol{\Xi}\| \leq \gamma$ on j$\mathbb{B}$. The minimal γ is $\gamma_\circ = \|\boldsymbol{\Xi}\|_\infty^{\mathbb{B}}$, and is independent of matrix $\mathscr{H}$. The filter (32) is a solution.

Proof: We prove only the independence of $\gamma_\circ$. Indeed, $\gamma_\circ$ is the minimal number such that the inertia of RM $\boldsymbol{\Pi}_\gamma$ $\widetilde{\text{on}}$ j$\mathbb{B}$ is $(n_f,\ 0,\ n_y - n_f)$. Since $\boldsymbol{\Pi}_\gamma = \boldsymbol{M}(\boldsymbol{G}_d\boldsymbol{G}_d^\# - \gamma^2\boldsymbol{G}_f\boldsymbol{G}_f^\#)\boldsymbol{M}^\#$, the independence of $\gamma_\circ$ on $\mathscr{H}$ follows by the invariance of the inertia of Hermitian matrices under the congruence. ■

If we replace Assumption 2 by Assumption 2:$\{0\}$ in Theorem 3, we can find a state-space realization of $\boldsymbol{F}_\ell(s^{-1})$ of order $2n$, similarly as in Appendix B of [16]. Under this new set of assumptions, we transform the condition $\|\boldsymbol{\Xi}\| \leq \gamma$ on j$\mathbb{B}$ of Theorem 3 to a solvability of LMIs. Denote by $(\mathcal{A},\ \mathcal{B},\ \mathcal{C},\ \mathcal{D})$ at least an observable realization of $\boldsymbol{\Xi}(s^{-1})$. Denote by $\widehat{\mathbb{B}}$ the region of the real axis obtained by mapping with the function $s \rightarrow 1/s$ of the region $\mathbb{B}$. Let Φ and Ψ be 2×2-dimensional Hermitian matrices depending on $\widehat{\mathbb{B}}$, constructed as shown in [8].

Corollary 2: Under Assumptions 1 and 2:$\{0\}$, Problem 3:$\mathbb{B}$ admits a solution if and only if $\boldsymbol{N}_f(0)$ is a full column rank matrix and there are Hermitian matrices $\mathcal{P}$ and $\mathcal{Q}$, such that:

$$\begin{aligned} &\begin{bmatrix} \mathcal{A} & I \\ \mathcal{C} & 0 \end{bmatrix}(\Phi\otimes\mathcal{P} + \Psi\otimes\mathcal{Q})\begin{bmatrix} \mathcal{A}^{\mathrm{T}} & \mathcal{C}^{\mathrm{T}} \\ I & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathcal{B}\mathcal{B}^{\mathrm{T}} & \mathcal{B}\mathcal{D}^{\mathrm{T}} \\ \mathcal{D}\mathcal{B}^{\mathrm{T}} & \mathcal{D}\mathcal{D}^{\mathrm{T}} - \gamma^2 I \end{bmatrix} \leq 0, \quad \mathcal{Q} \geq 0. \end{aligned} \quad (33)$$

Proof: The proof is a consequence of the application of the generalized KYP lemma [8] on the inequality (18). ■

Example 1: Consider the plant (7) given by $E = I_4$,

$$A = \begin{bmatrix} -1.5 & -0.7 & 0.2 & 0.3 \\ -0.2 & -1.9 & -0.1 & 0.1 \\ -0.1 & 0.2 & -1.8 & -0.2 \\ -0.1 & 0.1 & -0.2 & -2 \end{bmatrix}, B_f = \begin{bmatrix} 1 & -1.2 \\ -4 & -1.2 \\ -2.6 & 0 \\ 0.4 & -1.2 \end{bmatrix}$$

$$B_d = \begin{bmatrix} -0.4 & -1.1 \\ -1.1 & -1.3 \\ -1.8 & -1.00 \\ -0.1 & -0.5 \end{bmatrix},$$

$$C = \begin{bmatrix} -5 & 2 & 1 & 2.5 \\ 3 & 5 & 2.5 & 1.5 \\ 1 & 15 & 2.5 & 3 \end{bmatrix}$$

$$D_f = \begin{bmatrix} 2 & 0.7 \\ -2.4 & 1 \\ -2.6 & 2.2 \end{bmatrix}, D_d = \begin{bmatrix} 1.4 & 0.5 \\ 1.3 & 1.3 \\ -0.5 & 1.1 \end{bmatrix}.$$

We elaborate on the first filter of the bank (for the fault f_1), by introducing the disturbance d_3, which is equal to the fault f_2. A justification is the fact that the minimal γ for Problem 3 with the original matrices is 2.42978924, while for the first filter in the bank, it is only 0.21839718.

We shall obtain six FE filters for f_1: filters solving Problems 1, 2, deconvolution problem and filters from Theorems 1–3, and compare the FE by numerical simulation. We consider the initial value of the plant $x(0) = [1, 1, 1, 1]^{\mathrm{T}}$, zero initial values for the filters, and the following test signal. The disturbance d_2 is such that $d_2(t) = 0, t \leq 3$ and $d_2(t) = 0.3 \sin(\omega_0(t-3)), t > 3$, where $\omega_0 = 16\pi$. The disturbance $d_1(t)$ is zero. The fault signal $f_1(t)$ is a step with magnitude 1 appearing at $t = 5$. The fault signal $f_2(t)$ ($= d_3(t)$) satisfies $f_2(t) = 0$, $t \leq 3$, and $f_2(t) = \sin(\omega_0(t-3)), t > 3$.

Using the command hinfsyn of MATLAB, we find an optimal filter solving Problem 1, of realization order 4.

Then, we obtain an optimal filter solving Problem 2. We apply its transformation into the problem of finding proper stable $\boldsymbol{X}$ such that $\|\boldsymbol{X}\|_\infty \leq \gamma$ and $\boldsymbol{X}\boldsymbol{V}_b = \boldsymbol{U}_b$, for some RMs $\boldsymbol{V}_b$ and $\boldsymbol{U}_b$. The latter problem is elaborated in [18]. The minimal γ for Problem 2 is 0.3763776. By Corollary 4.9 of [18], the filter is unique. It is given by an order 3 realization.

Next, we obtain optimal filters of Theorems 1 and 3.

By Theorem 1, the minimal γ for Problem 3 is $\|\Xi\|_\infty = 0.21839718$, which is less than 0.3763776; therefore, we expect that the FE will be better than the FE obtained with the filter solving Problem 2.

By Theorem 3 with $\gamma = 0.15$, frequency region $\mathbb{B} = [0, 5.2798] \cup [36.78257, \infty]$ is feasible for Problem 3:$\mathbb{B}$. If we need to widen the frequency region, we have to increase γ.

The algorithms of Theorems 1, 3 (and 2) require pole assignment of $\boldsymbol{T}_{rf}$ by static state feedback. It is well known that this task is not reliable, but there is no need to assign the poles; it suffices for the poles to be in $\Re[s] \leq -\alpha$, for some predefined $\alpha > 0$. The latter problem we solve by finding a solution of an algebraic Riccati equation with shifted matrix "A," by αI, for some $\alpha > 0$.

Taking $\alpha = 0$ and using Theorem 1, we find a filter realization of order 4. Taking $\alpha = 0.1$ and using Theorem 3, we find a universal filter realization of order 8.

In order to apply the deconvolution algorithm from Section 8.3 in [15], the matrix B_d has to be zero. Another drawback of that algorithm is that the dimension of the matrix that has to be inverted is proportional with the number of samples (see [15, identity (8.8)]). For that reason we apply the standard discrete $\mathscr{H}_2$ minimization, after we discretize the model, with sampling period 0.01. We take the controlled output $z_k = u_k - f_k + \gamma(f_k - f_{k-1})$, $k = 0, 1, \ldots$, where "the control" u_k is the estimate of f_k. Then, we construct a deconvolution filter of realization order 5. For the particular test signal, the best estimate is obtained for $\gamma = 1.12$.

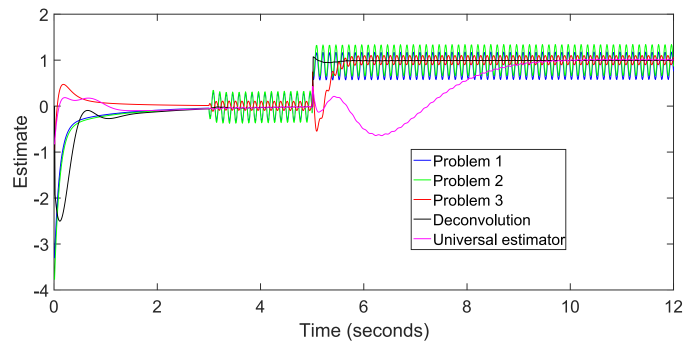

Fig. 1. Estimate of Problems 1, 2, 3, deconvolution, and universal estimator.

The response of the residual of the filters solving Problems 1, 2, 3, deconvolution, and universal estimator to the test signal is given in Fig. 1 (in blue, green, red, black and magenta, respectively). It is seen that the disturbance pattern with a big amplitude appears in the residual for the first two filters. In the stationary regime, we see that there is an offset of $r(t)$ in respect to $f_1(t)$ for the filter solving Problem 1. We see that the amplitude of the disturbance pattern for the filter solving Problem 3 is lesser than that of the filters solving Problems 1 and 2. It is seen that the disturbance attenuation with the universal estimator is satisfactory, but the time for FE is longer than the FE times of the previous filters. Note that if we increase α, the FE time will decrease, but the ripples in $r(t)$ will increase.

Finally, we obtain a filter of Theorem 2. For that purpose, at first we find a factorization (29) of $\boldsymbol{\Pi}_\gamma$ with $\gamma = 0.15$. The RM $\boldsymbol{\Pi}_\gamma$ has two zeros on the positive EIA: j5.2798 and j36.78257. Following the generalized spectral factorization theory of [19], we obtain $\omega_1 = -5.2798$ and $\omega_2 = 36.78257$. Using the inequalities $\omega_1 < 0$ and $\omega_2 > 0$, we obtain $\mathbb{B} = [36.78257, \infty]$, and $\boldsymbol{\Phi}_1(\mathrm{j}\omega) = \mathrm{diag}\{(\omega_1^2 - \omega^2)/\omega_1, \omega_2\} \geq 0$, $\boldsymbol{\Phi}_2(\mathrm{j}\omega) = (\omega_2^2 - \omega^2)/\omega_2 \leq 0, \omega \in \mathbb{B}$. We compute a balanced realization of the filter $\boldsymbol{F}$ solving Problem 3:$\mathbb{B}$, with the following matrices "A," "B," "C," and "D :"

$$\begin{bmatrix} -1.368 & 5.011 & -0.6924 & -6.502 & -0.1411 \\ -4.77 & -1.812 & 1.809 & 9.828 & 0.2184 \\ -0.1031 & -1.203 & -0.3006 & -5.755 & -0.1166 \\ 2.979 & 1.954 & 5.658 & -313.4 & -14.81 \\ -0.1039 & -0.1686 & -0.08106 & 12.15 & -1.772 \end{bmatrix}$$

$$\begin{bmatrix} -1.038 & -0.5271 & 0.1206 \\ -0.9388 & 0.054 & 0.3568 \\ -0.00319 & -0.2651 & 0.1612 \\ 0.1172 & 2.024 & -1.859 \\ -0.03401 & -0.02163 & 0.04502 \end{bmatrix}$$

$$\begin{bmatrix} 1.17 & -1.006 & 0.3103 & 2.75 & 0.06043 \end{bmatrix}$$

$$\begin{bmatrix} 0.2738 & -0.2596 & 0.06564 \end{bmatrix}. \quad (34)$$

Since the spectra of d_2 and d_3 are located at the frequency $16\pi \in \mathbb{B} = [36.78257, \infty]$, the attenuation of the disturbance will be at least $\gamma = 0.15$. Indeed, the response of the residual with this filter to the test signal is given in Fig. 2 . The FE is reliable, because the amplitude of the disturbance pattern in the residual is very small. Unlike the universal

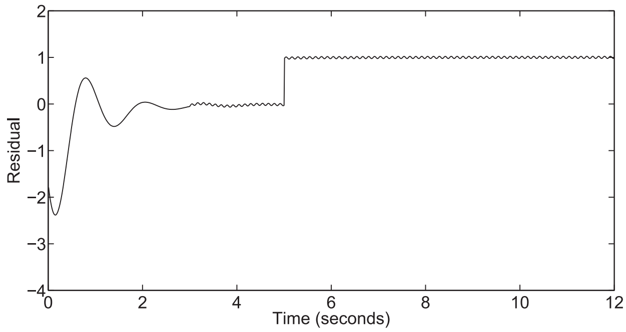

Fig. 2. Residual $r(t)$ for the filter (34).

estimator, for which $\mathbb{B} = [0, 5.2798] \cup [36.78257, \infty]$, the filter (34) is "dedicated" to the frequency region $\mathbb{B} = [36.78257, \infty]$.

Note that we can construct analogously a filter over frequency region $\mathbb{B} = [0, 5.2798]$ with the same $\gamma = 0.15$, with $\mathbf{\Phi}_1 = \mathrm{diag}\{(\omega_2^2 - \omega^2)/\omega_2, \omega_2\}$ and $\mathbf{\Phi}_2 = (\omega_1^2 - \omega^2)/\omega_1$.

III. Conclusion

We have motivated and formulated a new FE problem over a frequency region. Three solutions/filters are given, with a greater attenuation of disturbances, in respect to the existing filters. The numerical design of the filters relies on standard and well-accepted computer routines.

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