On Lyapunov and Upper Bohl Exponents of Diagonal Discrete Linear Time-Varying Systems

Adam Czornik, Alexander Konyukh, Iryna Konyukh, Michał Niezabitowski, and Justyna Orwat

***Abstract*—In this paper, we present necessary and sufficient conditions for two given functions to be the Lyapunov and the upper Bohl exponent of a certain discrete linear system with diagonal coefficients. The obtained conditions have a form of easily verifiable algebraic conditions.**

***Index Terms*—Discrete linear time-varying system, Lyapunov exponent, upper Bohl exponent.**

I. Introduction

The most frequently used numerical quantities to characterize the dynamic properties of a discrete linear time-varying system are the Lyapunov and Bohl exponents (see, for example, [11], [16], and [20] and the references therein). The Lyapunov exponents were introduced in Lyapunov's work [22]. The Bohl exponents were introduced by Bohl [8]. Within hard sciences, where there is a long-standing tradition of quantitative studies, Lyapunov exponents are naturally used in a large number of fields, such as control theory, fluid dynamics, chemical reactions, and laser physics. More recently, they started to be used also in disciplines, such as biology and sociology, where nowadays processes can be accurately monitored (e.g., the propagation of electric signals in neutral cells and population dynamics). In context of our research, the Lyapunov and Bohl exponents characterize the exponential and uniform stability, respectively. These exponents were introduced in the context of linear time-varying differential equations, and since then, they are an object of intensive research both for continuous and discrete times (see, for example, [4], [5], [7], [9], [17]–[19], [21], and [26]).

It is well known that the theory of Lyapunov exponents is much more developed[6], [12], [20]. There exists the necessity of building up an analog of the Lyapunov exponent theory for Bohl exponents because Bohl exponents carry an important information about solutions' behavior, which could be used in Lyapunov exponent theory as well. Bohl exponents were applied, for example, by Bylov in his theory of almost reducibility [10] and by Millionshchikov in his investigations of linear differential systems with almost periodic or uniformly continuous coefficients and properties of systems with integral separation.

In [24], [25] (here the upper Bohl exponent is called generalized spectral radius), and [27], a discrete version of upper Bohl exponents has been studied. Bohl exponents of the discrete-time varying linear system were also studied in [2] and [13].

It is well known that if we have the set of initial conditions such that the corresponding solutions have different Lyapunov exponents, then the initial conditions are linearly independent. Upper Bohl exponents do not possess this Lyapunov exponent property [1], [23]. Therefore, the problem of the structure of the set of Bohl exponents is much more complicated than the problem for Lyapunov exponents. These issues for the diagonal discrete-time system have been studied in [1] and [14].

Natural continuation of these studies is a description of all functions that may be simultaneously the Lyapunov and upper Bohl exponents of a certain system. In this paper, as in [6], by the Lyapunov exponent, we understand a function that assigns to each nonzero initial condition the value of the classical Lyapunov exponent (cf., Definition 1). Analogically, we understand the upper Bohl exponent (cf., Definition 2). This problem for the Lyapunov exponent is relatively easy, and its solution is given by the so-called theorem of filtration [6]. For continuous-time systems, this issue for upper and lower Bohl exponents is completely solved in [3]. The first step toward characterizing all pairs of functions being at the same time the Lyapunov and Bohl exponents has been made in [2] and [15], where we presented necessary and sufficient conditions for a single function to be the lower and upper Bohl exponents of a diagonal discrete-time system, respectively (in [15], the upper Bohl exponent is called the upper Bohl function).

In this paper, we are considering a pair of functions and provide necessary and sufficient conditions that guarantee that one of them is the Lyapunov exponent and the other is the Bohl exponent of the same discrete linear system with diagonal coefficients.

Apart from the natural cognitive motivation presented above, this problem is also interesting from the control theory point of view. Namely, the relations between the values of the Lyapunov/Bohl exponents and dynamic properties of a system are well known and deeply understood. Therefore, one may formulate demands for a control system by defining the Lyapunov and Bohl exponents (understood as functions), and then, the question about existence of a system realizing these demands arises naturally.

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A. Czornik is with the Faculty of Automatic Control, Electronics and Computer Science, Silesian University of Technology, 44-100 Gliwice, Poland (e-mail: Adam.Czornik@polsl.pl).

A. Konyukh is with Belarus State Economic University, 220070 Minsk, Belarus (e-mail: al3128@gmail.com).

I. Konyukh is with the Belarusian State University of Informatics and Radioelectronics, 220013, Minsk, Belarus (e-mail: airisster@gmail.com).

M. Niezabitowski is with the Faculty of Automatic Control, Electronics and Computer Science, Silesian University of Technology, 44-100 Gliwice, Poland, and also with the Faculty of Mathematics, Physics and Chemistry, Institute of Mathematics, University of Silesia, 40-007 Katowice, Poland (e-mail: michal.niezabitowski@polsl.pl).

J. Orwat is with the Faculty of Mining and Geology and the Faculty of Civil Engineering, Silesian University of Technology, 44-100 Gliwice, Poland (e-mail: justyna.orwat@polsl.pl).

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II. Main Results

In this paper, we will study a discrete time-varying linear system of the form

$$x(n+1) = A(n)x(n) \tag{1}$$

where $A = (A(n))_{n\in\mathbb{N}}$ is a bounded sequence of s-by-s real, invertible, and diagonal matrices such that $(A^{-1}(n))_{n\in\mathbb{N}}$ is bounded. For $x \in \mathbb{R}^s$, the solution of the system (1) with the initial condition $x_A(0,x) = x$ will be denoted by $(x_A(n,x))_{n\in\mathbb{N}}$. By $\mathbb{R}^s_*$, we will denote the set $\mathbb{R}^s\backslash\{0\}$. The symbol $\|\cdot\|$ denotes an arbitrary norm in $\mathbb{R}^s$. Sometimes, we will also use the maximum norm of a vector $x = [x_1,\ldots,x_s] \in \mathbb{R}^s$, denoted by the symbol $\|\cdot\|_\infty$:

$$\|x\|_\infty = \max_{i=1,\ldots,s} |x_i|.$$

Definition 1: The function $\lambda_A : \mathbb{R}^s_* \to [0,\infty)$ defined as

$$\lambda_A(x) = \limsup_{n\to\infty} \|x_A(n,x)\|^{\frac{1}{n}} \tag{2}$$

is called the Lyapunov exponent of the system (1).

Definition 2: The function $\overline{\beta}_A : \mathbb{R}^s_* \to [0,\infty)$ defined as

$$\overline{\beta}_A(x) = \limsup_{n-m\to\infty} \left(\frac{\|x_A(n,x)\|}{\|x_A(m,x)\|}\right)^{1/(n-m)} \tag{3}$$

is called the upper Bohl exponent of the system (1).

Due to the equivalence of all norms in $\mathbb{R}^s$, we do not have to specify the norm in formulas (2) and (3).

To formulate our main results, we introduce the following notation: for a vector $x \in \mathbb{R}^s$, $x = [x_1,\ldots,x_s]$, we denote

$$I(x) = \{i = 1,\ldots,s : x_i \neq 0\}$$

and

$$x_i^0 = [\underbrace{0,\ldots,0}_{i-1},1,0,\ldots,0] \in \mathbb{R}^s_*.$$

Theorem 1: Functions $f_1,\ f_2 : \mathbb{R}^s_* \to [0,\infty)$ are the Lyapunov and upper Bohl exponents of a certain diagonal system (1), respectively, if and only if the following conditions hold.

1) $f_1(x) = \max_{i\in I(x)} f_1(x_i^0)$ for every $x \in \mathbb{R}^s_*$.
2) $f_2(x) = f_2(y)$ for all $x, y \in \mathbb{R}^s_*$ such that $I(x) = I(y)$.
3) For every $x \in \mathbb{R}^s_*$, there exists a number $i_x \in I(x)$ such that for any $y \in \mathbb{R}^s_*$ satisfying the conditions $I(y) \subset I(x)$ and $i_x \in I(y)$, the inequality $f_2(x) \le f_2(y)$ holds.
4) $f_1(x) \le f_2(x)$ for every $x \in \mathbb{R}^s_*$.

Proof: Necessity: The necessity of Condition 1 means that for any nonzero solution $x_A(n,x)$ of the diagonal system (1) with the initial vector $x = [x_1,\ldots,x_s]$, the equality

$$\lambda_A(x) = \max_{i\in I(x)} \lambda_A\left(x_i^0\right) \tag{4}$$

is valid. Indeed, let

$$\max_{i\in I(x)} \lambda_A\left(x_i^0\right) = \lambda_A\left(x_{i_0}^0\right),\ i_0 \in I(x).$$

Since the obvious inequality

$$\|x_A(n,x)\|_\infty \ge |x_{i_0}| \cdot \left\|x_A\left(n,x_{i_0}^0\right)\right\|_\infty$$

we have

$$\lambda_A(x) \ge \limsup_{n\to\infty} (|x_{i_0}| \cdot \left\|x_A\left(n,x_{i_0}^0\right)\right\|_\infty)^{\frac{1}{n}}$$
$$= \limsup_{n\to\infty} \left\|x_A\left(n,x_{i_0}^0\right)_\infty\right\|^{\frac{1}{n}} = \lambda_A\left(x_{i_0}^0\right).$$

The opposite inequality

$$\lambda_A(x) \le \lambda_A\left(x_{i_0}^0\right)$$

is a consequence of one of the general properties of the Lyapunov exponents (see [6]), i.e.,

$$\lambda_A(x+y) \le \max\{\lambda_A(x),\lambda_A(y)\}.$$

Equality (4) is proved.

Conditions 2 and 3 are established in [15, Th. 3].

Finally, the inequality $\overline{\beta}_A(x) \ge \lambda_A(x)$, which follows straight from the definitions of these functions, proves the necessity of Condition 4.

Sufficiency: Let functions f_1 and f_2 satisfy Conditions 1–4 of Theorem 1. We will construct a diagonal system (1) such that $\lambda_A \equiv f_1$ and $\overline{\beta}_A \equiv f_2$.

Let

$$A^1(n) = \operatorname{diag}\left[a_1^{(1)}(n),\ldots,a_s^{(1)}(n)\right]$$

be matrices of some diagonal system (1) with the upper Bohl exponent satisfying the identity $\overline{\beta}_{A^1} \equiv f_2$. The existence of such a system was proved in [15, Th. 2]. In the proof of that theorem, two properties of the coefficients $a_i^{(1)}(n), i = 1,2,\ldots,s$, were established.

1) There exists some sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty}(n_{k+1} - n_k) = \infty$$

and

$$\prod_{p=0}^{n_k-1} a_i^{(1)}(p) = \alpha^{n_k} \tag{5}$$

for all $i = 1,2,\ldots,s$, $k \in \mathbb{N}$ and some in advance given number α such that

$$0 < \alpha < \min_{x\in\mathbb{R}^s_*} f_2(x).$$

For the convenience of further reasonings, we will suppose that

$$\alpha < \min_{x\in\mathbb{R}^s_*} f_1(x). \tag{6}$$

2) The inequalities

$$\prod_{p=0}^{n-1} a_i^{(1)}(p) \le \alpha^n \tag{7}$$

hold for all $i = 1,2,\ldots,s$, $n = 2,3,\ldots$.

As far as the members of the sequence $(n_k)_{k\in\mathbb{N}}$ can be as large as we need, we will also assume that the inequality

$$\left|\max_{\substack{n,m\in[0;n_k]\\ n-m>k}} \left(\frac{\|x_{A^1}(n,x)\|}{\|x_{A^1}(m,x)\|}\right)^{\frac{1}{n-m}} - f_2(x)\right| < \frac{1}{k} \tag{8}$$

holds for every $x \in \mathbb{R}^s_*$, $k = 1,2,\ldots$.

Let us denote $T_k = (\tau_{k-1};\tau_k]$, where the sequence $(\tau_k)_{k\in\mathbb{N}}$ of natural numbers is such that

$$\tau_0 = 0$$

and

$$\tau_{3k-2} = \tau_{3k-3} + n_k \tag{9}$$

$$\tau_{3k-1} = k \cdot \tau_{3k-2} \tag{10}$$

$$\tau_{3k} = 2\tau_{3k-1} - \tau_{3k-2} \tag{11}$$

for $k = 1,2,\ldots$.

Now, we will define the coefficients of the system under construction

$$A(n) = \operatorname{diag}[a_1(n), \ldots, a_s(n)]$$

in the following way: $a_i(0) = 1$ and

$$a_i(n) = \begin{cases} a_i^{(1)}(n - \tau_{3k-3}), & \text{if} \quad n \in T_{3k-2} \\ \lambda_i, & \text{if} \quad n \in T_{3k-1} \\ \frac{\alpha^2}{\lambda_i}, & \text{if} \quad n \in T_{3k} \end{cases} \tag{12}$$

for every $k = 1, 2, \ldots$, where $\lambda_i = f_1(x_i^0), i = 1, 2, \ldots, s$.

According to Condition 1 of Theorem 1, in order to prove the identity $\lambda_A \equiv f_1$, we need to show that

$$\lambda_A\left(x_i^0\right) = f_1\left(x_i^0\right) = \lambda_i$$

for every $i = 1, 2, \ldots, s$.

Let us estimate the value of $\|x_A(n, x_i^0)\|_\infty$. Due to (5) and (12), the equality

$$\left\|x_A\left(\tau_{3k-3}, x_i^0\right)\right\|_\infty = \alpha^{\tau_{3k-3}}$$

holds for every $k = 1, 2, \ldots$. Then, from (6) and (7), we have

$$\left\|x_A\left(n, x_i^0\right)\right\|_\infty = \alpha^{\tau_{3k-3}} \cdot \prod_{p=0}^{n-\tau_{3k-3}-1} a_i^{(1)}(p) \le \alpha^n < \lambda_i^n$$

if $n \in T_{3k-2}$,

$$\left\|x_A\left(n, x_i^0\right)\right\|_\infty = \alpha^{\tau_{3k-2}} \cdot \lambda_i^{n-\tau_{3k-2}} < \lambda_i^n$$

if $n \in T_{3k-1}$, and

$$\begin{aligned} &\left\|x_A\left(n, x_i^0\right)\right\|_\infty \\ &= \alpha^{\tau_{3k-2}} \cdot \lambda_i^{\tau_{3k-1}-\tau_{3k-2}} \cdot \left(\frac{\alpha^2}{\lambda_i}\right)^{n-\tau_{3k-1}} < \lambda_i^n \end{aligned}$$

if $n \in T_{3k-1}$.

It follows from the above inequalities that

$$\lambda_A\left(x_{i_0}^0\right) = \limsup_{n\to\infty} \left\|x_A\left(n, x_{i_0}^0\right)\right\|_\infty^{\frac{1}{n}} \le \lambda_i$$

for $i = 1, 2, \ldots, s$. From the other side, we have

$$\begin{aligned} &\lim_{k\to\infty} \left\|x_A\left(\tau_{3k-1}, x_i^0\right)\right\|_\infty^{\frac{1}{\tau_{3k-1}}} \\ &= \lim_{k\to\infty} \left(\alpha^{\tau_{3k-2}} \cdot \lambda_i^{\tau_{3k-1}-\tau_{3k-2}}\right)^{\frac{1}{\tau_{3k-1}}} \\ &= \lim_{k\to\infty} \left(\alpha^{\tau_{3k-2}} \cdot \lambda_i^{(k-1)\tau_{3k-2}}\right)^{\frac{1}{k\cdot\tau_{3k-2}}} \\ &= \lim_{k\to\infty} \left(\alpha^{\frac{1}{k}} \cdot \lambda_i^{\frac{k-1}{k}}\right) = \lambda_i,\ i = 1, 2, \ldots, s. \end{aligned}$$

Hence, we have

$$\begin{aligned} \lambda_A\left(x_i^0\right) &= \limsup_{n\to\infty} \|x_A(n, x)\|_\infty^{\frac{1}{n}} \\ &= \lambda_i = f_1(x_i^0),\ i = 1, 2, \ldots, s \end{aligned}$$

and, according to Condition 1 of Theorem 1, we obtain the necessary identity $\lambda_A \equiv f_1$.

To prove the identity $\overline{\beta}_A \equiv f_2$, we will use the Lemma 1 from [1].

Lemma 1: For a natural number $z \ge 1$, consider any infinite subsets

$$Q_r = \{\omega(k, r) : k = 1, 2, \ldots,\ r = 0, 1, \ldots, z\}$$

of the set of natural numbers $\mathbb{N}$ such that $Q_i \cap Q_j = \emptyset$, when $i \ne j$ and $\bigcup_{i=0}^{z} Q_i = \mathbb{N}$. Then, for any nonzero solution $(x_A(n, x))_{n\in\mathbb{N}}$ of the system (1), the following equality

$$\begin{aligned} &\overline{\beta}_A(x) \\ &= \max_{r\in\{0,1,\ldots,z\}} \left(\limsup_{\substack{n-m\to\infty \\ n,m\in T_{\omega(k,r)}}} \left(\frac{\|x_A(n, x)\|}{\|x_A(m, x)\|}\right)^{\frac{1}{n-m}} \right) \end{aligned} \tag{13}$$

is valid.

It is worth to note that the upper limit in (13) is taken for all n, m, satisfying the condition $n - m \to \infty$ and belonging to the same interval $T_{\omega(k,r)}$.

Let $\omega(k, r) = 3k - r, r = 0, 1, 2$; then, according to Lemma 1, we have

$$\begin{aligned} &\overline{\beta}_A(x) \\ &= \max_{r=0,1,2,} \left(\limsup_{\substack{n-m\to\infty \\ n,m\in T_{3k-r}}} \left(\frac{\|x_A(n, x)\|}{\|x_A(m, x)\|}\right)^{\frac{1}{n-m}} \right). \end{aligned}$$

Let us introduce the following quantity:

$$\chi_A(n, m, x) = \left(\frac{\|x_A(n, x)\|_\infty}{\|x_A(m, x)\|_\infty}\right)^{\frac{1}{n-m}},\ n > m.$$

According to (5)–(12), for every $x \in \mathbb{R}_*^s$, we have

$$\chi_A(n, m, x) = \max_{i=1,\ldots,s} a_i^{(1)}(n - \tau_{3k-3})$$

if $n \in T_{3k-2}$, and due to (8), we obtain

$$\lim_{k\to\infty} \left(\max_{\substack{n,m\in[n_k; n_{k-1}] \\ n-m>k}} (\chi_A(n, m, x))^{\frac{1}{n-m}} \right) = f_2(x).$$

From (12), we have

$$\chi_A(n, m, x) = \begin{cases} \max\limits_{i=1,\ldots,s} \lambda_i, & \text{if } n, m \in T_{3k-1} \\ \max\limits_{i=1,\ldots,s} \frac{\alpha^2}{\lambda_i}, & \text{if } \ n, m \in T_{3k}. \end{cases}$$

According to Lemma 1, taking into consideration inequality (6) and Condition 4 of the theorem, we obtain the required equality $\overline{\beta}_A(x) = f_2(x)$ for every $x \in \mathbb{R}_*^s$. The theorem is proved.

III. Example

Consider the following two functions $f_1, f_2 : \mathbb{R}_*^2 \to [0, \infty)$:

$$f_1(x_1, x_2) = \begin{cases} 2, & \text{for } x_1 \ne 0 \text{ and } x_2 = 0 \\ 3, & \text{for } \qquad x_2 \ne 0 \end{cases}$$

$$f_2(x_1, x_2) = \begin{cases} 5, & \text{for} \quad x_1 \ne 0 \text{ and } x_2 = 0 \\ 6, & \text{for} \quad x_1 = 0 \text{ and } x_2 \ne 0 \\ 4, & \text{for} \quad x_1 \ne 0 \text{ and } x_2 \ne 0 \end{cases}$$

satisfying Conditions 1–4 of Theorem 1. Using the above-described method, we will construct a diagonal two-dimensional discrete time-varying system, for which f_1 and f_2 are the Lyapunov and upper Bohl exponents, respectively. From [15], we obtain a system (1) with

$$A^1(n) = \operatorname{diag}\left[a_1^{(1)}(n), a_2^{(1)}(n)\right]$$

for which the identity $\overline{\beta}_{A^1} \equiv f_2$ holds. We define the coefficients $a_1^{(1)}(n), a_2^{(1)}(n)$ as follows: $a_1^{(1)}(0) = a_2^{(1)}(0) = 1$

and

$$a_1^{(1)}(n) = \begin{cases} 1, & \text{if} \quad n \in \Delta_k^1 \cup \Delta_k^2 \\ \frac{1}{5}, & \text{if} \quad n \in \Delta_k^3 \\ 5, & \text{if} \quad n \in \Delta_k^4 \\ 4, & \text{if} \quad n \in \Delta_k^5 \\ \frac{1}{4}, & \text{if} \quad n \in \Delta_k^6 \end{cases}$$

$$a_2^{(1)}(n) = \begin{cases} \frac{1}{6}, & \text{if} \quad n \in \Delta_k^1 \\ 6, & \text{if} \quad n \in \Delta_k^2 \\ 1, & \text{if} \quad n \in \bigcup_{i=3}^{6} \Delta_k^i \end{cases}$$

where $\Delta_k^l = (3(k-1)k + (l-1)k; 3(k-1)k + lk]$, $l = 1, \ldots, 6$, $k \in \mathbb{N}$.

Let us denote $T_k = (\tau_{k-1}; \tau_k]$, where $\tau_0 = 0$, $\tau_{3k-2} = \tau_{3k-3} + 6k$, $\tau_{3k-1} = k \cdot \tau_{3k-2}$, $\tau_{3k} = 2\tau_{3k-1} - \tau_{3k-2}$, and $S_k^l = (\tau_{3k-3} + (l-1k; \tau_{3k-3} + lk]$ for $k = 1, 2, \ldots$, $l = 1, \ldots, 6$.

Now, we will define the coefficients $a_1(n), a_2(n)$ of the system under construction in the following way: $a_1(0) = a_2(0) = 1$ and

$$a_1(n) = \begin{cases} 1, & \text{if } n \in S_k^1 \cup S_k^2 \\ \frac{1}{5}, & \text{if} \quad n \in S_k^3 \\ 5, & \text{if} \quad n \in S_k^4 \\ 4, & \text{if} \quad n \in S_k^5 \\ \frac{1}{4}, & \text{if} \quad n \in S_k^6 \\ 2, & \text{if} \quad n \in T_{3k-1} \\ \frac{1}{2}, & \text{if} \quad n \in T_{3k} \end{cases}$$

$$a_2(n) = \begin{cases} \frac{1}{6}, & \text{if} \quad n \in S_k^1 \\ 6, & \text{if} \quad n \in S_k^2 \\ 1, & \text{if} \quad n \in \bigcup_{i=3}^{6} S_k^i \\ 3, & \text{if} \quad n \in T_{3k-1} \\ \frac{1}{3}, & \text{if} \quad n \in T_{3k} \end{cases}$$

for $k = 1, 2, \ldots$.

Proof of identities $\lambda_A \equiv f_1$ and $\overline{\beta}_A \equiv f_2$ for the constructed system can be carried out in the same way as in the proof of Theorem 1.

IV. Conclusion

In this paper, we investigate the Lyapunov and upper Bohl exponents of a discrete time-varying linear system with diagonal coefficients. The main result provides necessary and sufficient conditions for two functions to be simultaneously the Lyapunov and upper Bohl exponents of a certain diagonal discrete time-varying system. It is worth to notice that there exists a large class of systems that maybe reduced to diagonal ones by appropriate state transformation, which does not change the values of the Lyapunov and Bohl exponents (diagonalizable systems). Therefore, our results can also be applied to them. Natural continuation of the problem solved in this paper is to extend the results for a not necessarily diagonal linear systems. Partial results in this research direction are available for continuous-time systems (see [4] and [5] and the references therein). However, the conditions are not only purely algebraic, but also of topological nature. Obtaining such conditions for discrete-time systems will be a subject of our further research.

References

[1] A. Babiarz, A. Czornik, and M. Niezabitowski, "On the number of upper Bohl exponents for diagonal discrete time-varying linear system," *J. Math. Anal. Appl.*, vol. 429, no. 1, pp. 337–353, 2015.

[2] A. Babiarz, A. Czornik, A. Konyukh, and M. Niezabitowski, "What types of functions may be the lower Bohl function of a diagonal discrete linear time-varying systems?" *J. Franklin Inst.—Eng. Appl. Math.*, vol. 354, no. 13, pp. 5131–5144, 2017.

[3] E. Barabanov, "The structure of the ranges of the lower and the upper Bohl exponents of a linear differential system," *Differ. Equ.*, vol. 32, no. 3, pp. 295–303, 1996. (Translation from *Differ. Uravneniya*, vol. 32, no. 3, pp. 291–300, 1996).

[4] E. A. Barabanov and A. V. Konyukh, "Uniform exponents of linear-systems of differential-equations," *Differ. Equ.*, vol. 30, no. 10, pp. 1536–1545, 1994. (Translation from *Differ. Uravneniya*, vol. 30, no. 10, pp. 1665–1676, 1994).

[5] E. A. Barabanov and A. V. Konyukh, "Bohl exponents of linear differential systems," *Memoirs Differ. Equ. Math. Phys.*, vol. 24, pp. 151–158, 2001.

[6] L. Barreira and Y. Pesin, "Lectures on Lyapunov exponents and smooth ergodic theory," in *Proceedings of Symposia in Pure Mathematics*, vol. 69. Providence, RI, USA: Amer. Math. Soc., 2001.

[7] L. Barreira and C. Valls, "Stability of the Lyapunov exponents under perturbations," *Ann. Funct. Anal.*, vol. 8, no. 3, pp. 398–410, 2017.

[8] P. Bohl, "Uber differentialungleichungen," (in German), *J. Reine Angew. Math.*, vol. 144, pp. 284–318, 1913.

[9] V. V. Bykov, A. Ilchmann, and F. Wirth, "Bohl exponents and Baire classes of functions," *J. Math. Sci.*, vol. 210, no. 2, pp. 168–185, 2015.

[10] B. F. Bylov, "On almost reducibility for a system of linear differential equations having different characteristic exponents," (in Russian), *Sibirsk. Mat. Z.*, vol. 4, pp. 1241–1262.

[11] B. F. Bylov, R. E. Vinograd, D. M. Grobman, and V. V. Nemytskii, *Theory of Lyapunov Exponents and Its Applications to Stability Theory* (in Russian). Moscow, Russia: Nauka, 1966.

[12] A. Czornik, *Perturbation Theory for Lyapunov Exponents of Discrete Linear Systems*. Cracow, Poland: AGH Univ. Sci. Technol. Press, 2012.

[13] A. Czornik, "The relations between the senior upper general exponent and the upper Bohl exponents," in *Proc. 19th Int. Conf. Methods Models Autom. Robot.*, 2014, pp. 897–902.

[14] A. Czornik, J. Klamka, and M. Niezabitowski, "About the number of the lower Bohl exponents of diagonal discrete linear time-varying systems," in *Proc. 11th IEEE Int. Conf. Control Autom.*, Taichung, Taiwan, Jun. 18–20, 2014, pp. 461–466.

[15] A. Czornik, A. Konyukh, and M. Niezabitowski, "Which functions may be the upper Bohl function of the diagonal discrete linear time-varying system?" *J. Math. Anal. Appl.*, vol. 452, no. 2, pp. 1420–1433, 2017.

[16] J. L. Dalecki and M. G. Krein, *Stability of Solutions of Differential Equations in Banach Space*, vol. 43. Providence, RI, USA: Amer. Math. Soc., 2002.

[17] T. S. Doan, K. J. Palmer, and M. Rasmussen, "The Bohl spectrum for linear nonautonomous differential equations," *J. Dyn. Differ. Equ.*, vol. 29, pp. 1459–1485, 2017.

[18] N. H. Du, V. H. Linh, and N. T. T. Nga, "On stability and Bohl exponent of linear singular systems of difference equations with variable coefficients," *J. Differ. Equ. Appl.*, vol. 22, no. 9, pp. 1350–1377, 2016.

[19] A. Ghomi Taheri, F. Setoudeh, and M. Najafi, "A novel method for computing the Lyapunov exponent based on differential transform method," *Int. Rev. Autom. Control*, vol. 9, no. 5, pp. 316–323, 2016.

[20] N. A. Izobov, *Lyapunov Exponents and Stability (Stability Oscillations and Optimization of Systems)*. Cambridge, U.K.: Cambridge Scientific, 2013.

[21] N. V. Kuznetsov, T. A. Alexeeva, and G. A. Leonov, "Invariance of Lyapunov exponents and Lyapunov dimension for regular and irregular linearizations," *Nonlinear Dyn.*, vol. 85, no. 1, pp. 195–201, 2016.

[22] A. M. Lyapunov, *Stability of Motion*. New York, NY, USA: Academic, 1892.

[23] M. Niezabitowski, "About the properties of the upper Bohl exponents of diagonal discrete linear time-varying systems," in *Proc. 19th Int. Conf. Methods Models Autom. Robot.*, 2014, pp. 880–884.

[24] K. M. Przyluski, "On asymptotic stability of linear time-varying infinite-dimensional systems," *Syst. Control Lett.*, vol. 6, pp. 147–152, 1985.

[25] K. M. Przyluski, "Remarks on the stability of linear infinite-dimensional discrete-time systems," *J. Differ. Equ.*, vol. 72, pp. 189–200, 1988.

[26] A. J. Steyer and E. S. Van Vleck, "A step-size selection strategy for explicit Runge-Kutta methods based on Lyapunov exponent theory," *J. Comput. Appl. Math.*, vol. 292, pp. 703–719, 2019.

[27] F. Wirth, "On stability of infinite-dimensional discrete inclusions," *J. Math. Syst., Estimation, Control*, vol. 8, pp. 507–510, 1998.