Discontinuous Nash Equilibria in a Two-Stage Linear-Quadratic Dynamic Game With Linear Constraints

Rajani Singh and Agnieszka Wiszniewska-Matyszkiel

***Abstract*—In this note, we study a simple example of a two-stage linear-quadratic dynamic game in which the presence of simple linear state dependent constraints results in nonexistence of continuous symmetric feedback Nash equilibria and the existence of continuum of discontinuous symmetric feedback Nash equilibria. The example is not an abstract model—it has obvious applications in economics of resource extraction.**

***Index Terms*—Bellman equation, common renewable resources, constraints, discontinuous solutions, discrete time, feedback, linear-quadratic (LQ) dynamic game, Nash equilibrium, state-dependent constraints.**

I. Introduction

Dynamic games are the tool to model decision making by independent but coupled agents in an external environment changing in response to their decisions that encompass both aspects of the problem. Although more than a half of century has passed since the seminal book by Isaacs [1], the results in nonzero sum games are very limited. The reason is that dynamic games—at least two simultaneous dynamic optimization problems in which each of the decision makers best responds to the choice of their opponents—are much more compound than analogous single-agent dynamic optimization problems. Finding a Nash equilibrium in a dynamic game requires finding a fixed point, in a functional space, of a multivalued correspondence defined by solving a set of coupled dynamic optimization problems.

Therefore, in the restricted class of nonzero sum dynamic games for which any results have been derived, a vast majority of works concerns only open-loop Nash equilibria, in which strategies of the players are just predetermined functions of time only, which is not realistic.

Linear quadratic (LQ) dynamic games, formulated in both discrete and continuous time, are the best researched class of nonzero sum dynamic games. The formulas are now a textbook material (see e.g., Basar and Olsder [2], Engwerda [3], Haurie *et al.* [4], and Long [5]). This class of dynamic games is very important because of their potential applicability in modeling decision making in real-life problems and solutions can be found more easily than in the other nonlinear games.

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Nash equilibria for a feedback[1] information structure have been extensively analyzed for continuous time in LQ dynamic games without constraints. In vast majority of applications in which a feedback or closed-loop information structure has been considered, linear or affine Nash equilibria have been obtained.

A two-stage LQ dynamic game with a closed-loop information structure, but without constraints, is considered by Basar [6], in which the existence of nonlinear and nonunique closed-loop Nash equilibria has been proven. That paper has been written as a counterexample to the common belief that Nash equilibria in LQ dynamic games are linear and unique. Papavassilopoulos and Olsder [7] also consider an analogous game but with continuous time, both in the finite and infinite time horizon cases, and they show a class of such LQ games with a unique feedback Nash equilibrium for any finite time horizon, whereas none or a unique or multiple feedback Nash equilibria for the infinite time horizon.

So, nonuniqueness of Nash equilibria have already been observed in LQ dynamic games. Nevertheless, discontinuity[2] has never appeared. However, the discontinuity of Nash equilibria in differential games with concave current payoff has already appeared in Dockner and Sorger [8], in which nonlinear dynamics is considered.

Compared to continuous-time models, discrete-time LQ dynamic games have limited literature. There are very few theoretical papers on closed-loop or feedback Nash equilibria. Besides, Basar [6], LQ dynamic games with discrete time, and finite time horizon are considered by Hamalainen: for open-loop and feedback information structures [9] and for an open–closed information structure [10], where an algorithm is provided for finding Nash and Stackelberg solutions to such games without constraints.

So, in standard LQ dynamic games, there are no constraints. On the other hand, constraints play an important role in a vast majority of real-life applications. For example, state variables, such as the biomass of fish in games of exploitation of fisheries, the state of physical capital in economic problems, or the stock of pollutant in pollution games are always nonnegative. Control variables in the corresponding problems, such as the catch, the production, or the emission of pollutant, are also nonnegative, whereas in the first case, also a constraint by the amount of biomass available has to be taken into account.

The number of papers in which constraints, especially state-dependent constraints that may be active at equilibrium, are considered is small. LQ dynamic games with discrete-time and linear constraints appear in Reddy and Zaccour [11, 12] in a finite time horizon; in which the open-loop [11] and feedback sets of strategies [12] are considered. However, the class of games considered in both papers is restricted to games in which the controls for which the state-dependent constraints are considered do not influence the state variable, whereas for the re-

[1] We use the terminology of Haurie *et al.* [4]. This information structure is also called closed-loop no-memory or Markovian.

[2] Obviously, we consider the continuity with respect to the state variable.

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maining controls, there are no state-dependent constraints. So, they do not encompass our game.

Piecewise affine controls and piecewise quadratic value function have been obtained in discrete-time dynamic optimization problems with constraints (e.g., the social optimum problem in Singh and Wiszniewska-Matyszkiel [13]). In continuous time, they appear in, e.g., Rantzer and Johansson [14], for a problem which is originally piecewise LQ. However, in an LQ differential games even with a state independent control constraint, the value function does not have to be piecewise quadratic (see, e.g., Wiszniewska-Matyszkiel *et al.* [15]).

Because two-period dynamic games are simple and they illustrate well most of phenomena that appear in dynamic games, they have already appeared in applications. An example of such applications is a model of capacity investment in a duopoly market by Genc and Zaccour [16], with state-dependent constraints on controls, but without a common state variable (the only coupling is by price).

Here, we present a two-stage LQ dynamic game with linear constraints. This game has obvious applications to modeling of extraction of a marine fishery, with players representing countries or firms that sell their catch at a common market. As Reddy and Zaccour [12], we also consider feedback information structure and we introduce linear state-dependent constraints on decisions, but we do not assume that the control variables that are constrained by the state variable do not influence the state. Therefore, our model does not belong to the class of games considered by Reddy and Zaccour [11, 12]. On the other hand, we do not consider explicit constraints on the state variable (besides its nonnegativity).

Our game is a two-period truncation of the game-theoretic model studied before in an infinite horizon version by Singh and Wiszniewska-Matyszkiel [13], in which the calculation of feedback Nash equilibria has turned out to be possible only for the continuum of players case. While solving the n-player symmetric Nash equilibrium problem in that game, the authors have only been able to prove a negative result that there is no Nash equilibrium of assumed regularity[3] with respect to the state variable. On the other hand, the cooperative common dynamic optimization problem (the social optimum problem) for any number of players is relatively simple in that game. Further analysis of the n-player Nash equilibrium problem in that game suggested that the irregularity is inherited from finite horizon truncations of the game, which was the motivation of the analysis of this paper.

To sum up: in this paper, we present an example of a symmetric deterministic LQ dynamic game with concave payoff but no continuous symmetric feedback Nash equilibrium and a continuum of discontinuous symmetric feedback Nash equilibria.

Abbreviations and Acronyms

LQ for linear quadratic; r.h.s. for the right-hand side.

II. Formulation of the Problem

We consider a discrete-time game with two stages, two players, and identical quadratic instantaneous payoffs with payoff of player i, $P_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, defined by

$$P_i(s_i, s_{\sim i}) = \left(A - \frac{1}{2}(s_i + s_{\sim i})\right) s_i - \frac{s_i^2}{2} \tag{1}$$

[3]The nonexistence of a fixed point of the best response correspondence in the space of continuous piecewise linear strategies with at most three pieces is caused by the state-dependent constraints on control.

where s_i and $s_{\sim i}$ are decisions of player i and their opponent, respectively, while $A > 0$.

There is no terminal payoff.

The trajectory X of the state variable resulting from the choices of decisions of the players given an initial state $x_0 \geq 0$ is defined by

$$X(t+1) = (1+\xi) \cdot X(t) - \frac{1}{2}(s_1(t) + s_2(t)) \text{ with } X(0) = x_0 \tag{2}$$

where $s_1(t)$ and $s_2(t)$ denote the decisions of players at time t, and $\xi \in (0, 1)$ represents the net rate of growth of the state variable (fertility minus natural mortality). We explain the constraint on ξ in Section II-A.

Although, generally, the decision sets are $\mathbb{R}_+$, given x, there are linear state-dependent *constraints* for decisions

$$s_i \in [0, (1+\xi)x]. \tag{3}$$

This results in an implicit state constraint $X(t) \geq 0$. This constraint does not have to be explicitly stated because it is automatically fulfilled.

In this note, we are interested in feedback strategies $S_i : \{1, 2\} \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for every state x, $S_i(t, x) \in [0, (1+\xi) \cdot x]$. Therefore, at each time instant t, player i chooses decision $S_i(t, X(t))$, given $X(t)$, with

$$S_i(t, X(t)) \in [0, (1+\xi)X(t)].$$

We consider discounting by $\frac{1}{1+\xi}$, so, for a profile S, the payoff in the game is

$$\Pi_i((S_i, S_{\sim i}), x_0) = P_i\left((S_i, S_{\sim i})(1, X(1))\right) + \frac{P_i((S_i, S_{\sim i})(2, X(2)))}{1+\xi}. \tag{4}$$

We are interested in analyzing symmetric feedback Nash equilibria.

Definition 2.0.1: A profile $\bar{S}$ is a Nash equilibrium iff for every $i \in \{1, 2\}$, every strategy S_i of player i and every initial state $x_0 \geq 0$, we have

$$\Pi_i((\bar{S}_i, \bar{S}_{\sim i}), x_0) \geq \Pi_i((S_i, \bar{S}_{\sim i}), x_0).$$

A. Economic Interpretation of the Game

This game can be used to model various economic problems, including exploitation of a common marine fishery divided into exclusive economic zones. In this game, the catch is sold at a common market. The state variable is the biomass of fish in the whole fishery, whereas s_i represents the catch of player i. Fish are assumed to spread uniformly over the fishery.

The price at the market is given by the following:

$\text{Price}(s_1, s_2) = a - \frac{1}{2}(s_1 + s_2)$ (in economics, it is called the inverse demand function) for some positive a, which is dependent on the aggregate catch.

Therefore, the current payoff functions are equal to the revenue from sales minus the cost of fishing $P_i(s_1, s_2) = \text{Price}(s_1, s_2) \cdot s_i - \text{Cost}(s_i)$.

The cost that we consider is quadratic, and $\text{Cost}(s_i) = f s_i + \frac{1}{2} s_i^2$ is identical for both players.

Obviously, we assume that a is substantially greater than f. Then, we obtain our model with $A = a - f$.

The constraint $s_i \leq (1+\xi)x$ corresponds to the possibility of catching at most all the fish in the player's region, including the last year offspring—the inherent constraint of physical availability.

The discount factor equal to $\frac{1}{1+\xi}$ is sometimes referred to as the golden rule (see, e.g., Ramsey [17]). In models with variable rate of growth of the resource, this fact, i.e., the fact of equality between the rate of growth of the resource and the rate used for discounting, is usually obtained as a result.

Hence, the constraint $\xi < 1$ is implied by the fact that ξ plays the double role of both the resource net growth rate and the real interest rate, which is assumed to be close to 0.

III. Calculation of Feedback Nash Equilibria

Calculating Nash equilibria in a dynamic game requires solving a dynamic optimization problem for each player given arbitrary strategy of the opponent and finding a fixed point of the resultant joint best response correspondence—guaranteeing that players' strategies are best responses to each other. In finite time horizon multistage games with feedback strategies, this can be decomposed to solving coupled static problems using auxiliary value functions and the Bellman equation.

Consider a player i and a strategy of the opponent $S_{\sim i}$. The value function $\bar{V}_i : \{1,2,3\} \times \mathbb{R}_+ \to \mathbb{R}$ is the function that represents the optimal payoff of player i in the game given this strategy of the opponent. To be more specific, $\bar{V}_i(t,x)$ represents the optimal payoff of player i given a strategy of the opponent if the game starts at time t from the initial state x. These functions are auxiliary to calculate the Nash equilibrium strategies.

The following theorem is a direct application of the standard Bellman method (see, e.g., Bellman [18]) to our problem.

Theorem 3.1: Consider player i and the strategy of the opponent $S_{\sim i}$.

a) If a function $V_i : \{1,2,3\} \times \mathbb{R}_+ \to \mathbb{R}$ fulfills

$$V_i(t,x) = \sup_{s_i \in [0,(1+\xi)x]} P_i(s_i, S_{\sim i}(t,x)) + \frac{1}{1+\xi} \cdot V_i\left(t+1, (1+\xi)x - \frac{1}{2}(s_i + S_{\sim i}(t,x))\right) \quad (5)$$

for $t = 1,2$ and all $x \geq 0$, with the terminal condition

$$V_i(3,x) = 0 \quad (6)$$

then it is the value function given $S_{\sim i}$, whereas if a strategy S_i of player i fulfills for $t = 1,2$ and all $x \geq 0$

$$S_i(t,x) \in \operatorname*{Argmax}_{s_i \in [0,(1+\xi)x]} P_i(s_i, S_{\sim i}(t,x)) + \frac{1}{1+\xi} \cdot V_i\left(t+1, (1+\xi)x - \frac{1}{2}(s_i + S_{\sim i}(t,x))\right) \quad (7)$$

for arbitrary function V_i fulfilling (5) and (6), then it is the best response to $S_{\sim i}$.

b) The value function $\bar{V}_i$ fulfills (5) and (6), whereas the strategy $\bar{S}_i$ being the best response to $S_{\sim i}$ fulfills (7).

Therefore, each profile fulfilling (7) for both i for V_i defined by (5) and (6) is a Nash equilibrium.

Equations (5)–(7) are called the Bellman optimality principle (see, e.g., Bellmann [18]), and (5) is called the Bellman equation.

We start solving the game backward. By (5)–(7), if the state at $t = 2$ is x, then the best choice of a player at $t = 2$ given a strategy of their opponent is independent of previous decisions and it depends only on the current opponent's decision and state. Therefore, we can consider a static game.

Proposition 3.2: Consider any fixed $x \geq 0$ and the one-stage game with strategies $s_i \in [0,(1+\xi)x]$ and the payoff functions P_i given by (1). The strategy profile given by

$$\bar{s}_i = \bar{S}_i(2,x) := \begin{cases} (1+\xi)x & \text{if } x \leq \hat{x}_1 \\ \hat{s} & \text{if } x \geq \hat{x}_1 \end{cases} \quad (8)$$

$$\text{where } \hat{s} = \frac{2}{5}A \text{ and } \hat{x}_1 = \frac{\hat{s}}{1+\xi} \quad (9)$$

for $i \in \{1,2\}$, is the unique Nash equilibrium in this game.

Proof: The Nash equilibrium strategy of player i fulfills

$$\bar{s}_i \in \operatorname*{Argmax}_{s_i \in [0,(1+\xi)x]} \left(\left(A - \frac{1}{2}(s_i + s_{\sim i}) \right) s_i - \frac{s_i^2}{2} + 0 \right). \quad (10)$$

We calculate the zero derivative of the r.h.s. of (10) w.r.t. s_i to get the first-order condition $s_i = \frac{2A - s_{\sim i}}{4}$.

If $\frac{2A - s_{\sim i}}{4} \geq (1+\xi)x$, then the maximum is at $(1+\xi)x$. Therefore, the unique Nash equilibrium is $\bar{s} = \bar{S}(2,x)$. ■

Corollary 3.3: For every Nash equilibrium, for our (two stage) LQ game, at terminal time 2, players' strategies fulfill $S_i(2,x) = \bar{S}_i(2,x)$ for $\bar{S}_i(2,x)$ given by (8), and they are nondecreasing in x while the value functions fulfill

$$\bar{V}_i(2,x) = \begin{cases} \left(A - \frac{3}{2}(1+\xi)x\right)(1+\xi)x & \text{if } x \leq \hat{x}_1 \\ \left(A - \frac{3}{2}\hat{s}\right)\hat{s} & \text{if } x \geq \hat{x}_1 \end{cases} \quad (11)$$

Next, we proceed backward to solve the problem at time 1, given Nash equilibrium strategies and the value functions at time 2 from Proposition 3.2.

Lemma 3.4: Consider any $x \geq 0$ and an LQ dynamic game with only one time instant 1, the current payoff given by (1), the sets of strategies $[0,(1+\xi)x]$, and the terminal payoff equal to $\bar{V}_i(2,x)$ defined by (11). With notation

$$x_{\text{next}}(s_i, s_{\sim i})(x) = (1+\xi)x - \frac{1}{2}(s_i + s_{\sim i}) \quad (12)$$

denote the point s at which $x_{\text{next}}(s_i, s_{\sim i}, x) = \hat{x}_1$ for $\hat{x}_1$ defined in (9), by $s_{\text{Bd}}(s_{\sim i})$.

a) Given a strategy of the opponent $s_{\sim i}$, the best response of player i can be at one of the points: $\{0, d_{\text{I}}(s_{\sim i}), d_{\text{II}}(s_{\sim i}), s_{\text{Bd}}(s_{\sim i}), (1+\xi)x\}$ where

$$d_{\text{I}}(s_{\sim i}) = \frac{6(1+\xi)^2 x + 2A - s_{\sim i}(5+3\xi)}{11+3\xi} \quad (13a)$$

$$d_{\text{II}}(s_{\sim i}) = \frac{2A - s_{\sim i}}{4} \quad (13b)$$

$$s_{\text{Bd}}(s_{\sim i}) = 2(1+\xi)x - 2\hat{x}_1 - s_{\sim i}. \quad (14)$$

Moreover, the best response is at most $\frac{A}{2}$.

b) If we have a symmetric Nash equilibrium, then for every x, we have four possible values of Nash equilibrium strategy $\{(1+\xi)x, \quad s_{\text{I}}(x), \quad \hat{s}, \quad s_{\text{Bd}}^{\text{sym}}(x)\}$ for

$$s_{\text{I}}(x) = \frac{A + 3(1+\xi)^2 x}{8+3\xi}, \quad \hat{s} \text{ defined by (9) and} \quad (15)$$

$$s_{\text{Bd}}^{\text{sym}}(x) = (1+\xi)x - \hat{x}_1. \quad (16)$$

In Lemma 3.4, we restrict the set of strategies for which we calculate the best responses excluding those strategies that cannot appear at a symmetric Nash equilibrium. This simplifies further work: since the maximization problem that has to be solved at stage 1 is a maximization of a function that may be nonconcave and nondifferentiable and it may have two local maxima, the comparison of the resulting two values is needed to find the global maximum and the final result depends on the strategy of the opponent in quite a complicated way.

Proof: The proof is based on properties of optima of strictly concave differentiable functions on compact intervals. The KKT sufficient condition can be used instead but in this case our method is analytically simpler.

a) Let us fix player i and a strategy $s_{\sim i}$ of their opponent. For brevity, given strategy profile $(s_i, s_{\sim i})$, we write $x_{\text{next}}(s_i, s_{\sim i}, x)$ given by (12) as x_{next}, whenever it does not lead to confusion. So, for $t=1$, given x and $s_{\sim i}$, (5) becomes

$$\bar{V}_i(1,x) = \sup_{s_i \in [0,(1+\xi)x]} \text{RBE}(s_i) \tag{17}$$

$$\text{for } \text{RBE}(s_i) := \left(A - \frac{s_{\sim i}}{2} - s_i\right) s_i + \begin{cases} \frac{\hat{s}}{1+\xi}\left(A - \frac{3}{2}\hat{s}\right) & \text{if } x_{\text{next}} \geq \hat{x}_1 \\ \left(A - \frac{3}{2}(1+\xi)\cdot x_{\text{next}}\right)\cdot x_{\text{next}} & \text{if } 0 \leq x_{\text{next}} \leq \hat{x}_1 \end{cases} \tag{18}$$

while the Nash equilibrium strategy has to fulfill

$$\bar{S}_i(1,x) \in \operatorname*{Argmax}_{s_i \in [0,(1+\xi)x]} \text{RBE}(s_i). \tag{19}$$

By calculating the zero derivative points, we obtain d_{I} and d_{II}. The maximum can also be attained either at one of the boundary points 0 or $(1+\xi)x$ or at the switching point $s_{\text{Bd}}(s_{\sim i})$, at which the function is usually nondifferentiable.

The global maximum of $P_i(\cdot, s_{\sim i})$ is attained at $\frac{A - \frac{s_{\sim i}}{2}}{2} \leq \frac{A}{2}$. Obviously, $d_{\text{II}}(s_{\sim i}) \leq \frac{A}{2}$. If the maximum of RBE is attained at $d_{\text{I}}(s_{\sim i}) > \frac{A}{2}$, then player i can increase their current payoff by reducing s_i by ϵ and the other part of $\text{RBE}(s_i)$ does not decrease, since x_{next} increases and $\bar{V}_i(2,x)$ is increasing in x over the set of possible x_{next}, which contradicts s_i is a Nash equilibrium strategy. Similarly, for $(1+\xi)x$ (maximal only if not greater than $d_{\text{I}}(s_{\sim i})$) and s_{Bd} (maximal only if not greater than $d_{\text{II}}(s_{\sim i})$).

b) Assuming that a strategy has to be equal to a best response to it for results of a), we obtain five possible candidates for the optimal strategy of player i with

$$s_{\text{I}}(x) \text{ for } d_{\text{I}}(s) = s, \quad \hat{s} \text{ for } d_{\text{II}}(s) = s, \quad s_{\text{Bd}}^{\text{sym}}(x) \text{ for } s_{\text{Bd}}(s) = s$$

$(1+\xi)$ and 0. For 0, the best response is greater than 0. ■

Lemma 3.5: Consider any $x \geq 0$ and the one-stage LQ dynamic game as in Lemma 3.4. The best response correspondence $\text{BR}_i : [0,(1+\xi)x] \to [0,(1+\xi)x]$ restricted to strategies $s^*_{\sim i} \in \{s_{\text{I}}(x), \hat{s}, s_{\text{Bd}}^{\text{sym}}(x), (1+\xi)x\}$ and with $s^*_{\sim i} \leq \frac{A}{2}$ is

$$\text{BR}_i(s^*_{\sim i}) = \begin{cases} \{(1+\xi)x\} & \text{if } x \leq \hat{y}_1(s^*_{\sim i}) \\ \{d_{\text{I}}(s^*_{\sim i})\} & \text{if } \hat{y}_1(s^*_{\sim i}) < x < y_{\text{bd}}(s^*_{\sim i}) \\ \{d_{\text{I}}(s^*_{\sim i}), d_{\text{II}}(s^*_{\sim i})\} & \text{if } x = y_{\text{bd}}(s^*_{\sim i}) \\ \{d_{\text{II}}(s^*_{\sim i})\} & \text{if } x > y_{\text{bd}}(s^*_{\sim i}) \end{cases} \tag{20}$$

for $d_{\text{I}}(s^*_{\sim i})$, $d_{\text{II}}(s^*_{\sim i})$, and $\hat{s}$ defined in (13) and (9), where

$$y_{\text{bd}}(s^*_{\sim i}) = \frac{45(1+\xi)s^*_{\sim i} + 2A(35 + 15\xi + \sqrt{2(11+3\xi)})}{120(1+\xi)^2},$$

$$\hat{y}_1(s^*_{\sim i}) = \frac{2A - s^*_{\sim i}(5+3\xi)}{5 + 2\xi - 3\xi^2}. \tag{21}$$

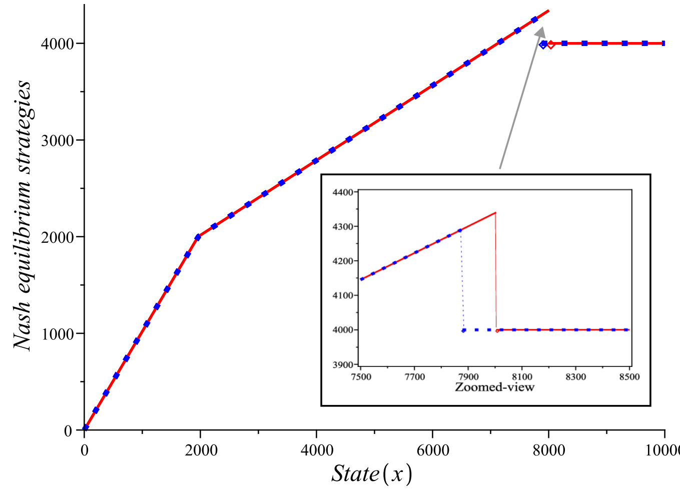

Fig. 1. Two symmetric Nash equilibria—the decision at stage 1 depending on the initial state.

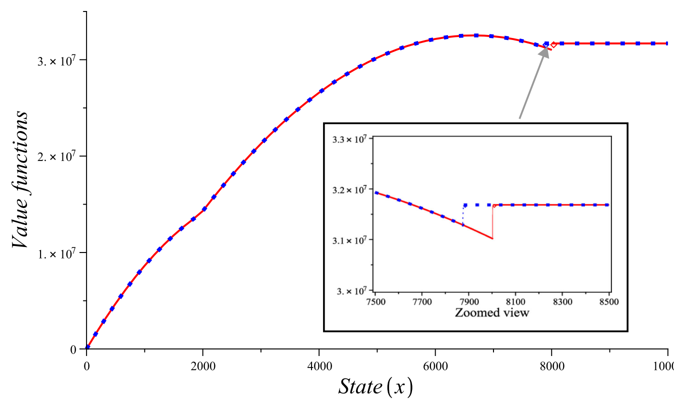

Fig. 2. Value functions at stage 1 for Nash equilibria from Fig. 1, depending on the initial state.

In Lemma 3.5, we calculate the best response of player i to a strategy of the opponent with properties restricted by Lemma 3.4 b).

The technical proof of Lemma 3.5 is in the Appendix.

Theorem 3.6: Any profile S with $S_i(1,x) = \bar{S}_i^{\text{L}}(1,x)$ or $S_i(1,x) = \bar{S}_i^{\text{R}}(1,x)$, where

$$\bar{S}_i^{\text{L}}(1,x) = \begin{cases} (1+\xi)x & \text{if } x \leq Y_1 \\ s_{\text{I}}(x) & Y_1 < x \leq Y_2 \\ \hat{s} & \text{if } x > Y_2 \end{cases} \tag{22}$$

$$\bar{S}_i^{\text{R}}(1,x) = \begin{cases} (1+\xi)x & \text{if } x \leq Y_1 \\ s_{\text{I}}(x) & Y_1 < x < Y_2 \\ \hat{s} & \text{if } x > Y_2, \end{cases} \tag{23}$$

for Y_1 being the solution of $\hat{y}_1((1+\xi)x) = \hat{y}_1(s_{\text{I}}(x))$ and an arbitrary $Y_2 \in [y_{\text{bd}}(\hat{s}), y_{\text{bd}}(s_{\text{I}}(Z))]$ for Z being the unique solution of $y_{\text{bd}}(s_{\text{I}}(Z)) = Z$ [for $\hat{y}_1$ and y_{bd} defined by (21)] and $S_i(2,x) = \bar{S}_i(2,x)$ for $\bar{S}_i(2,x)$ defined by (8), is a symmetric Nash equilibrium for our two-stage game and only such profiles can be symmetric Nash equilibria.

In Fig. 1, we present two symmetric Nash equilibrium strategies at $t=1$ (as a function of x), whereas in Fig. 2, we present the corresponding value functions at $t = 1$, both for the parameter values $A = 10\,000$ and $\xi = 0.02$.

As we can see, the strategy at stage 1 is piecewise linear in state and increasing on the first two intervals. We can interpret the results for the fishery model from Section II-A. The first interval corresponds to immediate catching of all the fish, in the second one, this happens at stage 2, whereas the third one corresponds to a certain level of sustainability and at stage 2 the global unconstrained maximum is reached for each player. If we compare the points near to the jump point Y_2, then for $x_0 < Y_2$, both players are more greedy than for $x_0 > Y_2$. The reason is that facing increased fishing of the opponent; a player also fishes more since the global unconstrained optimum in the second stage is either not available any more or reaching it requires substantial decrease of payoff at stage 1. This greediness results in substantially decreased state at stage 2, however, the price in stage 2 increases compared to that obtained for the case $x_0 > Y_2$.

Proof: By Corollary 3.3, for every Nash equilibrium, its profile of decisions at time 2 coincides with the Nash equilibrium of the one-stage game considered in Proposition 3.2, with identical Bellman equations. So, the value functions of both players for Nash equilibria at stage 2 are equal to $\bar{V}_i(2, x)$. So, the Nash equilibrium problem at stage 1 is equivalent to the Nash equilibrium problem in a one-stage game from Lemmas 3.4 and 3.5. The maximal set of x on which symmetric Nash equilibrium can be equal to $(1 + \xi)x$ is $[0, Y_1]$.

The maximal set of x in which a symmetric Nash equilibrium can be equal to $s_{\mathrm{I}}(x)$ is $[Y_1, y_{\mathrm{bd}}(s_{\mathrm{I}}(x))]$.

The maximal set of x in which a symmetric Nash equilibrium can be equal to $\hat{s}$ is $[y_{\mathrm{bd}}(\hat{s}), +\infty)$.

Note that $y_{\mathrm{bd}}(s_{\mathrm{I}}(x)) > y_{\mathrm{bd}}(\hat{s})$ since $s_{\mathrm{I}}(x) > \hat{s}$ for $x \geq y_{\mathrm{bd}}(\hat{s})$. Let us take any $Y_2 \in [y_{\mathrm{bd}}(\hat{s}), y_{\mathrm{bd}}(s_{\mathrm{I}}(Z))]$ and a profile S with $S_i(2, x) = \bar{S}_i(2, x)$ and $S_i(1, x) = \bar{S}_i^{\mathrm{L}}(1, x)$. The Bellman equation is fulfilled and $\bar{S}_i^{\mathrm{L}}(1, x)$ is in the best response to $\bar{S}_i^{\mathrm{L}}(1, x)$.

For $S_i(1, x) = \bar{S}_i^{\mathrm{R}}(1, x)$, the proof is analogous. ∎

Corollary 3.7: For our LQ dynamic game, there is a continuum of discontinuous symmetric feedback Nash equilibria but no continuous symmetric feedback Nash equilibrium (continuity with respect to the state variable). These equilibria are functions differing by the state Y_2 at which the jump appears at stage 1 and whether the function is left or right continuous.

The value functions corresponding to these symmetric feedback Nash equilibria are discontinuous at stage 1 at the same points Y_2.

We can also derive implications about different possible behavior of the players at symmetric feedback Nash equilibria given fixed x_0. If $x_0 \in [y_{\mathrm{bd}}(\hat{s}), Z]$ (for Z from Theorem 3.6), then, depending on Y_2, players choose either $S_i(1, x_0) = s_{\mathrm{I}}(x_0)$, which results in depletion of the resource after stage 2, or $S_i(1, x_0) = \hat{s}$, which results in some resources left. In this interval, an arbitrarily small change of the initial state may result in a substantial change of action. Besides this interval, $S_i(1, x_0)$ is unique.

IV. Conclusion

In this note, we have studied a symmetric two-stage LQ dynamic game with a feedback information structure and linear state-dependent constraints on strategies. This game has an obvious application in economics. We have proven that a continuous (with respect to the state variable) symmetric Nash equilibrium does not exist, whereas there are a continuum of discontinuous symmetric Nash equilibria although the instantaneous payoffs are concave while the sets of available decisions are convex. So, our result is a counterexample to the common belief in continuity of equilibria for LQ dynamic games with concave payoffs.

Since all the symmetric Nash equilibria are discontinuous and nonunique, the extension of the game to more than two periods poses several technical challenges: because of the discontinuity of the Nash equilibrium value functions in the two-stage game, calculation of the strategies at the first-stage even in a three-stage game requires solving the Bellman equation with discontinuous r.h.s., besides there is a problem which Nash equilibrium in the two-stage game to choose (since, e.g., the fact that there is no symmetric Nash equilibrium assuming a specific Nash equilibrium in the last two stages is not a proof that a symmetric Nash equilibrium does not exist).

Appendix
Technical Proofs

To prove Lemma 3.5, we state the following preliminary results on properties of the best response correspondence.

Lemma A.1: Consider the game of Lemma 3.4. For fixed x and a strategy $s^*_{\sim i} \in \{s_{\mathrm{I}}(x), \hat{s}, s^{\mathrm{sym}}_{\mathrm{Bd}}(x), (1+\xi)x\} \cap [0, \frac{A}{2}]$, the best response fulfills

$$\mathrm{BR}_i(s^*_{\sim i}) \subset \begin{cases} \{(1+\xi)x\} & \text{if } x \leq \hat{y}_1(s^*_{\sim i}) \\ \{d_{\mathrm{I}}(s^*_{\sim i})\} & \text{if } \hat{y}_1(s^*_{\sim i}) < x \leq \hat{x}_2(s^*_{\sim i}) \\ \{d_{\mathrm{I}}(s^*_{\sim i}), d_{\mathrm{II}}(s^*_{\sim i})\} & \text{if } \hat{x}_2 < x < \hat{y}_2(s^*_{\sim i}) \\ \{d_{\mathrm{II}}(s^*_{\sim i})\} & \text{if } x \geq \hat{y}_2(s^*_{\sim i}) \end{cases} \tag{24}$$

where, besides the symbols from (13) and (21)

$$\hat{x}_2(s^*_{\sim i}) = \frac{(15(1+\xi)s^*_{\sim i} + 2A(13+5\xi))}{40(1+\xi)^2} \tag{25}$$

$$\hat{y}_2(s^*_{\sim i}) = \frac{(15(1+\xi)s^*_{\sim i} + A(27+11\xi))}{40(1+\xi)^2}. \tag{26}$$

Proof: First, let us note that the function RBE is piecewise concave with at most two pieces. The switching points $\hat{y}_1$, $\hat{y}_2$, and $\hat{x}_2$ are defined as follows: $\hat{y}_1$ by $d_{\mathrm{I}}(s^*_{\sim i}) = (1+\xi)x$, $\hat{y}_2$ by $d_{\mathrm{I}}(s^*_{\sim i}) = s_{\mathrm{Bd}}(s^*_{\sim i})$, and $\hat{x}_2$ by $d_{\mathrm{II}}(s^*_{\sim i}) = s_{\mathrm{Bd}}(s^*_{\sim i})$.

We consider the four consecutive cases from (24). Given $s^*_{\sim i}$ and x, our optimization problem given in (17) can be decomposed into two optimization problems of differentiable and strictly concave functions: RBE_1 over the interval $[s_{\mathrm{Bd}}(s^*_{\sim i}), (1+\xi)x] \cap [0, (1+\xi)x]$ and RBE_2 over the interval $[0, s_{\mathrm{Bd}}(s^*_{\sim i})] \cap [0, (1+\xi)x]$ defined as follows:

$$\mathrm{RBE}_1(s_i) := \left(A - \frac{s^*_{\sim i}}{2} - s_i\right) s_i + \frac{\hat{s}}{1+\xi}\left(A - \frac{3}{2}\hat{s}\right) \tag{27}$$

$$\mathrm{RBE}_2(s_i) := \left(A - \frac{s^*_{\sim i}}{2} - s_i\right) s_i + \left(A - \frac{3(1+\xi)x_{\mathrm{next}}}{2}\right) \cdot x_{\mathrm{next}}. \tag{28}$$

By Lemma 3.4 a), the maximum of RBE can be attained at 0, $(1+\xi)x$, $s_{\mathrm{Bd}}(s^*_{\sim i})$, $d_{\mathrm{I}}(s^*_{\sim i})$, or $d_{\mathrm{II}}(s^*_{\sim i})$.

Note that for $s^*_{\sim i} \leq \frac{A}{2}$, both $d_{\mathrm{I}}(s^*_{\sim i})$ and $d_{\mathrm{II}}(s^*_{\sim i})$ are positive, so 0 is not the best response.

By meticulous checking, we obtain that all the functions $d_{\mathrm{II}}(s^*_{\sim i}) - s_{\mathrm{Bd}}(s^*_{\sim i})$, $d_{\mathrm{I}}(s^*_{\sim i}) - s_{\mathrm{Bd}}(s^*_{\sim i})$, $d_{\mathrm{II}}(s^*_{\sim i}) - (1+\xi)x$, $d_{\mathrm{I}}(s^*_{\sim i}) - (1+\xi)x$, and $-s_{\mathrm{Bd}}(s^*_{\sim i})$ for $(s^*_{\sim i}) \in \{s_{\mathrm{I}}(x), \hat{s}, s^{\mathrm{sym}}_{\mathrm{Bd}}(x), (1+\xi)x\}$ are strictly decreasing in x. So, to prove that, e.g., $d_{\mathrm{II}}(s^*_{\sim i}) \geq s_{\mathrm{Bd}}(s^*_{\sim i})$ on some interval of state variables, it is enough to check it at the upper bound of the interval only. Similarly, the function $(1+\xi)x - s_{\mathrm{Bd}}(s^*_{\sim i})$ is either strictly decreasing in x (for $s^*_{\sim i} = s_{\mathrm{I}}(x)$ and $s^*_{\sim i} = \hat{s}$) or it is a positive constant (for $s^*_{\sim i} = (1+\xi)x$ and $s^*_{\sim i} = s^{\mathrm{sym}}_{\mathrm{Bd}}(x)$).

We consider the following division of the state set. For brevity, in the sequel, we use s_{Bd} for $s_{\mathrm{Bd}}(s_{\sim i})$ given in (14).

Case 1: If $x \leq \hat{y_1}(s^*_{\sim i})$, then $d_{\mathrm{I}} \geq (1+\xi)x$ and $d_{\mathrm{II}} > s_{\mathrm{Bd}}$. So, the maximum of RBE_2 over $[0, s_{\mathrm{Bd}}]$ (if it is nonempty) is at s_{Bd} while RBE_1 is strictly increasing on $[s_{\mathrm{Bd}}, (1+\xi)x]$. So, the maximum of RBE is attained at $(1+\xi)x$.

Let us also note that for $x = \hat{y_1}(s^*_{\sim i})$, $d_{\mathrm{I}}(s^*_{\sim i}) = (1+\xi)x$.

Case 2: If $\hat{y_1}(s^*_{\sim i}) < x \leq \hat{x}_2(s^*_{\sim i})$ then $d_{\mathrm{II}}(s^*_{\sim i}) \geq s_{\mathrm{Bd}}$ and $s_{\mathrm{Bd}} \in [0, (1+\xi)x]$. So, the supremum of RBE_2 on $[0, s_{\mathrm{Bd}}]$ is attained at s_{Bd}. Since the zero derivative point of RBE_1, $d_{\mathrm{I}}(s^*_{\sim i}) \in [s_{\mathrm{Bd}}, (1+\xi)x]$, the maximum of RBE is attained at $d_{\mathrm{I}}(s^*_{\sim i})$. So, $d_{\mathrm{I}}(s^*_{\sim i})$ is the unique best response.

Case 3: If $\hat{x}_2(s^*_{\sim i}) < x < \hat{y}_2(s^*_{\sim i})$ then both $d_{\mathrm{I}}(s^*_{\sim i}) \in (s_{\mathrm{Bd}}, (1+\xi)x]$ and $d_{\mathrm{II}}(s^*_{\sim i}) \in (0, s_{\mathrm{Bd}})$ and $s_{\mathrm{Bd}} \in (0, (1+\xi)x)$. Therefore, the supremum of RBE_2 on $[0, s_{\mathrm{Bd}}]$ is attained at $d_{\mathrm{II}}(s^*_{\sim i})$, whereas the supremum of RBE_1 on $[s_{\mathrm{Bd}}, (1+\xi)x]$ is attained at $d_{\mathrm{I}}(s^*_{\sim i})$. So, the supremum of RBE can be attained either at $d_{\mathrm{I}}(s^*_{\sim i})$ or $d_{\mathrm{II}}(s^*_{\sim i})$, depending on whether $\mathrm{RBE}_1(d_{\mathrm{I}}(s^*_i))$ or $\mathrm{RBE}_2(d_{\mathrm{II}}(s^*_i))$ is greater. So, only $d_{\mathrm{I}}(s^*_{\sim i})$ and $d_{\mathrm{II}}(s^*_{\sim i})$ can be in the best response.

Case 4: If $x \geq \hat{y}_2(s^*_{\sim i})$, then $d_{\mathrm{I}}(s^*_{\sim i}) \leq s_{\mathrm{Bd}}$ while $d_{\mathrm{II}}(s^*_{\sim i}) \in [0, \min\{s_{\mathrm{Bd}}, (1+\xi)x\}]$. So, the maximum of RBE_1 on $[s_{\mathrm{Bd}}, (1+\xi)x)]$ is either at s_{Bd} and s_{Bd} is not the maximum of RBE_2 on $[0, s_{\mathrm{Bd}}]$, or the interval $[s_{\mathrm{Bd}}, (1+\xi)x]$ is empty. Therefore, the maximum of RBE over $[0, (1+\xi)x]$ is attained at $d_{\mathrm{II}}(s^*_{\sim i})$. So, $d_{\mathrm{II}}(s^*_{\sim i})$ is the unique best response. ∎

Proof of Lemma 3.5: By Lemma (A.1) and the fact that the best response is nonempty (as a maximum of continuous function over a compact set), we know exactly what the best response is besides the interval $[\hat{x}_2(s^*_{\sim i}), \hat{y}_2(s^*_{\sim i})]$. So, consider $x \in [\hat{x}_2(s^*_{\sim i}), \hat{y}_2(s^*_{\sim i})]$.

For $x < y_{\mathrm{bd}}(s^*_{\sim i})$, $\mathrm{RBE}(d_{\mathrm{I}}(s^*_{\sim i})) > \mathrm{RBE}(d_{\mathrm{II}}(s^*_{\sim i}))$, whereas for $x > y_{\mathrm{bd}}(s^*_{\sim i})$, $\mathrm{RBE}(d_{\mathrm{II}}(s^*_{\sim i})) > \mathrm{RBE}(d_{\mathrm{I}}(s^*_{\sim i}))$. For $x = y_{\mathrm{bd}}(s^*_{\sim i})$, they are equal. ∎

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