Direct and Indirect Model Reference Adaptive Control for Multivariable Piecewise Affine Systems

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***Abstract*—This article proposes direct and indirect model reference adaptive control strategies for multivariable piecewise affine systems, which constitute a popular tool to model hybrid systems and approximate nonlinear systems. A chosen reference model, which can be linear or also piecewise affine, describes the desired closed-loop system behavior that is to be achieved by the adaptive controllers for unknown system dynamics. Each subsystem acquires its own set of control gains, which is tuned under careful consideration of the switching behavior. In the indirect approach, the use of dynamic gain adjustment avoids singularities in the certainty equivalence principle. It is shown for both algorithms that the state of the reference model is tracked asymptotically given a common Lyapunov function for the switched reference model is available. Furthermore, parameter convergence in both the direct and indirect approach is proven for sufficiently rich reference signals. Finally, both algorithms are evaluated in numerical simulations and their advantages and disadvantages are discussed.**

***Index Terms*—Model reference adaptive control (MRAC), piecewise affine (PWA) systems, switching.**

I. Introduction

INCREASING complexity in technical systems lead to a growing interest in hybrid systems to efficiently handle switching behavior or approximate nonlinearities. Hybrid systems consist of continuous dynamics and a switching mechanism [1]. Different subclasses for hybrid systems exist, such as mixed logical dynamical systems, linear complementarity systems, and piecewise affine (PWA) systems. In PWA systems, the state-input space is partitioned into convex polytopes and the system dynamics are governed by different linear subsystems in each polytope. Hence, PWA systems also constitute a subclass of linear switched systems in which the switching signal is state-dependent instead of exogenous. Recent examples for the application of PWA systems can be found in switching power converters [2], biosystems [3], pneumatic systems [4], or cruise control systems [5]. While this list is by no means complete, it gives a good understanding of the ubiquitous presence of PWA systems due to their universal approximation capabilities for nonlinear systems.

Switching generally complicates the analysis of a dynamical system and the controller design. The research on hybrid systems extended the linear system theory and developed new stability concepts [6]–[8], definitions for observability and controllability [9] as well as observer [10] and controller designs [11]. In the next step, it is to be noted that hybrid systems may contain unknown or time-varying parameters, which sets the stage for adaptive control of hybrid systems. Model reference adaptive control (MRAC) forces the unknown system to track a specified reference system. By choosing a suitable reference system, this technique may, thus, remove undesired switching and nonlinearities from the open-loop hybrid systems, resulting in a simple linear closed-loop system. Existing approaches towards this goal can be divided into MRAC for general switched systems and MRAC for PWA systems.

For the general class of switched systems various adaptive controllers have been proposed in recent years [12]–[15]. A model reference robust adaptive control algorithm for uncertain switched linear systems is proposed in [12]. The proof of uniform boundedness under arbitrary switching is based on multiple Lyapunov functions constructed from a sufficient LMI condition. Furthermore, a dead band is introduced in order to suppress the bursting phenomenon in adaptive systems. A critical assumption in MRAC for switched systems is exact knowledge of the switching signal. The case in which the reference system and controller do not switch synchronously with the plant is investigated in [13]. Dwell-time constraints are derived that ensure practical global stability of the switched error system. While work in [13] is restricted to switched linear systems, [14] extends the proposed framework to nonlinear switched systems with nonlinear reference systems. This extension enables more sophisticated performance requirements as shown by the example of a highly maneuverable aircraft. Wang and Zhao point out that deriving dwell-time constraints for switched systems may be infeasible without knowledge of the actual subsystem parameters [15]. Instead, their adaptive state tracking algorithm

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is based on a hyper-stability criterion which relates to passivity. The proposed control, however, assumes that also the switching signal of the system can be controlled. This is unsuitable for PWA systems where the switching is always state-dependent.

This brings us to the class of PWA systems, for which there are various MRAC approaches available in the literature [16]–[24]. Early work by di Bernardo *et al.* extends the minimal control synthesis algorithm to bimodal PWA systems [16]–[18]. Later, di Bernardo *et al.* proposed a hybrid MRAC strategy for multimodal PWA systems [19] in which the state-input space partitioning of the controlled system may be different form the reference system. Excluding sliding modes, stability in [19] was proven by passivity-based arguments as well as Lyapunov theory. For the Lyapunov-based proof, a common Lyapunov-function of the reference system's subsystems is required. The extension of this hybrid MRAC algorithm in [20] is concerned with potential instabilities of the original algorithms due to disturbances at the input in form of the affine terms of the plant and reference system. The solution is based on a common Lyapunov function and can guarantee convergence of the error system even when the system enters a sliding mode. Note that the referenced work by di Bernardo *et al.* focuses on single input systems in control canonical form which restricts the class of applicable systems.

Work by Sang and Tao focuses on MRAC for piecewise linear (PWL) systems which approximate nonlinear systems by linearization at multiple operating points [21]–[24]. The state tracking algorithm proposed in [21] and [22] ensures bounded signals and asymptotic tracking for arbitrary fast switching in case a common quadratic Lyapunov function (CQLF) exists. In case no such Lyapunov function exists, signals are shown to be bounded, and an upper bound on the tracking error is given. Asymptotic tracking can be ensured through parameter projection combined with PE and slow switching. A more detailed analysis of the proposed algorithm is given in [23] with careful treatment of various special cases related to the existence of a common Lyapunov function or the level of excitation in the reference signal. A multivariable extension of the algorithm is proposed in [24].

Closely related to adaptive control of PWA systems is their recursive identification. In particular, our previous works [25], [26] provide algorithms for the subsystem identification of continuous-time PWL and PWA systems in state-space form. Furthermore, the achievable convergence rates were improved by using concurrent learning [27]. A recent survey of PWA system identification was given in [28].

Despite the revised list of publications on adaptive control of switched systems, there still is a lack of MRAC algorithms for multivariable PWA systems [29]. Our contribution in this paper is to fill this gap by means of two approaches: One direct and one indirect MRAC algorithm. First, we build upon the direct approach by Sang and Tao [23] to define direct MRAC for PWA systems. Especially for parameter convergence, the additional affine terms require further analysis. By adding one additional assumption and revising the proof for persistence of excitation, all gains are guaranteed to converge to their nominal values. This extension provides great advantages compared to the state of the art. It will be shown that working with affine subsystems requires less previous knowledge about the system. As opposed to [21]–[24], this enables application of the algorithm to systems with uncertain or unknown operating points. Then, we derive an indirect MRAC algorithm which is based on our previous work on parameter identifiers [26]. Parameter identifiers generate estimates of the subsystem parameters and the control gains are dynamically adjusted based on these estimates. The unique advantage of this approach is that estimates of all system parameters are obtained in parallel to the control task. For both the direct and the indirect approach, the stability proofs for asymptotic state tracking rely on Lyapunov theory and dwell-time assumptions. Moreover, parameter convergence is guaranteed under PE. Overall, this paper yields the first MRAC laws for multivariable PWA systems, which overcomes restrictions to PWL systems, single-input systems or systems in control canonical form, postulated by state of the art algorithms [16]–[24].

The remainder of this paper is structured as follows. In Section II, we provide required preliminaries such as the definition of PWA systems and the control structure. Also the formal MRAC problem is stated. The direct and indirect MRAC algorithms are presented in Sections III and IV, respectively. In both cases, we consider reference system tracking separately from the additional requirement of parameter convergence. The application of direct and indirect MRAC is demonstrated in Section V by means of numerical examples. Finally, Section VI concludes the paper.

II. Preliminaries and Problem Formulation

This section defines PWA systems and discusses briefly how they naturally arise from the linearization of a nonlinear system at multiple operating points. Furthermore, a motivation is given, why PWA systems are favorable for MRAC compared to PWL systems. Thereafter, the reference system is introduced and the MRAC problem is formulated. The section concludes with some preliminaries on signal properties and the Filippov concept for sliding mode solutions.

A. PWA Systems

The considered PWA system consists of a total of $s \in \mathbb{N}$ affine subsystems with dynamics

$$\dot{x}(t) = A_i x(t) + B_i u(t) + f_i, \quad i = 1, \ldots, s \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^p$ is the control input, and $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times p}$, and $f_i \in \mathbb{R}^n$ are the unknown, constant system parameters. The pairs (A_i, B_i) are assumed to be controllable and each B_i be of full column rank. For the common case $p \leq n$, the full column rank is usually satisfied by physical systems.

The state-input space $[x^\top, u^\top]^\top \in \mathbb{R}^{n+p}$ is partitioned into s polyhedral regions Ω_i. The i-th polyhedral region is defined by a set of $\mu_i \in \mathbb{N}$ linear inequalities

$$\Omega_i = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \mathbb{R}^{n+p} \;\middle|\; \begin{bmatrix} h_{1,i} \\ \vdots \\ h_{\mu_i,i} \end{bmatrix} \begin{bmatrix} x \\ u \\ 1 \end{bmatrix} \preceq_{[i]} 0 \right\} \quad (2)$$

where $\preceq_{[i]}$ represents an element-wise list of operators $<$ and $\leq$. Each row vector $h_{(\cdot,\cdot)}$ defines a hyperplane which divides the state-input space into two half spaces and is of the form $h = [\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_p, \gamma] \in \mathbb{R}^{1\times(n+p+1)}$. Hence, Ω_i is the intersection of μ_i half spaces. The discrimination between $<$ and $\leq$ specifies whether the half space is open or closed and hence whether the hyperplane does not or does belong to the partition, respectively. Each hyperplane must belong to either one of the two neighboring partitions. This is necessary to obtain a complete partition of the state space without overlapping regions, i.e., $\Omega_i \cap \Omega_j = \emptyset$, $\forall i \neq j$.

Each of the subsystems in (1) belongs to one of the regions Ω_i. Thus, the region which contains the current state-input vector determines which subsystem is active. This is expressed with the indicator functions

$$\chi_i(t) = \begin{cases} 1, & \text{if } (x(t), u(t)) \in \Omega_i \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

As the regions do not overlap, it follows that $\sum_{i=1}^{s} \chi_i(t) = 1$ and $\chi_i(t)\chi_j(t) = 0$, $i \neq j$. The dynamics of the PWA system are then given by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t) \quad (4)$$

where $A(t) = \sum_{i=1}^{s} A_i\chi_i(t)$, $B(t) = \sum_{i=1}^{s} B_i\chi_i(t)$, and $f(t) = \sum_{i=1}^{s} f_i\chi_i(t)$. Hence, in the following, a matrix without index refers to the currently active matrix, i.e., $A \to A_i$ with $\chi_i = 1$.

The adaptive control algorithms derived in this paper are based on the PWA system representation (4). In order to demonstrate the benefit of PWA systems compared to PWL systems, a brief revision of how they are obtained by linearizing a nonlinear system around multiple operating points is appropriate. Consider the nonlinear system

$$\dot{x}(t) = g\left(x(t), u(t)\right) \quad (5)$$

where $g : \mathbb{R}^{n+p} \to \mathbb{R}^n$ is a smooth (continuously differentiable) nonlinear vector function. Note that, throughout the paper, we omit the dependency on time t as long as it is clear from the context.

Neglecting higher order terms in the linearization of g around an operating point (x_i^*, u_i^*) yields the affine model

$$\begin{aligned} \dot{x} &\approx g(x_i^*, u_i^*) + A_i(x - x_i^*) + B_i(u - u_i^*) \\ &= A_i x + B_i u + g(x_i^*, u_i^*) - A_i x_i^* - B_i u_i^* \\ &= A_i x + B_i u + f_i \end{aligned} \quad (6)$$

where $A_i = \frac{\partial g(x,u)}{\partial x}\big|_{(x_i^*, u_i^*)}$, $B_i = \frac{\partial g(x,u)}{\partial u}\big|_{(x_i^*, u_i^*)}$, and $f_i = g(x_i^*, u_i^*) - A_i x_i^* - B_i u_i^*$. A PWA model is obtained by linearizing around various operating points $\{(x_i^*, u_i^*)\}_{i=1,\ldots,s}$ and partitioning the state space such that the active subsystem in each region is characterized by a minimal error with respect to the original nonlinear system. Considering only the deviations $\Delta x_i = x - x_i^*$ and $\Delta u_i = u - u_i^*$, and assuming additionally equilibrium operating points with $g(x_i^*, u_i^*) = 0$, the same considerations yield a PWL model with linear subsystems

$$\dot{x} \approx A_i \Delta x_i + B_i \Delta u_i. \quad (7)$$

As both the affine subsystems (6) and the linear subsystems (7) constitute first order approximations of the nonlinear dynamics g, they share the same approximation capabilities and result in the same partitioning of the state-input space. The central difference is that PWA models replace the local deviations from the operating points, i.e., $\Delta x_i, \Delta u_i$, by the globally valid measurements x and u at the expense of the additional uncertain terms f_i. That means, working with PWA models, as opposed to working with PWL models, requires no knowledge of the operating points x_i^* and u_i^* as well as the values of $g(x_i^*, u_i^*)$. Hence, PWA models are more favorable in the design and application of adaptive algorithms as less knowledge about the system has to be introduced *a priori*.

B. Reference System Modeling

The aim of this paper is to design MRAC laws for PWA systems. In MRAC, a reference system, which defines the desired behavior of the controlled system, is chosen by the designer. The adaptive control law causes the controlled system to track the reference system despite parameter uncertainties in the controlled system.

The reference system in this paper is chosen to be PWA in order to give enough flexibility to the designer. Let the reference system and the controlled PWA system share the same state-space partitions Ω_i. Hence, we consider the reference system

$$\dot{x}_\mathrm{m} = A_\mathrm{m} x_\mathrm{m} + B_\mathrm{m} r + f_\mathrm{m} \quad (8)$$

where $x_\mathrm{m} \in \mathbb{R}^n$ and $r \in \mathbb{R}^p$ are the state of the reference system and the reference signals, respectively. The parameters of the reference system switch according to the same indicator functions χ_i in (3) and are therefore given by $A_\mathrm{m} = \sum_{i=1}^{s} A_{\mathrm{m}i}\chi_i$, $B_\mathrm{m} = \sum_{i=1}^{s} B_{\mathrm{m}i}\chi_i$, and $f_\mathrm{m} = \sum_{i=1}^{s} f_{\mathrm{m}i}\chi_i$ with $A_{\mathrm{m}i} \in \mathbb{R}^{n\times n}$, $B_{\mathrm{m}i} \in \mathbb{R}^{n\times p}$, and $f_{\mathrm{m}i} \in \mathbb{R}^n$, $\forall i = 1, \ldots, s$.

Note that simply applying the partitions of the controlled PWA system also for the reference system would unnecessarily limit the choice of reference systems. Take for example the case in which the designer wants to introduce certain thresholds into the behavior of the closed-loop system at which the dynamics are supposed to change. In that case, the reference system may need to feature a switch at states where the controlled system is continuous. Hence, to gain additional freedom, an original partition Ω_i of the controlled PWA system may be divided into convex subsets Ω_k and Ω_l with $\Omega_i = \Omega_k \cup \Omega_l$. While the parameters of the controlled PWA system remain the same in both regions, e.g., $A_k = A_l = A_i$, the reference system parameters can be chosen differently in both sets, i.e., $A_{\mathrm{m}k} \neq A_{\mathrm{m}l}$. In practice, this enables to switch the reference system independently from the controlled system.

Stability of the reference system (8) is essential for the tracking problem. Assuming that each subsystem $A_{\mathrm{m}i}$ is stable, it follows that there exists a symmetric, positive definite Lyapunov matrix $P_i \in \mathbb{R}^{n\times n}$ for every symmetric, positive definite matrix $Q_{\mathrm{m}i} \in \mathbb{R}^{n\times n}$ such that

$$A_{\mathrm{m}i}^{\top} P_i + P_i A_{\mathrm{m}i} = -Q_{\mathrm{m}i} \quad \forall i = 1, \ldots, s. \tag{9}$$

Hence, $V_i = x_{\mathrm{m}}^{\top} P_i x_{\mathrm{m}}$ is a quadratic Lyapunov function for the i-th subsystem of the reference system. For each subsystem i, there also exist constants $a_{\mathrm{m}i}, \lambda_{\mathrm{m}i} > 0$ such that $\|\mathrm{e}^{A_{\mathrm{m}i}t}\| \leq a_{\mathrm{m}i}\mathrm{e}^{-\lambda_{\mathrm{m}i}t}$. If all subsystems satisfy (9) for a common matrix P, i.e., $P_i = P, \forall i$, then $V = x_{\mathrm{m}} P x_{\mathrm{m}}$ is referred to as a CQLF . For a CQLF, the reference system (8) is stable, even for arbitrary fast switching. If, however, no CQLF exists, stability of the reference system (8) can be ensured through dwell-time considerations. In other words, the reference system must reside in one mode for a sufficiently long (dwell) time T_0. The dwell-time constraint for the switched reference system (8) is captured in the following Lemma, where $\lambda_{\min}(P_i)$ and $\lambda_{\max}(P_i)$ refer to the minimum and maximum eigenvalue of P_i.

Lemma 1 (see[23]): "The switched system $\dot{x}(t) = A_{\mathrm{m}}(t)x(t)$ is exponentially stable with decay rate $\sigma \in (0, 1/2\alpha)$ if T_0 satisfies

$$T_0 \geq T_{\mathrm{m}} = \frac{\alpha}{1-2\sigma\alpha}\ln(1+\mu\Delta_{A_{\mathrm{m}}}) \;\; \mu = \frac{a_{\mathrm{m}}^2}{\lambda_{\mathrm{m}}\beta}\max_i \|P_i\|$$

where $\Delta_{A_{\mathrm{m}}} = \max_{i,j}\|A_{\mathrm{m}i} - A_{\mathrm{m}j}\|$, $\alpha = \max_i \lambda_{\max}(P_i)$, $\beta = \min_i \lambda_{\min}(P_i)$, $a_{\mathrm{m}} = \max_i a_{\mathrm{m}i}$, and $\lambda_{\mathrm{m}} = \max_i \lambda_{\mathrm{m}i}$."

Proof: The detailed proof of Lemma 1 can be found in [23]. Here, it suffices to point out that the proof is based on the Lyapunov function $V = x^{\top}\left(\sum_i^s P_i\chi_i\right)x$. The indicator functions χ_i cause discontinuities in the Lyapunov function at every switching time t_k. For later analysis in this paper, the following approximation of the potential increase in V associated with each switch is of greater importance:

$$V(t_k) \leq (1+\mu\Delta_{A_{\mathrm{m}}})V(t_k^-) \tag{10}$$

where $V(t_k^-)$ and $V(t_k)$ are the values of the Lyapunov function immediately before and after the switch, respectively. ■

Finally, given that the autonomous switched system $\dot{x} = A_{\mathrm{m}}(t)x(t)$ is exponentially stable for a dwell-time T_0, it can be concluded, with the same dwell-time constraint, that the state of the switched reference system (8) remains bounded for bounded reference signals [30].

C. Problem Formulation and Controller Design

The formal problem statement addressed by this paper reads as follows.

Problem 1 (State tracking): Given a PWA system (4) with known regions Ω_i and unknown subsystem parameters A_i, B_i, and f_i, design a state feedback control law $u(t)$ which stabilizes the PWA system and forces the state $x(t)$ to asymptotically track the state $x_{\mathrm{m}}(t)$ generated by the reference system (8).

In the following two sections, we propose a direct and an indirect approach to Problem 1. In both cases, the control law takes the form

$$u(t) = K_x(t)x(t) + K_r(t)r(t) + k_f(t) \tag{11}$$

with estimated control gains $K_x \in \mathbb{R}^{p\times n}$, $K_r \in \mathbb{R}^{p\times p}$, and $k_f \in \mathbb{R}^p$. Note that there is a different set of control gains $\{K_{xi}, K_{ri}, k_{fi}\}$ for each subsystem i and the control gains need to switch in synchrony with the system parameters. Hence, the control gain switching relies on the same indicator functions χ_i as the controlled system and the reference system, i.e., $K_x = \sum_{i=1}^s K_{xi}\chi_i$, $K_r = \sum_{i=1}^s K_{ri}\chi_i$, and $k_f = \sum_{i=1}^s k_{fi}\chi_i$. Therefore, as common in related works [16]–[27], Ω_i and χ_i are assumed to be known throughout the paper. Despite this technical assumption, the later simulation studies will suggest some degree of robustness against asynchronous switching. This also implies that, given the partitions of the PWA system are unknown, one may define a sufficiently large number of fictitious partitions, which deliver a satisfactory approximation of the actual switching behavior. In combination with the robustness to asynchronous switching this can yield an acceptable control performance.

Controlling the PWA system (4) with the state feedback controller (11) yields the closed-loop system

$$\dot{x} = \sum_{i=1}^{s}\chi_i\Big((A_i + B_iK_{xi})x + B_iK_{ri}r + B_ik_{fi} + f_i\Big) \tag{12}$$

which in order to solve Problem 1 should equal (8) for suitably chosen control gains. Therefore, to ensure feasibility of the tracking problem, it is assumed, as usual, that there exist nominal control gains K_{xi}^*, K_{ri}^*, and k_{fi}^* which fulfill the matching conditions

$$\begin{aligned} A_{\mathrm{m}i} &= A_i + B_iK_{xi}^* \\ B_{\mathrm{m}i} &= B_iK_{ri}^* \qquad\qquad \forall i = 1, \ldots, s \\ f_{\mathrm{m}i} &= f_i + B_ik_{fi}^* \end{aligned} \tag{13}$$

One practical implication of these matching conditions is that the structure of the controlled system needs to be known to some extend. Otherwise the designer cannot guarantee that the chosen reference systems yield feasible matching conditions.

The proposed direct and indirect adaptive control laws produce and update estimates of the nominal control gains in different ways, but both solve Problem 1. A well-known advantage of MRAC is that solutions to Problem 1 need not converge to the nominal control gains. However, parameter convergence improves robustness and transient performance in adaptive systems [31], [32]. Hence, in cases where the reference signal is not periodic or where it is desirable to monitor the nominal parameters, the convergence of estimated parameters according to the following problem is required.

Problem 2 (Convergence of parameter estimates): Besides the tracking according to Problem 1, ensure that all estimated parameters converge to their nominal values.

The proposed algorithms comply with Problem 2 under the assumption of suitable reference signals.

D. Signal Properties

In order to characterize suitable reference signals, we need to define some signal properties. First, persistence of excitation plays a fundamental role in the convergence of adaptive systems. The general idea is that if some internal signals are rich enough in their frequency content they are able to excite all system modes. We consider the standard notation presented in [33, Def. 4.3.1] and refer to a piecewise continuous signal vector $z : \mathbb{R}^+ \to \mathbb{R}^n$ as persistently exciting (PE) in $\mathbb{R}^n$ with a level of excitation $\alpha_0 > 0$ if there exist constants $\alpha_1, T_0 > 0$ such that $\alpha_1 I_n \geq \frac{1}{T_0}\int_t^{t+T_0} z(\tau)z^\top(\tau)d\tau \geq \alpha_0 I_n,\ \forall t \geq 0$. While, we rely on this PE definition in combination with dwell-time assumptions, a first notion of PE for switched systems was introduced in [34].

A common approach to ensure PE of a signal vector is to relate it to the spectral properties of the reference signals. As will be shown throughout the paper, the sufficiently rich property given in [33, Def. 5.2.1] serves well for this purpose. A signal $r : \mathbb{R}^+ \to \mathbb{R}$ is referred to as sufficiently rich of order $2n$, if it consists of at least n distinct frequencies. Therefore, a signal with n sinusoidal components, for instance, is sufficiently rich of order $2n$. Finally, a frequently applied relationship is that a reference signal sufficiently rich of order $n+1$ ensures PE of a controllable linear state-space system with state dimension n.

While the above properties can characterize the amount of information contained in signals, additional definitions are needed to analyze the input-output properties of a system. For this purpose, it is conventional to use $\mathcal{L}^p$ spaces defined for instance in [35, Section 2.7]. As $z \in \mathcal{L}^\infty$ implies boundedness of $z(t)$ and vice versa, we use the two statements synonymously.

E. Sliding Mode Solutions

One important aspect in the problem formulation has not been considered so far. That is, if the vector fields of two neighboring regions both point towards the switching hyperplane, the system trajectories cannot simply cross from one region to the other. Instead, the state of the system is unable to leave the hyperplane and an additional mode—the sliding mode—is created. As the solutions of such sliding modes may evolve in a substantially different manner from the individual subsystems, they must be carefully examined.

In the Filippov concept [36], the sliding mode solution is described by a unique convex combination of the contributing vector fields. In order to obtain such a convex combination in the presented framework, replace all indicator functions χ_i by $\bar{\chi}_i$ with the modified property that $\bar{\chi}_i$ now takes values in the interval $[0, 1]$. That means the property $\chi_i\chi_j = 0$ for $i \neq j$ does not hold for sliding modes. But for convexity $\sum_{i=1}^s \bar{\chi}_i = 1$ still holds. In the later proofs of Theorems 1 and 3, the Filippov concept is applied to show that the presented results also hold in the presence of sliding modes.

III. Direct MRAC

As pointed out in the previous section, the advantage of PWA systems over PWL systems for MRAC is the reduced *a priori* knowledge about the system. Hence, this section builds upon the framework in [23] and derives a direct MRAC algorithm for PWA systems. This generalization modifies the PE conditions in [23] and, hence, demands for an extended proof of parameter convergence.

We begin by defining errors between the current control gains and the nominal values: $\tilde{K}_{xi} = K_{xi} - K_{xi}^*$, $\tilde{K}_{ri} = K_{ri} - K_{ri}^*$, and $\tilde{k}_{fi} = k_{fi} - k_{fi}^*$. Let us rewrite the dynamics of the closed-loop system in terms of the reference system and the gain errors by adding and subtracting the expression $B_iK_{xi}^*x + B_iK_{ri}^*r + B_ik_{fi}^*$ in (12), which yields

$$\begin{aligned}\dot{x} &= \sum_{i=1}^{s}\chi_i\Big((A_i + B_iK_{xi}^*)x + B_iK_{ri}^*r + f_i + B_ik_{fi}^*\\&\quad + B_i\big((K_{xi}-K_{xi}^*)x + (K_{ri}-K_{ri}^*)r + (k_{fi}-k_{fi}^*)\big)\Big)\\&= A_\mathrm{m}x + B_\mathrm{m}r + f_\mathrm{m} + \sum_{i=1}^{s}\chi_iB_i\left(\tilde{K}_{xi}x + \tilde{K}_{ri}r + \tilde{k}_{fi}\right).\end{aligned} \tag{14}$$

Following from (14) and (8), the dynamics of the tracking error $e(t) = x(t) - x_\mathrm{m}(t)$ are

$$\dot{e} = A_\mathrm{m}e + \sum_{i=1}^{s}\chi_iB_i\left(\tilde{K}_{xi}x + \tilde{K}_{ri}r + \tilde{k}_{fi}\right). \tag{15}$$

If the control gains K_{xi}, K_{ri}, and k_{fi} take the nominal values defined in (13), then the error dynamics in (15) reduce to $\dot{e} = A_\mathrm{m}e$, which indicates that the tracking error e converges to zero exponentially for a stable reference system.

Consider now, however, the case in which the system parameters A_i, B_i, and f_i are unknown, which at the same time implies unknown nominal control gains. We make the usual assumption in multivariable MRAC [29] that there exist known quadratic matrices $S_i \in \mathbb{R}^{p\times p}$ for which $K_{ri}^*S_i$ is symmetric and positive definite: $K_{ri}^*S_i = (K_{ri}^*S_i)^\top \succ 0$. Then, initial estimates $K_{xi}(0) = K_{xi0}$, $K_{ri}(0) = K_{ri0}$, and $k_{fi}(0) = k_{fi0}$ are introduced and updated over time. Lyapunov analysis—which will be conducted shortly—suggests the following update laws for the estimated control gains

$$\begin{aligned}\dot{K}_{xi} &= -\Gamma_{xi}S_i^\top B_{\mathrm{m}i}^\top P_i e x^\top \chi_i\\\dot{K}_{ri} &= -\Gamma_{ri}S_i^\top B_{\mathrm{m}i}^\top P_i e r^\top \chi_i\\\dot{k}_{fi} &= -\Gamma_{fi}S_i^\top B_{\mathrm{m}i}^\top P_i e\, \chi_i\end{aligned} \tag{16}$$

where $\Gamma_{xi}, \Gamma_{ri}, \Gamma_{fi} \in \mathbb{R}^+$ are positive scaling constants, and where $P_i \in \mathbb{R}^{n\times n}$ is the symmetric, positive definite Lyapunov matrix of the i-th reference system satisfying $A_{\mathrm{m}i}^\top P_i + P_iA_{\mathrm{m}i} = -Q_{\mathrm{m}i}$ for some symmetric, positive definite matrix $Q_{\mathrm{m}i} \in \mathbb{R}^{n\times n}$. The tracking abilities under the adaptation (16) are summarized in the following theorem.

Theorem 1 (Direct MRAC for PWA systems with CQLF): Consider a reference system (8) for which a CQLF with $P = P_i, \forall i$ is known. Let the PWA system (4) with known regions Ω_i be controlled by the state feedback (11) with gains updated according to (16). Then, the state of the PWA system asymptotically tracks the state of the reference system.

Proof: Consider the following candidate Lyapunov function, which is quadratic in the tracking error e and the deviations $\tilde{K}_{xi}$, $\tilde{K}_{ri}$, $\tilde{k}_{fi}$ from the nominal control gains

$$V = \frac{1}{2}e^\top Pe + \frac{1}{2}\sum_{i=1}^{s}\Big(\text{tr}\left(\tilde{K}_{xi}^\top M_{\text{s}i}\tilde{K}_{xi}\right) + \text{tr}\left(\tilde{K}_{ri}^\top M_{\text{s}i}\tilde{K}_{ri}\right) + \tilde{k}_{fi}^\top M_{\text{s}i}\tilde{k}_{fi}\Big) \tag{17}$$

where $M_{\text{s}i} = (K_{ri}^* S_i)^{-1} \in \mathbb{R}^{p\times p}$. As mentioned before, the variables S_i are chosen such that the inverse exists and is positive definite. The time derivative of (17) along (15) is

$$\begin{aligned}\dot{V} = e^\top &\left(\frac{1}{2}\sum_{i=1}^{s}\chi_i(A_{\text{m}i}^\top P + PA_{\text{m}i})\right)e \\ &+ \sum_{i=1}^{s}\left(\chi_i e^\top PB_i\left(\tilde{K}_{xi}x + \tilde{K}_{ri}r + \tilde{k}_{fi}\right)\right) \\ &+ \sum_{i=1}^{s}\left(\text{tr}\left(\tilde{K}_{xi}^\top M_{\text{s}i}\dot{\tilde{K}}_{xi}\right) + \text{tr}\left(\tilde{K}_{ri}^\top M_{\text{s}i}\dot{\tilde{K}}_{ri}\right) + \tilde{k}_{fi}^\top M_{\text{s}i}\dot{\tilde{k}}_{fi}\right).\end{aligned} \tag{18}$$

At this point, the rationale behind the update laws in (16) can be seen as a suitable choice to cancel all terms in the last two lines of (18). Note that, for simplicity and without loss of generality, the scaling constants Γ_{xi}, Γ_{ri}, Γ_{fi} are neglected throughout the proof. Inserting the update laws (16) with $P_i = P$ in (18) yields

$$\begin{aligned}\dot{V} = -e^\top &\left(\frac{1}{2}\sum_{i=1}^{s}Q_{\text{m}i}\chi_i\right)e \\ &+ \sum_{i=1}^{s}\chi_i\Big(e^\top PB_i\tilde{K}_{xi}x - \text{tr}\left(\tilde{K}_{xi}^\top M_{\text{s}i}S_i^\top B_{\text{m}i}^\top Pex^\top\right) \\ &+ e^\top PB_i\tilde{K}_{ri}r - \text{tr}\left(\tilde{K}_{ri}^\top M_{\text{s}i}S_i^\top B_{\text{m}i}^\top Per^\top\right) \\ &+ e^\top PB_i\tilde{k}_{fi} - \tilde{k}_{fi}^\top M_{\text{s}i}S_i^\top B_{\text{m}i}^\top Pe\Big)\end{aligned} \tag{19}$$

for which the terms under the second summation cancel out. To see this, note that $M_{\text{s}i}S_i^\top B_{\text{m}i}^\top = M_{\text{s}i}S_i^\top (K_{ri}^*)^\top B_i^\top = M_{\text{s}i}M_{\text{s}i}^{-1}B_i^\top = B_i^\top$ simplifies the first trace to $\text{tr}(\tilde{K}_{xi}^\top B_i^\top Pex^\top)$. As $\text{tr}(X) = \text{tr}(X^\top)$, this is equivalent to $\text{tr}(xe^\top PB_i\tilde{K}_{xi})$, which can be rewritten as $\text{tr}(e^\top PB_i\tilde{K}_{xi}x)$ due to the property $\text{tr}(\text{XYZ}) = \text{tr}(\text{YZX})$. Next, note that the obtained expression inside the trace operator is a scalar and cancels the corresponding term in (19). The same steps can be performed to cancel the other terms in (19).

It follows that the derivative of V is negative semidefinite:

$$\dot{V} = -e^\top\left(\frac{1}{2}\sum_{i=1}^{s}Q_{\text{m}i}\chi_i\right)e \le 0. \tag{20}$$

Considering additionally the case of potential sliding modes, one must furthermore evaluate the derivative of V along the sliding mode solution. Hence, with the Filippov solution concept all convex combinations of the vector fields contributing to the sliding modes are analyzed by replacing $\chi_i \in \{0,1\}$ with $\bar{\chi}_i \in [0,1]$ in the above derivation. As in [20], doing so confirms negative semidefiniteness of V as $\dot{V} = -e^\top\left(\frac{1}{2}\sum_{i=1}^{s}Q_{\text{m}i}\bar{\chi}_i\right)e \le 0$. Note that this derivation shows stability of the sliding mode solution without determining the uniquely defined Fillipov vector field of the actual sliding mode.

In summary, the error system is shown to be stable under arbitrary fast switching and sliding modes, and the errors e, $\tilde{K}_{xi}$, $\tilde{K}_{ri}$, and $\tilde{k}_{fi}$ are bounded ($\in \mathcal{L}^\infty$). Boundedness of e, together with a stable reference system (i.e., $x_\text{m} \in \mathcal{L}^\infty$), yields $x \in \mathcal{L}^\infty$. Moreover, from (15) it follows that $\dot{e} \in \mathcal{L}^\infty$. This permits application of Barbalat's Lemma on (20) and yields $\lim_{t\to\infty} e(t) = 0$, which completes the proof. ∎

In most cases, it is desirable to choose an LTI reference system as it defines the same reference system for all subsystems or equivalently over the entire state-input space. This choice causes the nonlinear system (4) to behave like a linear system and naturally satisfies the assumption of a CQLF in Theorem 1. Depending on the PWA system, however, the matching conditions (13) might not allow for an LTI reference system. In such cases, a PWA reference system must be employed, for which a CQLF might not exist. In this case, the following theorem considers a PWA reference system and shows that under a certain dwell-time assumption and under sufficiently rich reference signals, all tracking errors and parameter errors converge to zero.

Theorem 2 (Direct MRAC and parameter convergence for PWA systems without CQLF): Consider the reference system (8) without CQLF and let the PWA system (4) with known regions Ω_i be controlled by the state feedback (11) with gains updated according to (16). Let the reference signals in r be sufficiently rich of order $n+1$ with distinct frequencies. Furthermore, let the resulting switching signal be sufficiently slow with dwell time T_{dwell} and cause repeated activation of all subsystems. If the input matrices B_i have full column rank, if the system matrices $A_{\text{m}i}$ are invertible, and if the the pairs $(A_{\text{m}i}, B_{\text{m}i})$ are controllable, then all errors e, $\tilde{K}_{xi}$, $\tilde{K}_{ri}$, and $\tilde{k}_{fi}$ asymptotically converge to zero for $t\to\infty$.

Proof: The proof of Theorem 2 begins with analyzing the convergence of the active subsystem parameters. Consider a time interval $t \in [t_{k-1}, t_k)$ and let i be such that $\chi_i(t) = 1$. Thus, the error (15) reads as

$$\dot{e} = A_{\text{m}i}e + B_i\left[\,\tilde{K}_{xi}\;\tilde{K}_{ri}\;\tilde{k}_{fi}\,\right]\begin{bmatrix}x\\r\\1\end{bmatrix}. \tag{21}$$

We make use of the Kronecker product $\otimes$ and define

$$\Psi_r := \begin{bmatrix}x\\r\\1\end{bmatrix}\otimes I_n \qquad \tilde{\vartheta}_i^B := \text{vec}\left(B_i\left[\,\tilde{K}_{xi}\;\tilde{K}_{ri}\;\tilde{k}_{fi}\,\right]\right) \tag{22}$$

where the operator $\text{vec}(\cdot)$ concatenates the columns of a matrix $(\cdot)$ in a single column vector. With (22), rewrite (21) as

$$\dot{e} = A_{\text{m}i}e + \Psi_r^\top\tilde{\vartheta}_i^B. \tag{23}$$

In the next step, the time derivative

$$\dot{\tilde{\vartheta}}_i^B = \operatorname{vec}\left(B_i \left[\dot{\tilde{K}}_{xi}\ \dot{\tilde{K}}_{ri}\ \dot{\tilde{k}}_{fi}\right]\right) \tag{24}$$

is formulated in terms of the update laws (16) (with neglected scaling constants for simplicity). After inserting (16) and performing various steps of algebraic reformulation, the derivative (24) relates to the tracking error e and Ψ_r in the following way:

$$\dot{\tilde{\vartheta}}_i^B = -\Psi_r \underbrace{B_i M_{\mathrm{s}i}^{-1} B_i^\top P_i}_{=:P_{i2}} e = -\Psi_r P_{i2} e. \tag{25}$$

Equations (23) and (25) can be combined in the following dynamic system, which is a standard form frequently appearing in adaptive systems

$$\begin{bmatrix} \dot{e} \\ \dot{\tilde{\vartheta}}_i^B \end{bmatrix} = \begin{bmatrix} A_{\mathrm{m}} & \Psi_r^\top \\ -\Psi_r P_{i2} & 0 \end{bmatrix} \begin{bmatrix} e \\ \tilde{\vartheta}_i^B \end{bmatrix}. \tag{26}$$

Exponential stability of the equilibrium $e = 0$ and $\tilde{\vartheta}_i^B = 0$ can thus be concluded if $[x^\top, r^\top, 1]^\top$ is PE [33, Lemma 5.6.3]. Following the same arguments as in [26, Theorem 1], we determine that $[x_{\mathrm{m}}^\top,\ r^\top,\ 1]^\top$ is PE under the given assumptions of controllable pairs $(A_{\mathrm{m}i}, B_{\mathrm{m}i})$, invertible $A_{\mathrm{m}i}$, and sufficiently rich reference signals with distinct frequencies. Finally, according to [37], PE of the reference system ensures PE of the controlled system. An alternative reasoning is that for $e \to 0$, we have $x = x_{\mathrm{m}}$ and, thus, if the reference system is PE also the controlled system is PE. Hence, on the interval $t \in [t_{k-1}, t_k)$, the errors e and $\tilde{\vartheta}_i^B$ converge towards zero exponentially.

Note that due to its assumed full column rank, the null space of B_i is empty. Therefore, the exponential convergence of $\tilde{\vartheta}_i^B$ also implies exponential convergence of $\tilde{\vartheta}_i$, where $\tilde{\vartheta}_i = \operatorname{vec}([\tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{k}_{fi}])$. Thus, for $z_i = [e^\top, \tilde{\vartheta}_i^\top]^\top$, there exist constants $a_i > 0$ and $\lambda_i > 0$ such that

$$\|z_i(t)\| \le a_i \mathrm{e}^{-\lambda_i (t - t_{k-1})} \|z_i(t_{k-1})\|. \tag{27}$$

It is well known that exponential stability of all subsystems alone does not imply stability of the switched system. In order to conclude asymptotic stability, or convergence in the present case, it has to be shown that the Lyapunov function values at switching times form a decreasing sequence [38]. Therefore, in the next step, the worst-case (i.e., minimal) decrease of the Lyapunov function (17) over the interval $t \in [t_{k-1}, t_k)$, for which $\chi_i(t) = 1$, is compared with the worst-case (i.e., maximal) increment of the Lyapunov function associated with the switch at time instance t_k. Note that the control gains of inactive subsystems are constant over the considered interval. Therefore, the term

$$C_{k-1} = \sum_{j=1, j\neq i}^{s} \operatorname{tr}(\tilde{K}_{xj}^\top M_{sj} \tilde{K}_{xj}) + \operatorname{tr}(\tilde{K}_{rj}^\top M_{sj} \tilde{K}_{rj}) + \tilde{k}_{fj}^\top M_{sj} \tilde{k}_{fj}$$

is also constant over $[t_{k-1}, t_k)$. Since V is quadratic in z_i, (27) can bound the decrease of V for the adapted gains and tracking error from above by

$$\begin{aligned} V(t_k^-) - C_{k-1} &\le \lambda_{\max}(P_i) \|z(t_k^-)\|^2 \\ &\le \alpha_i (a_i \mathrm{e}^{-\lambda_i (t_k^- - t_{k-1})} \|z_i(t_{k-1})\|)^2 \\ &\le \frac{\alpha_i}{\beta_i} a_i^2 \mathrm{e}^{-2\lambda_i (t_k^- - t_{k-1})} (V(t_{k-1}) - C_{k-1}) \end{aligned} \tag{28}$$

with $\alpha_i = \lambda_{\max}(P_i)$ and $\beta_i = \lambda_{\min}(P_i)$, and where the last step follows from $V(t_{k-1}) - C_{k-1} \ge \beta_i \|z_i(t_{k-1})\|^2$. In order for the Lyapunov function to form a decreasing sequence at switching times, the decrease in (28) must be greater than the increment induced by the switching at time t_k. Note that the switch does not affect the terms in C_{k-1}. Hence, the bound given in (10) only applies to $V(t_k) - C_{k-1}$ and $V(t_k^-) - C_{k-1}$, which yields the worst-case increase due to switching at t_k

$$V(t_k) - C_{k-1} \le (1 + \mu \Delta_{A_{\mathrm{m}}})(V(t_k^-) - C_{k-1}). \tag{29}$$

Combining the decrease in (28) with the increase in (29) characterizes the worst-case evolution between consecutive switches as

$$V(t_k) - C_{k-1} \le \rho (V(t_{k-1}) - C_{k-1}) \tag{30}$$

with

$$\rho := \frac{\alpha_i}{\beta_i} a_i^2 (1 + \mu \Delta_{A_{\mathrm{m}}}) \mathrm{e}^{-2\lambda_i (t_k - t_{k-1})}. \tag{31}$$

For a decreasing sequence, it must hold that $\rho \le 1$ or equivalently

$$t_k - t_{k-1} \ge \frac{1}{2\lambda_i} \ln\left(\frac{\alpha_i}{\beta_i} a_i^2 (1 + \mu \Delta_{A_{\mathrm{m}}})\right) =: T_i \tag{32}$$

which defines a dwell time T_i for the i-th subsystem. We conclude that stability and convergence of the direct adaptive control algorithm is given if the switching of the controlled system obeys a dwell time $T_{\text{dwell}} > \max\{T_{\mathrm{m}}, T_1, \ldots, T_s\}$, i.e., $t_k - t_{k-1} \ge T_{\text{dwell}}, \forall k > 1$. ■

Remark 1: Note that the assumption of known S_i constitutes—besides the known regions of the PWA system—the biggest limitation of the proposed algorithm. In the multiinput case the retrieval of suitable S_i may become nontrivial. For the single-input case, however, it reduces to $S_i = \operatorname{sign}(K_{ri}^*)$. Hence, S_i corresponds to the usual assumption of known input effectiveness. Relaxing the required knowledge about S_i is an important topic and has received some attention in the multivariable adaptive control community [29]. In [33], it is shown that for positive definite and symmetric nominal gain matrices K_{ri}^*, the design matrix can be chosen as identity matrix: $S_i = I$. Another approach, discussed in [39, Section 9.1.2], parametrizes K_{ri}^* in terms of an LDU decomposition with lower and upper triangular matrices L and U, and a diagonal matrix D. For the resulting stable convergent adaptive controller, only L and the sign of the diagonal entries in D need to be known.

Remark 2: When considering single input systems in canonical form, it is interesting to note that the direct MRAC algorithm presented here also relates to the extended MCS algorithm in [19] and [20]. Two characteristics differentiate the algorithms.

First, Di Bernardo *et al.* define the state-space partitions for the controlled PWA system and the reference system separately. Therefore, their control gains consist of two parts, one switching according to the controlled system and one switching according to the reference system. Second, in the MCS framework, the control gains depend on the tracking error through a proportional and an integral term. Due to the differential equations in (16), the algorithm presented here only contains the integral term. As the proportional term has a beneficial effect on the convergence of the tracking error, it is interesting to investigate in future work whether adding such a term in the presented adaptation schemes can further improve the control performance.

This section introduced MRAC with direct gain adaptation according to (16). Compared to its PWL counterpart in [23], the consideration of affine terms enlarges the class of applicable systems. Also, it is not restricted to single input system in canonical form as [19], [20]. It was determined in Theorem 1 that the state tracking task can be achieved without imposing additional requirements on the excitation of the reference system. Theorem 2 on the other side suggests excitation with sufficiently rich reference signals in case convergence to the nominal control gains is required. This is for instance the case if a nonperiodic reference signal needs to be tracked precisely or the nominal gains are to be monitored. For monitoring purposes it might, however, be more interesting to analyze the system parameters A_i, B_i, and f_i instead of the control gains. This is the motivation for the indirect MRAC algorithm derived in the next section, which has the advantage that estimates of the system parameters are generated in parallel to the control task.

IV. Indirect MRAC

In the direct MRAC approach, the control gains are directly tuned based on the tracking error. In the indirect case, which is discussed next, the adaptation of control gains is based on time-varying estimates $\hat{A}_i(t)$, $\hat{B}_i(t)$, and $\hat{f}_i(t)$ of the system parameters A_i, B_i, and f_i. Under the certainty equivalence principle, the estimates are handled as if they are the true parameters at all times [29]. Based on the estimates, the control gains K_{xi}, K_{ri}, and k_{fi} are derived from the matching conditions (13).

The estimates $\hat{A}_i$, $\hat{B}_i$, and $\hat{f}_i$ are determined by parameter identifiers, which are a standard tool for the adaptive identification of LTI systems, see [33, chapter 5]. Recently in [25] and [26], parameter identifiers were extended to the setting of switched linear and switched affine systems. Parameter identifiers in [26] predict the state x in terms of the parameter estimates ($\hat{A}_i$, $\hat{B}_i$, $\hat{f}_i$) of the currently active subsystem ($\chi_i = 1$). The dynamics of the predicted state $\hat{x} \in \mathbb{R}^n$ are

$$\dot{\hat{x}} = A_{\mathrm{m}}\hat{x} + \sum_{i=1}^{s}\left((\hat{A}_i - A_{\mathrm{m}i})x + \hat{B}_i u + \hat{f}_i\right)\chi_i \tag{33}$$

where $A_{\mathrm{m}} = \sum_{i=1}^{s} A_{\mathrm{m}i}\chi_i$ is the switched system matrix of the stable reference system (8). Let $\tilde{x} = \hat{x} - x$ be the prediction error between the predictor (33) and the controlled system (4). This prediction error contains information about the mismatch between parameter estimates and true parameters and is, hence, an essential element in the update law. The parameter update laws proposed in [26] are

$$\dot{\hat{A}}_i = -\chi_i P_i \tilde{x} x^\top \quad \dot{\hat{B}}_i = -\chi_i P_i \tilde{x} u^\top \quad \dot{\hat{f}}_i = -\chi_i P_i \tilde{x}$$

where the indicator functions χ_i ensure that only the currently active subsystem parameters are updated. Let $\tilde{A}_i = \hat{A}_i - A_i$, $\tilde{B}_i = \hat{B}_i - B_i$, and $\tilde{f}_i = \hat{f}_i - f_i$ be the parameter errors. Then, the derivative of the prediction error in terms of these parameter errors is

$$\dot{\tilde{x}} = A_{\mathrm{m}}\tilde{x} + \sum_{i=1}^{s}\left(\tilde{A}_i x + \tilde{B}_i u + \tilde{f}_i\right)\chi_i. \tag{34}$$

Following the certainty equivalence principle, one can rearrange the matching conditions (13) and obtain the following algebraic equations for the controller gains:

$$\begin{aligned} K_{xi} &= \hat{B}_i^{+}(A_{\mathrm{m}i} - \hat{A}_i) \\ K_{ri} &= \hat{B}_i^{+} B_{\mathrm{m}i} \\ k_{fi} &= \hat{B}_i^{+}(f_{\mathrm{m}i} - \hat{f}_i) \end{aligned}$$

where $\hat{B}_i^{+}$ represents the Moore–Penrose pseudoinverse of $\hat{B}_i$. This pseudo inverse constitutes the main obstacle in this approach as it introduces singularities into the system. In case of singularities, the control gains take unbounded values which violates the stability requirement. Hence, additional measures need to be taken to ensure stability. One approach is the use of projection operators to avoid singularities [33, Sect. 8.5.5]. In order to use projection operators, however, additional knowledge about the system (more precisely K_{ri}^*) as compared to the direct adaptive control algorithm in Section III would be needed.

Instead of introducing projection operators, we follow ideas previously discussed in [35], [40], and define dynamic update laws for the control parameters (K_{xi}, K_{ri}, k_{fi}) which are based on the current parameter estimates ($\hat{A}_i, \hat{B}_i, \hat{f}_i$). This preserves the indirect nature of the adaptation algorithm (hence provides estimates of the system parameters) and at the same time relies on the same previous knowledge about the system as the direct approach. Our contribution in this context is the extension of [35, Section 3.3] and [40] to switched affine systems. We also present a thorough proof of parameter convergence, which is not given in [35] and [40].

We begin by defining the following closed-loop estimation errors, which can be interpreted as errors in the matching conditions (13):

$$\begin{aligned} \varepsilon_{Ai} &= \hat{A}_i + \hat{B}K_{xi} - A_{\mathrm{m}i} \\ \varepsilon_{Bi} &= \hat{B}_i K_{ri} - B_{\mathrm{m}i} \\ \varepsilon_{fi} &= \hat{f}_i + \hat{B}k_{fi} - f_{\mathrm{m}i}. \end{aligned} \tag{35}$$

The closed-loop estimation errors (35) characterize the mismatch between the desired system dynamics and the estimated closed-loop system dynamics, which are obtained for the estimated system parameters and control gains.

Lyapunov-based stability analysis (which will be presented shortly) suggests the next steps. Let the closed-loop estimation errors drive the adaptation of the control gains as follows:

$$\begin{aligned}\dot{K}_{xi} &= -S_i^\top B_{\mathrm{m}i}^\top \varepsilon_{Ai}\\ \dot{K}_{ri} &= -S_i^\top B_{\mathrm{m}i}^\top \varepsilon_{Bi}\\ \dot{k}_{fi} &= -S_i^\top B_{\mathrm{m}i}^\top \varepsilon_{fi}\end{aligned} \tag{36}$$

where S_i is the same—assumed to be known—matrix as in the direct adaptive control algorithm, for which $K_{ri}^* S_i$ is symmetric and positive definite. In order to maintain convergence for the new control gain adaptation (36), also the adaptation of system parameters needs to include the closed-loop estimation errors. Stability analysis suggests

$$\begin{aligned}\dot{\hat{A}}_i &= -\chi_i P_i \tilde{x} x^\top - \varepsilon_{Ai}\\ \dot{\hat{B}}_i &= -\chi_i P_i \tilde{x} u^\top - \varepsilon_{Ai} K_{xi}^\top - \varepsilon_{Bi} K_{ri}^\top - \varepsilon_{fi} k_{fi}^\top\\ \dot{\hat{f}}_i &= -\chi_i P_i \tilde{x} - \varepsilon_{fi}.\end{aligned} \tag{37}$$

We are now ready to state the following theorem, which characterizes indirect MRAC for PWA systems for reference systems whose subsystems have a CQLF.

Theorem 3 (Indirect MRAC for PWA systems with CQLF): Consider a reference system (8) for which a CQLF with $P = P_i, \forall i$ is known. Let the PWA system (4) with known regions Ω_i be controlled by the state feedback (11) with gains updated according to (36), which is based on (33), (35), and (37). Then, the state of the PWA system asymptotically tracks the state of the reference system.

Proof: The proof is based on the candidate Lyapunov function

$$\begin{aligned}V = \frac{1}{2}\tilde{x}^\top P \tilde{x} + \frac{1}{2}\sum_{i=1}^{s}\Big(&\mathrm{tr}\left(\tilde{A}_i^\top \tilde{A}_i\right) + \mathrm{tr}\left(\tilde{B}_i^\top \tilde{B}_i\right) + \tilde{f}_i^\top \tilde{f}_i\\ &+ \mathrm{tr}\left(\tilde{K}_{xi}^\top M_{\mathrm{s}i} \tilde{K}_{xi}\right) + \mathrm{tr}\left(\tilde{K}_{ri}^\top M_{\mathrm{s}i}\tilde{K}_{ri}\right) + \tilde{k}_{fi}^\top M_{\mathrm{s}i}\tilde{k}_{fi}\Big),\end{aligned}$$

whose time derivative along $\dot{\tilde{x}}$ in (34) is

$$\begin{aligned}\dot{V} = \tilde{x}^\top &\left(\frac{1}{2}\sum_{i=1}^{s}(A_{\mathrm{m}i}^\top P + P A_{\mathrm{m}i})\chi_i\right)\tilde{x}\\ &+ \sum_{i=1}^{s}\Big(\tilde{x}^\top P\left(\tilde{A}_i x + \tilde{B}_i u + \tilde{f}_i\right)\chi_i\\ &+ \mathrm{tr}\left(\tilde{A}_i^\top \dot{\hat{A}}_i\right) + \mathrm{tr}\left(\tilde{B}_i^\top \dot{\hat{B}}_i\right) + \tilde{f}_i^\top \dot{\hat{f}}_i\\ &+ \mathrm{tr}\left(\tilde{K}_{xi}^\top M_{\mathrm{s}i}\dot{\tilde{K}}_{xi}\right) + \mathrm{tr}\left(\tilde{K}_{ri}^\top M_{\mathrm{s}i}\dot{\tilde{K}}_{ri}\right) + \tilde{k}_{fi}^\top M_{\mathrm{s}i}\dot{\tilde{k}}_{fi}\Big).\end{aligned}$$

At this point, with equalities of the form $\tilde{x}^\top P \tilde{A}_i x = \mathrm{tr}(\tilde{A}_i^\top P \tilde{x} x^\top)$, the first elements in the update laws (37) for $\hat{A}$, $\hat{B}$, and $\hat{f}$ prove suitable to cancel out $\tilde{x}^\top P(\tilde{A}_i x + \tilde{B}_i u + \tilde{f}_i)\chi_i$. The elements containing $\varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi}$ on the other side will turn out to cancel those terms associated with the control gains $\tilde{K}_{xi}$, $\tilde{K}_{ri}$, $\tilde{k}_{fi}$. To see this, insert the update laws (37) with $P_i = P$, which yields

$$\begin{aligned}\dot{V} = &-\tilde{x}^\top\left(\frac{1}{2}\sum_{i=1}^{s} Q_{\mathrm{m}i}\chi_i\right)\tilde{x} + \sum_{i=1}^{s}\Big(-\mathrm{tr}\left(\tilde{A}_i^\top \varepsilon_{Ai}\right)\\ &- \mathrm{tr}\Big(\tilde{B}_i^\top\left(\varepsilon_{Ai}K_{xi}^\top + \varepsilon_{Bi}K_{ri}^\top + \varepsilon_{fi}k_{fi}^\top\right)\Big) - \tilde{f}_i^\top \varepsilon_{fi}\\ &+ \mathrm{tr}\left(\tilde{K}_{xi}^\top M_{\mathrm{s}i}\dot{\tilde{K}}_{xi}\right) + \mathrm{tr}\left(\tilde{K}_{ri}^\top M_{\mathrm{s}i}\dot{\tilde{K}}_{ri}\right) + \tilde{k}_{fi}^\top M_{\mathrm{s}i}\dot{\tilde{k}}_{fi}\Big).\end{aligned}$$

With some algebraic reformulations, this expression suggest the proposed adaptation of control gains in (36). Inserting the update laws (36) for $\dot{\tilde{K}}_{xi}$, $\dot{\tilde{K}}_{ri}$, and $\dot{\tilde{k}}_{fi}$ gives

$$\begin{aligned}\dot{V} = &-\tilde{x}^\top\left(\frac{1}{2}\sum_{i=1}^{s} Q_{\mathrm{m}i}\chi_i\right)\tilde{x} + \sum_{i=1}^{s}\Big(-\mathrm{tr}\left(\tilde{A}_i^\top \varepsilon_{Ai}\right)\\ &- \mathrm{tr}\Big(\tilde{B}_i^\top\left(\varepsilon_{Ai}K_{xi}^\top + \varepsilon_{Bi}K_{ri}^\top + \varepsilon_{fi}k_{fi}^\top\right)\Big) - \tilde{f}_i^\top \varepsilon_{fi}\\ &- \mathrm{tr}\left(\tilde{K}_{xi}^\top M_{\mathrm{s}i}S_i^\top B_{\mathrm{m}i}^\top \varepsilon_{Ai}\right) - \mathrm{tr}\left(\tilde{K}_{ri}^\top M_{\mathrm{s}i}S_i^\top B_{\mathrm{m}i}^\top \varepsilon_{Bi}\right)\\ &- \tilde{k}_{fi}^\top M_{\mathrm{s}i}S_i^\top B_{\mathrm{m}i}^\top \varepsilon_{fi}\Big)\end{aligned}$$

which by exploiting

$$\begin{aligned}M_{\mathrm{s}i}S_i^\top B_{\mathrm{m}i}^\top &= M_{\mathrm{s}i}S_i^\top (B_i K_{ri}^*)^\top = M_{\mathrm{s}i}S_i^\top {K_{ri}^*}^\top B_i^\top\\ &= M_{\mathrm{s}i}M_{\mathrm{s}i}^{-1}B_i^\top = B_i^\top\end{aligned}$$

and grouping traces of matrices with equal dimension, results in

$$\begin{aligned}\dot{V} = &-\tilde{x}^\top\left(\frac{1}{2}\sum_{i=1}^{s} Q_{\mathrm{m}i}\chi_i\right)\tilde{x}\\ &+ \sum_{i=1}^{s}\Big(-\mathrm{tr}\left(\tilde{A}_i^\top \varepsilon_{Ai} + \tilde{K}_{xi}^\top B_i^\top \varepsilon_{Ai}\right) - \tilde{f}_i^\top \varepsilon_{fi} - \tilde{k}_{fi}^\top B_i^\top \varepsilon_{fi}\\ &- \mathrm{tr}\Big(\tilde{B}_i^\top(\varepsilon_{Ai}K_{xi}^\top + \varepsilon_{Bi}K_{ri}^\top + \varepsilon_{fi}k_{fi}^\top) + \tilde{K}_{ri}^\top B_i^\top \varepsilon_{Bi}\Big)\Big).\end{aligned} \tag{38}$$

Inside the trace operators, one can justify the transformations

$$\begin{aligned}\left(\tilde{A}_i^\top + \tilde{K}_{xi}^\top B_i^\top\right)\varepsilon_{Ai} &= \left(\hat{A}_i^\top - A_i^\top + K_{xi}^\top B_i^\top - {K_{xi}^*}^\top B_i^\top\right)\varepsilon_{Ai}\\ &= \left(\hat{A}_i^\top - A_{\mathrm{m}i}^\top + K_{xi}^\top B_i^\top\right)\varepsilon_{Ai}\\ &= \left(\hat{A}_i^\top - A_{\mathrm{m}i}^\top + K_{xi}^\top \hat{B}_i^\top - K_{xi}^\top \tilde{B}_i^\top\right)\varepsilon_{Ai}\\ &= \left(\varepsilon_{Ai} - K_{xi}^\top \tilde{B}_i^\top\right)\varepsilon_{Ai}\end{aligned}$$

and

$$\begin{aligned}&\tilde{B}_i^\top(\varepsilon_{Ai}K_{xi}^\top+\varepsilon_{Bi}K_{ri}^\top+\varepsilon_{fi}k_{fi}^\top)+\tilde{K}_{ri}^\top B_i^\top\varepsilon_{Bi}\\&\quad=\tilde{B}_i^\top\varepsilon_{Ai}K_{xi}^\top+\tilde{B}_i^\top\varepsilon_{fi}k_{fi}^\top+\hat{B}_i^\top\varepsilon_{Bi}K_{ri}^\top\\&\qquad-B_i^\top\varepsilon_{Bi}K_{ri}^\top+K_{ri}^\top B_i^\top\varepsilon_{Bi}-K_{ri}^{*\top}B_i^\top\varepsilon_{Bi}\\&\quad=\tilde{B}_i^\top\varepsilon_{Ai}K_{xi}^\top+\tilde{B}_i^\top\varepsilon_{fi}k_{fi}^\top+\hat{B}_i^\top\varepsilon_{Bi}K_{ri}^\top-K_{ri}^{*\top}B_i^\top\varepsilon_{Bi}\\&\quad=\tilde{B}_i^\top\varepsilon_{Ai}K_{xi}^\top+\tilde{B}_i^\top\varepsilon_{fi}k_{fi}^\top+\hat{B}_i^\top\varepsilon_{Bi}K_{ri}^\top-B_{\mathrm{m}i}^\top\varepsilon_{Bi}\\&\quad=\tilde{B}_i^\top\varepsilon_{Ai}K_{xi}^\top+\tilde{B}_i^\top\varepsilon_{fi}k_{fi}^\top+\varepsilon_{Bi}^\top\varepsilon_{Bi}.\end{aligned}$$

Hence, together with

$$\begin{aligned}\left(\tilde{f}_i^\top+\tilde{k}_{fi}^\top B_i^\top\right)\varepsilon_{fi}&=\left(\hat{f}_i^\top-f_i^\top+k_{fi}^\top B_i^\top-k_{fi}^{*\top}B_i^\top\right)\varepsilon_{fi}\\&=\left(\hat{f}_i^\top-f_{\mathrm{m}i}^\top+k_{fi}^\top B_i^\top\right)\varepsilon_{fi}\\&=\left(\hat{f}_i^\top-f_{\mathrm{m}i}^\top+k_{fi}^\top\hat{B}_i^\top-k_{fi}^\top\tilde{B}_i^\top\right)\varepsilon_{fi}\\&=\varepsilon_{fi}^\top\varepsilon_{fi}-k_{fi}^\top\tilde{B}_i^\top\varepsilon_{fi}\end{aligned}$$

the derivative of the Lyapuonv function in (38) can be rewritten as

$$\begin{aligned}\dot{V}=&-\tilde{x}^\top\left(\frac{1}{2}\sum_{i=1}^{s}Q_{\mathrm{m}i}\chi_i\right)\tilde{x}+\sum_{i=1}^{s}\Bigg(-\operatorname{tr}\left(\varepsilon_{Ai}^\top\varepsilon_{Ai}\right)\\&-\operatorname{tr}\left(\varepsilon_{Bi}^\top\varepsilon_{Bi}\right)-\varepsilon_{fi}^\top\varepsilon_{fi}+\operatorname{tr}\left(K_{xi}^\top\tilde{B}_i^\top\varepsilon_{Ai}\right)\\&-\operatorname{tr}\left(\tilde{B}_i^\top\varepsilon_{Ai}K_{xi}^\top+\tilde{B}_i^\top\varepsilon_{fi}k_{fi}^\top\right)+k_{fi}^\top\tilde{B}_i^\top\varepsilon_{fi}\Bigg)\\=&-\tilde{x}^\top\left(\frac{1}{2}\sum_{i=1}^{s}Q_{\mathrm{m}i}\chi_i\right)\tilde{x}\\&-\sum_{i=1}^{s}\left(\operatorname{tr}\left(\varepsilon_{Ai}^\top\varepsilon_{Ai}\right)+\operatorname{tr}\left(\varepsilon_{Bi}^\top\varepsilon_{Bi}\right)+\varepsilon_{fi}^\top\varepsilon_{fi}\right)\end{aligned}$$

which is negative semidefinite.

As in the direct case, special attention must be given to potential sliding modes. In order to analyze all convex combinations of vector fields contributing to the sliding mode, replace the binary indicator functions χ_i by $\bar{\chi}_i$ in the above derivation. Since all steps are also valid for $\bar{\chi}_i$, the property $\dot{V}<0,\forall\tilde{x}\neq0$ is verified for all possible sliding mode solutions.

It follows that V is a Lyapunov function decreasing also along possible sliding mode solutions and therefore $\tilde{x}$, $\hat{A}_i$, $\hat{B}_i$, $\hat{f}_i$, K_{xi}, K_{ri}, and k_{fi} are all bounded (equivalently $\in\mathcal{L}^\infty$). At the same time, according to (35), this implies boundedness of ε_{Ai}, ε_{Bi}, and ε_{fi}. Furthermore, the Lyapunov function and its derivative imply that $\varepsilon_{Ai},\varepsilon_{Bi},\varepsilon_{fi}\in\mathcal{L}^2$. It remains to be shown that also x, $\hat{x}$, and u are bounded, which cannot be concluded directly from boundedness of $\tilde{x}$. Instead, rewrite (33) with feedback (11) in terms of ε_{Ai}, ε_{Bi}, and ε_{fi} as

$$\begin{aligned}\dot{\hat{x}}&=A_{\mathrm{m}}\hat{x}+(\hat{A}-A_{\mathrm{m}})x+\hat{B}u+\hat{f}\\&=A_{\mathrm{m}}\hat{x}+(\hat{A}-A_{\mathrm{m}})x+\hat{B}(K_xx+K_rr+k_f)+\hat{f}\\&=A_{\mathrm{m}}\hat{x}+\varepsilon_Ax+\hat{B}(K_rr+k_f)+\hat{f}\\&=(A_{\mathrm{m}}+\varepsilon_A)\hat{x}-\varepsilon_A\tilde{x}+(B_{\mathrm{m}}-\varepsilon_B)r+f_{\mathrm{m}}-\varepsilon_f\end{aligned}\tag{39}$$

where $\varepsilon_A=\sum_{i=1}^{s}\varepsilon_{Ai}\chi_i$, $\varepsilon_B=\sum_{i=1}^{s}\varepsilon_{Bi}\chi_i$, and $\varepsilon_f=\sum_{i=1}^{s}\varepsilon_{fi}\chi_i$. By design, reference signals are assumed to be bounded: $r\in\mathcal{L}^\infty$. With stable A_{m}, boundedness of $(B_{\mathrm{m}}-\varepsilon_B)r+f_{\mathrm{m}}-\varepsilon_f$, and $\varepsilon_A\in\mathcal{L}^2$, it can be concluded from (39) that $\hat{x}\in\mathcal{L}^\infty$. Since $x=\hat{x}-\tilde{x}$, and since both $\hat{x}$ and $\tilde{x}$ are bounded, also the system state is bounded ($x\in\mathcal{L}^\infty$). This implies boundedness of u due to (11). Finally, bounded x and u imply boundedness of $\dot{\tilde{x}}$, $\dot{\varepsilon}_A$, $\dot{\varepsilon}_B$, and $\dot{\varepsilon}_f$ due to $\dot{\tilde{x}}=A_{\mathrm{m}}\tilde{x}+\tilde{A}x+\tilde{B}u+\tilde{f}$ and (35). It follows that $\ddot{V}\in\mathcal{L}^\infty$ and $\dot{V}\in\mathcal{L}^2\cap\mathcal{L}^\infty$ which makes Barbalat's Lemma applicable and, therefore $\lim_{t\to\infty}\dot{V}(t)=0$ [35, Corollary 2.9]. Hence, $\tilde{x},\varepsilon_A,\varepsilon_B,\varepsilon_f\to0$ as $t\to\infty$.

Inspecting the dynamics of $\hat{x}$ in (39), we conclude that

$$\lim_{t\to\infty}\dot{\hat{x}}(t)=A_{\mathrm{m}}\hat{x}+B_{\mathrm{m}}r+f_{\mathrm{m}}$$

and therefore the predicted state $\hat{x}$ approaches x_{m} asymptotically. As $\tilde{x}\to0$ it follows that also the state x of the controlled system asymptotically tracks the state x_{m} of the reference system, i.e., $\lim_{t\to\infty}e(t)=0$.

The advantage of indirect approaches lies in the ability to estimate the actual system parameters. As can be seen from the proof of Theorem 3, however, convergence of the parameter estimates is not necessary for the tracking task. Before we show which additional assumptions need to be imposed on the reference signals in order to achieve parameter converge, we need the following lemma on the PE condition of some signal vector in the reference system.

Lemma 2: Let the reference system (8) be realized as the unknown system $\dot{x}_{\mathrm{m}}=A_ix_{\mathrm{m}}+B_iu_{\mathrm{m}}+f_i$ with nominal controller $u_{\mathrm{m}}=K_{xi}^*x_{\mathrm{m}}+K_{ri}^*r+k_{fi}^*$. Furthermore, let the reference signals in r be sufficiently rich of order $n+1$ with distinct frequencies. If the system matrices $A_{\mathrm{m}i}$ are invertible, and if the pairs $(A_{\mathrm{m}i},B_{\mathrm{m}i})$ are controllable, then the signal vector $z_{\mathrm{m}}=[x_{\mathrm{m}}^\top,u_{\mathrm{m}}^\top,1]^\top$ is PE.

The technical proof of Lemma 2 is given in Appendix A. Based on Lemma 2, the following theorem specifies that all estimated parameters converge to the nominal values under the assumption of sufficiently rich reference signals.

Theorem 4 (Indirect MRAC and parameter convergence for PWA systems with CQLF): Consider a reference system (8) for which a CQLF with $P=P_i$, $\forall i$ is known. Let the PWA system (4) with known regions Ω_i be controlled by the state feedback (11) with gains updated according to (36), which is based on (33), (35), and (37). Let the reference signals in r be sufficiently rich of order $n+1$ with distinct frequencies and such that all subsystems are repeatedly activated. If the input matrices B_i have full column rank, if the system matrices $A_{\mathrm{m}i}$ are invertible, and if the pairs $(A_{\mathrm{m}i},B_{\mathrm{m}i})$ are controllable, then the state of the PWA system asymptotically tracks the state of the reference system and the estimated parameters $\hat{A}_i$, $\hat{B}_i$, and $\hat{f}_i$ as well as the estimated gains K_{xi}, K_{ri}, and k_{fi} converge to their nominal values as $t \to \infty$.

Proof: The proof of Theorem 3 showed that the adaptation is stable and that state tracking is achieved. For the proof of Theorem 4, it is left to show that the parameter estimates of the active subsystem converge asymptotically to the true parameters. During this proof, we thus neglect the index i with the understanding that all steps shown here apply for the active subsystem. Hence, if all subsystems are activated in a persistent manner, all parameter estimates converge.

The proof consists of three steps. First, the convergence of estimated subsystem parameters is tied to a PE condition of internal signals. Then, Lemma 2 is applied to verify this PE requirement and, thus, show convergence of the estimated subsystem parameters. In the final step, it is shown that converging parameter estimates yield converging control gains.

Using a similar notations as in (22), begin by introducing

$$\Psi_u := \begin{bmatrix} x \\ u \\ 1 \end{bmatrix} \otimes I_n \quad \tilde{\theta} := \mathrm{vec}\left(\left[\, \tilde{A}\ \tilde{B}\ \tilde{f} \,\right]\right). \tag{40}$$

Then, from (34) and (37), likewise (26), the closed-loop system for both the prediction error $\tilde{x}$ and parameter errors $\tilde{\theta}$ can be written as

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} A_{\mathrm{m}} & \Psi_u^\top(t) \\ -\Psi_u(t)P & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} \tag{41}$$

where $\varepsilon = -\mathrm{vec}([\varepsilon_A,\, \varepsilon_A K_x^\top + \varepsilon_B K_r^\top + \varepsilon_f k_f^\top,\, \varepsilon_f])$. Note that from the proof of Theorem 3 it can be concluded that $\varepsilon \in \mathcal{L}^2$. Hence, convergence is derived from the analysis of the first summand in (41).

For (41) with $\varepsilon = 0$, exponential stability of the equilibrium $\tilde{x} = 0$ and $\tilde{\theta} = 0$ holds if $z_u = [x^\top, u^\top, 1]^\top$ is PE [33, Lemma 5.6.3]. As Theorem 3 showed $e \to 0$, it follows that z_u is PE if $z_{\mathrm{m}} = [x_{\mathrm{m}}^\top, u_{\mathrm{m}}^\top, 1]^\top$ is PE, where u_{m} can be thought of as an internal signal in the reference system if it was realized in terms of the matching conditions, i.e., $u_{\mathrm{m}} = K_x^* x_{\mathrm{m}} + K_r^* r + k_f^*$. Under the given assumptions, Lemma 2 guarantees PE of z_{m}. Hence, the PE condition on z_u is also satisfied, which—according to [33, Lemma 5.6.3] —implies asymptotic convergence of $\tilde{\theta}$ to zero and thus $\hat{A} \to A$, $\hat{B} \to B$ and $\hat{f} \to f$ as $t \to \infty$. Note that the exponential convergence stated in [33, Lemma 5.6.3] does not hold here due to the presence of ε in (41).

Finally, combine the convergence of $\hat{A}$, $\hat{B}$, and $\hat{f}$ with the convergence of ε_A, ε_B, and ε_f shown in the proof of Theorem 3. For the assumed full column rank of B, it can then be concluded that also the control gains converge to their nominal values: $K_x \to K_x^*$, $K_r \to K_r^*$, and $k_f \to k_f^*$ as $t \to \infty$. ■

After deriving direct and indirect MRAC control algorithms for PWA systems, let us briefly revise their preferred use cases as well as advantages and disadvantages. The direct MRAC algorithm presented in Section III is ideally suited for reference system tracking alone. In case a linear reference system is chosen, or a CQLF for the reference subsystems is known, Theorem 1 guarantees asymptotic state tracking without imposing additional assumptions on the reference signals. If, however, the choice of reference systems is restricted and no CQLF is found, one must resort to Theorem 2 and impose additional requirements on the system as well as the reference signals in order to achieve asymptotic state tracking. The dominant requirement are sufficiently rich reference signals to ensure PE and in turn guarantee convergence of the control gains to the nominal gains. As converged gains imply perfect reference system matching, direct MRAC according to Theorem 2 should also be applied if precise tracking under nonperiodic reference signals is required. Independent of the existence of a CQLF this task is only achievable for arbitrary reference signals if the nominal control gains are applied.

The ideal use case for the indirect MRAC algorithm presented in Section IV is when monitoring of subsystem parameters becomes the main goal besides reference system tracking alone. Applications can be the early detection of parameter drifts or aging components in the system. From this point of view, indirect MRAC according to Theorem 3 has limited practical relevance as it only achieves the asymptotic state tracking of a reference system for which a CQLF is known. No guarantees are given for the convergence of parameter estimates. Hence, in this case the benefit of a simpler structure in the direct MRAC law is more convincing. Yet, the advantage of the indirect MRAC strategy is that control gains are updated indirectly over the intermediate closed-loop estimation errors, such as ε_A. Therefore, the control gains are updated less aggressively which can reduce the wear associated with strongly oscillating control gains. The convincing full potential of indirect MRAC is given in form of Theorem 4. That is, by exciting the system with sufficiently rich reference signals, it is guaranteed that the estimated subsystem parameters converge to the true values, which is ideal for monitoring purposes. Unfortunately the application of indirect MRAC is restricted to reference systems with CQLF as the convergence of subsystem parameters is asymptotic. Therefore, a stability proof as carried out for direct MRAC is infeasible as the discontinuities in the Lyapunov function can not be compensated by a combination of exponential convergence and dwell-time constraints.

V. Numerical Validation

We consider the mass-spring-damper system in Fig. 1(a) for numerical validation of the proposed adaptation schemes. The system consists of two masses $m_1 = 5\,\mathrm{kg}$ and $m_2 = 1\,\mathrm{kg}$ connected to each other and the environment by springs and dampers. All dampers have fixed damping coefficients $d = 1\,\mathrm{Ns/m}$. The left cart is connected to the environment by a constant spring with stiffness $c_0 = 1\,\mathrm{N/m}$. Let the stiffness of the spring between the two masses be PWA with characteristic $F_c(p_1, p_2)$ shown in Fig. 1(b), which suggest a PWA system with three subsystems. The characteristic is assumed to be linear with slope $c_1 = 10\,\mathrm{N/m}$ for $|p_2 - p_1| < \gamma = 1\,\mathrm{m}$. If the spring is extended beyond γ, the slope of the spring characteristic decreases to $c_2 = 1\,\mathrm{N/m} < c_1$. For strong spring compression, i.e., $p_2 - p_1 < -\gamma$, the slope of the spring characteristic increases to $c_3 = 100\,\mathrm{N/m} > c_1$. As Fig. 1(b) indicates, the change in slope

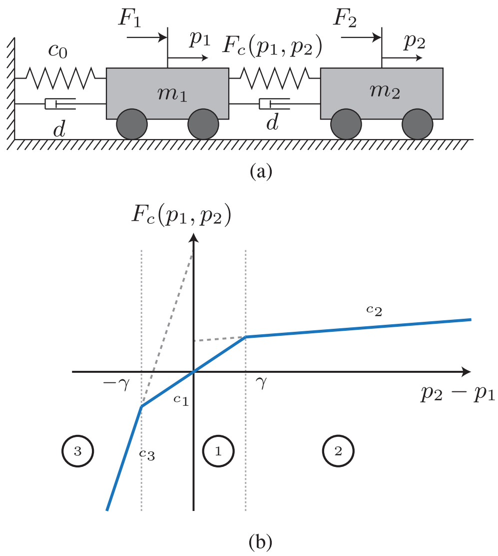

Fig. 1. The two-cart system in (a), whose connecting spring has the PWA stiffness characteristic shown in (b), can be described by a PWA model.

yields affine terms in regions Ω_2 and Ω_3 of the form $\gamma(c_1 - c_2)$ and $-\gamma(c_1 - c_3)$, respectively. The two-cart system in Fig. 1 can be modeled by a PWA system with three subsystems. Subsystem 2 for instance takes the form

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(c_0+c_2)}{m_1} & -\frac{2d}{m_1} & \frac{c_2}{m_1} & \frac{d}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{c_2}{m_2} & \frac{d}{m_2} & -\frac{c_2}{m_2} & -\frac{2d}{m_2} \end{bmatrix}}_{A_2} x$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}}_{B_2} u + \underbrace{\begin{bmatrix} 0 \\ \frac{\gamma(c_1-c_2)}{m_1} \\ 0 \\ -\frac{\gamma(c_1-c_2)}{m_2} \end{bmatrix}}_{f_2}$$

with state vector $x = [p_1, \dot{p}_1, p_2, \dot{p}_2]^\top$ and input vector $u = [F_1, F_2]^\top$. The two inputs, in form of forces F_1 and F_2 applied to the two carts, make the system multivariable. This paper proposes the first adaptive control laws for such multivariable PWA systems. For the following simulation studies, we assume all system parameters in A_i, B_i, and f_i, $i \in \{1, 2, 3\}$ to be unknown.

With the proposed algorithms, the system can be forced to track the dynamics of a reference system. For the reference system it is desirable to remove the nonlinearity as well as the coupling between the two masses. This motivates the following choice of reference system:

$$\dot{x}_\mathrm{m} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -25 & -10 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -25 & -10 \end{bmatrix}}_{A_\mathrm{m}} x_\mathrm{m} + \underbrace{\begin{bmatrix} 0 & 0 \\ 25 & 0 \\ 0 & 0 \\ 0 & 25 \end{bmatrix}}_{B_\mathrm{m}} r$$

which corresponds to a transfer function matrix with all poles located at -5

$$\begin{bmatrix} p_1(s) \\ p_2(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{(0.2s+1)^2} & 0 \\ 0 & \frac{1}{(0.2s+1)^2} \end{bmatrix} \begin{bmatrix} r_1(s) \\ r_2(s) \end{bmatrix}.$$

Hence, the reference signals r_1 and r_2 can be used as decoupled set points for the cart positions p_1 and p_2, respectively.

In the following, the direct and indirect MRAC algorithms are analyzed in three settings. First, the tracking ability according to Theorems 1 and 3 is validated for simple reference signals. Afterwards, the robustness against asynchronous switching is inspected, before finally, the convergence derived in Theorems 2 and 4 is confirmed for sufficiently rich reference signals.

A. Reference System Tracking

In order to test the tracking ability, let the reference signals be

$$r_1(t) = 3\sin(0.5t) \quad \text{and}$$

$$\bar{r}_2(t) = \begin{cases} -2, & \text{for } kT \le t < kT + 10\,\mathrm{s} \\ 0, & \text{for } kT + 10\,s \le t < kT + 20\,\mathrm{s} \\ 2, & \text{for } kT + 20\,s \le t < kT + 30\,\mathrm{s} \end{cases}$$

where $k \in \mathbb{N}$ and $T = 30\,\mathrm{s}$. These signals yield a sinusoidal set point for the left cart and a piecewise constant set point for the right one. Furthermore, the reference signals drive the controlled system through all three partitions of the state space. The state of the reference system and two-cart system are initialized with $x_\mathrm{m} = 0_n$ and $x = [-2, 0, 2, 0]^\top$, respectively. Moreover, Gaussian noise with zero mean and 0.1 variance is added to the state measurements x during simulation. All control gains and parameter estimates are initialized with zeros.

The design parameters were chosen as follows. Let all scaling constants $(\Gamma_{xi}, \Gamma_{ri}, \Gamma_{fi})$ be equal to one, and let the design matrices be

$$Q_\mathrm{m} = \begin{bmatrix} 100 & 10 & 0 & 0 \\ 10 & 100 & 0 & 0 \\ 0 & 0 & 100 & 10 \\ 0 & 0 & 10 & 100 \end{bmatrix}, \quad P = \begin{bmatrix} 140 & 2 & 0 & 0 \\ 2 & 5.2 & 0 & 0 \\ 0 & 0 & 140 & 2 \\ 0 & 0 & 2 & 5.2 \end{bmatrix}$$

which also incorporates the desired decoupling of the two carts. Since the input effectiveness of the forces F_1 and F_2 with respect to the accelerations $\ddot{p}_1$ and $\ddot{p}_2$ is known, and since each force does not directly affect the opposite cart, and since the same holds for the reference system, it can be anticipated that all K_{ri}^* are symmetric and positive definite. This, according to Remark 1, suggests the choice $S = I_2$.

Figs. 2 and 3 show the tracking performance of the direct and indirect MRAC law, respectively. As can be seen from the figures, both algorithms yield the desired performance despite frequent switching and the considerable parameter variations between the subsystems. The same performance could not be obtained by a single adaptive controller operating in all partitions.

Note that the frequent switching is a consequence of noisy state measurements. The switching signal used in the controller is, therefore, not equal to the switching signal of the actual system. As the closed-loop system nonetheless exhibits the desired

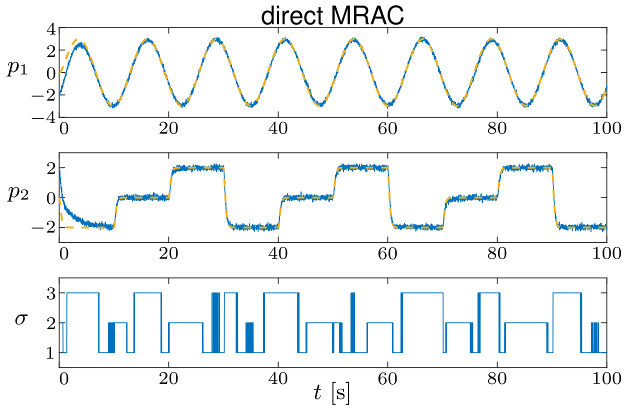

Fig. 2. Direct MRAC of the PWA two-cart system enables decoupled control of both carts despite frequent switching. Solid lines represent the measured states of the system and dashed lines indicate the states of the reference system.

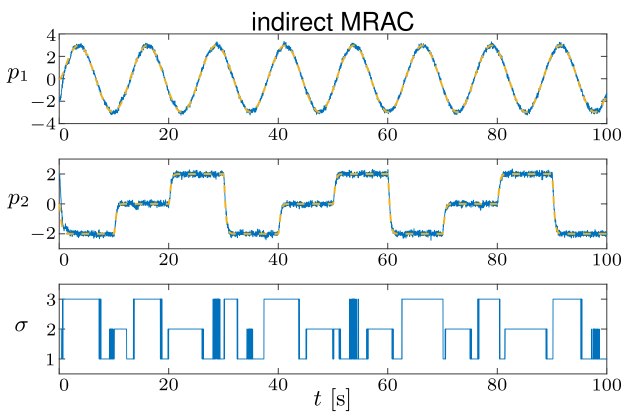

Fig. 3. Indirect MRAC of the PWA two-cart system enables decoupled control of both carts despite frequent switching. Solid lines represent the measured states of the system and dashed lines indicate the states of the reference system.

behavior, the simulation results suggest that the proposed algorithm contains some robustness against uncertainties in the assumed partitioned state space.

B. Robustness Against Delayed Switching

Robustness against delayed switching is further validated numerically by repeating the tracking experiment with increasing delays. Let Δt be the delay between the time instance at which the actual PWA system switches and the time instance at which controllers, reference systems, and predictors switch. Now, the tracking is performed over a time frame of 1000 s without measurement noise. In order to evaluate the tracking performance, we determine the integrated tracking error $E = \int_0^{1000} \sqrt{e^\top(\tau)e(\tau)}d\tau$ for each run withincreasing delay $\Delta t \in [0, 0.5]$. Fig. 4 shows $E(\Delta t)$ for the direct and indirect MRAC algorithm. As expected, delays increase the integrated tracking error. In the indirect approach this increase is stronger than in the direct approach. Nonetheless, the maximum tracking error during all trials stays within 3 % of the maximum reference signal amplitude for the direct approach, and within 7 % for the indirect approach. Hence, the direct approach is more suitable for applications with considerable delays.

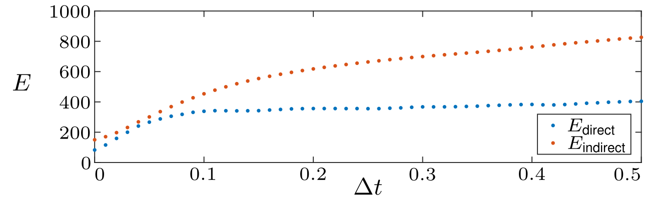

Fig. 4. Integrated tracking error E obtained for increasing delays Δt between switches of the system and the controller.

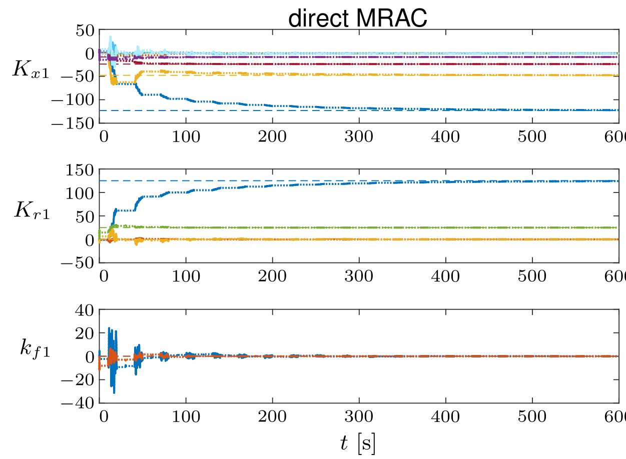

Fig. 5. Convergence of controller gains under direct MRAC for subsystem 1. Dashed lines represent ideal gains; solid lines correspond to the estimated gains while the subsystem 1 is active ($\chi_1 = 1$) and dotted lines visualize the estimated gains during inactive periods of subsystem 1 ($\chi_1 = 0$).

C. Parameter Estimation

Next, the derived convergence properties of the direct and indirect MRAC laws are validated with the reference signals

$$r_1(t) = 0.4\sin(t) + 0.3\sin(4t) + 0.2\sin(11t)$$

$$r_2(t) = \bar{r}_2(t) + 0.4\sin(2t) + 0.3\sin(10t) + 0.2\sin(14t)$$

where $\bar{r}_2(t)$ is the piecewise constant reference signal used in the tracking example above. Besides exciting the system in all modes, these reference signals ensure the required properties of being sufficiently rich of at least order 5 with distinct frequencies. All design parameters and initial values are the same as in the previous example. The added measurement noise has zero mean and variance 10^{-3}.

Fig. 5 exemplifies the evolution of gains for subsystem 1 under direct MRAC. Note that gains for region 2 and 3 evolve similarly and are, thus, not shown for clarity. The figure shows that adaptation pauses in time intervals in which the system evolves in region 2 and 3, visualized here by the dotted segments. Despite repeated switching, all gains converge to the ideal values visualized by dashed lines and specified by the matching conditions (13). This meets the expectations raised by Theorem 2.

The parameter trend for subsystem 1 under indirect MRAC is given in Fig. 6 (again, the performance in region 2 and 3 is

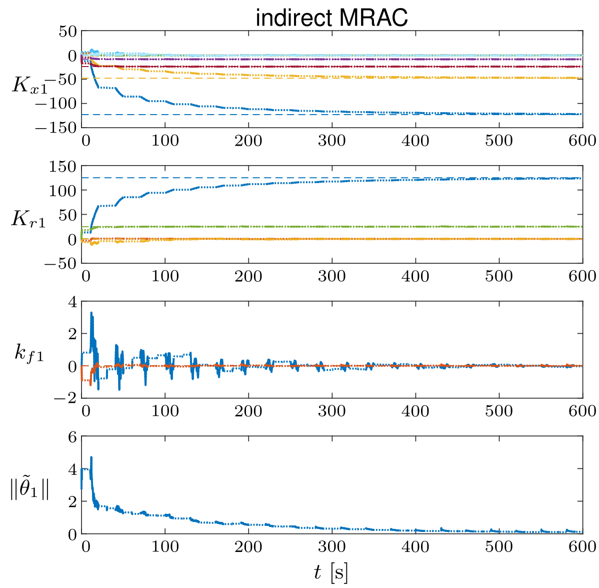

Fig. 6. Parameter convergence under indirect MRAC for subsystem 1. Solid lines represent parameters during the active periods of subsystem 1 ($\chi_1 = 1$) and dotted lines visualize the inactive periods of subsystem 1 ($\chi_1 = 0$). The ideal control gains are given by dashed lines.

similar and therefore omitted). Besides the control gains K_{x1}, K_{r1}, and k_{f1}, the figure also visualizes the norm of the error in the estimated parameters: $\|\tilde{\theta}_1\| = \|\mathrm{vec}([\tilde{A}_1, \tilde{B}_1, \tilde{f}_1])\|$. As derived in Theorem 4, all errors tend to zero asymptotically despite switching.

One difference between direct and indirect MRAC is the strength with which control gains are adjusted. Especially k_{f1} is adjusted less aggressively in the indirect case (note that the scales differ by a factor of ten).

VI. Conclusion

This paper proposes a direct and an indirect algorithm for MRAC of multivariable PWA systems. In both cases, asymptotic tracking of the reference system is shown for arbitrary fast switching in case a common Lyapunov function for the reference system is known. Without a common Lyapunov function, a dwell-time constraint on the switching signal is given for the direct approach. Furthermore, convergence of all parameter errors and tracking errors to zero is guaranteed for sufficiently rich reference signals. The limitations of the proposed algorithms are twofold. First, the full state feedback restricts the class of applicable systems and limits the choice of reference systems. This limitation can be overcome by output feedback. The second limitation of the proposed algorithms is the assumption of full state measurement and known state-space partitions. While the conducted simulation studies show some degree of robustness against asynchronous switching, the proposed algorithms become inapplicable for completely unknown state-space partitions. Therefore, future work focuses on extending the presented ideas to uncertain switching hyperplanes. From a practical point of view, also the reduction of discontinuities in the control inputs at switching times is an important topic for future work.

Appendix A
Proof of Lemma 2

Proof: The proof of Lemma 2 follows ideas presented in [33, pp. 306–308] and relates the PE condition of z_{m} to the spectral properties of r. Consider the two representations $\dot{x}_{\mathrm{m}} = A_{\mathrm{m}} x_{\mathrm{m}} + B_{\mathrm{m}} r_{\mathrm{m}} + f_{\mathrm{m}}$ and

$$\dot{x}_{\mathrm{m}} = A x_{\mathrm{m}} + B u_{\mathrm{m}} + f$$
$$u_{\mathrm{m}} = K_x^* x_{\mathrm{m}} + K_r^* r + k_f^*.$$

Note that the affine terms f_{m} and k_f^* can be understood to relate x_{m} and u_{m} to a constant input of 1. Therefore, let $z_{\mathrm{m}} = [x_{\mathrm{m}}^\top, u_{\mathrm{m}}^\top, 1]^\top$ be related to the constant input via the transfer function $H_f(s)$. Furthermore, let the transfer function $H_k(s)$ relate z_{m} to the k-th element of the reference signal vector $r = [r_1, \ldots, r_p]^\top$. Hence, $z_{\mathrm{m}}(s)$ is given by

$$z_{\mathrm{m}}(s) = \begin{bmatrix} x_{\mathrm{m}}(s) \\ u_{\mathrm{m}}(s) \\ 1(s) \end{bmatrix} = \underbrace{\begin{bmatrix} (sI - A_{\mathrm{m}})^{-1} f_{\mathrm{m}} \\ k_f^* \\ 1 \end{bmatrix}}_{=:H_f(s)} 1(s)$$
$$+ \sum_{k=1}^{p} \underbrace{\begin{bmatrix} (sI - A_{\mathrm{m}})^{-1} b_{\mathrm{m},k} \\ k_{r,k}^* + K_x^* (sI - A_{\mathrm{m}})^{-1} b_{\mathrm{m},k} \\ 0 \end{bmatrix}}_{=:H_k(s)} r_k(s)$$

where I is the identity matrix, and $b_{\mathrm{m},k}$ and $k_{r,k}^*$ denote the k-th column of B_{m} and K_r^*, respectively.

Under the assumption of distinct frequencies in the reference signals, the auto covariance of z_{m} is given by

$$R_{z_{\mathrm{m}}}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_f(-j\omega) S_1(\omega) H_f^\top(j\omega) d\omega$$
$$+ \frac{1}{2\pi} \sum_{k=1}^{p} \int_{-\infty}^{\infty} H_k(-j\omega) S_{r_k}(\omega) H_k^\top(j\omega) d\omega \quad (42)$$

where $S_1(\omega)$ and $S_{r_k}(\omega)$ are the spectral distributions of the constant input 1 and the reference signals r_k, respectively. Since r_k is sufficiently rich of order $n+1$, its spectrum has $n+1$ distinct peaks $\mathcal{F}_{r_k}(\omega_{kl})$ at frequencies $\omega_{kl}, l = 1, \ldots, n+1$. The constant input $1(s)$ results in a single spectral peak at zero. The spectral distributions are

$$S_1(\omega) = \delta(\omega) \quad S_{r_k}(\omega) = \sum_{l=1}^{n+1} \mathcal{F}_{r_k}(\omega_{kl}) \delta(\omega - \omega_{kl}) \quad (43)$$

where $\delta(\omega = 0) = 1$ and $\delta(\omega \neq 0) = 0$. With (43), the integrals in (42) are replaced by summations

$$R_{z_{\mathrm{m}}}(0) = \frac{1}{2\pi} H_f(-j0) H_f^\top(j0)$$
$$+ \frac{1}{2\pi} \sum_{k=1}^{p} \sum_{l=1}^{n+1} \mathcal{F}_{r_k}(\omega_{kl}) H_k(-j\omega_{kl}) H_k^\top(j\omega_{kl}). \quad (44)$$

According to [33, Lemma 5.6.1], the vector signal z_{m} is PE if $R_{z_{\mathrm{m}}}(0)$ is positive definite, i.e., the quadratic equation

$$v^\top R_{z_{\mathrm{m}}}(0)v = 0, \quad v \in \mathbb{R}^{n+p+1} \tag{45}$$

can only have a single solution at $v = 0_{n+p+1}$. Note that the outer products $H_k(-j\omega_{kl})H_k(j\omega_{kl})^\top$ of the column vectors $H_k(j\omega_{kl})$ are positive semidefinite. Therefore, each summand in (44) is positive semidefinite and (45) is equivalent to

$$v^\top R_{z_{\mathrm{m}}}(0)v = \underbrace{v^\top R_{z1}(0)v}_{\geq 0} + \underbrace{v^\top R_{z2}(0)v}_{\geq 0} = 0 \tag{46}$$

where

$$R_{z1}(0) := \frac{1}{2\pi}\sum_{k=1}^{p}\sum_{l=1}^{n+1}\mathcal{F}_{r_k}(\omega_{kl})H_k(-j\omega_{kl})H_k^T(j\omega_{kl})$$

$$R_{z2}(0) := \frac{1}{2\pi}H_f(-j0)H_f^T(j0).$$

First, $v^\top R_{z1}(0)v = 0$ is analyzed. Since all summands in $R_{z1}(0)$ are positive semidefinite, the equality holds, if and only if

$$v^\top H_k(-j\omega_{kl})H_k^\top(j\omega_{kl})v = 0 \quad \begin{matrix}\forall k = 1,\ldots,p \\ \forall l = 1,\ldots,n+1\end{matrix}$$

or equivalently

$$H_k^\top(j\omega_{kl})v = 0 \quad \begin{matrix}\forall k = 1,\ldots,p \\ \forall l = 1,\ldots,n+1.\end{matrix} \tag{47}$$

Next, rewrite the transfer function $H_k(s)$ with $a(s) = \det(sI - A_{\mathrm{m}})$ as

$$H_k(s) = \frac{1}{a(s)}\underbrace{\begin{bmatrix}\mathrm{adj}(sI - A_{\mathrm{m}})b_{\mathrm{m},k} \\ a(s)k_{r,k}^* + K_x^*\mathrm{adj}(sI - A_{\mathrm{m}})b_{\mathrm{m},k} \\ 0\end{bmatrix}}_{=:\bar{H}_k(s)} \tag{48}$$

and note that the equalities (47) are also equivalent to

$$\bar{H}_k^\top(j\omega_{kl})v = 0 \quad \begin{matrix}\forall k = 1,\ldots,p \\ \forall l = 1,\ldots,n+1.\end{matrix} \tag{49}$$

From (48), it can be seen that each element of the vector $\bar{H}_k(j\omega_{kl})$ is a polynomial in ω_{kl} of maximal order n. Hence, multiplying with v makes $g_k(s) := \bar{H}_k^\top(s)v$ also a polynomial in s of maximal order n. Now note that the requirement $\forall l$ in (49) implies that $g_k(s)$ vanishes at all $n+1$ frequencies ω_{kl} or corresponding values of s. As, however, $g_k(s)$ is only of order n, it follows that $g_k(s) = 0$, $\forall s \in \mathbb{C}$, $\forall k$. Combining the p equalities obtained for $k = 1,\ldots,p$ in matrix form yields

$$\begin{bmatrix}\mathrm{adj}(sI - A_{\mathrm{m}})B_{\mathrm{m}} \\ a(s)K_r^* + K_x^*\mathrm{adj}(sI - A_{\mathrm{m}})B_{\mathrm{m}} \\ 0_p^\top\end{bmatrix}^\top v = 0_p \tag{50}$$

where $0_p \in \mathbb{R}^p$ is a zero vector. Next, express the vector v as $v = [\mathcal{X}^\top, \mathcal{Y}^\top, \mathcal{Z}]^\top$, with $\mathcal{X} = [v_1,\ldots,v_n]^\top \in \mathbb{R}^n$, $\mathcal{Y} = [v_{n+1},\ldots,v_{n+p}]^\top \in \mathbb{R}^p$ and $\mathcal{Z} = v_{n+p+1} \in \mathbb{R}$, which leads to

$$\begin{aligned}&(\mathrm{adj}(sI - A_{\mathrm{m}})B_{\mathrm{m}})^\top\mathcal{X} \\ &+ (a(s)K_r^* + K_x^*\mathrm{adj}(sI - A_{\mathrm{m}})B_{\mathrm{m}})^\top\mathcal{Y} + 0_p\mathcal{Z} = 0_p.\end{aligned} \tag{51}$$

The following expressions for $\mathrm{adj}(sI - A)B$ and $a(s)$ are applied to (51)

$$\begin{aligned}\mathrm{adj}(sI - A)B &= Bs^{n-1} + (AB + a_{n-1}B)s^{n-2} \\ &\quad + (A^2B + a_{n-1}AB + a_{n-2}B)s^{n-3} +,\ldots, \\ &\quad + (A^{n-1}B + a_{n-1}A^{n-2}B +,\ldots,+ a_1B) \\ a(s) &= s^n + a_{n-1}s^{n-1} +,\ldots,+ a_1s + a_0\end{aligned}$$

which results in a polynomial of order n. All coefficients of the resulting polynomial must be zero in order for the polynomial to be zero independent of s. For the coefficient of s^n, one obtains

$$\mathcal{Y}^\top K_r^* = 0 \tag{52}$$

which by multiplication of SM_s from the right yields $\mathcal{Y} = 0_p$. Note that the assumption of full column ranks in B and B_{m} ensures that both S and M_s have empty null spaces such that their multiplication does not affect the solution of (52).

For $\mathcal{Y} = 0_p$, the n equalities related to the coefficients from s^0 to s^{n-1} can be combined in the compact form

$$\begin{bmatrix}B_{\mathrm{m}} & A_{\mathrm{m}}B_{\mathrm{m}} & \ldots & A_{\mathrm{m}}^{n-1}B_{\mathrm{m}}\end{bmatrix}^\top\mathcal{X} = 0_{np}$$

which due to controllable $(A_{\mathrm{m}}, B_{\mathrm{m}})$ leads to $\mathcal{X} = 0_n$.

This shows that (51) holds for $\mathcal{X} = 0_n$, $\mathcal{Y} = 0_p$, and an arbitrary $\mathcal{Z} \in \mathbb{R}$. Consequently, $R_{z1}(0)$ is only positive semidefinite. Therefore, $R_{z2}(0)$ in (46) must restrict the choice of $\mathcal{Z}$ further. In other words, for $\mathcal{X} = 0_n$ and $\mathcal{Y} = 0_p$ the equality $v^\top R_{z2}(0)v = 0$ must only hold for $\mathcal{Z} = 0$. The same arguments as above lead to

$$\bar{H}_f^\top(j0)v = \begin{bmatrix}\mathrm{adj}(-A_{\mathrm{m}})f_{\mathrm{m}} \\ a(j0)k_f^* \\ a(j0)\end{bmatrix}^\top v = 0$$

which with $v = [0_p^\top, 0_n^\top, \mathcal{Z}]^\top$ and $a(j0) = a_0$ implies $a_0\mathcal{Z} = 0$. Since A_{m} is assumed to be invertible, we have $a_0 \neq 0$. Therefore, $\mathcal{Z} \overset{!}{=} 0$ in order to ensure $\bar{H}_f^\top(j0)v = 0$ with $\mathcal{X} = 0_n$ and $\mathcal{Y} = 0_p$. This proves positive definiteness of $R_{z_{\mathrm{m}}}(0)$ and according to [33, Lemma 5.6.1] makes z_{m} PE.

References

[1] W. P. M. H. Heemels, B. De Schutter, and A. Bemporad, "Equivalence of hybrid dynamical models," *Automatica*, vol. 37, no. 7, pp. 1085–1091, 2001.

[2] H. Molla-Ahmadian, F. Tahami, A. Karimpour, and N. Pariz, "Hybrid control of DC–DC series resonant converters: The direct piecewise affine approach," *IEEE Trans. Power Electron.*, vol. 30, no. 3, pp. 1714–1723, Mar. 2015.

[3] S.-i. Azuma, E. Yanagisawa, and J.-i. Imura, "Controllability analysis of biosystems based on piecewise-affine systems approach," *IEEE Trans. Automat. Control*, vol. 53, no. Special Issue, pp. 139–152, Jan. 2008.

[4] G. Andrikopoulos, G. Nikolakopoulos, I. Arvanitakis, and S. Manesis, "Piecewise affine modeling and constrained optimal control for a pneumatic artificial muscle," *IEEE Trans. Ind. Electron.*, vol. 61, no. 2, pp. 904–916, Feb. 2014.

[5] D. Corona and B. De Schutter, "Adaptive cruise control for a SMART car: A comparison benchmark for MPC-PWA control methods," *IEEE Trans. Control Syst. Technol.*, vol. 16, no. 2, pp. 365–372, Mar. 2008.
[6] L. Hai and P. Antsaklis, "Stability and stabilizability of switched linear systems: A survey of recent results," *IEEE Trans. Automat. Control*, vol. 54, no. 2, pp. 308–322, Feb. 2009.
[7] R. DeCarlo, M. S. Branicky, S. Pettersson, and B. Lennartson, "Perspectives and results on the stability and stabilizability of hybrid systems," *Proc. IEEE*, vol. 88, no. 7, pp. 1069–1082, Jul. 2000.
[8] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King, "Stability criteria for switched and hybrid systems," *SIAM Rev.*, vol. 49, no. 4, pp. 545–592, 2007.
[9] A. Bemporad, G. Ferrari-Trecate, and M. Morari, "Observability and controllability of piecewise affine and hybrid systems," *IEEE Trans. Automat. Control*, vol. 45, no. 10, pp. 1864–1876, Oct. 2000.
[10] A. Tanwani, H. Shim, and D. Liberzon, "Observability for switched linear systems: Characterization and observer design," *IEEE Trans. Automat. Control*, vol. 58, no. 4, pp. 891–904, Apr. 2013.
[11] B. Passenberg, P. E. Caines, M. Leibold, O. Stursberg, and M. Buss, "Optimal control for hybrid systems with partitioned state space," *IEEE Trans. Automat. Control*, vol. 58, no. 8, pp. 2131–2136, Aug. 2013.
[12] Q. Wang, Y. Hou, and C. Dong, "Model reference robust adaptive control for a class of uncertain switched linear systems," *Int. J. Robust Nonlinear Control*, vol. 22, no. 9, pp. 1019–1035, 2012.
[13] C. Wu, J. Zhao, and X.-M. Sun, "Adaptive tracking control for uncertain switched systems under asynchronous switching," *Int. J. Robust Nonlinear Control*, vol. 25, no. 17, pp. 3457–3477, 2015.
[14] J. Xie and J. Zhao, "Model reference adaptive control for nonlinear switched systems under asynchronous switching," *Int. J. Adapt. Control Signal Process.*, vol. 31, pp. 3–22, 2016.
[15] X. Wang and J. Zhao, "Adaptive state tracking of switched systems based on a hyperstability criterion," *Int. J. Adapt. Control Signal Process.*, vol. 28, no. 1, pp. 28–39, 2014.
[16] M. Di Bernardo, U. Montanaro, and S. Santini, "Minimal control synthesis adaptive control of continuous bimodal piecewise affine systems," in *Proc. 51st IEEE Annu. Conf. Decision Control*, 2012, pp. 2637–2642.
[17] M. Di Bernardo, U. Montanaro, and S. Santini, "Minimal control synthesis adaptive control of continuous bimodal piecewise affine systems," *SIAM J. Control Optim.*, vol. 48, no. 7, pp. 4242–4261, 2010.
[18] M. Di Bernardo, C. I. H. Velasco, U. Montanaro, and S. Santini, "Experimental implementation and validation of a novel minimal control synthesis adaptive controller for continuous bimodal piecewise affine systems," *Control Eng. Practice*, vol. 20, pp. 269–281, 2012.
[19] M. Di Bernardo, U. Montanaro, and S. Santini, "Hybrid model reference adaptive control of piecewise affine systems," *IEEE Trans. Automat. Control*, vol. 58, no. 2, pp. 304–316, Feb. 2013.
[20] M. Di Bernardo, U. Montanaro, R. Ortega, and S. Santini, "Extended hybrid model reference adaptive control of piecewise affine systems," *Nonlinear Anal., Hybrid Syst.*, vol. 21, pp. 11–21, 2016.
[21] Q. Sang and G. Tao, "Adaptive control of piecewise linear systems: The state tracking case," in *Proc. Amer. Control Conf.*, 2010, pp. 4040–4045.
[22] Q. Sang and G. Tao, "Adaptive control of piecewise linear systems with applications to NASA GTM," in *Proc. Amer. Control Conf.*, 2011, pp. 1157–1162.
[23] Q. Sang and G. Tao, "Adaptive control of piecewise linear systems: The state tracking case," *IEEE Trans. Automat. Control*, vol. 57, no. 2, pp. 522–528, Feb. 2012.
[24] Q. Sang, G. Tao, and J. Guo, "Multivariable state feedback for output tracking MRAC for piecewise linear systems with relaxed design conditions," in *Proc. Amer. Control Conf.*, 2014, pp. 703–708.
[25] S. Kersting and M. Buss, "Adaptive identification of continuous-time switched linear and piecewise linear systems," in *Proc. Eur. Control Conf.*, 2014, pp. 31–36.
[26] S. Kersting and M. Buss, "Online identification of piecewise affine systems," in *Proc. UKACC 10th Int. Conf. Control*, 2014, pp. 90–95.
[27] S. Kersting and M. Buss, "Concurrent learning adaptive identification of piecewise affine systems," in *Proc. 53rd IEEE Conf. Decision Control*, 2014, pp. 3930–3935.
[28] A. Garulli, S. Paoletti, and A. Vicino, "A survey on switched and piecewise affine system identification," in *Proc. 16th IFAC Symp. Syst. Identif.*, 2012, pp. 344–355.
[29] G. Tao, "Multivariable adaptive control: A survey," *Automatica*, vol. 50, no. 11, pp. 2737–2764, 2014.
[30] G. Michaletzky and L. Gerencser, "BIBO stability of linear switching systems," *IEEE Trans. Automat. Control*, vol. 47, no. 11, pp. 1895–1898, Nov. 2002.
[31] M. Duarte and K. Narendra, "Combined direct and indirect approach to adaptive control," *IEEE Trans. Automat. Control*, vol. 34, no. 10, pp. 1071–1075, Oct. 1989.
[32] E. Lavretsky, "Combined/composite model reference adaptive control," *IEEE Trans. Automat. Control*, vol. 54, no. 11, pp. 2692–2697, Nov. 2009.
[33] P. Ioannou and J. Sun, *Robust Adaptive Control*. Upper Saddle River, NJ, USA: Prentice-Hall, 1996.
[34] M. Petreczky and L. Bako, "On the notion of persistence of excitation for linear switched systems," in *Proc. 50th IEEE Conf. DecisionControl Eur. Control Conf.*, 2011, pp. 1840–1847.
[35] K. Narendra and A. M. Annaswamy, *Stable Adaptive Systems*. New York, NY, USA: Dover, 2005.
[36] A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, vol. 18. Dordrecht, The Netherlands: Springer, 1988.
[37] S. Boyd and S. Sastry, "On parameter convergence in adaptive control," *Syst. Control Lett.*, vol. 3, pp. 311–319, 1983.
[38] M. S. Branicky, "Multiple lyapunovfunctions and other analysis tools for switched and hybrid systems," *IEEE Trans. Automat. Control*, vol. 43, no. 4, pp. 475–482, Apr. 1998.
[39] G. Tao, *Adaptive Control Design and Analysis* (Adaptive and learning systems for signal processing, communications, and control Series). Hoboken, NJ, USA: Wiley, 2003.
[40] M. Duarte and K. Narendra, "Indirect model reference adaptive control with dynamic adjustment of parameters," *Int. J. Adapt. Control Signal Process.*, vol. 10, pp. 603–621, 1996.

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