Control Contraction Metrics: Convex and Intrinsic Criteria for Nonlinear Feedback Design

Ian R. Manchester and Jean-Jacques E. Slotine

***Abstract*—We introduce the concept of a control contraction metric, extending contraction analysis to constructive nonlinear control design. We derive sufficient conditions for exponential stabilizability of all trajectories of a nonlinear control system. The conditions have a simple geometrical interpretation, can be written as a convex feasibility problem, and are invariant under coordinate changes. We show that these conditions are necessary and sufficient for feedback linearizable systems and also derive novel convex criteria for exponential stabilization of a nonlinear submanifold of state space. We illustrate the benefits of convexity by constructing a controller for an unstable polynomial system that combines local optimality and global stability, using a metric found via sum-of-squares programming.**

***Index Terms*—Contraction, control system synthesis, linear matrix inequalities, lyapunov methods, nonlinear control systems.**

I. INTRODUCTION

The concept of a Lyapunov function is central in nonlinear system analysis and builds upon the intuitive notion of a system's energy dissipating over time [1], [2]. For the nonlinear control design, the natural extension is the control Lyapunov function (CLF): a generalized measure of energy that can be made to decrease by choice of control action, first formalized in [3] and [4] though implicit in earlier works.

If a CLF can be found, then, surprisingly, simple formulas yield stabilizing feedback designs for quite broad classes of systems [5], [6, Ch. 4]. However, the fundamental problem of finding a CLF remains challenging. For mechanical and electrical systems, physical energy often yields effective choices [2], while for systems of particular "triangular" structures, backstepping and related methods can be applied [1], [6].

For linear systems, a straightforward change of variables converts the CLF criteria to a linear matrix inequality (LMI) [7], [8], but for nonlinear systems, the set of CLFs for a particular system is not necessarily convex or even connected [9]. The density functions of [9], the occupation measures of [10], and the Lyapunov measures of [11] each offer "dual" representations that yield convex (but generally infinite-dimensional) searches guaranteeing almost-everywhere

stability. Computationally tractable finite-dimensional approximations have been based on gridding or the sum-of-squares relaxation [12].

The main result of this paper is that if the nonlinear stabilization problem is studied differentially, then the simple convexification results for linear systems are recovered by generalizing the concept of a contraction metric.

Contraction analysis is based on the study of a nonlinear system by way of its differential dynamics (also known as variational system) along solutions [13]. Roughly speaking, since the differential dynamics are linear time varying, many techniques from linear systems theory can be directly applied. A central result is that if all solutions of a smooth nonlinear system are locally exponentially stable in a common metric, then all solutions are globally exponentially stable. Historically, basic convergence results on contracting systems can be traced back to the results of [14] in terms of Finsler metrics, further explored in [15], while convex conditions for existence and robustness of limit cycles were given in [16]. In contraction analysis, question of stability is decoupled from the specification of particular solutions, and this property is also relevant for control design: it is common in industrial plant-wide control and robot motion control to have a "layered architecture," in which a higher layer generates target trajectories and a lower layer guarantees accurate tracking.

In this paper, we introduce universal stabilizability: the property that every forward-complete solution of a system can be globally stabilized. We then define a control contraction metric (CCM) for a nonlinear system and show that existence of a CCM is sufficient for universal exponential stabilizability. We also give extensions for stabilization of submanifolds.

The CCM stabilizability condition has a simple geometric interpretation: small displacements in directions orthogonal to the span of the control inputs must be "naturally" contracting. Here, both the notions of orthogonality and contraction depend on the choice of metric. This can be thought of as a differential version of the CLF condition of [3] and [5]. While the resulting conditions are arguably quite strong, in contrast to a CLF, our criteria for the existence for a CCM can be formulated as a convex feasibility problem. Furthermore, unlike, e.g., backstepping, they are invariant under smooth changes of coordinates and affine feedback transformations, and necessary and sufficient for feedback linearizable systems.

The feedback controllers we propose will generally involve real-time optimization to find a minimal-length path with respect to the metric (a geodesic) joining the current state to the desired state. This problem is generally simpler than that in nonlinear model-predictive control (MPC), since it has lower dimension and lacks dynamic constraints, and minimal geodesics are guaranteed to exist [17]. If a state-independent metric exists, then geodesics are just straight lines, and our method is closely related to well-known methods using quadratic Lyapunov functions, e.g., [7, p. 99] and [18].

The main feedback controller we propose is smooth almost everywhere, but as with a nonlinear MPC, our controller may be

I. R. Manchester is with the Australian Centre for Field Robotics and the School of Aerospace, Mechanical and Mechatronic Engineering, University of Sydney, Sydney, NSW 2006, Australia (e-mail: ian.manchester@ sydney.edu.au).

J.-J. E. Slotine is with the Nonlinear Systems Laboratory, Massachusetts Institute of Technology, Cambridge, MA 02139 USA (e-mail: jjs@mit.edu).

discontinuous at some points in state space. To ensure existence of solutions at such points, we also propose a sampled-data controller that runs open-loop short intervals, a common strategy in nonlinear MPC [19], and similar to the notion of s-stability introduced in [20].

II. PRELIMINARIES AND PROBLEM SETUP

For symmetric matrices A and B, the notation $A \geq B$ ($A > B$) means that $A - B$ is positive semidefinite (positive definite). The nonnegative reals are denoted $\mathbb{R}^+ := [0, \infty)$. Given a smooth matrix function $M(x, t)$ and vector field $v : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ defined for $x \in \mathbb{R}^n$, $t \in \mathbb{R}^+$, we use the following notation for directional derivative $\partial_v M := \sum_j \frac{\partial M}{\partial x_j} v_j$. A set of sample times is a sequence $t_0, t_1, \ldots$ with $t_0 = 0$ and $t_i < t_{i+1}$ for all i and $t_i \to \infty$ as $i \to \infty$.

In this paper, we consider control-affine nonlinear systems:

$$\dot{x} = f(x, t) + B(x, t)u \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are state and control, respectively, at time $t \in \mathbb{R}^+$, and f and B are smooth functions of their arguments. We denote the ith column of $B(x, t)$ by $b_i(x, t)$. We assume $u(t)$ is at least piecewise continuous, and (1) holds with the right derivative at points of discontinuity.

We define a target trajectory to be a forward-complete solution of (1), i.e., a pair $(x^\star, u^\star)$ with $x^\star : \mathbb{R}^+ \to \mathbb{R}^n$ piecewise differentiable and $u^\star : \mathbb{R}^+ \to \mathbb{R}^m$ piecewise continuous satisfying (1) for all $t \in \mathbb{R}^+$.

We will consider open-loop, sampled-data, and continuous feedback controllers. An open-loop controller is a mapping $(x(0), x^\star(\cdot), u^\star(\cdot), t) \mapsto u(t)$. Given a set of sample times, a sampled-data feedback controller has the property that on each interval $[t_i, t_{i+1})$, the control law is a mapping $(x(t_i), x^\star(t), u^\star(t), t) \mapsto u(t)$, while a continuous feedback controller is a mapping $(x(t), x^\star(t), u^\star(t), t) \mapsto u(t)$.

A target trajectory $x^\star, u^\star$ is said to be globally exponentially controllable (respectively, stabilizable) if one can construct an open-loop (respectively, feedback) controller such that for any initial condition $x(0) \in \mathbb{R}^n$, a unique solution $x(t)$ of (1) exists for all t and satisfies

$$|x(t) - x^\star(t)| \leq e^{-\lambda t} R |x(0) - x^\star(0)| \qquad (2)$$

where rate $\lambda > 0$ and overshoot $R > 0$ are constants independent of initial conditions. If every target trajectory is globally exponentially controllable (respectively, stabilizable), then the system is said to be universally exponentially controllable (respectively, stabilizable). The following example illustrates the distinction with global stabilizability of a particular solution.

Example 1: Consider the planar system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2x_1 + x_1^2 - x_2^2 \\ -6x_2 + 2x_1 x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

which has four equilibria [9]: $(x_1, x_2) = (0, 0)$, $(2, 0)$, and $(3, \pm\sqrt{3})$. The origin is globally exponentially stabilized by the feedback law $u = -x_1^2 - x_2^2$, since the quadratic Lyapunov function $V(x) = x_1^2 + x_2^2$ verifies $\dot{V} = -4x_1^2 - 12x_2^2 \leq -4V$. However, notice that if $x_2(0) = 0$, then $x_2(t) = 0, \forall t \geq 0$ regardless of x_1, u, so it is impossible for a control input to move the state from the line $x_2 = 0$ to the equilibria at $(3, \pm\sqrt{3})$. Therefore, this system is not universally stabilizable.

We utilize the following standard results of Riemannian geometry, see, e.g., [21] for details. A Riemannian metric is a smoothly varying inner product $\langle \cdot, \cdot \rangle_x$ on the tangent space of a state manifold $\mathcal{X}$; this defines local notions of length, angle, and orthogonality. In this paper, $\mathcal{X} = \mathbb{R}^n$ and the tangent space can also be identified with $\mathbb{R}^n$. We allow metrics to be smoothly time varying and use the following notation:

$\langle \delta_1, \delta_2 \rangle_{x,t} = \delta_1' M(x, t) \delta_2$ and $\|\delta\|_{x,t} = \sqrt{\langle \delta, \delta \rangle_{x,t}}$. We call a metric uniformly bounded if $\exists \alpha_2 \geq \alpha_1 > 0$ such that $\alpha_1 I \leq M(x, t) \leq \alpha_2 I$ for all x, t. For a smooth curve $c : [0, 1] \to \mathbb{R}^n$, we use the notation $c_s(s) := \frac{\partial c(s)}{\partial s}$ and define the Riemannian length and energy functionals as

$$L(c, t) := \int_0^1 \|c_s\|_{c,t} ds, \quad E(c, t) := \int_0^1 \|c_s\|_{c,t}^2 ds$$

respectively, with integration interpreted as the summation of integrals for each smooth piece. Let Γ be the set of piecewise-smooth curves $[0, 1] \to \mathbb{R}^n$, and for a pair of points $x, y \in \mathbb{R}^n$, let $\Gamma(x, y)$ be the subset of Γ connecting x to y, i.e. curves $c \in \Gamma(x, y)$ if $c \in \Gamma$, $c(0) = x$, and $c(1) = y$. A smooth curve $c(s)$ is regular if $\frac{\partial c}{\partial s} \neq 0$ for all $s \in [0, 1]$. The Riemannian distance $d(x, y, t) := \inf_{c \in \Gamma(x,y)} L(c, t)$, and we define $E(x, y, t) := d(x, y, t)^2$. Under the conditions of the Hopf–Rinow theorem, a smooth regular minimum-length curve (a geodesic) γ exists connecting every such pair, and the energy and length satisfy the following inequalities: $E(x, y, t) = E(\gamma, t) = L(\gamma, t)^2 \leq L(c, t)^2 \leq E(c, t)$, where c is any curve joining x and y. For time-varying paths $c(t, s)$, we also write $c(t) := c(t, \cdot) : [0, 1] \to \mathbb{R}^n$.

A central result of [13] is that if there exists a uniformly bounded metric $M(x, t)$ such that $\dot{M} + \frac{\partial f'}{\partial x} M + M \frac{\partial f}{\partial x} \leq -2\lambda M$, where $\dot{M} = \frac{\partial M}{\partial t} + \partial_f M$, then the system is contracting with rate λ, i.e., $\frac{d}{dt} \|\delta_x\|_{x,t} \leq -\lambda \|\delta_x\|_{x,t}$. By integrating along minimizing geodesics, we see that $d(x, y, t)$ and $E(x, y, t)$ between any pair of points x, y both decrease exponentially under the flow of the system and, thus, can serve as incremental Lyapunov functions. Such systems are called contracting systems and M is a contraction metric. Similarly, we call a system strictly contracting with rate λ if $\frac{d}{dt} \|\delta_x\|_{x,t} < -\lambda \|\delta_x\|_{x,t}$ for $\delta_x \neq 0$. Since $M(x, t) > 0$, a system that is contracting with rate $\lambda > 0$ is strictly contracting with any rate less than λ.

III. CONTROL CONTRACTION METRICS

To analyze stabilizability, we utilize the "extended" system consisting of (1) paired with its differential dynamics:

$$\dot{\delta}_x = A(x, u, t)\delta_x + B(x, t)\delta_u \qquad (3)$$

defined along solutions $x(t), u(t)$, where $A := \frac{\partial f}{\partial x} + \sum_{i=1}^m \frac{\partial b_i}{\partial x} u_i$.

Let us begin by examining the case when a known controller makes the system contracting and yet is flexible enough that any target trajectory of (1) remains possible in closed loop.

Proposition 1: Suppose there exists a smooth feedback control law $u = k(x, t) + v$ that makes the closed-loop system strictly contracting with rate λ in some metric $M(x, t)$ for any piecewise-continuous signal $v(t)$. Then, for all x, u, t

$$\dot{M} + (A + BK)'M + M(A + BK) < -2\lambda M \qquad (4)$$

where $K = \frac{\partial k}{\partial x}$, and for $\delta_x \neq 0$, the following is true:

$$\delta_x' M B = 0 \Rightarrow \delta_x' (\dot{M} + A'M + MA + 2\lambda M)\delta_x < 0. \qquad (5)$$

It is clear that (4) $\Longrightarrow$ (5); the proof of (4) and all subsequent results in this paper are collected in the Appendix.

Since (5) is independent of the particular control law k, it describes an intrinsic property of the system (1): if $\langle \delta_x, b_i(x) \rangle_{x,t} = 0$ for all $i = 1, 2, \ldots, m$, then $\frac{d}{dt} \|\delta_x\|_{x,t} < -\lambda \|\delta_x\|_{x,t}$. That is, every tangent vector δ_x orthogonal to the span of actuated directions $b_i(x)$ is naturally contracting with rate λ. It is interesting to ask whether (5) implies the existence of some form of stabilizing control for any target trajectory. Our main theoretical result is that this is indeed the case.

Theorem 1: If there exists a uniformly bounded metric $M(x,t)$, i.e., $\alpha_1 I \leq M(x,t) \leq \alpha_2 I$, for which (5) holds for all $\delta_x \neq 0, x, u, t$, then System (1) is

1) universally exponentially open-loop controllable;
2) universally exponentially stabilizable via sampled-data feedback with arbitrary sample times;
3) universally exponentially stabilizable via continuous feedback defined almost everywhere, and everywhere in a neighborhood of the target trajectory;

all with rate λ and overshoot $R = \sqrt{\frac{\alpha_1}{\alpha_2}}$.

We refer to a metric satisfying the conditions of this theorem as a CCM for the system (1). The proof of this theorem is given in the Appendix, but here we briefly describe the main idea and construction of controllers.

Given a CCM, Lemma 2 in the Appendix establishes the existence of a differential feedback controller $\delta_u = k_\delta(x, \delta_x, u, t)$ that achieves closed-loop exponential stabilization of the differential dynamics (3) along all solutions:

$$\frac{d}{dt}(\delta_x' M \delta_x) = \delta_x' \dot{M} \delta_x + 2\delta_x' M(A\delta_x + Bk_\delta) < -2\lambda \delta_x' M \delta_x$$

and furthermore, is path-integrable, so that for any smooth path $c \in \Gamma$ and any $u^\star \in \mathbb{R}^m$ and $t \in \mathbb{R}^+$, the following integral equation has a unique solution:

$$k_p(c, u^\star, t, s) := u^\star + \int_0^s k_\delta(c(\mathfrak{s}), c_s(\mathfrak{s}), k_p(c, u^\star, t, \mathfrak{s}), t)d\mathfrak{s}. \quad (6)$$

The motivation for this construction is to give a smooth path of control signals k_p with tangent vectors $\frac{\partial k_p}{\partial s} = k_\delta$ for all $s \in [0,1]$ and boundary condition $k_p = u^\star$ at $s = 0$.

The path-integrability condition is a significantly weaker requirement than the k_δ being completely integrable, i.e., of the form $k_\delta = K(x,t)\delta_x$, with K the Jacobian of a feedback controller, as was assumed in Proposition 1. This distinction will be important for our convex conditions in Section III-B.

1) *Open-loop control:* For $t_i \geq 0$, consider a bounded or unbounded time interval in one of the following forms: $\mathcal{T} = [t_i, t_{i+1})$, $\mathcal{T} = [t_i, t_{i+1}]$ or $\mathcal{T} = [t_i, \infty)$.

   a) At the initial time t_i, measure $x(t_i)$ and construct a smooth path $c(t_i) \in \Gamma(x^\star(t_i), x(t_i))$.
   b) For each $t \in \mathcal{T}$, apply the control signal $u(t) = k_p(c(t), u^\star(t), t, 1)$, where $c(t)$ is the forward image of the path $c(t_i)$ with the path of controls (6), i.e., for each $s \in [0,1]$ and $t \in \mathcal{T}$, $c(t, s)$ satisfies

$$\frac{d}{dt}c(t,s) = f(c(t,s),t) + B(c(t,s),t)$$
$$k_p(c(t), u^\star(t), t, s). \quad (7)$$

When this strategy is applied on an interval $\mathcal{T} = [t_i, \infty)$, the length of the curve $c(t)$ shrinks exponentially and allows us to establish claim 1 of Theorem 1.

The bound $R = \sqrt{\frac{\alpha_2}{\alpha_1}}$ given in the theorem is achieved if the initial path c_0 is a minimal geodesic joining $x^\star(t_i)$ to $x(t_i)$, existence of which is established in Lemma 1 in the Appendix. With any other initial path, exponential stability is still achieved with the same rate but perhaps with larger overshoot. Note that when $c(t_i)$ is a geodesic, it is in general not the case that $c(t)$ is a geodesic for $t > t_i$ (see Fig. 1).

2) *Sampled-data feedback controller:* The open-loop controller can be extended to sampled-data feedback by recomputing geodesics at the sampling instants. To be precise:
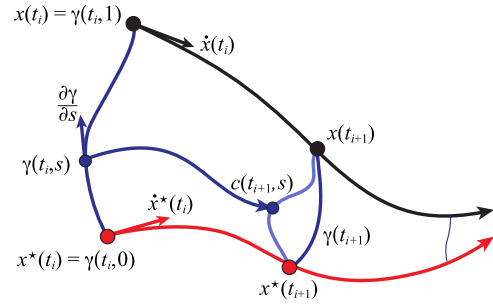

Fig. 1. Illustration of the geometry of solutions using the open-loop or sampled-data CCM-based control over an interval $[t_i, t_{i+1}]$. The target trajectory $x^\star(t)$ is shown in red, and system trajectory $x(t)$ in black. Paths joining $x^\star(t)$ to $x(t)$ are shown in blue.

   a) At each sample time t_i, measure the state $x(t_i)$ and compute a minimal geodesic $\gamma_i \in \arg\min_{c \in \Gamma_i} E(c, t_i)$ where $\Gamma_i := \Gamma(x^\star(t_i), x(t_i))$.
   b) On the interval $\mathcal{T} = [t_i, t_{i+1})$, apply the open-loop control described above with $c(t_i) = \gamma_i$.

Note that this is stabilizing with any choice of sample times, including uniform sampling: $t_i = it_s$ for some fixed $t_s > 0$.

3) *Smooth feedback, uniquely defined almost everywhere and in a neighborhood of $x^\star$:* By taking the limit as sampling interval goes to zero, one can obtain a continuous-time controller, which does away with the need to solve (7) over the intersample intervals. Specifically:

   a) measure the state $x(t)$ and a minimal geodesic $\gamma = \arg\min_{c \in \Gamma(x^\star(t), x(t))} E(c, t_i)$;
   b) apply the control signal $u(t) = k_p(\gamma, u^\star(t), t, 1)$.

This defines a mapping $(x(t), x^\star(t), u^\star(t), t \mapsto u(t)$; however, a difficulty is that it may be multiply defined or nonsmooth at some states $x(t)$ and specifically points on the cut locus, denoted by $\mathfrak{C}(x^\star, t)$, which is the set of points for which nonunique minimizing geodesics exist from $x^\star$ (cut points) and/or the first-order minimality condition fails (conjugate points).

This set is known to have zero Lebesgue measure. Let us define $\mathfrak{D}(x^\star, t) := \mathbb{R}^n / (\mathfrak{C}(x^\star, t) \cup x^\star)$, which is diffeomorphic to punctured open ball. For every $x \in \mathfrak{D}(x^\star, t)$, there is a unique minimal geodesic γ joining x and $x^\star$ [21, Ch 13], and we show in the Appendix that the above controller is smooth on $\mathfrak{D}(x^\star, t)$ and continuous at $x = x^\star(t)$. This controller is universally exponentially stabilizing under the technical assumption that the set of times at which $x(t) \in \mathfrak{C}(x^\star, t)$ has zero measure.

A. Stronger Conditions Giving Simpler Controllers

Since the differential dynamics are linear, it is tempting to look for an admissible differential feedback controller of the form $\delta_u = K(x,t)\delta_x$ satisfying (4). We will show that this is possible under the following slightly stronger conditions:

C1: if $\delta_x \neq 0$ satisfies $\delta_x' MB = 0$, then

$$\delta_x' \left(\frac{\partial M}{\partial t} + \partial_f M + \frac{\partial f}{\partial x}' M + M\frac{\partial f}{\partial x} + 2\lambda M \right) \delta_x < 0$$

C2: for each $i = 1, 2, \ldots, m$, $\partial_{b_i} M + \frac{\partial b_i}{\partial x}' M + M\frac{\partial b_i}{\partial x} = 0$.

These stronger conditions also hold under the assumptions of Proposition 1, as is clear from the proof in the Appendix. Condition C1 says that the uncontrolled system is contracting in directions orthogonal to the span of the control inputs. Condition C2 ensures that

large u of unknown sign cannot cause expansion of $\|\delta_x\|$. Formally, it states that the vector fields b_i are Killing fields for the metric M.

In particular, if B is of the form $[0, I]'$, with 0 and I the zero and identity matrices of appropriate dimension, then Condition C2 states that M must not depend on the last m state variables. By applying Finsler's theorem (see, e.g., [22]) pointwise in x and t to condition C1, we immediately obtain the following.

Proposition 2: Condition C1 is equivalent to the existence of a scalar multiplier $\rho(x, t)$ such that for all x, t:

$$\frac{\partial M}{\partial t} + \partial_f M + \frac{\partial f'}{\partial x} M + M \frac{\partial f}{\partial x} - \rho MBB'M + 2\lambda M < 0. \quad (8)$$

One can then construct the differential feedback gain $K(x, t) = -\frac{1}{2}\rho(x, t)B(x, t)'M(x, t)$, which satisfies (4) and is always path integrable since it is independent of u.

Remark 1: If (8) holds for some multiplier $\rho(x, t) = \rho_0(x, t)$, then it clearly holds for any $\rho(x, t) \geq \rho_0(x, t)$ for all x, t since $MBB'M \geq 0$. This can be interpreted as the differential feedback having infinite up-side gain margin and also implies that one can construct a smooth $\rho(x, t)$.

B. Dual Metrics and Convexity of Synthesis

It is known that the search for a CLF for a linear system is convexified by a simple change of variables, leading to an LMI representation of stabilizability [7], [8]. In this section, we show that essentially the same transformation makes the search for a CCM convex.

Consider the change of variables $\eta = M(x, t)\delta_x$ and $W(x, t) = M(x, t)^{-1}$. This is related to the "musical isomorphism" to the dual space of cotangent vectors, and the function $\eta'W\eta$ is the Fenchel dual of $\delta_x'M\delta_x$, so we refer to W as a dual CCM. Under this change of variables, the CCM condition (5) now states that $\eta'(-\dot{W} + AW + WA' + 2\lambda W)\eta < 0$ whenever $\eta'B = 0$, which can be written as

$$B'_\perp \left(-\dot{W} + AW + WA' + 2\lambda W \right) B_\perp < 0 \quad (9)$$

for all x, u, t, where $B'_\perp(x, t)$ is any matrix function satisfying $B'_\perp B = 0$ for all x, t. Since differentiation is a linear operation, inequality (9) is linear (and hence convex) in the unknown matrix function W.

One can search directly for differential feedback $\delta_u = K(x, u, t)\delta_x$ by way of W and $Y(x, u, t) \in R^{m \times n}$ satisfying

$$-\dot{W} + \frac{\partial f}{\partial x}W + W\frac{\partial f'}{\partial x} + BY + Y'B' + 2\lambda W < 0 \quad (10)$$

giving the differential feedback gain $K = YW^{-1}$. If Y, and hence K, are at most affine in u, then the resulting differential control will be path integrable, similarly to Lemma 2.

Condition C1 can be written similarly to (9) and, by Finsler's theorem, is equivalent to the existence of a scalar function $\rho(x, t)$ satisfying the inequality

$$-\frac{\partial W}{\partial t} - \partial_f W + \frac{\partial f}{\partial x}W + W\frac{\partial f'}{\partial x} - \rho BB' + 2\lambda W < 0. \quad (11)$$

which is jointly convex in W and ρ and gives an explicit construction of a differential feedback gain $K = -\frac{1}{2}\rho B'W^{-1}$.

Condition C2 also transforms to a linear constraint on W: $\partial_{b_i} W - \frac{\partial b_i}{\partial x}W - W\frac{\partial b_i}{\partial x}' = 0$.

The above conditions are all convex but infinite dimensional: they are inequalities that must hold over all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$, and the decision variables are sets of smooth matrix functions. Finite-dimensional LMI approximations can be constructed by building W and ρ or Y as linear combinations of a finite basis set (e.g., polynomials

up to some order), and verifying the inequalities either by gridding over states and times, or by the sum-of-squares relaxation [12].

Remark 2: Note that complete integrability of $k_\delta(x, \delta_x, t) = K(x, t)\delta_x$ could be imposed by requiring that each row of K satisfies the Schwarz condition, i.e., $\frac{\partial K_{i,j}}{\partial x_k} = \frac{\partial K_{i,k}}{\partial x_j}$. While this constraint is linear and hence convex in K, it is not convex jointly in the decision variables W, Y for (10), since $K = YW^{-1}$, or the decision variables W, ρ for (11), since $K = -\frac{1}{2}\rho B'W^{-1}$. This is essentially the same problem as the well-known nonconvexity of structured feedback synthesis for linear systems, e.g., static output feedback [23].

IV. PROPERTIES OF CCMs

A. Riemannian Energy as a CLF

The proof of Theorem 1 uses an explicit construction of a particular stabilizing controller, but in doing so, we have actually shown that the Riemannian energy $E(x, x^\star, t)$ can always be decreased and, hence, be used as CLF for any target trajectory of the system.

The formula for first variation of energy [21, p. 195] gives a particularly convenient expression for the time derivative of the energy functional as an affine function of u:

$$\frac{1}{2}\frac{d}{dt}E(x, x^\star, t) = \langle \gamma_s(t, 0), \dot{x}^\star \rangle_{x^\star, t} - \langle \gamma_s(t, 1), f(x, t) \rangle_{x, t}$$
$$- \langle \gamma_s(t, 1), B(x, t)u \rangle_{x, t} + \frac{1}{2}\frac{\partial E}{\partial t} \quad (12)$$

When $x(t) \in \mathfrak{C}(x^\star, t)$, the above formula still holds with $=$ replaced by $\leq$ and $\frac{d}{dt}$ replaced by the Dini derivative.

In proving Theorem 1, we have also proven that for any $x^\star, u^\star, t$, the convex set (either a half-space or all of $\mathbb{R}^m$):

$$\mathcal{U} = \left\{ u \in R^m : \frac{d}{dt}E(x, x^\star, t) \leq -2\lambda E(x, x^\star, t) \right\}$$

where $\frac{d}{dt}E(x, x^\star, t)$ is given by (12), is always nonempty.

This opens up the possibility of using many other particular controllers based on CLFs that may have further desirable properties. For example, pointwise min-norm control [6] $u(t) = \arg\min_{\tilde{u} \in \mathcal{U}} \|\tilde{u}\|^2$ would have reduced control magnitude and can be generalized to provide approximate optimality with guaranteed stability [24].

B. Invariance Under Coordinate Change and Feedback

Metrics and dual metrics are tensors: geometrical objects that are 'intrinsic" and have coordinate representations that transform appropriately under smooth coordinate changes. In the following theorem, we establish that the CCM criteria are invariant under such coordinate changes and, additionally, under affine feedback laws.

Theorem 2: If the CCM condition (5) [or equivalently (9)] is satisfied for system (1), then (5) and (9) still hold under:
1) affine feedback transformations $u(x, v) = \alpha(x) + \beta(x)v$ with β a smooth nonsingular $n \times n$ matrix function;
2) differential coordinate changes $\delta_\xi = \Phi(x)\delta_x$, in which $\Phi(x)$ is a nonsingular matrix for all x, with the new CCM $M_\xi(x, t) := \Psi'(x)M(x, t)\Psi(x)$ and dual CCM $W_\xi(x, t) := \Phi(x)W(x, t)\Phi(x)'$, where $\Psi(x) = \Phi(x)^{-1}$;
3) coordinate changes $\xi = \phi(x)$, ϕ a smooth diffeomorphism, with the new CCM and dual CCM M_ξ, W_ξ as above with $\Phi(x) = \frac{\partial\phi}{\partial x}$ evaluated at $x = \phi^{-1}(\xi)$.

Remark 3: If $\Phi(x)$ has bounded singular values over all x, t within $[\sigma_{\min}, \sigma_{\max}]$, then the uniform bounds on M and W are also preserved under coordinate change.

C. Necessity for Feedback Linearizable Systems

A corollary of Theorem 2 is that for feedback linearizable systems, existence of a CCM is guaranteed. A system of the form (1) is feedback linearizable if there exists a change of variables and feedback transformation such that the transformed system is linear time invariant: $\dot{\xi} = G\xi + Hv$, where the pair of constant matrices (G, H) is controllable [1].

Corollary 1: For any feedback linearizable system, there is a CCM that verifies universal stabilizability, given by $W(x, t) = \Phi(x, t)P\Phi(x, t)'$, where P is any constant symmetric positive-definite matrix satisfying $H_\perp(GP + PG')H'_\perp < 0$.

The proof is immediate from Theorem 2, and we note that such a P is guaranteed to exist if (G, H) is stabilizable [8].

The converse is not true: the necessary and sufficient conditions for feedback-linearizability consist of a controllability condition and an involutivity (complete integrability) condition to find an appropriate coordinate change [1]. In contrast, existence of a CCM depends only on stabilizability and does not require complete integrability of the differential control. This is because a metric corresponds to a differential change of coordinates $\delta_z = \Theta(x, t)\delta_x$, i.e., $M = \Theta'\Theta$, but there is no requirement that this should be integrable to an explicit change of coordinates $z = \theta(x)$ (see also discussion in [Sec 3, 13]).

Example 2: The following system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 - x_1^3 + x_2^2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u =: f(x) + Bu$$

is not feedback linearizable in the sense of [25], since the vector fields B and $ad_f B := \frac{\partial f}{\partial x}B - \frac{\partial B}{\partial x}f = [2x_2, 0]'$ are not linearly independent when $x_2 = 0$. However, it is universally stabilizable as verified by (11) with $W = I$ and multiplier $\rho(x) = 1 + 2x_2^2$. Additionally, since we can take $B_\perp = [1\ 0]'$, condition (9) reduces to the fact that $\frac{\partial f_1}{\partial x_2} = -1 - 3x_1^2 < 0\ \forall x$.

V. STABILITY AND STABILIZATION OF SUBMANIFOLDS

Convergence of a nonlinear system to a submanifold of state space is a requirement that appears in many applications, including coordination of multiagent systems [26], synchronization of oscillators [27], computational neuroscience [28], and nonlinear control design [29]. Design of controllers to stabilize submanifolds has been investigated recently using transverse feedback linearization [30], reduction and backstepping [31], and modifying controllers for drift-free systems [32].

Suppose a submanifold is defined by a level set of some smooth funciton $Z(t) = \{x : z(x, t) = c\}$, where $z : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^q$ and $\frac{\partial z}{\partial x}$ has rank q for all x. The sets $Z(t)$ are called controlled invariant if there exists a smooth mapping $u^\star : Z(t) \times \mathbb{R}^+ \to \mathbb{R}^m$ such that

$$\frac{\partial z(x, t)}{\partial t} + \frac{\partial z(x, t)}{\partial x}(f(x, t) + B(x, t)u^\star(x, t)) = 0$$

for all $x \in Z(t)$ for all $t \in \mathbb{R}^+$. In the case of an uncontrolled system, this reduces to the condition for a manifold to be flow invariant: $\frac{\partial z}{\partial t} + \frac{\partial z}{\partial x}f(x, t) = 0$.

The objective is to design a controller guaranteeing exponential convergence to $Z(t)$, i.e., for each $x(0)$, one can construct a control signal $u(t)$ such that the solution $x(t)$ of (1) satisfies $\inf_{y \in Z(t)}|x(t) - y| \le e^{-\lambda t}\bar{R}$ for some $\bar{R} > 0$.

Assume that we can construct a smooth matrix function $G(x, t)$ with columns that form a basis for the null space of $\frac{\partial z(x,t)}{\partial x}$. In order to study stability and stabilization of $Z(t)$, we construct a "virtual control

system":

$$\dot{x} = f(x, t) + \bar{B}(x, t)\bar{u} \tag{13}$$

where $\bar{B}(x, t) = [B(x, t)\ G(x, t)]$ and $\bar{u} = [u'\ v']'$, with u the actual control input and v a newly introduced "virtual control."

Theorem 3: If there exists a CCM for the virtual control system (13) satisfying the strong conditions C1 and C2, then any time-varying submanifold of the form $Z(t) = \{x : z(x, t) = c\}$ can be exponentially stabilized (open-loop, sampled-data, or continuously almost everywhere) with rate λ.

The proof uses the concept of a "shadow state" $\bar{x}(t)$, which has the property that $\bar{x}(t) \in Z(t)\ \forall t$ and can be thought of as a generalized projection of $x(t)$ onto $Z(t)$. The virtual control system (13) is constructed so that it can represent dynamics of the real system when $v = 0$, but can also represent $\bar{x}(t) \in Z(t)$ when $u = x^\star(\bar{x}, t)$ and v is arbitrary. The idea is to ensure $x(t)$ converges to $\bar{x}(t)$ and, therefore, to $Z(t)$.

The following corollary gives simple convex criteria for an uncontrolled system to converge to a submanifold.

Corollary 2: Consider an uncontrolled system of the form (1) with $B = 0\ \forall x, t$. Suppose that there exists a uniformly bounded dual metric W, invariant on level sets of $z(x, t)$, satisfying

$$\frac{\partial z}{\partial x}\left(-\frac{\partial W}{\partial t} - \partial_f W + \frac{\partial f}{\partial x}W + W\frac{\partial f}{\partial x}' + 2\lambda W\right)\frac{\partial z}{\partial x}' < 0. \tag{14}$$

Then, all solutions of the system converge exponentially with rate λ to the set $Z(t)$.

Note that this corollary is based solely on the existence of a universally stabilizing controller. Actual computation of a control signal is not required. Corollary 2 generalizes results on partial contraction in [33] and [34]. The latter showed that convergence to a linear manifold defined by $z(x) = Vx = 0$, with V a constant matrix, is guaranteed by the condition that $V(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}')V'$ is uniformly negative definite. A similar notion of horizontal contraction was studied in [15].

VI. ILLUSTRATIVE EXAMPLE

One of the advantages of convex criteria is that it is possible to mix and match different design objectives for one controller. In [35] and references therein, the problem of "uniting" locally optimal and globally stabilizing control was considered. This problem is nontrivial in a Lyapunov framework, since the set of CLFs for a system is nonconvex, but in the CCM framework, it is straightforward. Let us illustrate this with a particular example system taken from [35], with state $x = [x_1, x_2, x_3]'$ and dynamics (1) with

$$f(x) = \begin{bmatrix} -x_1 + x_3 \\ x_1^2 - x_2 - 2x_1 x_3 + x_3 \\ -x_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{15}$$

Note that this system is not feedback linearizable, since the matrix $[B, ad_f B, ad_f^2 B]$ drops rank at the origin.

We first solve the linear quadratic regulator (LQR) problem for the system linearized at the origin with cost function $\int_0^\infty (x'x + ru^2)dt$ with $r = 1$, obtaining a solution $P = P' > 0$ of the algebraic Riccati equation, and the locally optimal controller $u = -r^{-1}B'Px$. Then, we can search for W and ρ satisfying (11) and the additional linear constraints $W(0) = P^{-1}$ and $\rho(0) = 2r^{-1}$, so that locally the LQR and CCM controllers are the same. To satisfy Condition C2, entries of W were allowed to be quadratic functions of x_1 and x_2. The resulting metric is not uniformly bounded, but still satisfies the conditions of Lemma 1. Similarly, ρ was a quadratic polynomial in x_1, and $\lambda = 0.5$.
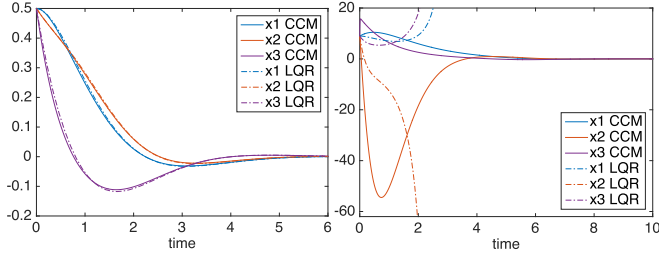
Fig. 2. Response of system (15) with CCM and LQR control to initial state $x(0) = [0.5, 0.5, 0.5]'$ (left) and $x(0) = [9, 9, 9]'$ (right). This exhibits the "locally optimal" and "globally stabilizing" behavior of the CCM controller.

As an optimization objective, we chose the l^1 norm of the coefficients of the polynomial entries of W to encourage sparsity. The metric was found using sum-of-squares programming [12] via the parser Yalmip [36] and solver Mosek; the resulting semidefinite program took about 0.4 s to solve on a standard desktop computer.

It can be seen in Fig. 2 that for small initial conditions, the CCM controller and the LQR are virtually identical. This is because the minimal geodesic is close to a straight line, and ρ and W are almost unchanged, so the CCM control law approximates a simple linear feedback on $x - x^\star$. In contrast, for larger initial conditions, the LQR controller was not stabilizing, while the CCM controller was. Simulations under LQR diverge rapidly after about 2 s. Further results on CCMs for problems in robotics can be found in [37], and a method for computing geodesics can be found in [17].

APPENDIX

Proof of Proposition 1: By assumption that the closed-loop system is strictly contracting, we have $\frac{d}{dt}(\delta_x' M(x, t)\delta_x) < -2\lambda \delta_x' M(x, t)\delta_x$ for all x, v, t and $\delta_x \neq 0$. Expanding the left-hand side, $\frac{d}{dt}(\delta_x' M(x, t)\delta_x) = \delta_x'[\frac{\partial M}{\partial t} + \partial_f M + (\frac{\partial f}{\partial x} + BK)'M + M(\frac{\partial f}{\partial x} + BK). + \sum_{i=1}^m (k_i(x, t) + v_i)(\partial_{b_i} M + \frac{\partial b_i}{\partial x}'M + M \frac{\partial b_i}{\partial x})]\delta_x$. Note that this is affine in v_i, so if this to be bounded for all $v \in \mathbb{R}^m$, it is clearly required that $\partial_{b_i} M + \frac{\partial b_i}{\partial x}'M + M \frac{\partial b_i}{\partial x} = 0$ for each i. Furthermore, if $\delta_x' MB = 0$, then terms involving K vanish, and hence, $\delta_x'(\frac{\partial M}{\partial t} + \partial_f M + \frac{\partial f}{\partial x}'M + M \frac{\partial f}{\partial x} + 2\lambda M)\delta_x < 0$, and the result follows from direct calculation of A and $\dot{M}$.

Lemma 1: Suppose a dual metric $W(x) = M(x)^{-1}$ satisfies a quadratic bound on its largest eigenvalue: $\lambda_{\max}(W(x)) \leq |\mathfrak{A}x + \mathfrak{B}|^2$ for all $x \in \mathbb{R}^n$, for some fixed matrices $\mathfrak{A}, \mathfrak{B}$, where $|\cdot|$ is the Euclidean norm. Then, there exists a minimal geodesic between any pair of points in $\mathbb{R}^n$.

Proof: By the Hopf–Rinow theorem, the result follows if any geodesic segment can be extended indefinitely [21]. By assumption, the metric $M(x)$ satisfies the lower bound $\lambda_{\min}(M(x)) \geq \frac{1}{|\mathfrak{A}x + \mathfrak{B}|^2}$. Take any geodesic $\gamma(s)$ defined on some interval $s \in (a, b) \subset \mathbb{R}$. Geodesics have constant speed: $\gamma_s' M(\gamma)\gamma_s = c$ for some $c > 0$, so $|\gamma_s| \leq c|\mathfrak{A}\gamma + \mathfrak{B}|$. This implies that $|\gamma|$ grows at-worst exponentially as the parameter $s \to \pm\infty$. Therefore, solutions do not exhibit finite escape in either direction, so the interval of existence is $(-\infty, \infty)$. ∎

Lemma 2: Consider the system (1), (3), and smooth real-valued "differential storage function" $V(x, \delta_x, t)$ and "differential supply rate" $\kappa(x, \delta_x, t)$. If for all $x, u, \delta_x \neq 0, t$

$$\frac{\partial V}{\partial \delta_x} B = 0 \implies \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}(f + Bu) + \frac{\partial V}{\partial \delta_x} A\delta_x < \kappa. \quad (16)$$

then a "differential feedback controller" $k_\delta(x, \delta_x, u, t) \in \mathbb{R}^m$ exists that satisfies the following two properties:
1) *Closed-loop dissipativity*: for all $x, u, \delta_x \neq 0, t$

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}(f + Bu) + \frac{\partial V}{\partial \delta_x}(A\delta_x + Bk_\delta) < \kappa.$$

2) *Path integrability*: For any regular curve c, and any $u_0 \in \mathbb{R}^m, t \in \mathbb{R}^+$, a unique solution of the following integral equation exists on $s \in [0, 1]$:

$$v(s) = u_0 + \int_0^s k_\delta(c(\mathfrak{s}), c_s(\mathfrak{s}), v(\mathfrak{s}), t)d\mathfrak{s}. \quad (17)$$

Proof: For brevity of notation, let us define

$$\mathfrak{a}(x, \delta_x, u, t) := \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}(f + Bu) + \frac{\partial V}{\partial \delta_x} A\delta_x - \kappa$$

$$\mathfrak{b}(x, \delta_x, t) := \frac{\partial V}{\partial \delta_x} BB' \frac{\partial V}{\partial \delta_x}'.$$

Note that by construction $\mathfrak{b} \geq 0$, and by assumption (16), for all $x, \delta_x \neq 0, u, t$, either $\mathfrak{a} < 0$ or $\mathfrak{b} > 0$. Now, define

$$\rho(x, \delta_x, u, t) := \begin{cases} 0, & \text{if } \mathfrak{a} < 0 \\ \dfrac{\mathfrak{a} + \sqrt{\mathfrak{a}^2 + \mathfrak{b}^2}}{\mathfrak{b}}, & \text{otherwise.} \end{cases} \quad (18)$$

It follows from [5, Th. 1] that ρ is a smooth for all $x, \delta_x \neq 0, u, t$. Now, we construct the control:

$$k_\delta(x, \delta_x, u, t) = -\rho(x, \delta_x, u, t)B(x, t)' \frac{\partial V(x, \delta_x, t)'}{\partial \delta_x}. \quad (19)$$

Substituting into (3) establishes closed-loop dissipativity: $\frac{d}{dt} V = \mathfrak{a} - \rho\mathfrak{b} + \kappa = \kappa - \sqrt{\mathfrak{a}^2 + \mathfrak{b}^2} < \kappa$.

We now prove path integrability, i.e., that a solution of (17) exists. We will prove this by contradiction. By assumption, the curve c is regular, so $c_s(\mathfrak{s}) \neq 0$ for all $\mathfrak{s}$, so k_δ is a smooth function of its third argument for all $\mathfrak{s}$. Hence, integrability follows unless there is finite escape at some $s = \bar{s} \leq 1$. It is clear from (19) that this would imply $\rho \to \infty$ as $s \to \bar{s}$.

First, we observe that $\mathfrak{b} > 0$ in a neighborhood of $\bar{s}$, since if $\mathfrak{b} = 0$, then $\rho = 0$, but we require ρ to blow up.

Second, we note that the only dependence ρ has on u is via $\mathfrak{a}$, which by construction is an s-dependent affine function of u. It follows from (18) that if there is a closed interval $\mathfrak{S} \subset [0, 1]$ such that $\mathfrak{b} > 0$ for $s \in \mathfrak{S}$, then ρ is a globally Lipschitz function of u on $\mathfrak{S}$. By standard comparison results, e.g., [1, Th. 3.2], a unique solution to (17) exists on this interval, which contradicts finite escape at $s = \bar{s}$. ∎

Lemma 3: Given a CCM $M(x, t)$, a time interval $\mathcal{T} = [t_i, t_{i+1}] \subset \mathbb{R}^+$, and a path $c(t_i)$ connecting $x^\star(t_i)$ to $x(t_i)$, suppose the open-loop control signal in point 1 in Section III is applied on $\mathcal{T}$ initialized with c, then for all $t \in \mathcal{T}$

$$d(x^\star(t), x(t), t) \leq e^{-\lambda(t-t_i)} L(c(t_i), t_i). \quad (20)$$

Proof: By construction, $c(t, 0) = x^\star(t)$ and $c(t, 1) = x(t)$ for all $t \in \mathcal{T}$. Furthermore, $c_s(t, s) = \frac{\partial c}{\partial s}(t, s)$ satisfies $\frac{d}{dt} c_s(t, s) = A(c(t, s), k_p, t)c_s + B(c(t, s), t)k_\delta(t, s)$ for all $t \in \mathcal{T}, s \in [0, 1]$. ∎

By construction of k_δ, we have $\frac{d}{dt}(c_s' M(c, t)c_s) < -2\lambda c_s' M(c, t)c_s$. Integrating with respect to s gives $\frac{d}{dt} E(c(t), t) < -2\lambda E(c(t), t)$ and integrating with respect to t gives $E(c(t), t) \leq e^{-2\lambda(t-t_i)} E(c(t_i), t_i)$ for $t \in \mathcal{T}$, with strict inequality for $t > t_i$. Taking square roots gives $L(c(t), t) \leq e^{-\lambda(t-t_i)} L(c(t_i), t_i)$. Now, for each t, the curve $c(t)$ connects $x^\star(t)$ to $x(t)$, and the Riemannian distance between these points is the infimum of lengths of such curves, so we obtain the bound (20).

Proof of Theorem 1: We will show that for each type of controller, the Riemannian distance between $x(t)$ and $x^\star(t)$ decreases exponentially:

$$d(x^\star(t), x(t), t) \leq e^{-\lambda t} d(x^\star(0), x(0), 0). \qquad (21)$$

Exponential convergence in the Euclidean metric is then implied by uniform boundedness of $M(x, t)$. In particular, it is straightforward to show that $\sqrt{\alpha_1}|x(t) - x^\star(t)| \leq d(x^\star(t), x(t), t)$ and $d(x(0), x^\star(0), 0) \leq \sqrt{\alpha_2}|x(0) - x^\star(0)|$. Combining with (21) gives

$$\sqrt{\alpha_1}|x(t) - x^\star(t)| \leq e^{-\lambda t}\sqrt{\alpha_2}|x(0) - x^\star(0)|$$

i.e., (2) holds with $R = \sqrt{\frac{\alpha_2}{\alpha_1}}$.

For the open-loop control, Lemma 3 implies that for any $t > 0$, we have $d(x^\star(t), x(t), t) \leq e^{-\lambda t}L(c(0), 0)$. If the minimal geodesic is chosen for $c(0)$, then $L(c(0), 0) = d(x^\star(0), x(0), 0)$, and so we obtain (21).

Similarly, for the sampled-data controller, on each interval $[t_i, t_{i+1})$, we have

$$d(x^\star(t), x(t), t) \leq L(c(t), t) \leq e^{-\lambda(t-t_i)}L(c(t_i), t_i)$$

and then, at time t_{i+1}, a minimal geodesic γ_{i+1} is computed, which has length $L(\gamma_{i+1}) = d(x^\star(t_{i+1}), x(t_{i+1}), t_{i+1}) \leq \lim_{t \to t_{i+1}} L(c(t), t)$, where the limit in t is from the left.

For continuous feedback, we first show that the given controller is smooth on $\mathfrak{D}(x^\star, t)$. It follows from [21, Prop. 3.5, p. 117] that the mapping $(x, x^\star, t) \mapsto \gamma$ is smooth. Now, it follows from Lemma 2 and the smoothness of ρ defined in (18) that the mapping $(x, x^\star, t) \mapsto u$ is smooth.

To show continuity at $x = x^\star$, we first note that since $V = \delta_x' M \delta_x$ is quadratic in δ_x, and the differential dynamics (3) are linear in δ_x, the small control property of [5] holds, i.e., for any $\epsilon > 0$, there exists $\eta > 0$ such that for $\delta_x \neq 0$ with $|\delta_x| \leq \eta$, there exists δ_u with $|\delta_u| < \epsilon$ satisfying $\dot{V} < 0$. This implies that k_δ in (19) is continuous in δ_x at $\delta_x = 0$.

Let us show convergence of the control law to $u^\star$ for a sequence of states x^k with $\lim_{k \to \infty} x^k = x^\star(t)$. It suffices to construct the sequence x^k along a particular but arbitrary geodesic γ^0, e.g., $x^k = \gamma^0(1/k)$, since all states in a neighborhood of $x^\star$ lie on such a geodesic [21, Th. 3.7]. Let γ^k denote the segment from $x^\star(t)$ to x^k. Now, the constant-speed property of geodesics states that $\|\gamma_s^k\|_s = L(\gamma^k, t)$ for all s, and so by uniform boundedness of the metric, $|\gamma_s^k| \to 0$ uniformly in $s \in [0, 1]$. Therefore, by continuous dependence of (6) on k_δ (e.g., [38, Lemma 3.1]), the sequence of solutions $k_p(\gamma^k, u^\star(t), t, 1)$ of (6) converges to $u^\star(t)$ as $k \to \infty$.

To show exponential decrease in distance, recall [21, Prop 2.4, p. 195] that on $\mathfrak{D}(x^\star, t)$, the energy is a smooth function of its endpoints, i.e., $(a, b) \mapsto E(a, b, t)$ is smooth for each t. Now, consider the open-loop control initialized with $t_i = t$ and γ for $c(t_i)$. This open-loop control is identical to the proposed continuous feedback at time t, and therefore, $\dot{x}(t)$ is also identical. Hence, there exists a (nonminimal) path $c(\cdot)$ defined on $[t, t+\epsilon)$ with $c(t) = \gamma_i$ satisfying $\frac{d}{dt}E(c(t), t) < -2\lambda E(\gamma, t)$; therefore, $\frac{d}{dt}E(x(t), x^\star(t), t) < -2\lambda E(x(t), x^\star(t), t)$. Integrating with respect to time gives the result.

Proof of Theorem 2: For the first statement, under the feedback transformation, we have a new control system affine in v: $\dot{x} = (f + B\alpha) + B\beta v$, and the associated differential dynamics are of the form (3) with $A_v(x, v) = A(x, u(x, v)) + B(x)\frac{\partial u(x, v)}{\partial x}$ and $B_v(x) = B(x)\beta(x)$. Now, consider the dual CCM condition (9). The same annihilator matrix $B_\perp$ can be used since $B_\perp' B_v = B_\perp B\beta = 0$. Now, substitute A_v for A in (9), and notice that since $B_\perp' B = 0$, the second term $(B\frac{\partial u}{\partial x})$ in A_v has no effect. The first term in A_v is just $A(x, u)$

evaluated a particular value of u, and (9) holds for all u and, hence, can be applied under feedback transformation.

For the second statement, the differential dynamics transforms as $\dot{\delta}_\xi = A_\xi \delta_\xi + B_\xi \delta_u$, with $A_\xi = \dot{\Phi}\Psi + \Phi A\Psi$ and $B_\xi := \Phi B$. Taking $M_\xi = \Psi' M \Psi$ and the identity $\dot{\Psi} = -\Psi\dot{\Phi}\Psi$, we have $\dot{M}_\xi = -\Psi'\dot{\Phi}'\Psi' M\Psi + \Psi'\dot{M}\Psi - \Psi' M\Phi\dot{\Phi}\Psi$; then, straightforward calculation gives $\dot{M}_\xi + A_\xi' M_\xi + M_\xi A_\xi + 2\lambda M_\xi = \Psi'(\dot{M} + A'M + MA + 2\lambda M)\Psi$ and the result follows from the fact that $\delta_x = \Psi\delta_\xi$. The third statement then follows by considering points $x = \phi^{-1}(\xi)$.

Proof of Theorem 3: We will construct the open-loop controller, but others are analogous to the construction for Theorem 1. Since there exists a CCM $M(x, t)$ satisfying the strong conditions, we can construct a differential feedback controller for (13) of the form $k_\delta(x, \delta_x, t) = -\frac{1}{2}\rho(x, t)\bar{B}(x, t)'M(x, t)\delta_x$. By construction of $\bar{B}$, this decomposes as $\delta_u = -\frac{1}{2}\rho B'M\delta_x$, $\delta_v = -\frac{1}{2}\rho G'M\delta_x$. Now, for a given path $c(t) \in \Gamma(\bar{x}(t), x(t))$, we construct the following paths of signals for the real and virtual control inputs:

$$u(t, s) = u^\star(\bar{x}, t) + \int_0^s \delta_u(c(\mathfrak{s}), c_s(\mathfrak{s}), t)d\mathfrak{s} \qquad (22)$$

$$v(t, s) = -\int_s^1 \delta_v(c(\mathfrak{s}), c_s(\mathfrak{s}), t)d\mathfrak{s} \qquad (23)$$

noting that $\frac{\partial u}{\partial s} = \delta_u$ and $\frac{\partial v}{\partial s} = \delta_v$. Therefore, by an analogous argument to Theorem 1, one can construct open-loop, sampled-data, or continuous almost-everywhere controllers such that the length of $c(t)$ shrinks exponentially.

Now, notice that at $s = 1$, we have $v(t, 1) = 0$, so $\dot{c}(t, 1) = f(c(t, 1), t) + B(c(t, 1), t)u(t)$ and so $c(t, 1) = x(t)$ for all t, i.e., the "real" dynamics are recovered. On the other hand, at $s = 0$, we have the $c(t, 0) = \bar{x}(t)$ (the "shadow" state) with dynamics $\dot{\bar{x}} = f(\bar{x}, t) + B(\bar{x}, t)u^\star(\bar{x}, t) + G(\bar{x}, t)k_v(t, 0)$. But by the assumption that Z is invariant for $\dot{x} = f + Bu^\star$, and that $\frac{\partial z}{\partial x}G = 0$, it follows that $\bar{x}(t) \in Z(t)$ for all t.

So, since $L(c(t)) \to 0$ exponentially and $c(t, 1) = x(t)$ and $c(t, 0) \in Z(t)$, it follows that $x(t) \to Z(t)$ exponentially.

REFERENCES

[1] H. Khalil, *Nonlinear Systems*. Englewood Cliffs, NJ, USA: Prentice-Hall, 2002.

[2] J. Slotine and W. Li, *Applied Nonlinear Control*. Englewood Cliffs, NJ, USA: Prentice-Hall, 1991.

[3] Z. Artstein, "Stabilization with relaxed controls," *Nonlinear Anal., Theory, Methods Appl.*, vol. 7, no. 11, pp. 1163–1173, 1983.

[4] E. D. Sontag, "A Lyapunov-like characterization of asymptotic controllability," *SIAM J. Control Optim.*, vol. 21, no. 3, pp. 462–471, 1983.

[5] E. D. Sontag, "A 'universal' construction of Artstein's theorem on nonlinear stabilization," *Syst. Control Lett.*, vol. 13, no. 2, p. 117, 1989.

[6] R. A. Freeman and P. V. Kokotovic, *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques*. Basel, Switzerland: Birkhäuser, 2008.

[7] S. Boyd, L. el Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA, USA: SIAM, 1994.

[8] G. Dullerud and F. Paganini, *A Course in Robust Control Theory: A Convex Approach*. New York, NY, USA: Springer-Verlag, 2000.

[9] A. Rantzer, "A dual to Lyapunov's stability theorem," *Syst. Control Lett.*, vol. 42, no. 3, pp. 161–168, 2001.

[10] J. B. Lasserre, D. Henrion, C. Prieur, and E. Trélat, "Nonlinear optimal control via occupation measures and LMI-relaxations," *SIAM J. Control Optim.*, vol. 47, no. 4, pp. 1643–1666, 2008.

[11] U. Vaidya, P. G. Mehta, and U. V. Shanbhag, "Nonlinear stabilization via control Lyapunov measure," *IEEE Trans. Autom. Control*, vol. 55, no. 6, pp. 1314–1328, Jun. 2010.

[12] P. A. Parrilo, "Semidefinite programming relaxations for semialgebraic problems," *Math. Program.*, vol. 96, no. 2, p. 293, 2003.

[13] W. Lohmiller and J.-J. E. Slotine, "On contraction analysis for non-linear systems," *Automatica*, vol. 34, no. 6, pp. 683–696, Jun. 1998.

[14] D. Lewis, "Metric properties of differential equations," *Amer. J. Math.*, vol. 71, pp. 294–312, 1949.

[15] F. Forni and R. Sepulchre, "A differential Lyapunov framework for contraction analysis," *IEEE Trans. Autom. Control*, vol. 59, no. 3, pp. 614–628, Mar. 2014.

[16] I. R. Manchester and J.-J. E. Slotine, "Transverse contraction criteria for existence, stability, and robustness of a limit cycle," *Syst. Control Lett.*, vol. 62, pp. 32–38, 2014.

[17] K. Leung and I. R. Manchester, "Nonlinear stabilization via control contraction metrics: A pseudospectral approach for computing geodesics," in *Proc. Amer. Control Conf.*, Seattle, WA, USA, May 2017.

[18] A. V. Pavlov, N. van de Wouw, and H. Nijmeijer, *Uniform Output Regulation of Nonlinear Systems: A Convergent Dynamics Approach*. New York, NY, USA: Springer, 2006.

[19] F. Fontes, "Discontinuous feedbacks, discontinuous optimal controls, and continuous-time model predictive control," *Int. J. Robust Nonlinear Control*, vol. 13, nos. 3/4, pp. 191–209, 2003.

[20] F. H. Clarke, Y. S. Ledyaev, E. D. Sontag, and A. Subbotin, "Asymptotic controllability implies feedback stabilization," *IEEE Trans. Autom. Control*, vol. 42, no. 10, pp. 1394–1407, Oct. 1997.

[21] M. P. Do Carmo, *Riemannian Geometry*. New York, NY, USA: Springer, 1992.

[22] F. Uhlig, "A recurring theorem about pairs of quadratic forms and extensions: A survey," *Linear Algebra Appl.*, vol. 25, pp. 219–237, 1979.

[23] V. Syrmos, C. Abdallah, P. Dorato, and K. Grigoriadis, "Static output feedback— A survey," *Automatica*, vol. 33, no. 2, pp. 125–137, 1997.

[24] J. A. Primbs, V. Nevistic, and J. C. Doyle, "A receding horizon generalization of pointwise min-norm controllers," *IEEE Trans. Autom. Control*, vol. 45, no. 5, pp. 898–909, May 2000.

[25] A. Isidori, *Nonlinear Control Systems*. New York, NY, USA: Springer, 1995.

[26] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, "Flocking in fixed and switching networks," *IEEE Trans. Autom. Control*, vol. 52, no. 5, pp. 863–868, May 2007.

[27] F. Dörfler and F. Bullo, "Synchronization in complex networks of phase oscillators: A survey," *Automatica*, vol. 50, no. 6, pp. 1539–1564, 2014.

[28] Y. Burak and I. R. Fiete, "Accurate path integration in continuous attractor network models of grid cells," *PLoS Comput. Biol.*, vol. 5, no. 2, 2009, Art. no. e1000291.

[29] A. Astolfi, D. Karagiannis, and R. Ortega, *Nonlinear and Adaptive Control with Applications*. New York, NY, USA: Springer, 2007.

[30] C. Nielsen and M. Maggiore, "On local transverse feedback linearization," *SIAM J. Control Optim.*, vol. 47, no. 5, pp. 2227–2250, 2008.

[31] M. I. El-Hawwary and M. Maggiore, "Reduction theorems for stability of closed sets with application to backstepping control design," *Automatica*, vol. 49, no. 1, pp. 214–222, 2013.

[32] J. M. Montenbruck, M. Burger, and F. Allgower, "Compensating drift vector fields with gradient vector fields for asymptotic submanifold stabilization," *IEEE Trans. Autom. Control*, vol. 61, no. 2, pp. 388–399, Feb. 2016.

[33] W. Wang and J.-J. E. Slotine, "On partial contraction analysis for coupled nonlinear oscillators," *Biological Cybern.*, vol. 92, no. 1, pp. 38–53, 2005.

[34] Q. C. Pham and J.-J. E. Slotine, "Stable concurrent synchronization in dynamic system networks," *Neural Netw.*, vol. 20, pp. 62–77, 2007.

[35] V. Andrieu and C. Prieur, "Uniting two control Lyapunov functions for affine systems," *IEEE Trans. Autom. Control*, vol. 55, no. 8, pp. 1923–1927, Aug. 2010.

[36] J. Löfberg, "YALMIP: A toolbox for modeling and optimization in MATLAB," in *Proc. IEEE Int. Symp. Comput. Aided Control Syst. Des.*, 2004, pp. 284–289.

[37] I. R. Manchester, J. Z. Tang, and J.-J. E. Slotine, "Unifying robot trajectory tracking with control contraction metrics," in *Robot. Res.*, A. Bicchi and W. Burgard, Eds. (Springer Proceedings in Advanced Robotics Series), ISBN 978-3-319-51531-1. Springer International Publishing, 2017 (forthcoming). [Online]. Available: http://www.springer.com/gp/book/9783319515311

[38] J. K. Hale, *Ordinary Differential Equations*. New York, NY, USA: Robert E. Krieger, 1980.