Discrete-Valued Control of Linear Time-Invariant Systems by Sum-of-Absolute-Values Optimization

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***Abstract*—In this paper, we propose a new design method of discrete-valued control for continuous-time linear time-invariant systems based on sum-of-absolute-values (SOAV) optimization. We first formulate the discrete-valued control design as a finite-horizon SOAV optimal control, which is an extended version of $\mathbb{L}^1$ optimal control. We then give simple conditions that guarantee the existence, discreteness, and uniqueness of the SOAV optimal control. Also, we show that the value function is continuous, by which we prove the stability of infinite-horizon model predictive SOAV control systems. We provide a fast algorithm for the SOAV optimization based on the alternating direction method of multipliers (ADMM), which has an important advantage in real-time control computation. A simulation result shows the effectiveness of the proposed method.**

***Index Terms*—Convex optimization, discrete-valued control, model predictive control, optimal control.**

I. Introduction

Discrete-valued control is a control mechanism that achieves control objectives (e.g. stability) with control inputs taking values in a finite alphabet (e.g. bang-bang control: 1-bit control taking ± 1). Discrete-valued control has a significant advantage in networked control in which control signals are quantized and transmitted through networks (see e.g. [1]); since discrete-valued control signals need not be quantized, no quantization error may occur. Also, discrete-valued control has important applications in DC-DC conversion [2], class D amplifier [3], hybrid power system [4], train control [5], hormone therapy [6], to name a few.

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A standard design method for discrete-valued control is mixed-integer programming [7]. Although this directly gives discrete-valued control, this method requires heavy computation, and hence it can be used only for relatively slow systems such as a gas supply system reported in [7]. A more tractable method is dynamic quantization proposed in [8], [9]. In this approach, a dynamic quantizer is designed such that the quantizer mimics the ideal (i.e. no quantization) continuous output, and the state space representation of the dynamic quantizer is given in a closed form. This method, however, assumes an infinite alphabet (e.g. the set of integers, $\mathbb{Z}$). Another approach is the control parametrization enhancing transform proposed by [10], in which the optimal switching times of a piecewise-constant (i.e. discrete-valued) control input are computed. This approach assumes that the number of switching is previously known, which is in practice hard to obtain.

Alternatively, we propose a novel method for discrete-valued control based on the idea of the *sum-of-absolute-values* (SOAV) optimization [11]. The proposed optimal control, which we call the *SOAV optimal control*, is an extended version of $\mathbb{L}^1$ optimal control [12] (also known as the minimum fuel control [13]). The SOAV optimization is convex and hence the solution can be obtained efficiently. In fact, as shown in Section V-B, the optimization is solved, after time-discretization, by the *alternating direction method of multipliers* (ADMM) [14]–[16], which is a simple but much faster algorithm for large scale problems than the standard interior point method [17, Chap 11].

For theoretical analysis, we prove the existence, discreteness, and uniqueness of the (finite-horizon) SOAV optimal control under simple conditions (e.g. the system model is controllable, the A-matrix is non-singular, and the finite alphabet for the control includes 0). The obtained discrete-valued control is a piecewise constant signal, and we prove the number of discontinuities, or *switching times*, is bounded. This property is very important in particular for networked control since the upper bound of the number of switching times, which can be given before optimization, ensures the upper bound of the data rate required to transmit the discrete-valued control.

We also prove that the value function, which is defined as the optimal value of the cost function of the optimal control problem, is a continuous and convex function of initial states. This property is applied to prove the stability of the model predictive control (MPC) feedback system based on the finite-horizon SOAV optimal control. As mentioned above, the SOAV optimal control can be obtained by the fast ADMM algorithm, and hence the control is well-adapted for MPC.

The remainder of this paper is organized as follows: In Section II, we give mathematical preliminaries for our subsequent discussion. In Section III, we formulate optimal control problem so that optimal controls have the desired discrete values. After that, we examine optimal controls, and lead the existence, discreteness, and uniqueness of the SOAV optimal control. A numerical optimization algorithm based on ADMM is also presented in this section. Section IV investigates the continuity and the convexity of the value function in SOAV optimal control. Section V gives the model predictive control formulation and shows the stability. Section VI presents an example of model predictive control to illustrate the effectiveness of the proposed method. In Section VII, we offer concluding remarks.

II. Mathematical Preliminaries

This section reviews basic definitions, facts, and notation that will be used throughout the paper.

Let n be a positive integer. For a vector $x \in \mathbb{R}^n$ and a scalar $\varepsilon > 0$, the *ε-neighborhood* of x is defined by $\mathcal{B}(x,\varepsilon) \triangleq \{y \in \mathbb{R}^n : \|y - x\| < \varepsilon\}$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$. Let $\mathcal{X}$ be a subset of $\mathbb{R}^n$. A point $x \in \mathcal{X}$ is called an *interior point* of $\mathcal{X}$ if there exists $\varepsilon > 0$ such that $\mathcal{B}(x,\varepsilon) \subset \mathcal{X}$. The *interior* of $\mathcal{X}$ is the set of all interior points of $\mathcal{X}$, and we denote the interior of $\mathcal{X}$ by $\mathrm{int}\mathcal{X}$. A set $\mathcal{X}$ is said to be *open* if $\mathcal{X} = \mathrm{int}\mathcal{X}$. A point $x \in \mathbb{R}^n$ is called an *adherent point* of $\mathcal{X}$ if $\mathcal{B}(x,\varepsilon) \cap \mathcal{X} \neq \emptyset$ for every $\varepsilon > 0$, and the *closure* of $\mathcal{X}$, denoted by $\overline{\mathcal{X}}$, is the set of all adherent points of $\mathcal{X}$. A set $\mathcal{X} \subset \mathbb{R}^n$ is said to be *closed* if $\mathcal{X} = \overline{\mathcal{X}}$. The *boundary* of $\mathcal{X}$, denoted by $\partial\mathcal{X}$, is the set of all points in the closure of $\mathcal{X}$, not belonging to the interior of $\mathcal{X}$, i.e., $\partial\mathcal{X} = \overline{\mathcal{X}} - \mathrm{int}\mathcal{X}$, where $\mathcal{X}_1 - \mathcal{X}_2$ is the set of all points that belong to the set $\mathcal{X}_1$ but not to the set $\mathcal{X}_2$. In particular, if $\mathcal{X}$ is closed, then $\partial\mathcal{X} = \mathcal{X} - \mathrm{int}\mathcal{X}$, since $\mathcal{X} = \overline{\mathcal{X}}$. A set $\mathcal{X} \subset \mathbb{R}^n$ is said to be *convex* if, for any $x, y \in \mathcal{X}$ and any $\lambda \in [0,1]$, $(1-\lambda)x + \lambda y$ belongs to $\mathcal{X}$.

A real-valued function f defined on $\mathbb{R}^n$ is said to be *lower semi-continuous* on $\mathbb{R}^n$ if for every $\alpha \in \mathbb{R}$ the set $\{x \in \mathbb{R}^n : f(x) > \alpha\}$ is open. It is known that if a function f is lower semi-continuous on $\mathbb{R}^n$, then

$$f(x) \leq \liminf_{y \to x} f(y)$$

for every $x \in \mathbb{R}^n$ [18, pp. 32]. A real-valued function f defined on a convex set $\mathcal{C} \subset \mathbb{R}^n$ is said to be *convex* if

$$f\big((1-\lambda)x + \lambda y\big) \leq (1-\lambda)f(x) + \lambda f(y),$$

for all $x, y \in \mathcal{C}$ and all $\lambda \in (0,1)$.

Let $T > 0$ and m be a positive integer. For a continuous-time signal $u(t) \in \mathbb{R}^m$ over a time interval $[0,T]$, we define its $\mathbb{L}^1$ *and* $\mathbb{L}^\infty$ *norms* respectively by

$$\|u\|_1 \triangleq \sum_{j=1}^m \int_0^T |u_j(t)|dt, \quad \|u\|_\infty \triangleq \max_{1\leq j\leq m} \operatorname*{ess\ sup}_{t\in[0,T]} |u_j(t)|,$$

where $u_j(t)$ is the j-th component of $u(t)$. We denote by μ the Lebesgue measure on $\mathbb{R}$.

For a vector $x = [x_1, x_2, \ldots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$, we define its ℓ^1 *and* ℓ^∞ *norms* respectively by

$$\|x\|_{\ell^1} \triangleq \sum_{j=1}^n |x_j|, \quad \|x\|_{\ell^\infty} \triangleq \max_{1\leq j\leq n} |x_j|.$$

III. Discrete-Valued Control Problem

In this paper, we consider a linear time-invariant system represented by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0 \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$. For the system (1), we attempt to arrive at a *discrete-valued control*, that is, the control $u(t)$ can only take values in a fixed finite set (or finite alphabet)

$$\mathbb{U} \triangleq \{\pm U_1, \pm U_2, \ldots, \pm U_N\} \tag{2}$$

where $U_1, \ldots, U_N$ are non-negative real numbers satisfying

$$0 \leq U_{\min} \triangleq U_1 < U_2 < \cdots < U_N = 1. \tag{3}$$

Here we assume the maximum value $U_N = 1$ without loss of generality (otherwise, use B/U_N instead of B in (1)). An initial state $\xi \in \mathbb{R}^n$ and a finite time $T > 0$ are given. The control objective is to obtain a discrete-valued control $u(t) \in \mathbb{U}^m$ for $t \in [0,T]$ that steers the state $x(t)$ from the initial state ξ to the origin at time T. We will show in this paper that such a discrete-valued control can be efficiently obtained by *sum-of-absolute-values* (SOAV) optimal control described below.

SOAV optimal control is an extended version of $\mathbb{L}^1$ optimal control (also known as *minimum-fuel control* [13]). Denote by $\mathcal{U}(\xi)$ the set of all $\mathbb{L}^1$-integrable *feasible controls* that satisfy $x(0) = \xi$, $x(T) = 0$, and $\|u\|_\infty \leq 1$ for the system (1). We consider initial values ξ such that $\mathcal{U}(\xi)$ is non-empty. This assumption is satisfied if T is greater than some minimum time $T^\star$ [19]. Then the $\mathbb{L}^1$ optimal control is a control that minimizes the $\mathbb{L}^1$ cost function $\|u\|_1$ among all feasible $u \in \mathcal{U}(\xi)$. It is known that the $\mathbb{L}^1$ optimal control takes only 0 and ± 1 when the system (1) is *normal*, that is, the coefficient matrix A is non-singular and the pairs $(A, b_1), (A, b_2), \ldots, (A, b_m)$ are controllable [13, Theorem 6–13], where $B = [b_1, b_2, \ldots, b_m]$, $b_j \in \mathbb{R}^n$. In other words, if (1) is normal, then the $\mathbb{L}^1$ optimal control gives a discrete-valued control on $\mathbb{U}$ with $U_1 = 0$ and $U_2 = 1$ $(N = 2)$. To extend this idea to a general set $\mathbb{U}$ as in (2), we consider the following SOAV cost function:

$$J(u) \triangleq \sum_{i=1}^N \sum_{j=1}^m w_i \phi_{i,j}(u),$$

$$\phi_{i,j}(u) \triangleq \|u_j - U_i\|_1 + \|u_j + U_i\|_1 \tag{4}$$

where $w_1, \ldots, w_N$ are given weights satisfying $w_1 + w_2 + \cdots + w_N = 1$. The motivation for this cost function is based on the observation that if $u_j(t) = U_i$ on a set $\mathcal{I} \subset [0,T]$, then $u_j(t) - U_i = 0$ on $\mathcal{I}$, which is *sparse* and reduces the $\mathbb{L}^1$ norm $\|u_j - U_i\|_1$ as discussed in [12].

Let us formulate the associated optimal control problem as follows.

Problem 1 (SOAV Optimal Control Problem): For a given initial state $\xi \in \mathbb{R}^n$, find a feasible control $u \in \mathcal{U}(\xi)$ that minimizes the SOAV cost function $J(u)$ given in (4).

We call the optimal solution the *SOAV optimal control*. We will show that under some assumptions on the system (1) and the initial state ξ, the SOAV optimal control takes its values in the set $\mathbb{U}$.

A. Existence

Here we show the existence theorem for the SOAV optimal control.

Let us define the *reachable set* of initial values from which the state $x(t)$ in (1) is steered to the origin by some control $u(t)$, $t \in [0, T]$ with $\|u\|_\infty \leq 1$.

Definition 1 (Reachable Set): For the system (1), the reachable set $\mathcal{R}$ at time T is defined by

$$\mathcal{R} \triangleq \left\{ \int_0^T e^{-At} Bu(t)dt : \|u\|_\infty \leq 1 \right\} \subset \mathbb{R}^n. \tag{5}$$

Then we have the following existence theorem.

Theorem 1 (Existence): For each initial state in the reachable set $\mathcal{R}$, there exists an SOAV optimal control.

Proof: Let an initial state $\xi \in \mathcal{R}$ be fixed. The feasible control set $\mathcal{U}(\xi)$ can be described by

$$\mathcal{U}(\xi) = \left\{ u \in \mathbb{L}^1 : \int_0^T e^{-At} Bu(t)dt = -\xi, \|u\|_\infty \leq 1 \right\}. \tag{6}$$

Since the set $\mathcal{U}(\xi)$ is non-empty, we can define

$$\theta \triangleq \inf\{J(u) : u \in \mathcal{U}(\xi)\}.$$

Then there exists a sequence $\{u_l\}_{l \in \mathbb{N}} \subset \mathcal{U}(\xi)$ such that $\lim_{l\to\infty} J(u_l) = \theta$, $\|u_l\|_\infty \leq 1$, and

$$\xi = -\int_0^T e^{-At} Bu_l(t)dt. \tag{7}$$

Since the set $\{u \in L^\infty : \|u\|_\infty \leq 1\}$ is sequentially compact in the weak* topology of L^∞ [20], there exist a measurable function u_∞ with $\|u\|_\infty \leq 1$ and a subsequence $\{u_{l'}\}$ such that each component $\{u_{j,l'}\}$ converges to $u_{j,\infty}$, the j-th component of u_∞, in the weak* topology of L^∞, that is, we have

$$\lim_{l'\to\infty} \int_0^T \big(u_{j,l'}(t) - u_{j,\infty}(t)\big) f(t)dt = 0 \tag{8}$$

for any $f \in \mathbb{L}^1$ and $j = 1, 2, \ldots, m$. By (7) and (8),

$$\xi = -\int_0^T e^{-At} Bu_\infty(t)dt$$

and hence $u_\infty \in \mathcal{U}(\xi)$. Put

$$J_{l'}^\pm \triangleq \sum_{i=1}^N \sum_{j=1}^m w_i \int_0^T (u_{j,l'}(t) \pm U_i)\mathrm{sgn}(u_{j,\infty}(t) \pm U_i)dt \tag{9}$$

where the function sgn is defined by

$$\mathrm{sgn}(v) = \begin{cases} v/|v|, & \text{if } v \neq 0, \\ 0, & \text{if } v = 0. \end{cases}$$

From (8), we have

$$\lim_{l'\to\infty} J_{l'}^\pm = \sum_{i=1}^N \sum_{j=1}^m w_i \|u_{j,\infty} \pm U_i\|_1.$$

Let $J_{l'} \triangleq J_{l'}^+ + J_{l'}^-$. Then the above equation gives

$$\lim_{l'\to\infty} J_{l'} = \lim_{l'\to\infty} (J_{l'}^+ + J_{l'}^-) = J(u_\infty). \tag{10}$$

Also, from (4) and (9), we have

$$J_{l'} \leq |J_{l'}| \leq \sum_{i=1}^N \sum_{j=1}^m w_i \phi_{i,j}(u_{l'}) = J(u_{l'}) \tag{11}$$

for each $l' \in \mathbb{N}$. Since the sequence $\{J(u_l)\}$ converges to θ as $l \to \infty$, the subsequence $\{J(u_{l'})\}$ has the same limit θ. Therefore we have

$$J(u_\infty) = \lim_{l'\to\infty} J_{l'} \leq \lim_{l'\to\infty} J(u_{l'}) = \theta \tag{12}$$

from (10) and (11). On the other hand, since $u_\infty \in \mathcal{U}(\xi)$, we have $J(u_\infty) \geq \theta$. This with (12), we have $J(u_\infty) = \theta$, and u_∞ is an optimal control for the initial state ξ. ∎

B. Discreteness of SOAV Optimal Control

Here we show the SOAV optimal solution is a discrete-valued control on $\mathbb{U}$. The following theorem is one of the main results.

Theorem 2 (Discreteness): Assume that the coefficient matrix A is non-singular and the pairs $(A, b_1), (A, b_2), \ldots, (A, b_m)$ are controllable[1]. If an SOAV optimal control u^* exists, then at least one of the followings holds.

i) $u^*(t) \in \mathbb{U}^m$ for almost all $t \in [0, T]$.

ii) If u_j^* violates (i) for some j, then $\|u_j^*\|_\infty \leq U_{\min}$.

In particular, if $U_{\min} = 0$, then $u^*(t)$ takes values in $\mathbb{U}$ for almost all $t \in [0, T]$.

Proof: We give the proof based on the discussion in the proof of [13, Theorem 6–13], which is devoted to the discreteness of $\mathbb{L}^1$ optimal controls. The Hamiltonian H for the SOAV optimal control problem is defined by

$$H(x, u, p) \triangleq \sum_{j=1}^m L(u_j) + p^{\mathrm{T}} \left(Ax + \sum_{j=1}^m b_j u_j \right)$$

where

$$L(u_j) \triangleq \sum_{i=1}^N w_i(|u_j - U_i| + |u_j + U_i|)$$

and p is the costate vector. Let x^* denote the trajectory corresponding to u^*. From Pontryagin's minimum principle [13],

[1]The pair (A, b_i) is said to be controllable if the matrix $[b_i, Ab_i, \ldots, A^{n-1}b_i]$ is non-singular.

there exists a costate vector p^* satisfying $H(x^*, u^*, p^*) \leq H(x^*, u, p^*)$, or

$$L(u_j^*) + (p^*)^{\mathrm{T}} b_j u_j^* \leq L(u_j) + (p^*)^{\mathrm{T}} b_j u_j \tag{13}$$

for every u_j with $|u_j| \leq 1$, where $j = 1, 2, \ldots, m$. Therefore, each component of the optimal control is the minimizer of the right hand side of (13), which can be obtained analytically as follows.

Fix arbitrarily $j \in \{1, 2, \ldots, m\}$. An elementary computation yields

$$L(u_j) = \begin{cases} -a_k u_j + a_k', & \text{if } u_j \in [-U_{k+1}, -U_k], \\ 2\sum_{i=1}^{N} w_i U_i, & \text{if } u_j \in [-U_{\min}, U_{\min}], \\ a_k u_j + a_k', & \text{if } u_j \in [U_k, U_{k+1}] \end{cases} \tag{14}$$

for $k = 1, 2, \ldots, N-1$, where

$$a_k \triangleq 2\sum_{i=1}^{k} w_i, \quad a_k' \triangleq 2\sum_{i=k+1}^{N} w_i U_i, \quad k = 1, 2, \ldots, N-1. \tag{15}$$

Put $q_j(t) \triangleq p^*(t)^{\mathrm{T}} b_j \in \mathbb{R}$, $f(u_j) \triangleq L(u_j) + q_j u_j$, and

$$c_{j,k} \triangleq \begin{cases} -a_{N-k} + q_j, & k = 1, \ldots, N-1, \\ q_j, & k = N, \\ a_{k-N} + q_j, & k = N+1, \ldots, 2N-1. \end{cases}$$

Since f is continuous and $c_{j,1} < c_{j,2} < \cdots < c_{j,2N-1}$, we have the following.

1) If $c_{j,1} > 0$, then

$$\arg\min_{|u_j| \leq 1} f(u_j) = -U_N = -1.$$

2) If $c_{j,k} < 0$ and $c_{j,k+1} > 0$ for $k \in \{1, 2, \ldots, 2N-2\}$, then we have

$$c_{j,1} < \cdots < c_{j,k} < 0 < c_{j,k+1} < \cdots < c_{j,2N-1}.$$

This implies that

$$\arg\min_{|u_j| \leq 1} f(u_j) = \begin{cases} -U_{N-k}, & k = 1, \ldots, N-1, \\ U_{\min}, & k = N, \\ U_{k-N+1}, & k = N+1, \ldots, 2N-2. \end{cases}$$

3) If $c_{j,2N-1} < 0$, then we have

$$c_{j,1} < c_{j,2} < \cdots < c_{j,2N-1}$$

and hence

$$\arg\min_{|u_j| \leq 1} f(u_j) = U_N = 1.$$

4) If $c_{j,k} = 0$ for $k \in \{1, 2, \ldots, 2N-1\}$, then we have

$$\arg\min_{|u_j| \leq 1} f(u_j) \in \begin{cases} [-U_{N-k+1}, -U_{N-k}], & k = 1, \ldots, N-1, \\ [-U_{\min}, U_{\min}], & k = N, \\ [U_{k-N}, U_{k-N+1}], & k = N+1, \ldots, 2N-1. \end{cases}$$

In this case, the minimizer of $f(u_j)$ is not determined uniquely.

Note the statements 1) to 4) boil down to L being a piecewise linear function on intervals. In summary, the minimizer of $f(u_j)$, that is the j-th component of the SOAV optimal control u^*, is given by

$$u_j^*(t) = \begin{cases} -1, & \text{if } a_{N-1} < q_j(t), \\ -U_{N-k}, & \text{if } a_{N-k-1} < q_j(t) < a_{N-k}, \\ -U_{\min}, & \text{if } 0 < q_j(t) < a_1, \\ U_{\min}, & \text{if } -a_1 < q_j(t) < 0, \\ U_{k+1}, & \text{if } -a_{k+1} < q_j(t) < -a_k, \\ 1, & \text{if } q_j(t) < -a_{N-1} \end{cases}$$

where $k = 1, 2, \ldots, N-2$ and

$$u_j^*(t) \in \begin{cases} [-U_{N-k+1}, -U_{N-k}], & \text{if } q_j(t) = a_{N-k}, \\ [-U_{\min}, U_{\min}], & \text{if } q_j(t) = 0, \\ [U_k, U_{k+1}], & \text{if } q_j(t) = -a_k \end{cases} \tag{16}$$

where $k = 1, 2, \ldots, N-1$.

Next we claim that

$$\mu\big(\{t \in [0, T] : q_j(t) = \pm a_k\}\big) = 0 \tag{17}$$

for every $k \in \{1, 2, \ldots, N-1\}$ and $j \in \{1, 2, \ldots, m\}$, where μ denotes the Lebesgue measure. Fix arbitrarily $j \in \{1, 2, \ldots, m\}$, take any $k \in \{1, 2, \ldots, N-1\}$, and assume $\mu(\{t \in [0, T] : q_j(t) = a_k\}) > 0$. Then we have

$$q_j(t) = p^*(t)^{\mathrm{T}} b_j = a_k \tag{18}$$

on a set $E \subset [0, T]$ with $m(E) > 0$. From Pontryagin's minimum principle, we have

$$\dot{p}^*(t) = -A^{\mathrm{T}} p^*(t) \tag{19}$$

for $t \in [0, T]$, and hence we have $p^*(t)^{\mathrm{T}} A b_j = 0$ for $t \in E$ by differentiating (18). Again, by differentiating this equation, we also have $p^*(t)^{\mathrm{T}} A^2 b_j = 0$ for $t \in E$ from (19). Repeating this yields $p^*(t)^{\mathrm{T}} A^l b_j = 0$ on E for every $l \in \mathbb{N}$. Therefore we have

$$p^*(t)^{\mathrm{T}} A \big[\, b_j \quad Ab_j \quad \ldots \quad A^{n-1} b_j \,\big] = 0 \tag{20}$$

for $t \in E$. Since $a_k \neq 0$ for every $k \in \{1, \ldots, N-1\}$, it follows from (18) that $p^*(t)$ is not identically 0 on $[0, T]$, and hence the determinant of $A[b_j\ Ab_j \ldots A^{n-1} b_j]$ is 0. However, this contradicts the assumption that the matrix A is non-singular and the pair (A, b_j) is controllable. Therefore $\mu(\{t \in [0, T] :$

$q_j(t) = a_k\}) = 0$ holds for every $k \in \{1, \ldots, N-1\}$. Similarly, we can also prove that $\mu(\{t \in [0,T] : q_j(t) = -a_k\}) = 0$ for every $k \in \{1, 2, \ldots, N-1\}$, and hence (17) holds for every $k \in \{1, 2, \ldots, N-1\}$ and $j \in \{1, 2, \ldots, m\}$.

Next, let assume $\mu(\{t \in [0,T] : q_j(t) = 0\}) > 0$ for some j. Then we have $p^*(t)^{\mathrm{T}} b_j = 0$ on a set $F \subset [0,T]$. From (19), by a similar computation as above, we have the relation (20) for $t \in F$. Since the matrix $A[b_j \ Ab_j \ldots A^{n-1} b_j]$ is non-singular from the assumption, it follows that

$$p^*(t) = 0, \quad \forall t \in F. \tag{21}$$

Since we have $p^*(t) = e^{-A^{\mathrm{T}} t} p_0$ on $[0,T]$ for some $p_0 \in \mathbb{R}^n$ from (19), it follows from (21) that $p_0 = 0$, and hence $p^*(t) = 0$ on $[0,T]$. Then $q_j(t) = p^*(t)^{\mathrm{T}} b_j = 0$ on $[0,T]$, and we have $\|u_j^*\|_\infty \le U_1 = U_{\min}$ from (16). Therefore, if $q_j(t)$ is 0 on a set with positive measure for some j, then the j-th component of the optimal control satisfies $\|u_j^*\|_\infty \le U_{\min}$, and otherwise, the j-th component of the optimal control takes discrete values $\pm U_1, \ldots, \pm U_N$ for almost all $t \in [0,T]$. ■

Remark 1: Although we focus on the well-defined lattice set for $\mathbb{U}$ in (2), that is, $\mathbb{U}$ contains the pair $\pm U_i$, we note that this proof can be also applied to show the discreteness of the optimal control when we take general alphabet sets such as $\mathbb{U} = \{U_1, U_2, \ldots, U_N\}$ with $-1 = U_1 < U_2 < \cdots < U_N = 1$, as considered in [21]. The lattice structure is however used for the proof of the continuity of the value function (Theorem 6 in Section IV), with which we show the stability of the feedback system based on model predictive control (Theorem 7 in Section V).

Theorem 2 suggests that if $U_{\min} = 0$ then the SOAV optimal control is a discrete-valued control that takes values in $\mathbb{U}$. Otherwise, it is useful to derive a condition for the optimal control u^* to satisfy the statement (i) in Theorem 2. In fact, it will be shown that there exists a subset of $\mathbb{R}^n$ such that if an initial state ξ is in this set then the SOAV optimal control takes values in $\mathbb{U}$. To derive such a subset, we prepare the following lemmas.

Lemma 1: Define

$$J_j(u) \triangleq \sum_{i=1}^{N} w_i \phi_{i,j}(u)$$

for $j \in \{1, 2, \ldots, m\}$, which is the j-th part of the cost function $J(u)$. Then $J_j(u)$ has the minimum value

$$J_{\min} \triangleq 2T \sum_{i=1}^{N} w_i U_i$$

on the set $\{u \in \mathbb{L}^1 : \|u\|_\infty \le 1\}$.

Proof: See Appendix A. ■

Lemma 2: Let $\mathcal{R}_{j,\min}$ be the set of all initial states for which there exists at least one feasible control u such that $J_j(u) = J_{\min}$. Then we have

$$\mathcal{R}_{j,\min} = \left\{ \int_0^T e^{-At} Bu(t) dt : \|u_j\|_\infty \le U_{\min}, \|u\|_\infty \le 1 \right\}.$$

Proof: See Appendix B. ■

Remark 2: The cost function has the minimum value $mJ_{\min}$ on the set $\{u : \|u\|_\infty \le 1\}$ from Lemma 1. It follows from Lemma 2 that the optimal value for a fixed initial state in $\mathcal{R}$ is equal to $mJ_{\min}$ if and only if each component of the optimal controls satisfies $\|u_j\|_\infty \le U_{\min}$, since we have $J_j(u) > J_{\min}$ if $\|u_j\|_\infty > U_{\min}$ Hence, if we take $U_{\min} = 0$, then the set of all initial states for which the optimal value is $mJ_{\min}$ consists of only $\{0\}$. This will be used to prove the stability in Theorem 7.

Now let us state the following theorem on the discreteness of the SOAV optimal control.

Theorem 3 (Discreteness for Nonzero $U_{\min}$): Assume that A is non-singular and the pairs (A, b_1), (A, b_2), ..., (A, b_m) are controllable. Define

$$\mathcal{R}_{\min} \triangleq \bigcup_{j=1}^{m} \mathcal{R}_{j,\min}.$$

If $\xi \in \mathcal{R} - \mathcal{R}_{\min}$, then the optimal controls take values in $\mathbb{U}$ almost everywhere in $[0,T]$. Otherwise, if $\xi \in \mathcal{R}_{j,\min}$ for some j, then the j-th components of the optimal controls take values less than or equal to $U_{\min}$ on $[0,T]$.

Proof: This follows from Theorem 1, Theorem 2, Lemma 1 and Lemma 2, immediately. ■

Next, we show the uniqueness theorem of the SOAV optimal control.

Theorem 4 (Uniqueness): Assume that A is non-singular and the pairs $(A, b_1), (A, b_2), \ldots, (A, b_m)$ are controllable. Then the SOAV optimal control for the initial state $\xi \in \mathcal{R} - \mathcal{R}_{\min}$ is unique (up to null sets). In particular, if $U_{\min} = 0$, the SOAV optimal control is unique for any initial state $\xi \in \mathcal{R}$.

Proof: Fix an initial state $\xi \in \mathcal{R} - \mathcal{R}_{\min}$, and let u_1 and u_2 be optimal controls for ξ. Then we have

$$J(u_1) = J(u_2) \le J(u) \tag{22}$$

for all $u \in \mathcal{U}(\xi)$. For any $\lambda \in (0,1)$, the control $\lambda u_1 + (1-\lambda) u_2$ is feasible for ξ, and hence the convexity of the cost function J and (22) yields

$$\begin{aligned} J(u_1) \le \ & J(\lambda u_1 + (1-\lambda) u_2) \le \lambda J(u_1) + (1-\lambda) J(u_2) \\ = \ & J(u_1) \end{aligned}$$

which means $J(\lambda u_1 + (1-\lambda) u_2) = J(u_1)$. Therefore $\lambda u_1 + (1-\lambda) u_2$ is an optimal control for ξ.

Put

$$\begin{aligned} E_1 &= \{t \in [0,T] : u_1(t) \in \mathbb{U}^m\}, \\ E_2 &= \{t \in [0,T] : u_2(t) \in \mathbb{U}^m\}, \\ F_j &= \{t \in E_1 \cap E_2 : u_{1,j}(t) = u_{2,j}(t)\}, \\ G_j &= \{t \in E_1 \cap E_2 : u_{1,j}(t) \ne u_{2,j}(t)\} \end{aligned}$$

for $j \in \{1, 2, \ldots, m\}$. From Theorem 3, we have $\mu(E_1) = \mu(E_2) = T$, and then we also have $\mu(E_1 \cap E_2) = T$.

Here, there exist some $\lambda \in (0,1)$ such that $\lambda u_{1,j}(t) + (1-\lambda) u_{2,j}(t) \notin \mathbb{U}$ for any $t \in G_j$ and for any j, from the gap between the uncountability of $(0,1)$ and the countability of $\mathbb{U}$. It follows from the optimality of the control $\lambda u_1 + (1-\lambda) u_2$ for such λ and Theorem 3 that $\mu(G_j) = 0$ for any j. Therefore we have

$$\mu(F_j) = \mu(F_j) + \mu(G_j) = \mu(E_1 \cap E_2) = T,$$

and then

$$T = \mu(F_j) \leq \mu(\{t \in [0,T] : u_{1,j}(t) = u_{2,j}(t)\}) \leq T,$$

which yields

$$\mu(\{t \in [0,T] : u_{1,j}(t) = u_{2,j}(t)\}) = T$$

for any j. This means the uniqueness. ■

From the above discussion, the SOAV optimal control can give a discrete-valued control taking values in $\mathbb{U}$ under some assumptions on the system (1) and the initial state ξ. A discrete-valued control is a piecewise constant signal, and changes its value at switching instants. It is undesirable for real applications if the number of switches were infinite, however this never happens. The following theorem gives an upper bound on the number of switches.

Theorem 5 (Number of Switches): Assume that A is nonsingular and the pairs (A, b_1), (A, b_2), ..., (A, b_m) are controllable. Then the number M of switches. of the SOAV optimal control for each initial state $\xi \in \mathcal{R} - \mathcal{R}_{\min}$ satisfies

$$M < mn(2N-1)(1 + \Omega T/\pi), \tag{23}$$

where Ω is the largest imaginary part of the eigenvalues of A. In particular, if $U_{\min} = 0$, then the number M of switches for each initial state $\xi \in \mathcal{R}$ satisfies

$$M < 2mn(N-1)(1 + \Omega T/\pi). \tag{24}$$

Proof: Fix arbitrarily an initial state $\xi \in \mathcal{R} - \mathcal{R}_{\min}$, and let u^* be the SOAV optimal control for the initial state ξ. From the proof of Theorem 2, $u^*(t)$ has discontinuities at points such that $q_j(t) = p^*(t)^{\mathrm{T}} b_j \in S \triangleq \{0, \pm a_1, \ldots, \pm a_{N-1}\}$, where a_k is defined in (15). Take arbitrarily j and an element $a \in S$. Since $p^*(t) = e^{-A^{\mathrm{T}} t} p_0$ for some $p_0 \in \mathbb{R}^n$, we have $q_j(t) = p_0^{\mathrm{T}} e^{-At} b_j$. Therefore $q_j(t) = a$ implies $a - p_0^{\mathrm{T}} e^{-At} b_j = 0$, or

$$\begin{bmatrix} a & -p_0^{\mathrm{T}} \end{bmatrix} \exp\left(\begin{bmatrix} 0 & 0 \\ 0 & -A \end{bmatrix} t \right) \begin{bmatrix} 1 \\ b_j \end{bmatrix} = 0.$$

The number of zeros on $[0, T]$ of the function on the left hand side is less than $n(1 + T\Omega/\pi)$ according to [22]. Counting the elements of the sets $S = \{0, \pm a_1, \ldots, \pm a_{N-1}\}$ and $\{1, 2, \ldots, m\}$ yields the estimate (23). In particular, if $U_{\min} = 0$, the switching instants of the SOAV optimal control for $\xi \in \mathcal{R}$ consist of all t such that $q_j(t) \in \{\pm a_1, \ldots, \pm a_{N-1}\}$, and the estimate (24) holds. ■

IV. Value Function

In this section, we investigate the value function in the SOAV optimal control. The value function is the optimal value of the SOAV optimal control, defined by

$$V(\xi) \triangleq \min\{J(u) : u \in \mathcal{U}(\xi)\}, \quad \xi \in \mathcal{R}$$

where $J(u)$ is defined in (4). From the existence theorem (Theorem 1), this is well-defined. In this section, we will show the continuity of the value function $V(\xi)$. This property plays an important role to prove the stability when the optimal control is extended to model predictive control (see Section V below). To prove the continuity, the following lemmas are fundamental.

Lemma 3: The value function $V(\xi)$ is convex on $\mathcal{R}$.

Proof: See Appendix C. ■

Lemma 4: For $\alpha \geq mJ_{\min}$, let

$$\mathcal{R}_\alpha \triangleq \left\{ \int_0^T e^{-At} Bu(t) dt : \|u\|_\infty \leq 1, J(u) \leq \alpha \right\}.$$

Then the set $\mathcal{R}_\alpha$ is closed for every $\alpha \geq mJ_{\min}$.

Proof: See Appendix D. ■

Lemma 5: If the pairs (A, b_1), (A, b_2), ..., (A, b_m) are controllable, then we have

$$\mathcal{R} = \{\xi : V(\xi) \leq 2mT\}.$$

In particular,

$$\partial\mathcal{R} = \{\xi : V(\xi) = 2mT\}.$$

Proof: See Appendix E. ■

From these lemmas, we show the continuity of the value function $V(\xi)$ based on the discussion given in [23].

Theorem 6 (Continuity of V): If the pairs (A, b_1), (A, b_2), ..., (A, b_m) are controllable, then $V(\xi)$ is continuous on $\mathcal{R}$.

Proof: Define

$$\overline{V}(\xi) \triangleq \begin{cases} V(\xi), & \xi \in \mathcal{R}, \\ 2mT, & \xi \in \mathbb{R}^n - \mathcal{R}. \end{cases}$$

It is sufficient to show that $\overline{V}(\xi)$ is continuous on $\mathbb{R}^n$.

First, we show that $\overline{V}(\xi)$ is continuous at every $\xi \in \partial\mathcal{R}$. Fix $\xi \in \partial\mathcal{R}$. For α that satisfies $\alpha \geq 2mT$ or $\alpha < mJ_{\min}$, the set $\{\xi : \overline{V}(\xi) > \alpha\}$ is empty or $\mathbb{R}^n$, respectively. For α with $mJ_{\min} \leq \alpha < 2mT$, we have

$$\{\xi : \overline{V}(\xi) > \alpha\} = \mathbb{R}^n - \{\xi : \overline{V}(\xi) \leq \alpha\}$$

which is open from Lemma 4 since

$$\{\xi : \overline{V}(\xi) \leq \alpha\} = \{\xi : V(\xi) \leq \alpha\} = \mathcal{R}_\alpha.$$

It follows that the set $\{\xi : \overline{V}(\xi) > \alpha\}$ is open for every real number α, and hence $\overline{V}(\xi)$ is lower semi-continuous on $\mathbb{R}^n$. Then we have

$$\overline{V}(\xi) \leq \liminf_{\eta \to \xi} \overline{V}(\eta). \tag{25}$$

On the other hand, we have

$$\limsup_{\eta \to \xi} \overline{V}(\eta) \leq 2mT, \tag{26}$$

from Lemma 5. Therefore,

$$\overline{V}(\xi) \leq \liminf_{\eta \to \xi} \overline{V}(\eta) \leq \limsup_{\eta \to \xi} \overline{V}(\eta) \leq 2mT = \overline{V}(\xi)$$

from (25), (26), and Lemma 5. This yields

$$\overline{V}(\xi) = \lim_{\eta \to \xi} \overline{V}(\eta)$$

which means that $\overline{V}(\xi)$ is continuous at every $\xi \in \partial\mathcal{R}$.

Since $V(\xi)$ is convex on $\mathcal{R}$ from Lemma 3 and $\mathcal{R}$ contains the origin in its interior from the controllability of the pairs (A, b_1),

$(A, b_2), \ldots, (A, b_m)$, [19, Theorem 17.3, Corollary 17.1], $V(\xi)$ is continuous at every point in int$\mathcal{R}$ [24, Theorem 10.1, p. 44].

Therefore $\overline{V}(\xi)$ is continuous on $\mathbb{R}^n$, and then $V(\xi)$ is continuous on $\mathcal{R}$. ∎

V. An Extension to MPC

In this section, we extend the finite-horizon SOAV optimal control discussed above to infinite-horizon model predictive control (MPC).

Suppose that we are given a sequence $\{t_k\}_{k\in\mathbb{N}}$ of sampling instants. We assume that

$$0 = t_0 < t_1 < t_2 < \cdots \tag{27}$$

and there exists $\tau > 0$ such that

$$\tau \leq t_{k+1} - t_k \leq T, \quad k = 0, 1, 2, \ldots \tag{28}$$

We also assume that the initial state $x(0) = x_0 \in \mathcal{R}$ is given. At each sampling instant t_k, the SOAV optimal control, say u_k, with horizon length T is computed by solving the SOAV optimal control problem (Problem 1) with $\xi = x(t_k)$. We then apply $u_k(t)$ on the k-th time interval $[t_k, t_{k+1}]$. If each optimization has the optimal solution, then this process gives a control

$$u(t) = u_k(t - t_k), \quad t \in [t_k, t_{k+1}], \quad k = 0, 1, 2, \ldots \tag{29}$$

From the assumptions (27) and (28), the control $u(t)$ is defined for all $t \in [0, \infty)$.

A. Stability

Here we investigate the stability of the closed-loop system with the model predictive control given in (29). More precisely, the question here is whether the origin is stable in the sense of Lyapunov regardless of the choice of sampling instants $\{t_k\}$ with the control (29).

Note that the state $x(t)$ for $t \in [0, t_1]$ obviously exists in the reachable set $\mathcal{R}$ while the control u_0 is used, since every point out of the set $\mathcal{R}$ needs a time duration more than T to be steered to the origin by any control v with $\|v\|_\infty \leq 1$. Therefore, we have $x(t_1) \in \mathcal{R}$, and the next optimal control u_1 exists on the next interval $[t_1, t_2]$. Then the state $x(t)$ for $t \in [t_1, t_2]$ lies in the reachable set $\mathcal{R}$ while the control u_1 is used. It follows that the state $x(t)$ lies in the reachable set $\mathcal{R}$ for all $t \in [0, \infty)$ under this situation and each optimization has the optimal solution, and hence the control u is well defined. Then the continuity of the value function (Theorem 6) leads to the stability of the closed-loop system, as described in the following theorem. Our proof is based on the Lyapunov's stability theorem [25, Theorem 4.1].

Theorem 7 (stability): If the pairs (A, b_1), (A, b_2), ..., (A, b_m) are controllable and $U_{\min} = 0$, then the origin is stable in the sense of Lyapunov regardless of the choice of the sampling instants $t_0, t_1, \ldots$ that satisfy (27) and (28) when we use the control u defined in (29).

Proof: Fix a sequence $\{t_k\}_{k=0}^\infty$ of sampling instants that satisfy (27) and (28). Also fix a positive real number $\varepsilon > 0$. We can take a real number $r \in (0, \varepsilon)$ such that

$$\mathcal{B}_r \triangleq \{\xi \in \mathbb{R}^n : \|\xi\| \leq r\} \subset \mathcal{R}$$

since $\mathcal{R}$ contains the origin in its interior from the controllability of the pairs $(A, b_1), (A, b_2), \ldots, (A, b_m)$. From Theorem 6, $V(\xi)$ is continuous on $\partial\mathcal{B}_r$, and then we can define

$$\alpha \triangleq \min_{\|\xi\|=r} V(\xi).$$

From Remark 2, we have $V(\xi) > mJ_{\min}$ for the initial state $\xi \neq 0$, and hence $\alpha > mJ_{\min}$. Take $\beta \in (mJ_{\min}, \alpha)$. Then the set $\mathcal{R}_\beta \cap \partial\mathcal{B}_r$ is empty, and $\mathcal{R}_\beta$ contains the origin and is convex. Hence we have $\mathcal{R}_\beta \subset \text{int}\mathcal{B}_r$. From the continuity of $V(\xi)$ at the origin, there exists $\delta > 0$ such that $\|\xi\| \leq \delta$ implies

$$mJ_{\min} \leq V(\xi) \leq \beta. \tag{30}$$

When we use the control u defined in (29) for ξ with $\|\xi\| \leq \delta$, it is clear that we have

$$V(x_\xi(t)) \leq V(\xi), \quad \forall t \geq 0 \tag{31}$$

where $x_\xi(t)$ is the state with $x_\xi(0) = \xi$ and is obtained by using u. Therefore for ξ with $\|\xi\| \leq \delta$ we have $V(x_\xi(t)) \leq \beta$ for all $t \geq 0$ from (30) and (31). Since $\mathcal{R}_\beta \subset \mathcal{B}_r \subset \mathcal{B}_\varepsilon$, for any initial state ξ with $\|\xi\| \leq \delta$ we have $x_\xi(t) \in \mathcal{B}_\varepsilon$ for all $t \geq 0$, which means that the origin is stable in the sense of Lyapunov. ∎

Remark 3 (Practical Stability): As noted above, the state starting from any initial state in the reachable set $\mathcal{R}$ remains in $\mathcal{R}$ regardless of the choice of alphabet set $\mathbb{U}$. Hence even if $U_{\min} > 0$, we can guarantee at least the boundness of the state for all time $t \geq 0$, since the set $\mathcal{R}$ is compact for any finite $T > 0$ [26]. This guarantees the *practical stability* discussed in [21].

B. Numerical Optimization

Here we propose a numerical computation algorithm to solve the (finite-horizon) SOAV optimal control problem to obtain a discrete-valued control input.

For simple systems, such as single or double integrators, the discrete-valued control can be obtained in a closed form via Pontryagin's minimum principle as the discussion in [13, Chap. 8] for $\mathbb{L}^1$ optimal control. However, for general linear time-invariant systems, one should rely on numerical computation. For this, we adopt a time discretization approach to solve the SOAV control problem. This approach is standard for numerical optimization; see e.g. [27, Sec. 2.3]. We then derive an algorithm for the optimization based on the *alternating direction method of multipliers* (ADMM) [14]–[16]. This algorithm is simple but much faster than the standard interior point method.

We first divide the interval $[0, T]$ into ν subintervals, $[0, T] = [0, h) \cup \cdots \cup [(\nu - 1)h, \nu h]$, where h is the discretization step chosen such that $T = \nu h$. We here assume (or approximate) that the state $x(t)$ and the control $u(t)$ are constant over each subinterval. On the discretization grid, $t = 0, h, \ldots, \nu h$, the continuous-time system (1) is described as

$$x_d[l + 1] = A_d x_d[l] + B_d u_d[l], \quad l = 0, 1, \ldots, \nu - 1$$

where $x_d[l] \triangleq x(lh)$, $u_d[l] \triangleq u(lh)$, and

$$A_d \triangleq e^{Ah}, \quad B_d \triangleq \int_0^h e^{At} B dt.$$

Set the control vector

$$z \triangleq \left[u_d[0]^{\mathrm{T}}, u_d[1]^{\mathrm{T}}, \dots, u_d[\nu-1]^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{m\nu}.$$

Let ξ be the initial state, that is, $x(0) = \xi$. Then the final state $x(T)$ can be described as

$$x(T) = x_d[\nu] = \zeta + \Phi z$$

where $\zeta \triangleq A_d^{\nu}\xi$ and

$$\Phi \triangleq \left[\, A_d^{\nu-1}B_d, \quad A_d^{\nu-2}B_d, \quad \dots, \quad B_d \,\right] \in \mathbb{R}^{n\times m\nu}.$$

Rename the discrete values in $\mathbb{U}$ as

$$r_1 \triangleq -U_N, r_2 \triangleq -U_{N-1}, \dots, r_L \triangleq U_N$$

where $L \triangleq 2N$ and the weights for $J(u)$ in (4) as

$$p_1 = p_L \triangleq w_N, p_2 = p_{L-1} \triangleq w_{N-1}, \dots, p_N = p_{N+1} \triangleq w_1.$$

Then the SOAV optimal control problem is approximated by

$$\begin{aligned} &\underset{z\in\mathbb{R}^{m\nu}}{\text{minimize}} && \sum_{i=1}^{L} p_i\|z - r_i\|_{\ell^1} \\ &\text{subject to} && \|z\|_{\ell^\infty} \le 1,\ \Phi z + \zeta = 0, \end{aligned} \tag{32}$$

where $\|\cdot\|_{\ell^1}$ and $\|\cdot\|_{\ell^\infty}$ are the ℓ^1 and ℓ^∞ norms in $\mathbb{R}^{m\nu}$, respectively. The optimization problem (32) is reducible to linear programming [11], and can be solved by standard numerical software packages, such as cvx with MATLAB [28], [29], based on the interior point method. However, for large scale problems, the computational burden of such an algorithm becomes heavy, and hence we give a more efficient algorithm based on ADMM.

1) Alternating Direction Method of Multipliers (ADMM): We here briefly review the ADMM algorithm. The ADMM solves the following type of convex optimization.

$$\begin{aligned} &\underset{z\in\mathbb{R}^{N_1}, y\in\mathbb{R}^{N_2}}{\text{minimize}} && f(z) + g(y) \\ &\text{subject to} && y = \Psi z \end{aligned} \tag{33}$$

where $f : \mathbb{R}^{N_1} \mapsto \mathbb{R}\cup\{\infty\}$ and $g : \mathbb{R}^{N_2} \mapsto \mathbb{R}\cup\{\infty\}$ are proper lower semi-continuous convex functions, and $\Psi \in \mathbb{R}^{N_2\times N_1}$. The algorithm of ADMM is given, for $y[0], w[0] \in \mathbb{R}^{N_2}$ and $\gamma > 0$, by

$$\begin{cases} z[j+1] \leftarrow \arg\min_{z\in\mathbb{R}^{N_1}} \left\{ f(z) + \frac{1}{2\gamma}\left\|y[j] - \Psi z - w[j]\right\|^2 \right\} \\ y[j+1] \leftarrow \mathrm{prox}_{\gamma g}\left(\Psi z[j+1] + w[j]\right) \\ w[j+1] \leftarrow w[j] + \Psi z[j+1] - y[j+1] \end{cases} \tag{34}$$

for $j = 0, 1, 2, \dots$, where $\mathrm{prox}_{\gamma g}$ denotes the *proximity operator* of γg defined by

$$\mathrm{prox}_{\gamma g}(z) \triangleq \underset{y\in\mathbb{R}^{N_2}}{\arg\min}\, \gamma g(y) + \frac{1}{2}\|z - y\|^2.$$

We recall a convergence analysis of ADMM by Eckstein-Bertsekas [15].

Theorem 8 (Convergence of ADMM [15]): Consider the optimization problem (33). Assume that $\Psi^{\mathrm{T}}\Psi$ is invertible and that a saddle point of its unaugmented Lagrangian $\mathcal{L}_0(z, y, w) \triangleq f(z) + g(y) - (\Psi z - y)^{\mathrm{T}} w$ exists. Then the sequence $\{(z[j], y[j])\}_{j\in\mathbb{N}}$ generated by Algorithm (34) converges to a solution of (33).

2) Reformulation Into ADMM-Applicable Form: In what follows, we reformulate our optimization problem described in (32) into the standard form in (33) to apply ADMM.

Let $\Omega_1 \triangleq \{z \in \mathbb{R}^{m\nu} : \|z\|_{\ell^\infty} \le 1\}$ be the unit-ball of the infinity norm, and $\Omega_2 \triangleq \{-\zeta\}$ be the singleton consisting of the vector $-\zeta$. Define the indicator function of a nonempty closed convex set by

$$\iota_\Omega(z) \triangleq \begin{cases} 0, & \text{if } z \in \Omega, \\ \infty, & \text{otherwise.} \end{cases}$$

Then, we can rewrite the optimization problem (32) as

$$\underset{z\in\mathbb{R}^{m\nu}}{\text{minimize}} \sum_{i=1}^{L} p_i\|z - r_i\|_{\ell^1} + \iota_{\Omega_1}(z) + \iota_{\Omega_2}(\Phi z). \tag{35}$$

Introducing new variables $y_1, \dots, y_{L+2}$ such that $y_i = z$ $(i = 1, \dots, L+1)$, and $y_{L+2} = \Phi z$, we can translate (35) into

$$\begin{aligned} &\underset{z\in\mathbb{R}^{N_1}, y\in\mathbb{R}^{N_2}}{\text{minimize}} && \sum_{i=1}^{L} p_i\|y_i - r_i\|_{\ell^1} + \iota_{\Omega_1}(y_{L+1}) + \iota_{\Omega_2}(y_{L+2}) \\ &\text{subject to} && y = \Psi z \end{aligned} \tag{36}$$

where $N_1 \triangleq m\nu, N_2 \triangleq (L+1)m\nu + n, y \triangleq [y_1^{\mathrm{T}} \cdots y_{L+2}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{N_2}$, and

$$\Psi \triangleq \left[\, I \quad \dots \quad I \quad \Phi^{\mathrm{T}} \,\right]^{\mathrm{T}} \in \mathbb{R}^{N_2\times N_1}.$$

Finally, by setting

$$f(z) \triangleq 0,$$

$$g(y) \triangleq \sum_{i=1}^{L} p_i\|y_i - r_i\|_{\ell^1} + \iota_{\Omega_1}(y_{L+1}) + \iota_{\Omega_2}(y_{L+2})$$

the optimization problem (36) is reduced to the standard from of (33).

3) Computation: Since $f = 0$, the first step of (34) becomes strictly convex quadratic minimization, which boils down to solving linear equations, that is,

$$\begin{aligned} z[j+1] &= \underset{z\in\mathbb{R}^{m\nu}}{\arg\min}\, \frac{1}{2\gamma}\|y[j] - \Psi z - w[j]\|^2 \\ &= (\Psi^{\mathrm{T}}\Psi)^{-1}\Psi^{\mathrm{T}}(y[j] - w[j]) \\ &= \left((L+1)I + \Phi^{\mathrm{T}}\Phi\right)^{-1} v[j] \end{aligned}$$

where

$$v[j] \triangleq \sum_{i=1}^{L+1}(y_i[j] - w_i[j]) + \Phi^{\mathrm{T}}(y_{L+2}[j] - w_{L+2}[j]).$$

Note that the inverse matrix $\left((L+1)I + \Phi^{\mathrm{T}}\Phi\right)^{-1}$ can be computed off-line.

On the other hand, the second step of (34) can be separated with respect to each y_i. For y_i $(i = 1, \dots, L)$, we have to compute the proximity operator of the ℓ_1 norm with shift r_i, which is reduced to a simple soft-thresholding operation: for $l = 1, \dots, m\nu$,

$$\left[\mathrm{prox}_{\gamma p_i \|\cdot - r_i\|_1}(z)\right]_{(l)} = r_i + \mathrm{prox}_{\gamma p_i |\cdot|}(z_{(l)} - r_i)$$
$$= r_i + \mathrm{sgn}(z_{(l)} - r_{i(l)}) \max\{|z_{(l)} - r_i| - \gamma p_i, 0\}$$

where $(\cdot)_{(l)}$ denotes the l-th entry of a vector. Here we use the shift property of the proximity operator (see, e.g., [30]).

For y_{L+1} and y_{L+2}, the computation of the proximity operators of the indicator functions are required. Since the proximity operator of the indicator function of a nonempty closed convex set Ω equals to the metric projection P_Ω onto Ω, the updates of y_{L+1} and y_{L+2} are reduced to calculating P_{Ω_1} and P_{Ω_2}, respectively. We can compute P_{Ω_1} as follows:

$$P_{\Omega_1}(z) \triangleq \begin{cases} z, & \text{if } \|z\|_{\ell^\infty} \le 1, \\ \tilde{z}, & \text{otherwise} \end{cases}$$

where

$$\tilde{z} \triangleq \left[\mathrm{sgn}(z_{(1)}) \min\{|z_{(1)}|, 1\} \dots \mathrm{sgn}(z_{(m\nu)}) \min\{|z_{(m\nu)}|, 1\}\right]^{\mathrm{T}}.$$

Meanwhile, $P_{\Omega_2} = P_{\{-\zeta\}}$ is simply given by

$$P_{\Omega_2}(z) \triangleq -\zeta.$$

As addressed in [16], ADMM tends to converge to modest accuracy within a few tens of iterations. This property is favorable in real-time control systems.

Remark 4: For numerical computation of the SOAV optimal control, one may adapt the theory of Hamilton-Jacobi equations with viscosity solutions [18, Section 19.3]. We leave this to future work.

VI. Example

In this section, we give two examples of model predictive control (MPC) based on the SOAV optimal control.

A. Example 1

We first consider a single-input system represented in

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

with the initial state $x(0) = [5, -5.4]^{\mathrm{T}}$. For this system, at each sampling instant t_k, $k = 0, 1, 2, \dots$, we solve the (finite-horizon) SOAV optimal control problem with the cost function

$$J(u) = \sum_{i=1}^{3} w_i(|u - U_i| + |u + U_i|), \tag{37}$$

where $w_1 = 0.3$, $w_2 = 0.3$, $w_3 = 0.4$, $U_1 = U_{\min} = 0$, $U_2 = 0.5$, and $U_3 = 1$. Let the horizon length $T = 3$ and the sampling instants be given by $t_1 = 2.5, t_2 = 3.5, t_3 = 5.5, t_4 = 8, t_5 = 9$, and $t_6 = 10$. Here we note that the boundary of the reachable set

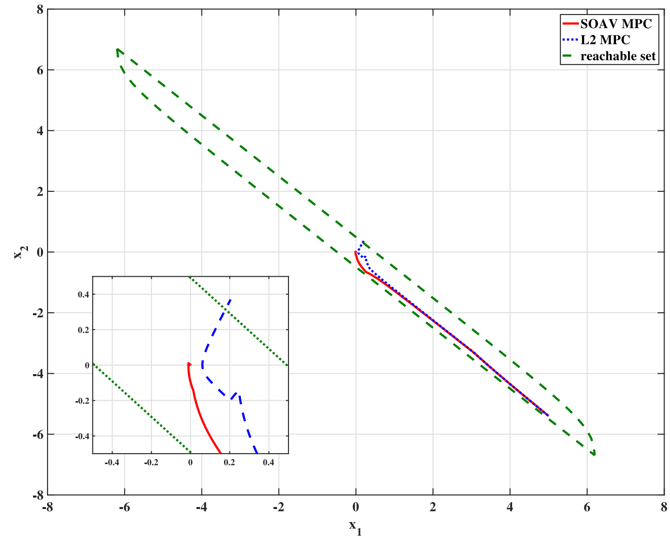

Fig. 1. Reachable set $\mathcal{R}$ and the state variable $x(t) = [x_1(t), x_2(t)]^{\mathrm{T}}$ according to the proposed control $u(t)$ based on SOAV (solid line) and the $\mathbb{L}^2$-based control (dash-dotted line). An enlarged figure is also shown.

$\mathcal{R}$ defined in (5) of this system is given by (see [13, Section 7.3])

$$\partial\mathcal{R} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_1 = \frac{2y_1 + y_2}{6}, x_2 = \frac{-y_1 + y_2}{3}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathcal{R}_1 \cup \mathcal{R}_2 \right\},$$

$$\mathcal{R}_1 = \left\{ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2 : y_2 = -2\left(\frac{2}{1 + e^T - y_1}\right)^2 + 1 + e^{-2T}, |y_1| \le e^T - 1 \right\},$$

$$\mathcal{R}_2 = \left\{ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2 : y_2 = 2\left(\frac{2}{1 + e^T + y_1}\right)^2 - 1 - e^{-2T}, |y_1| \le e^T - 1 \right\},$$

and Fig. 1 shows the set $\mathcal{R}$. Hence we have $x(0) \in \mathcal{R}$, and the MPC according to the SOAV optimization are well defined for all $t \ge 0$ from the discussion in Section V. For this example, we apply ADMM algorithm with $y[0] = w[0] = [0, 0, \dots, 0]^{\mathrm{T}} \in \mathbb{R}^{N_2}$, $\gamma = 0.5$, $N_1 = 100$, and $N_2 = 702$. The ADMM iteration is stopped when we have $\|z[j+1] - z[j]\| < 10^{-6}$, where $z[j]$ is defined by (34).

Fig. 2 shows the control u defined by (29) and Fig. 1 shows the state trajectory according to u. We used a standard desktop computer with a 3.2 GHz Intel Core i5 processor, and then (finite-horizon) optimal controls u_k are obtained in up to 0.037 second on average. Certainly, we can see that the control u takes

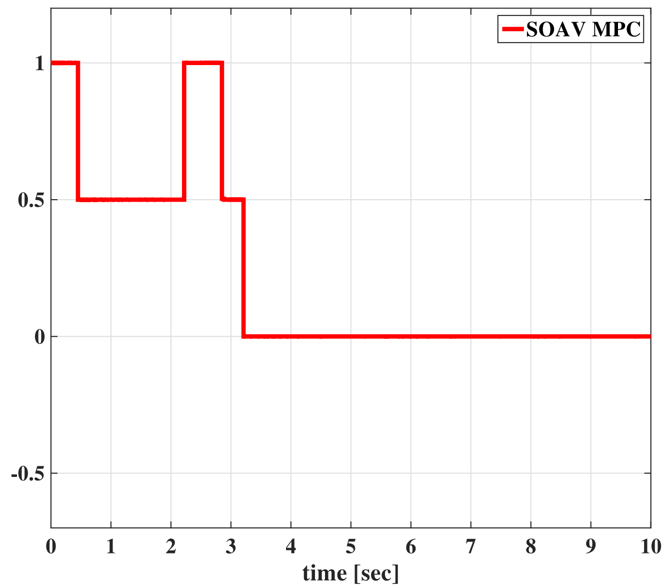

Fig. 2. Discrete-valued control by SOAV MPC.

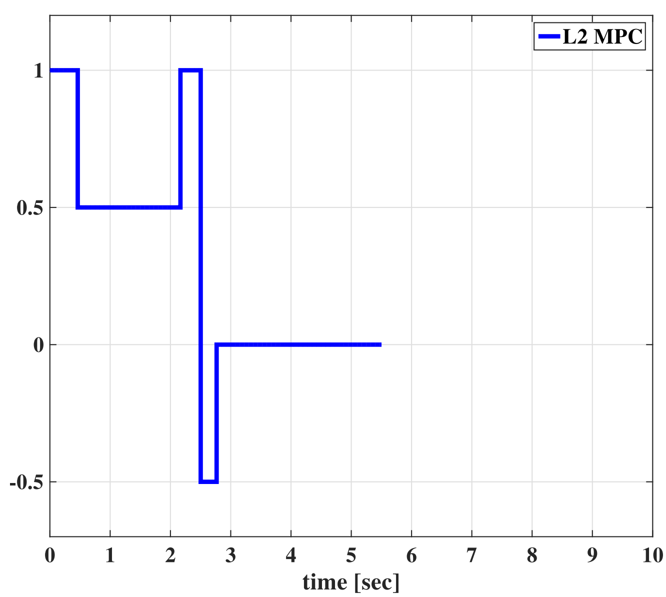

Fig. 3. Quantized MPC $v(t)$ by $\mathbb{L}^2$ optimal control: the recursive feasibility is lost at 3rd sampling time ($t = 5.5$).

only discrete values 0, 0.5, and 1, and the state converges to the origin, which are not inconsistent with Theorems 2 and 7.

For comparison, we also compute the quantized control based on $\mathbb{L}^2$ optimal control, that is, we optimize the following $\mathbb{L}^2$ cost function

$$J_2(v) \triangleq \|v\|_2^2 = \int_0^T |v(t)|^2 dt$$

among all feasible controls, instead of $J(u)$ in (37), and quantize the control value to the nearest value in $\mathbb{U}$ on each time interval $[t_k, t_k + T]$. The obtained MPC $v(t)$ is shown in Fig. 3, and the

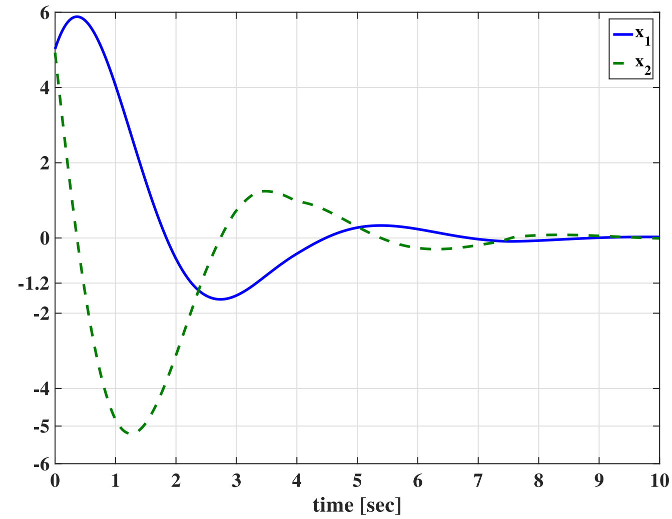

Fig. 4. State $x(t) = [x_1(t), x_2(t)]^{\mathrm{T}}$ according to the SOAV MPC without state constraints.

associated state is shown in Fig. 1. We observe that the state according to this control $v(t)$ diverges from the origin because of the quantization error of the control. Eventually, at the 3rd sampling instant ($t = 5.5$) the state goes out of the reachable set $\mathcal{R}$, where we can not take any finite-horizon feasible control, and hence the recursive feasibility is lost.

B. Example 2

We next take state constraints into account. Here we consider the following system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

Let the initial state be given by $x(0) = [5, 5]^{\mathrm{T}}$, and take $T = 5$, $w_i = 0.1i$ ($i = 1, 2, 3, 4$), $U_1 = U_{\min} = 0$, $U_2 = 0.3$, $U_3 = 0.6$ and $U_4 = 1$ in each finite-horizon SOAV optimization. The sampling instants are taken as $t_1 = 4$, $t_2 = 8$, $t_3 = 9$, and $t_4 = 10$.

Fig. 4 shows the resultant state trajectory according to the MPC based on the SOAV optimization. In the example, the lower bounds of $x_i(t)$ are about -1.62 and -5.19, respectively. Then we add the following state constraints:

$$-1.2 \le x_1(t) \le 6, \quad -5 \le x_2(t) \le 6, \quad t \ge 0, \qquad (38)$$

into each finite-horizon SOAV optimal control problem in MPC. Figs. 6 and 5 show the obtained MPC and the associated state trajectories, where we used a software package `cvx` with MATLAB. We can see that the obtained control takes only values in $\mathbb{U}$ while the state satisfies the constraints given in (38) and converges to the origin. In this paper, although we have analyzed the SOAV optimization with a magnitude constraint only for control inputs, this indicates compatibility of the SOAV optimization with state constraints.

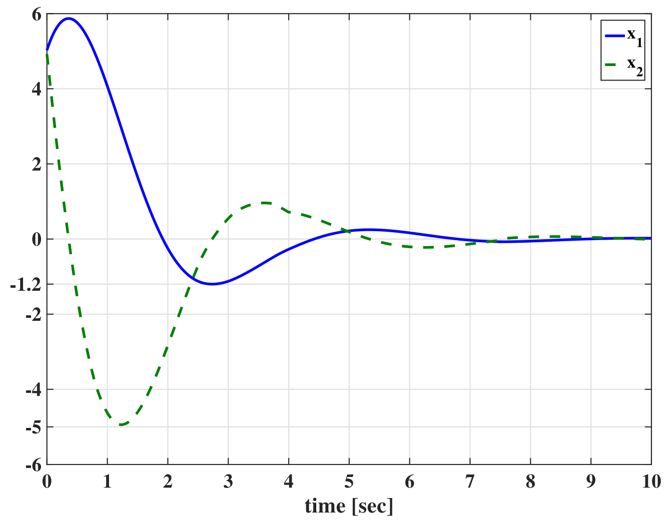

Fig. 5. State $x(t) = [x_1(t), x_2(t)]^{\mathrm{T}}$ according to the control in Fig. 6.

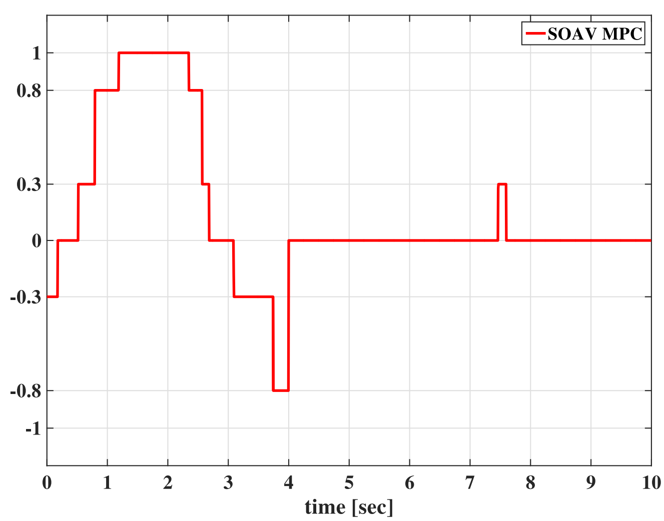

Fig. 6. Discrete-valued control by SOAV MPC with a state constraint.

VII. Conclusion

In this paper, we have proposed sum-of-absolute-values (SOAV) optimization for discrete-valued control. We have shown the existence and uniqueness theorems of the SOAV optimal control. We have also given conditions for the SOAV optimal control to generate a discrete-valued control signal. The obtained discrete-valued control has a finite number of switches, of which an upper bound has been derived. Furthermore we have investigated the continuity of the value function, by which the stability has been proved when the (finite-horizon) SOAV optimal control is extended to model predictive control (MPC). For MPC, a fast algorithm based on ADMM is proposed. A simulation result has been illustrated to show the effectiveness of the proposed method. Future work includes mathematical analysis when constraints are added to state variables.

Appendix A
Proof of Lemma 1

Fix any $j \in \{1, 2, \dots, m\}$. First, we show that the value of $J_j(u)$ for each $u \in \{u \in \mathbb{L}^1 : \|u\|_\infty \leq 1\}$ is greater than or equal to $J_{\min}$, and then we show the minimum $J_{\min}$ is achieved by $u = 0$.

Fix a control u with $\|u\|_\infty \leq U_N = 1$, and define

$$E_{0,j} \triangleq \{t \in [0, T] : -U_1 \leq u_j(t) \leq U_1\},$$

$$E_{k,j}^{+} \triangleq \{t \in [0, T] : U_k < u_j(t) \leq U_{k+1}\},$$

$$E_{k,j}^{-} \triangleq \{t \in [0, T] : -U_{k+1} \leq u_j(t) < -U_k\} \tag{39}$$

where $k = 1, 2, \dots, N-1$. Let $\gamma_j \triangleq \mu(E_{0,j})$ and $\gamma_{k,j}^{\pm} \triangleq \mu((E_{k,j}^{\pm})$. Since these sets are pairwise disjoint and satisfy

$$E_{0,j} \cup \bigcup_{k=1}^{N-1} (E_{k,j}^{+} \cup E_{k,j}^{-}) = [0, T]$$

we have

$$\gamma_j + \sum_{k=1}^{N-1} (\gamma_{k,j}^{+} + \gamma_{k,j}^{-}) = T \tag{40}$$

from the countable additivity of the Lebesgue measure. Let

$$\lambda_{k,j}^{\pm} \triangleq \pm \int_{E_{k,j}^{\pm}} u_j(t) dt.$$

An elementary computation yields

$$\phi_{i,j}(u) = \int_0^T \big(|u_j(t) - U_i| + |u_j(t) + U_i|\big) dt$$

$$= 2U_i \left(\gamma_j + \sum_{k=1}^{i-1} (\gamma_{k,j}^{+} + \gamma_{k,j}^{-})\right) + 2 \sum_{k=i}^{N-1} (\lambda_{k,j}^{+} + \lambda_{k,j}^{-})$$

for $i = 1, 2, \dots, N$, where we define $\sum_{k=1}^{0} = 0$ and $\sum_{k=N}^{N-1} = 0$. Then for $k = i, i+1, \dots, N-1$, we have

$$\lambda_{k,j}^{\pm} = \pm \int_{E_{k,j}^{\pm}} u_j(t) dt \geq U_k \gamma_{k,j}^{\pm} \geq U_i \gamma_{k,j}^{\pm}.$$

It follows from (40) that

$$\phi_{i,j}(u) \geq 2U_i \left(\gamma_j + \sum_{k=1}^{N-1} (\gamma_{k,j}^{+} + \gamma_{k,j}^{-})\right) = 2U_i T$$

and hence

$$J_j(u) \geq 2T \sum_{i=1}^{N} w_i U_i = J_{\min}.$$

Therefore each part of the cost function $J(u)$ takes values greater than or equal to $J_{\min}$, and $J_j(u)$ attains the minimum $J_{\min}$ when $u = 0$.

Appendix B
Proof of Lemma 2

Fix j. Take any initial state

$$\xi \in \left\{ \int_0^T e^{-At}Bu(t)dt : \|u_j\|_\infty \le U_{\min}, \|u\|_\infty \le 1 \right\}.$$

Then there exists a control u satisfying

$$\xi = \int_0^T e^{-At}Bu(t)dt, \quad \|u_j\|_\infty \le U_{\min}, \quad \|u\|_\infty \le 1.$$

Since

$$J_j(-u) = 2T\sum_{i=1}^N w_i U_i = J_{\min}$$

and $-u \in \mathcal{U}(\xi)$, we have $\xi \in \mathcal{R}_{j,\min}$.

Conversely, take any initial state $\xi \in \mathcal{R}_{j,\min}$ and let $u \in \mathcal{U}(\xi)$ denote a control such that $J_j(u) = J_{\min}$. Define sets $E_{k,j}^{\pm}$ and E_j as in the proof of Lemma 1. Then we can easily show that

$$\int_{E_{k,j}^{\pm}} (\pm u_j(t) - U_k)dt = 0$$

for every $k = 1, 2, \ldots, N-1$. Since $u_j(t) - U_k$ and $-u_j(t) - U_k$ are positive on $E_{k,j}^+$ and $E_{k,j}^-$ for every k respectively, we have $\mu((E_{k,j}^{\pm}) = 0$ for every k. Therefore $\mu((E_j) = T$ from (40), that is, $\|-u_j\|_\infty \le U_1 = U_{\min}$. Also, since the control u steers the initial state ξ to the origin at time T, we have

$$\xi = \int_0^T e^{-At}B\big(-u(t)\big)dt$$

and it follows that

$$\xi \in \left\{ \int_0^T e^{-At}Bu(t)dt : \|u_j\|_\infty \le U_{\min}, \|u\|_\infty \le 1 \right\}.$$

Appendix C
Proof of Lemma 3

Fix initial states $\xi, \eta \in \mathcal{R}$ and a scalar $\lambda \in (0,1)$. From Theorem 1, there exist optimal controls u_ξ and u_η for the initial states ξ and η, respectively. Then we have $\lambda\xi + (1-\lambda)\eta \in \mathcal{R}$ since $\mathcal{R}$ is convex, and the control $\lambda u_\xi + (1-\lambda)u_\eta$ is feasible for the initial state $\lambda\xi + (1-\lambda)\eta$. From the convexity of $\phi_{i,j}$ in $J(u)$ (see (4)), we have

$$\begin{aligned} V\big(\lambda\xi + (1-\lambda)\eta\big) &\le J\big(\lambda u_\xi + (1-\lambda)u_\eta\big) \\ &= \sum_{i=1}^N \sum_{j=1}^m w_i \phi_{i,j}\big(\lambda u_\xi + (1-\lambda)u_\eta\big) \\ &\le \sum_{i=1}^N \sum_{j=1}^m w_i\big(\lambda\phi_{i,j}(u_\xi) + (1-\lambda)\phi_{i,j}(u_\eta)\big) \\ &= \lambda J(u_\xi) + (1-\lambda)J(u_\eta) \\ &= \lambda V(\xi) + (1-\lambda)V(\eta). \end{aligned}$$

Appendix D
Proof of Lemma 4

First, we note that the set $\mathcal{R}_\alpha$ is well defined for $\alpha \ge mJ_{\min}$ since $mJ_{\min}$ is the minimum of the cost function from Lemma 1.

Fix $\alpha \ge mJ_{\min}$, and take a sequence $\{\xi_l\}$ in $\mathcal{R}_\alpha$ that converges to $\xi_\infty \in \mathbb{R}^n$. It is sufficient to show that $\xi_\infty \in \mathcal{R}_\alpha$.

For each $\xi_l \in \mathcal{R}_\alpha$, there exists a control u_l such that

$$\xi_l = \int_0^T e^{-At}Bu_l(t)dt, \quad \|u_l\|_\infty \le 1, \quad J(u_l) \le \alpha.$$

Since the set $\{u \in L^\infty : \|u\|_\infty \le 1\}$ is sequentially compact in the weak* topology of L^∞, there exist a measurable function u_∞ with $\|u_\infty\|_\infty \le 1$, and a subsequence $\{u_{l'}\}$ such that each component $\{u_{j,l'}\}$ converges to $u_{j,\infty}$, the j-th component of u_∞, in the weak* topology of L^∞. Clearly, we have

$$\xi_\infty = \int_0^T e^{-At}Bu_\infty(t)dt.$$

Define $J_{l'}^{\pm}$ as (9) and $J_{l'} \triangleq J_{l'}^+ + J_{l'}^-$. Then we have

$$J(u_\infty) = \lim_{l' \to \infty} J_{l'} \le \lim_{l' \to \infty} J(u_{l'}) \le \alpha$$

which is verified from (10) and (11). It follows that $\xi_\infty \in \mathcal{R}_\alpha$.

Appendix E
Proof of Lemma 5

First, we show

$$\partial\mathcal{R} = \{\xi : V(\xi) = 2mT\}. \tag{41}$$

Fix $\xi \in \partial\mathcal{R}$, then the feasible control for the initial state ξ is only the time optimal control, which is determined uniquely and takes only ± 1 for almost all $t \in [0,T]$ since the pairs (A, b_1), (A, b_2), …, (A, b_m) are controllable [31], [19, Theorem 12.1]. Let us denote the time optimal control by $u^\star$, and let $F_j^+, F_j^- \subset [0,T]$ be the set on which $u_j^\star$ takes 1 and -1, respectively, that is,

$$u_j^\star(t) = \begin{cases} 1, & \text{if } t \in F_j^+, \\ -1, & \text{if } t \in F_j^- \end{cases}$$

and $\mu((F^+) + \mu((F^-) = T$. Then we have

$$V(\xi) = J(u^\star) = 2\sum_{i=1}^N \sum_{j=1}^m w_i\big(\mu((F_j^+) + \mu((F_j^-)\big) = 2mT.$$

Conversely, fix an initial state $\xi \in \mathcal{R}$ such that $V(\xi) = 2mT$. If $\xi \in \operatorname{int}\mathcal{R}$, then there exist a scalar $\lambda \in [0,1)$ and a vector $\eta \in \partial\mathcal{R}$ such that $\xi = \lambda\eta$. As we proved above, we have $V(\eta) = 2mT$. It follows from the convexity of V that

$$\begin{aligned} V(\xi) = V(\lambda\eta) &\le \lambda V(\eta) + (1-\lambda)V(0) \\ &= 2\lambda mT + (1-\lambda)V(0) \end{aligned}$$

which yields

$$2mT \le V(0) \tag{42}$$

since $V(\xi) = 2mT$.

However, since $u = 0$ is feasible for the initial state 0, we have

$$2mT \sum_{i=1}^{N} w_i U_i \leq V(0) \leq J(0) = 2mT \sum_{i=1}^{N} w_i U_i$$

from Lemma 1. This implies

$$V(0) = 2mT \sum_{i=1}^{N} w_i U_i < 2mT \sum_{i=1}^{N} w_i = 2mT. \qquad (43)$$

Thus a contradiction occurs between (42) and (43), and hence $\xi \notin \text{int}\mathcal{R}$. Since $\mathcal{R}$ is closed [26], we have $\xi \in \partial\mathcal{R}$.

Next, we show

$$\mathcal{R} = \{\xi : V(\xi) \leq 2mT\}.$$

From (41), it is sufficient to show

$$\text{int}\mathcal{R} = \{\xi : V(\xi) < 2mT\}. \qquad (44)$$

First, fix an initial state $\xi \in \text{int}\mathcal{R}$, then there exist a scalar $\lambda \in [0, 1)$ and a vector $\eta \in \partial\mathcal{R}$ such that $\xi = \lambda\eta$, and $V(\eta) = 2mT$ from (41). It follows from (43) that

$$\begin{aligned} V(\xi) \leq\ & \lambda V(\eta) + (1-\lambda)V(0) \\ =\ & 2\lambda mT + (1-\lambda)V(0) < 2mT. \end{aligned}$$

Conversely, for any initial state ξ such that $V(\xi) < 2mT$, we have $\xi \in \text{int}\mathcal{R}$ from (41). Thus (44) follows, and the proof is completed.

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