The $\mathcal{H}_2$ Control Problem for Quadratically Invariant Systems With Delays

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Abstract—This technical note gives a new solution to the output feedback $\mathcal{H}_2$ problem for quadratically invariant communication delay patterns. A characterization of all stabilizing controllers satisfying the delay constraints is given and the decentralized $\mathcal{H}_2$ problem is cast as a convex model matching problem. The main result shows that the model matching problem can be reduced to a finite-dimensional quadratic program. A recursive state-space method for computing the optimal controller based on vectorization is given.

Index Terms—Decentralized control, optimal control, quadratic invariance.

I. INTRODUCTION

In decentralized control problems with delays, inputs to a dynamic system are chosen by multiple controllers that pass their local measurements over a communication network with delays. As a result, some controllers will have access to measurements before others. This technical note provides a new solution to the $\mathcal{H}_2$ optimal control problem, subject to quadratically invariant delay constraints, based on the Youla parametrization and vectorization.

A. Contributions

This technical note solves the decentralized $\mathcal{H}_2$ problem for a class of delay patterns arising from strongly-connected communication networks. The delay constraints are assumed to be quadratically invariant, which implies that the optimal control problem is convex. The main contribution of the technical note is a reformulation of the decentralized $\mathcal{H}_2$ problem for such delay patterns as a finite-dimensional quadratic program. This quadratic program, in turn, can be solved as a finite-horizon linear quadratic regulator problem.

To derive the quadratic program, a Youla parametrization framework developed for sparsity problems, [1], is adapted to communication delay patterns. The parametrization is then used to characterize all stabilizing controllers that satisfy a given delay pattern. It is then shown that for a doubly-coprime factorization based on the centralized LQG controller, the corresponding model matching problem reduces to a quadratic program. Finally, the quadratic program is cast as a finite-horizon linear quadratic regulator problem using vectorization.

B. Related Work

This technical note focuses on the $\mathcal{H}_2$ problem subject to a general class of quadratically invariant delay constraints. Existing approaches to this problem are based on vectorization [2] and linear matrix inequalities (LMIs) [3], [4]. In those works, the decentralized problems are reduced to centralized control problems with state dimensions that grow with the size of the delay. This technical note, on the other hand, shows that the solution can be computed in terms of the classical centralized solution and a quadratic program. This quadratic program, in turn, may be interpreted as a finite-horizon control problem with fixed dimension but horizon growing the with the size of the delay.

For specific delay patterns, dynamic programming techniques exist to solve output feedback decentralized LQG problems [5]–[8]. These delay patterns all satisfy the condition known as partial nestedness [9], which is closely related to quadratic invariance [10], and guarantees that the optimal policies are linear functions of the measurements. For more general partially nested delay constraints, dynamic programming methods for linear quadratic state feedback are known, [11], [12]. New results have identified sufficient statistics for dynamic programming in decentralized problems, without partial-nestedness assumptions, [13], [14], but they do not provide solutions to the corresponding LQG problems.

This technical note uses an operator theoretic approach to solve decentralized $\mathcal{H}_2$ problems with delays. It is an extension of [15], which uses spectral factorization to derive a similar quadratic program. Many of the calculations are modified from spectral factorization methods for sparsity constraints such as [16]–[18]. Another operator theoretic approach, based on loop-shifting [19], has also been developed for special quadratically invariant delay patterns [20].

C. Overview

The technical note is structured as follows. Section II defines the general problem studied in this technical note, the decentralized $\mathcal{H}_2$ problem with a strongly-connected delay pattern. Section III gives a parametrization of all stabilizing controllers that satisfy a given delay pattern, and presents the corresponding model matching problem. In Section IV, the decentralized $\mathcal{H}_2$ problem is reduced to a quadratic program, and this program is solved by vectorization. Numerical results are given in Section V and finally, conclusions are given in Section VI.

II. PROBLEM

This section introduces the basic notation and the control problem of interest. Section II-C describes how delayed information sharing patterns can be cast in the framework of this technical note.

A. Preliminaries

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc of complex numbers and let $\overline{\mathbb{D}}$ be its closure. Let $\mathcal{H}_2$ and $\mathcal{H}_\infty$ denote the Hardy spaces of matrix-valued functions that are analytic on $(\mathbb{C} \cup \{\infty\}) \setminus \overline{\mathbb{D}}$.

Let $\mathcal{R}_{\mathrm{p}}$ denote the space of proper real rational transfer matrices. Furthermore, denote $\mathcal{R}_{\mathrm{p}} \cap \mathcal{H}_2$ and $\mathcal{R}_{\mathrm{p}} \cap \mathcal{H}_\infty$ by $\mathcal{R}\mathcal{H}_2$ and $\mathcal{R}\mathcal{H}_\infty$,

Fig. 1.   The basic feedback loop.

Fig. 2.   The strictly proper N-step delay information pattern can be visualized as a two-node graph. The delay-1 self-loops specify computational delays of 1 at each node, while the delay-N edges specify communication delays. Self-loops are drawn as dashed arrows to distinguish them as denoting computational delays.
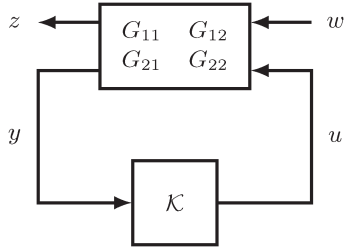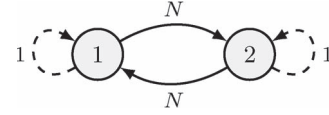
respectively. Note that $\mathcal{RH}_2 = \mathcal{RH}_\infty$, since both correspond to transfer matrices with no poles outside of $\mathbb{D}$.

A function $G(z) \in \mathcal{H}_2$ has a power series expansion given by $G(z) = \sum_{i=0}^{\infty}(1/z^i)G_i$. Furthermore, $\mathcal{H}_2$ is a Hilbert space with inner product defined by

$$\langle G, H \rangle = \lim_{r \downarrow 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{Tr}\left(G(re^{j\theta})H(re^{j\theta})^{\sim} \right) d\theta$$

$$= \sum_{i=0}^{\infty} \mathrm{Tr}(G_i H_i^*)$$

where the second equality follows from Parseval's identity.

Define the conjugate of G by $G(z)^{\sim} = \sum_{i=0}^{\infty} z^i G_i^*$. For $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \in \mathcal{R}_{\mathrm{p}}$, the conjugate is given by

$$\left(C(zI - A)^{-1}B + D \right)^{\sim} = B^{\mathsf{T}} \left(\frac{1}{z}I - A^{\mathsf{T}} \right)^{-1} C^{\mathsf{T}} + D^{\mathsf{T}}.$$

If $\mathcal{M}$ is a subspace of $\mathcal{H}_2$, denote the orthogonal projection onto $\mathcal{M}$ by $\mathbb{P}_{\mathcal{M}}$.

B. Formulation

This subsection introduces the generic problem of interest. Let G be a discrete-time plant given by

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right] = \left[\begin{array}{cc} G_{11} & G_{12} \\ G_{21} & G_{22} \end{array} \right]$$

with inputs of dimension p_1, p_2 and outputs of dimension q_1, q_2. Let $\mathcal{K}$ be a feedback controller connected to G as in Fig. 1.

For the existence of solutions of the appropriate Riccati equations, as well as simplicity of formulas, assume that

- (A, B_1, C_1) is stabilizable and detectable,
- (A, B_2, C_2) is stabilizable and detectable,
- $D_{12}^T[C_1 \ D_{12}] = [0 \ I]$,
- $D_{21}[B_1^T \ D_{21}^T] = [0 \ I]$.

For $N \geq 1$, define the spaces of proper and strictly proper finite impulse response (FIR) transfer matrices by $\mathcal{X}_{\mathrm{p}} = \bigoplus_{i=0}^{N}(1/z^i)\mathbb{C}^{p_2 \times q_2}$ and $\mathcal{X} = \bigoplus_{i=1}^{N}(1/z^i)\mathbb{C}^{p_2 \times q_2}$, respectively. Denote the corresponding spaces of real FIR transfer matrices by $\mathcal{RX}_{\mathrm{p}} = \bigoplus_{i=0}^{N}(1/z^i)\mathbb{R}^{p_2 \times q_2}$ and $\mathcal{RX} = \bigoplus_{i=1}^{N}(1/z^i)\mathbb{R}^{p_2 \times q_2}$. Note that $\mathcal{H}_2$ and $(1/z)\mathcal{H}_2$ can thus be decomposed into orthogonal subspaces as

$$\mathcal{H}_2 = \mathcal{X}_{\mathrm{p}} \oplus \frac{1}{z^{N+1}}\mathcal{H}_2 \quad \text{and} \quad \frac{1}{z}\mathcal{H}_2 = \mathcal{X} \oplus \frac{1}{z^{N+1}}\mathcal{H}_2. \quad (1)$$

Let $\mathcal{S} \subset (1/z)\mathcal{R}_{\mathrm{p}}$ be a subspace of the form

$$\mathcal{S} = \mathcal{Y} \oplus \frac{1}{z^{N+1}}\mathcal{R}_{\mathrm{p}}, \quad \text{where} \quad \mathcal{Y} = \bigoplus \frac{1}{z^i}\mathcal{Y}_i \quad (2)$$

and $\mathcal{Y}_i \subset \mathbb{R}^{p_2 \times q_2}$ defines a sparsity pattern over matrices. Delay patterns satisfying the decomposition in (2) will be called *strongly connected*, since delay patterns arising from strongly-connected communication networks always have this form. (See Section II-C.)

The set $\mathcal{S}$ is assumed to be *quadratically invariant* with respect to G_{22}, which means that for all $\mathcal{K} \in \mathcal{S}$, $\mathcal{K}G_{22}\mathcal{K} \in \mathcal{S}$. The key property of quadratic invariance is that $\mathcal{K} \in \mathcal{S}$ if and only if $\mathcal{K}(I - G_{22}\mathcal{K})^{-1} \in \mathcal{S}$ [2].

The decentralized $\mathcal{H}_2$ problem studied in this technical note is given by

$$\min_{\mathcal{K}} \left\| G_{11} + G_{12}\mathcal{K}(I - G_{22}\mathcal{K})^{-1}G_{21} \right\|_{\mathcal{H}_2}^2 \quad \text{s.t. } \mathcal{K} \in \mathcal{S}. \quad (3)$$

The quadratic invariance assumption guarantees that the corresponding model matching problem is convex [2]. Reduction to model matching is discussed in Section III-B.

The decomposition of $\mathcal{S}$ in (2) is crucial for the results of this technical note. The property that $(1/z^{N+1})\mathcal{R}_{\mathrm{p}} \subset \mathcal{S}$ implies that every measurement is available to all controller subsystems within N time steps. Concrete examples of delay patterns of this form are described in the next subsection.

C. Communication Delay Patterns

This subsection will discuss how (2) can be used to model delay patterns that arise from strongly connected graphs. As an example, consider an N-step delayed information pattern, represented by (2) with $\mathcal{Y}$ corresponding to block diagonal FIR matrices

$$\mathcal{Y} = \bigoplus_{i=1}^{N} \frac{1}{z^i} \left[\begin{array}{cc} \mathbb{R}^{p_{21} \times q_{21}} & 0 \\ 0 & \mathbb{R}^{p_{22} \times q_{22}} \end{array} \right].$$

The corresponding graph is given in Fig. 2. It was shown in [21] that the separation principle conjectured in [22] fails when $N \geq 2$, and appropriate sufficient statistics were given in [14], [23]. The special case of $N = 1$ was solved explicitly in [5]–[7].

More generally, assume that communication between the controller subsystems is specified by a strongly-connected graph (V, E) with self-loops at each node. Computational delays are specified by positive integers on the self-loops, while communication delays are represented by non-negative integers on the edges between distinct nodes. Requiring positive computational delays ensures that the controller is strictly proper.

A constraint space of the form (2) can be constructed as follows. For nodes i and j let c_i be the computational delay at node i and let $\tilde{d}_{ij}$ be the sum of communication delays along the directed path with shortest aggregate delay. Let the *delay matrix*, d, be the matrix with entries $d_{ij} = c_i + \tilde{d}_{ij}$. In the N-step delay example, the delay matrix is given by $d = \left[\begin{array}{cc} 1 & N+1 \\ N+1 & 1 \end{array} \right]$.
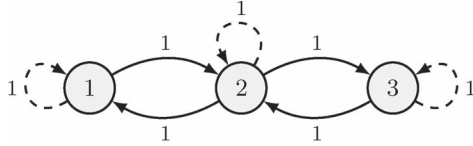
Fig. 3. The network graph for the three-player chain. The self-loops specify computational delays, while solid edges specify communication delays.

Let $N = \max\{d_{ij} : i, j \in V\} - 1$.[1] The corresponding constraint space is defined by

$$
\mathcal{S} = \begin{bmatrix} \frac{1}{z^{d_{11}}} \mathcal{R}_{\mathrm{p}} & \cdots & \frac{1}{z^{d_{1|V|}}} \mathcal{R}_{\mathrm{p}} \\ \vdots & & \vdots \\ \frac{1}{z^{d_{|V|1}}} \mathcal{R}_{\mathrm{p}} & \cdots & \frac{1}{z^{d_{|V||V|}}} \mathcal{R}_{\mathrm{p}} \end{bmatrix}.
$$

Thus, the $\mathcal{S}$ can be decomposed as in (2) by defining

$$
\mathcal{Y} = \bigoplus_{k=1}^{N} \frac{1}{z^k} \begin{bmatrix} \mathcal{Y}_k^{11} & \cdots & \mathcal{Y}_k^{1|V|} \\ \vdots & & \vdots \\ \mathcal{Y}_k^{|V|1} & \cdots & \mathcal{Y}_k^{|V||V|} \end{bmatrix}
$$

where $\mathcal{Y}_k^{ij} = \begin{cases} \mathbb{R}^{p_{2i} \times q_{2j}} & \text{if } d_{ij} \leq k \\ 0 & \text{if } d_{ij} > k \end{cases}$.

Let the blocks of G_{22} satisfy $(G_{22})_{ij} \in (1/z^{p_{ij}})\mathcal{R}_p$. It was shown in [24] that $\mathcal{S}$ defined above is quadratically invariant with respect to G_{22} if and only if

$$
d_{ki} + p_{ij} + d_{jl} \geq d_{kl} \text{ for all } i, j, k, l.
$$

This constraint guarantees that signals travel through the controller network at least as fast as through the plant.

As another example, consider the strictly proper version of the three-player chain problem discussed in [15], [25]. The graph describing the delays is given in Fig. 3, leading to a delay matrix and FIR constraint space

$$
d = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} \text{ and } \mathcal{Y} = \frac{1}{z} \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} \oplus \frac{1}{z^2} \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix} \quad (4)
$$

respectively. For compactness, $*$ is used to denote a space of appropriately sized real matrices.

III. DECENTRALIZED STABILIZATION

This section parametrizes the set of controllers $\mathcal{K} \in \mathcal{S}$ which internally stabilize the plant G. The parametrization naturally leads to a convex model matching formulation of $\mathcal{H}_2$ problem. In analogy with results on sparse transfer matrices [1], the parametrization is based on quadratic invariance and the classical Youla parametrization.

A. All Stabilizing Decentralized Controllers

A collection of stable transfer matrices, $\hat{M}, \hat{N}, \hat{X}, \hat{Y}, \tilde{M}, \tilde{N}, \tilde{X},$ and $\tilde{Y}$, defines a *doubly-coprime factorization* of G_{22} if $G_{22} = \hat{N}\hat{M}^{-1} = \tilde{M}^{-1}\tilde{N}$ and

$$
\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} \hat{M} & \hat{Y} \\ \hat{N} & \hat{X} \end{bmatrix} = I. \quad (5)
$$

As long as (A, B_2, C_2) is stabilizable and detectable, there are numerous ways to construct a doubly coprime factorization of G_{22}.

The following theorem is well known [26].

Theorem 1: Assume that G_{22} has a double doubly-coprime factorization of the form in (5). A controller $\mathcal{K} \in \mathcal{R}_{\mathrm{p}}$ internally stabilizes G if and only if there is a transfer matrix $Q \in \mathcal{RH}_\infty$ such that

$$
\mathcal{K} = (\hat{Y} - \hat{M}Q)(\hat{X} - \hat{N}Q)^{-1} = (\tilde{X} - Q\tilde{N})^{-1}(\tilde{Y} - Q\tilde{M}). \quad (6)
$$

From [2], if G_{22} is quadratically invariant under $\mathcal{S}$, then $\mathcal{K} \in \mathcal{S}$ if and only if $\mathcal{K}(I - G_{22}\mathcal{K})^{-1} \in \mathcal{S}$. As in [1], a straightforward calculation shows that

$$
\mathcal{K}(I - G_{22}\mathcal{K})^{-1} = (\hat{Y} - \hat{M}Q)\tilde{M} \quad (7)
$$

and thus $\mathcal{K} \in \mathcal{S} \iff (\hat{Y} - \hat{M}Q)\tilde{M} \in \mathcal{S}$.

Based on (1), $Q \in \mathcal{RH}_2 = \mathcal{RH}_\infty$ can be decomposed uniquely as $Q = U + V$ with $U \in (1/z^{N+1})\mathcal{RH}_2$ and $V \in \mathcal{X}_{\mathrm{p}}$. Recalling (2) and noting that $\hat{M}U\tilde{M} \in (1/z^{N+1})\mathcal{H}_2$ implies that

$$
(\hat{Y} - \hat{M}Q)\tilde{M} \in \mathcal{S} \iff \mathbb{P}_{\mathcal{X}_{\mathrm{p}}}\left((\hat{Y} - \hat{M}Q)\tilde{M}\right) \in \mathcal{Y}
$$
$$
\iff \mathbb{P}_{\mathcal{X}_{\mathrm{p}}}\left((\hat{Y} - \hat{M}V)\tilde{M}\right) \in \mathcal{Y}.
$$

Thus, the following characterization of all stabilizing decentralized controllers holds.

Theorem 2: A controller $\mathcal{K} \in \mathcal{S}$ internally stabilizes G_{22} if and only if there are transfer matrices $U \in (1/z^{N+1})\mathcal{RH}_\infty$ and $V \in \mathcal{RX}_{\mathrm{p}}$ such that $\mathcal{K} = (\hat{Y} - \hat{M}(U + V))(\hat{X} - \hat{N}(U + V))^{-1}$ and

$$
\mathbb{P}_{\mathcal{X}_{\mathrm{p}}}\left((\hat{Y} - \hat{M}V)\tilde{M}\right) \in \mathcal{Y}. \quad (8)
$$

Note that (8) reduces to a finite-dimensional linear constraint on the FIR term, $V \in \mathcal{RX}_{\mathrm{p}}$. The other term, U, is delayed, but otherwise unconstrained.

B. Model Matching

Given a doubly-coprime factorization, (7) implies that the closed-loop transfer matrix is given by

$$
G_{11} + G_{12}\mathcal{K}(I - G_{22}\mathcal{K})^{-1}G_{21} = P_{11} + P_{12}QP_{21}
$$

where

$$
P_{11} = G_{11} + G_{12}\hat{Y}\tilde{M}G_{21},
$$
$$
P_{12} = -G_{12}\hat{M}, \quad P_{21} = \tilde{M}G_{21}. \quad (9)
$$

Using the decomposition $Q = U + V$, with $V \in \mathcal{RX}_{\mathrm{p}}$ and $U \in (1/z^{N+1})\mathcal{RH}_2$, the decentralized $\mathcal{H}_2$ problem, (3), is equivalent to the following *model matching problem*:

$$
\min_{U,V} \quad \|P_{11} + P_{12}(U + V)P_{21}\|_{\mathcal{H}_2}^2
$$
$$
\text{s.t.} \quad U \in \frac{1}{z^{N+1}}\mathcal{RH}_2, \quad V \in \mathcal{RX}_{\mathrm{p}},
$$
$$
\mathbb{P}_{\mathcal{X}}\left((\hat{Y} - \hat{M}V)\tilde{M}\right) \in \mathcal{Y}. \quad (10)
$$

IV. RESULTS

This section gives the main result of the technical note, a reduction of the decentralized control problem, (3), to a quadratic program. A vectorization method for computing the optimal solution is also given.

A. Quadratic Programming Formulation

In the previous section, it was shown that the decentralized feedback problem is equivalent to a model matching problem, (10). It will be shown that for a special doubly-coprime factorization, the model matching problem reduces to a quadratic program.

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[1]Using this convention, all measurements, $y_j(t)$, are available to all controllers by time $t + N + 1$.

Let X and Y be the stabilizing solutions of the Riccati equations associated with the linear quadratic regulator and Kalman filter, respectively

$$X = C_1^\mathsf{T} C_1 + A^\mathsf{T} X A - A^\mathsf{T} X B_2 \left(I + B_2^\mathsf{T} X B_2\right)^{-1} B_2^\mathsf{T} X A \quad (11)$$

$$Y = B_1 B_1^\mathsf{T} + A Y A^\mathsf{T} - A Y C_2^\mathsf{T} \left(I + C_2 Y C_2^\mathsf{T}\right)^{-1} C_2 Y A^\mathsf{T}. \quad (12)$$

Define $\Omega = I + B_2^\mathsf{T} X B_2$ and $\Psi = I + C_2 Y C_2^\mathsf{T}$. The corresponding gains are given by

$$K = -\Omega^{-1} B_2^\mathsf{T} X A, \quad (13)$$
$$L = -A Y C_2^\mathsf{T} \Psi^{-1}. \quad (14)$$

Furthermore, $A + B_2 K$ and $A + L C_2$ are stable.

It is well known (e.g. [26]) that a doubly-coprime factorization of G_{22} is given by

$$\begin{bmatrix} \hat{M} & \hat{Y} \\ \hat{N} & \hat{X} \end{bmatrix} = \left[\begin{array}{c|cc} A + B_2 K & B_2 & -L \\ \hline K & I & 0 \\ C_2 & 0 & I \end{array} \right],$$

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A + L C_2 & B_2 & -L \\ \hline -K & I & 0 \\ -C_2 & 0 & I \end{array} \right]. \quad (15)$$

The following theorem is the main result of the technical note.

Theorem 3: Consider the doubly-coprime factorization of G_{22} defined by (15). The optimal solution to the decentralized $\mathcal{H}_2$ problem defined by (3) is given by

$$\mathcal{K}^* = (\hat{Y} - \hat{M} V^*)(\hat{X} - \hat{N} V^*)^{-1}$$

where V^* is the unique optimal solution to the quadratic program

$$\min_{V \in \mathcal{RX}} \quad \left\| \Omega^{\frac{1}{2}} V \Psi^{\frac{1}{2}} \right\|_{\mathcal{H}_2}^2$$
$$s.t. \quad \mathbb{P}_\mathcal{X}\left((\hat{Y} - \hat{M} V)\tilde{M}\right) \in \mathcal{Y}. \quad (16)$$

Furthermore, the optimal cost is given by $\|P_{11}\|_{\mathcal{H}_2}^2 + \|\Omega^{1/2} V^* \Psi^{1/2}\|_{\mathcal{H}_2}^2$.

Proof: For the doubly-coprime factorization given by (15) the model matching matrices, (9), have state space realizations given by

$$P_{11} = \left[\begin{array}{cc|c} A + B_2 K & -B_2 K & B_1 \\ 0 & A + L C_2 & B_1 + L D_{21} \\ \hline C_1 + D_{12} K & -D_{12} K & 0 \end{array} \right]$$

$$P_{12} = -\left[\begin{array}{c|c} A + B_2 K & B_2 \\ \hline C_1 + D_{12} K & D_{12} \end{array} \right],$$

$$P_{21} = \left[\begin{array}{c|c} A + L C_2 & B_1 + L D_{21} \\ \hline C_2 & D_{21} \end{array} \right]. \quad (17)$$

Note that since $\mathcal{Y} \subset \mathcal{RX}$, $\hat{Y}$ is strictly proper, and $\hat{M}$, $\tilde{M}$ have identity feed-through terms, the constraint in (8) implies that $V \in \mathcal{RX}$.

For a fixed $V \in \mathcal{RX}$, the optimal $U \in (1/z^{N+1})\mathcal{RH}_2$ is found by solving

$$\min_{U \in \frac{1}{z^{N+1}} \mathcal{RH}_2} \|P_{11} + P_{12} V P_{21} + P_{12} U P_{21}\|_{\mathcal{H}_2}^2.$$

A necessary condition for U to be optimal, given V, is

$$P_{12}^\sim P_{11} P_{21}^\sim + P_{12}^\sim P_{12} V P_{21} P_{21}^\sim$$
$$+ P_{12}^\sim P_{12} U P_{21} P_{21}^\sim \in \left(\frac{1}{z^{N+1}} \mathcal{H}_2\right)^\perp.$$

Lemma A.2 implies that $P_{12}^\sim P_{12} = \Omega$, $P_{21} P_{21}^\sim = \Psi$, and $\mathbb{P}_{(1/z)\mathcal{H}_2}(P_{12}^\sim P_{11} P_{21}^\sim) = 0$. Thus, the optimality condition becomes

$$P_{12}^\sim P_{11} P_{21}^\sim + \Omega V \Psi + \Omega U \Psi \in \left(\frac{1}{z^{N+1}} \mathcal{H}_2\right)^\perp.$$

Furthermore, U must satisfy

$$U = -\mathbb{P}_{\frac{1}{z^{N+1}} \mathcal{H}_2}\left(\Omega^{-1} P_{12}^\sim P_{11} P_{21}^\sim \Psi^{-1} + V\right) = 0.$$

Thus, the optimal U is 0 for any $V \in \mathcal{RX}$.

Plugging V into the cost of (10) and applying Lemma A.2 gives

$$\|P_{11} + P_{12} V P_{21}\|_{\mathcal{H}_2}^2 = \|P_{11}\|_{\mathcal{H}_2}^2 + \|P_{12} V P_{21}\|_{\mathcal{H}_2}^2$$
$$+ 2 \langle P_{11}, P_{12} V P_{21}\rangle$$
$$= \|P_{11}\|_{\mathcal{H}_2}^2 + \left\| \Omega^{\frac{1}{2}} V \Psi^{\frac{1}{2}} \right\|_{\mathcal{H}_2}^2.$$

Thus, Theorem 2 and the model matching formulation, (10), imply that the optimal V must solve (16). Note that

$$\left\| \Omega^{\frac{1}{2}} V \Psi^{\frac{1}{2}} \right\|_{\mathcal{H}_2}^2 = \sum_{i=1}^N \mathrm{Tr}\left(\Omega V_i \Psi V_i^\mathsf{T}\right)$$

is a positive definite quadratic function of V, while the constraint is linear. Thus (16) is a quadratic program and it must have a unique optimal solution. ∎

B. Vectorization

In this subsection, the quadratic program of Theorem 3 will be cast as a finite-horizon state-feedback problem using vectorization techniques. The vectorization approach is similar to method used in [8].

First, by defining $R = \Psi \otimes \Omega$, the vectorized form of the cost function becomes

$$\left\| \Omega^{\frac{1}{2}} V \Psi^{\frac{1}{2}} \right\|_{\mathcal{H}_2}^2 = \sum_{i=1}^N \mathrm{Tr}\left(\Omega V_i \Psi V_i^\mathsf{T}\right) = \sum_{i=1}^N \mathrm{vec}(V_i)^\mathsf{T} R \, \mathrm{vec}(V_i). \quad (18)$$

Define the FIR transfer matrix $J = \mathbb{P}_\mathcal{X}((-\hat{Y} + \hat{M} V)\tilde{M})$. If $\mathcal{Y}$ is defined by (2), then the model matching constraint of (10) is equivalent to $J_i \in \mathcal{Y}_i$. The vectorized form of J is computed from

$$\mathrm{vec}(-\hat{Y}\tilde{M} + \hat{M} V \tilde{M}) = -\mathrm{vec}(\hat{Y}\tilde{M}) + (\tilde{M}^\mathsf{T} \otimes \hat{M})\mathrm{vec}(V).$$

By Lemma A.3 in the Appendix, the terms of J_i can be computed by the recursion

$$x_{i+1} = A_v x_i + B_v \mathrm{vec}(V_i), \quad x_1 = \begin{bmatrix} \mathrm{vec}(L) \\ 0_{n p_2 \times 1} \end{bmatrix}$$

$$\mathrm{vec}(J_i) = C_v x_i + \mathrm{vec}(V_i) \quad (19)$$

where

$$\left[\begin{array}{c|c} A_v & B_v \\ \hline C_v & D_v \end{array} \right]$$

$$= \left[\begin{array}{cc|c} I_{q_2} \otimes (A + B_2 K) & 0_{n q_2 \times n p_2} & I_{q_2} \otimes B_2 \\ C_2^\mathsf{T} \otimes K & (A + L C_2)^\mathsf{T} \otimes I_{p_2} & C_2^\mathsf{T} \otimes I_{p_2} \\ \hline I_{q_2} \otimes K & L^\mathsf{T} \otimes I_{p_2} & I_{p_2 q_2} \end{array} \right].$$

Now let E_i and F_i be matrices with columns that form orthonormal bases of $\mathrm{vec}(\mathcal{Y}_i)$ and $\mathrm{vec}(\mathcal{Y}_i^\perp)$, respectively. The term $\mathrm{vec}(V_i)$ can then be decomposed as

$$\mathrm{vec}(V_i) = E_i u_i + F_i u_i^\perp,$$

for some vectors u_i and $u_i^\perp$.

Using (19), the constraint that $J_i \in \mathcal{Y}_i$ can be equivalently cast as

$$F_i^\mathsf{T}(C_v x_i + \mathrm{vec}(V_i)) = F_i^\mathsf{T} C_v x_i + u_i^\perp = 0. \tag{20}$$

Plugging (20) into the cost (18) and the recursion (19) leads to the following optimal control problem:

$$\min_u \quad \sum_{i=1}^{N} [x_i^\mathsf{T} \quad u_i^\mathsf{T}] \begin{bmatrix} C_i^\mathsf{T} \\ D_i^\mathsf{T} \end{bmatrix} [C_i \quad D_i] \begin{bmatrix} x_i \\ u_i \end{bmatrix}$$

$$\text{s.t.} \quad x_{i+1} = A_i x_i + B_i u_i, \quad x_1 = \begin{bmatrix} \mathrm{vec}(L) \\ 0_{np_1 \times 1} \end{bmatrix} \tag{21}$$

where the time-varying matrices are given by[2]

$$\begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} = \begin{bmatrix} A_v - B_v F_i F_i^\mathsf{T} C_v & B_v E_i \\ -R^{\frac{1}{2}} F_i F_i^\mathsf{T} C_v & R^{\frac{1}{2}} E_i \end{bmatrix}.$$

As is standard, [27], the optimal controller can be computed as $u_i = K_i x_i$, where

$$K_i = -\left(R_i + B_i^\mathsf{T} X_{i+1} B_i\right)^{-1}\left(B_i^\mathsf{T} X_{i+1} A_i + D_i^\mathsf{T} C_i\right)$$

and X_i is computed from backward recursion with $X_{N+1} = 0$ and

$$X_i = Q_i + A_i^\mathsf{T} X_{i+1} A_i$$
$$- \left(A_i^\mathsf{T} X_{i+1} B_i + S_i\right)\left(R_i + B_i^\mathsf{T} X_{i+1} B_i\right)^{-1}\left(B_i^\mathsf{T} X_{i+1} A_i + S_i^\mathsf{T}\right).$$

Furthermore, the optimal cost is given by $x_1^\mathsf{T} X_1 x_1$. The next theorem follows immediately from Theorem 3 and the preceding discussion.

Theorem 4: The optimal V is computed as

$$x_{i+1} = (A_i + B_i K_i)x_i, \quad x_1 = \begin{bmatrix} \mathrm{vec}(L) \\ 0_{np_1 \times 1} \end{bmatrix},$$

$$\mathrm{vec}(V_i) = \left(E_i K_i - F_i F_i^\mathsf{T} C_v\right) x_i.$$

Furthermore, the decentralized $\mathcal{H}_2$ problem of (3) has optimal value $\|P_{11}\|^2 + x_1^\mathsf{T} X_1 x_1$.

V. NUMERICAL EXAMPLE

This section gives a numerical example of optimal controllers computed using the vectorization method of the previous section.[3]

A. Increasing Delays

Consider the plant defined by

$$G = \begin{bmatrix} \begin{bmatrix} 0.9 & 0 \\ 0 & 1.1 \end{bmatrix} & 1_{2 \times 1} & 0_{2 \times 2} & 0.1 I_{2 \times 2} \\ \hline 1_{1 \times 2} & 0_{1 \times 1} & 0_{1 \times 2} & 0_{1 \times 2} \\ 0_{2 \times 2} & 0_{2 \times 2} & 0_{2 \times 2} & I_{2 \times 2} \\ \hline .1 I_{2 \times 2} & 0_{2 \times 1} & I_{2 \times 2} & 0_{2 \times 2} \end{bmatrix}$$

[2]This definition is a slight abuse of notation, since B_i here are distinct from the original input matrices B_1 and B_2, etc. Similarly for C_i are distinct from the original output matrices.

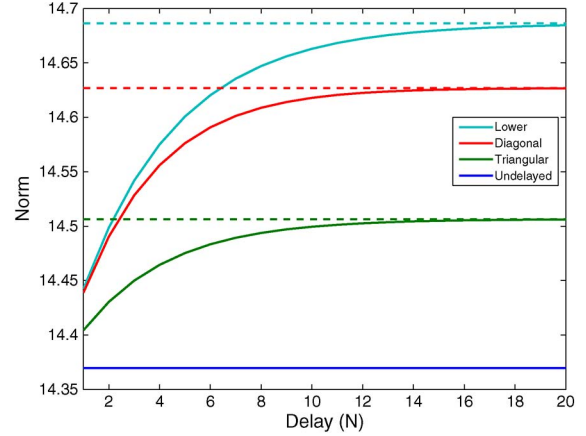[3]Code for these examples is available at http://www.ece.umn.edu/~alampers/code/decH2.php.

Fig. 4. This plot shows the closed-loop norm for $\mathcal{K}_{\mathrm{Tri}}^N$, $\mathcal{K}_{\mathrm{Di}}^N$, and $\mathcal{K}_{\mathrm{Low}}^N$. For a given N, the controllers with fewer sparsity constraints give rise to lower norms. As N increases, all of the norms increase monotonically since the controllers have access to less information. The dotted lines correspond to the optimal norms for sparsity structures given in (22).

where $1_{p \times q}$ is the $p \times q$ matrix of ones.

For $N \geq 1$, let $\mathcal{K}_{\mathrm{Tri}}^N$, $\mathcal{K}_{\mathrm{Di}}^N$, and $\mathcal{K}_{\mathrm{Low}}^N$ solve the decentralized $\mathcal{H}_2$ problem, (3), with constraints defined by

$$\mathcal{Y}_{\mathrm{Tri}}^N = \bigoplus_{i=1}^{N} \frac{1}{z^i} \begin{bmatrix} \mathbb{R} & 0 \\ \mathbb{R} & \mathbb{R} \end{bmatrix}, \quad \mathcal{Y}_{\mathrm{Di}}^N = \bigoplus_{i=1}^{N} \frac{1}{z^i} \begin{bmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{R} \end{bmatrix},$$

$$\mathcal{Y}_{\mathrm{Low}}^N = \bigoplus_{i=1}^{N} \frac{1}{z^i} \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{R} \end{bmatrix}.$$

The resulting norms are plotted in Fig. 4.

In this example, as N increases, the norms approach the optimal values given by the sparse controllers

$$\mathcal{K}_{\mathrm{Tri}}^s \in \begin{bmatrix} \frac{1}{z}\mathcal{R}_{\mathrm{p}} & 0 \\ \frac{1}{z}\mathcal{R}_{\mathrm{p}} & \frac{1}{z}\mathcal{R}_{\mathrm{p}} \end{bmatrix}, \quad \mathcal{K}_{\mathrm{Di}}^s \in \begin{bmatrix} \frac{1}{z}\mathcal{R}_{\mathrm{p}} & 0 \\ 0 & \frac{1}{z}\mathcal{R}_{\mathrm{p}} \end{bmatrix},$$

$$\mathcal{K}_{\mathrm{Low}}^s \in \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{z}\mathcal{R}_{\mathrm{p}} \end{bmatrix} \tag{22}$$

computed by the vectorization technique from [2]. Formally comparing the delayed solution to the sparse solution would be an interesting direction for future work.

VI. CONCLUSION

This technical note derives a novel solution for a class of output feedback $\mathcal{H}_2$ control problems with quadratically invariant communication delay patterns. First, all stabilizing decentralized controllers are characterized via doubly-coprime factorization. Then, by a standard change of variables, the $\mathcal{H}_2$ problem is cast as a convex model matching problem. The main theorem shows that for a doubly-coprime factorization based on the LQR and Kalman filter gains, the model matching problem reduces to a quadratic program. A solution to the quadratic program based on vectorization is also presented.

Many open problems remain for optimal control with quadratically invariant delay constraints. One limitation of the current work is that the solution is less explicit than those available for state feedback [11], [12] or special cases of output feedback [5]–[8], [20]. It is hoped that the method in this technical note can be utilized to derive more general explicit solutions. Furthermore, work is needed to understand how the controllers in this technical note could be realized [31], [32] and computed [33] in a distributed fashion.

APPENDIX

This appendix collects state-space formulas that are useful for deriving the results in the technical note. For compactness, the proofs are omitted or sketched.

Lemma A.1: Let P_{11}, P_{12}, and P_{21} be defined as in (17). The following equations hold:

$$P_{12}^{\sim}P_{12} = \Omega, \qquad P_{21}P_{21}^{\sim} = \Psi, \qquad \mathbb{P}_{\frac{1}{z}\mathcal{H}_2}\left(P_{12}^{\sim}P_{11}P_{21}^{\sim}\right) = 0.$$

Lemma A.2: For $\hat{Y}$, $\hat{M}$, and $\tilde{M}$ defined as in (15), the following equation holds:

$$\begin{bmatrix} -\text{vec}(\hat{Y}\tilde{M}) & \tilde{M}^{\mathsf{T}} \otimes \hat{M} \end{bmatrix} =$$

$$\left[\begin{array}{cc|cc} I_{q_2} \otimes (A+B_2K) & 0_{nq_2 \times np_2} & \text{vec}(L) & I_{q_2} \otimes B_2 \\ C_2^{\mathsf{T}} \otimes K & (A+LC_2)^{\mathsf{T}} \otimes I_{p_2} & 0_{np_2 \times 1} & C_2^{\mathsf{T}} \otimes I_{p_2} \\ \hline I_{q_2} \otimes K & L^{\mathsf{T}} \otimes I_{p_2} & 0_{p_2 q_2 \times 1} & I_{p_2 q_2} \end{array} \right].$$

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