Stability Criteria of Random Nonlinear Systems and Their Applications

Zhaojing Wu, *Member, IEEE*

***Abstract*—Stochastic differential equations (SDEs) are widely adopted to describe systems with stochastic disturbances, while they are not necessarily the best models in some specific situations. This paper considers the nonlinear systems described by random differential equations (RDEs). The notions and the corresponding criteria of noise-to-state stability, asymptotic gain and asymptotic stability are proposed, in the m-th moment or in probability. Several estimation methods of stochastic processes are presented to explain the reasonability of the assumptions used in theorems. As applications of stability criteria, some examples about stabilization, regulation and tracking are considered, respectively. A theoretical framework on stability of RDEs is finally constructed, which is distinguished from the existing framework of SDEs.**

***Index Terms*—Lyapunov stability, nonlinear systems, random differential equations.**

I. Introduction

STABILITY theory is one of the most important issues for RDEs that are defined as differential equations involving random elements. To show more specific stabilities as well as other properties, researchers pay their attentions to the affine form:

$$\dot{x} = f(x,t) + g(x,t)\xi(t), x(t_0) = x_0 \quad (1)$$

where $\xi(t) \in \mathbb{R}^l$ is a stochastic process. When $\xi(t)$ is a white noise, regarding $\xi(t)$ as the formal derivative of a Wiener process $W(t)$, (1) becomes the so called SDE described by

$$dx = f(x,t)dt + g(x,t)dW_t, x(t_0) = x_0. \quad (2)$$

In this paper, we only consider the narrow sense of RDEs that only refers to the affine form (1) where $\xi(t)$ is not a white noise. Nonlinear systems described by RDEs are called as random nonlinear systems.

For stability of SDEs, fruitful results have been obtained, which are applied in many fields such as theoretical physics, economics and finances. Not long after Lyapunov theory was introduced to the control community in [1], some important results appeared in 1960s, primarily by Khas'minskii and Kushner, respectively (see [2] and [3]). Since then, the stochastic Lyapunov's second method has been developed to deal with the stochastic stability by many authors, and here we only mention [4]–[8]. There appeared some stochastic versions of LaSalle-like theorem that locate limit sets of a system with linear growth condition (see, [9]) or without linear growth condition (see, [10]–[12]). Regarding the unknown covariance of Brown motion as the input, the notion of noise-to-state stability (NSS) was proposed by [10] and [11] with application in [13]. The notion of γ-input-to-state stability (γ-ISS) was introduced by Tsinias in [14], which corresponds to the stochastic robust stability in a special case where some global linear growth conditions were imposed. Different from the γ-ISS, the concept of stochastic input-to-state stability (SISS) was initiated in [15], in the form of $\beta + \gamma$ estimate, which is a generalization of the notion of NSS.

Although significant successes have been obtained for SDEs, some preliminary questions still need convincing reasons. For example, the white noise is viewed as the formal derivative of a Wiener process, while the latter has no derivative everywhere. A specific physical system with white noise can be described by Itô integral equation and Stratonovich integral equation, respectively, then how to decide which one is the best description? What is the physical meaning of Hessian term in Itô's formula? It seems that the models descried by SDEs are not suit very well for many control tasks in engineering. For example, road irregularities are often described by white noises, but their final effects to operation system of a car with spring absorber should be illustrated by stationary processes. For another example, in a circuit system with power noise filter, it is more reasonable to describe the final effects of stochastic disturbances to other electric elements by stationary processes than by white noises.

Stability results of nonlinear RDEs are very few and the existing results are somewhat conservative. In [16] and [17], stability and asymptotic stability were proved by using Lyapunov function $V(x,t) \geq a|x|$ for a constant $a > 0$, and $E\dot{V}(x,t) \leq 0$ (or < 0), which depends heavily on the analytical solutions to RDEs. In [2], the existence and boundedness in probability of solution were considered under the assumption $\sup_{x,t}\{g(x,t)\} < c$ (constant). For a Lyapunov function V with bounded Lipschitz coefficient i.e., $\sup_{t>0, x_i \in \mathbb{R}^n, x_1 \neq x_2}\{|V(x_2,t) - V(x_1,t)|/|x_2 - x_1|\} < c$, stability was analyzed. The almost surely asymptotic stability in the large and exponential p-stability were proved under the assumption $|g(x,t)| \leq V(x,t)$. Stability under small random perturbations was researched by giving the condition $|x(t_0)| + \sup_{x,t}\{\|g(x,t)\|\} < c$. For these assumptions, Khas'minskii gave the explanation: "unless certain restrictive assumptions are made concerning a given system, it is not likely that non-trivial and effective stability conditions can be

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The author is with School of Mathematics and Informational Science, Yantai University, Yantai, Shandong Province, 264005, China (e-mail: wuzhaojing00@188.com).

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found" (see [2]). These assumptions have prevented the wide applications of RDE models in the engineering.

The aim of this paper is to construct a framework of stability analysis for random nonlinear systems with second-order processes. In Section II, the existence and uniqueness of solutions are analyzed based on globally and locally Lipschitz conditions, respectively. In Section III, notions of noise-to-state stability are given in the m-th moment and in probability, respectively. In Section IV, notions of asymptotic gain and their criteria are presented by following lines of NSS. Suppose certain specific estimations of disturbances are known, some results of asymptotical stability are obtained in Section V. Some estimations of stochastic process $\xi(t)$ are addressed in Section VI, under some natural situations such as periodic, stationary, ergodic and normal conditions. As applications, some examples of stabilization, regulation and tracking are considered in Section VII. Comparison to SDEs is presented in Section VIII. Conclusion is presented in Section IX by summarizing the main idea of this paper and pointing out some issues deserving consideration in the future. It is expected that, by replacing of the restrictive assumptions [2], [16] and [17] with some weaker conditions, our research results will greatly widen the applications of RDE models, and themselves will motivate many issues deserved researching.

Notations: For a vector x, $|x|$ stands for its usual Euclidean norm and x^T denotes its transpose; $\mathbb{R}^n$ stands for the real n-dimensional space; $\|X\|$ is the 2-norm of a matrix X; $\mathbb{R}_+$ is the set of all nonnegative real numbers; U_R stands for the ball $|x| < R$ and U_R^C its complement in $\mathbb{R}^n$. C^i denotes the set of all functions with continuous i-th partial derivative; $\mathcal{K}$ stands for the set of all functions: $\mathbb{R}_+ \to \mathbb{R}_+$, which are continuous, strictly increasing and vanish at zero; $\mathcal{K}_\infty$ denotes the set of all functions which are of class-$\mathcal{K}$ and unbounded; $\mathcal{KL}$ stands for the set of all functions $\beta(s,t) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which is of class-$\mathcal{K}$ for each fixed t, and decreases to zero as $t \to \infty$ for each fixed s. For $a, b \in \mathbb{R}$, define $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. Function $\alpha : D \to \mathbb{R}$ is convex on D, if it satisfies $\alpha((s_1 + s_2)/2) \leq (\alpha(s_1) + \alpha(s_2))/2, \forall s_1, s_2 \in D$.

II. The Existence-and-Uniqueness of Solution to RDEs

Consider a random nonlinear system described by

$$\dot{x} = f(x,t) + g(x,t)\xi(t), \qquad x(t_0) = x_0 \tag{3}$$

where $x \in \mathbb{R}^n$ is the state, the underlying complete probability space is taken to be the quartet $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with a filtration $\mathcal{F}_t$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets).

Stochastic process $\xi(t) \in \mathbb{R}^l$ satisfies the following assumption.

A1: Process $\xi(t)$ is $\mathcal{F}_t$-adapted and piecewise continuous, and satisfies

$$\sup_{t_0 \leq s \leq t} E\,|\xi(s)|^2 < \infty, \qquad \forall\, t \geq 0. \tag{4}$$

A solution to system (3) on $[t_0, T]$ $(T < \infty)$ is a process $x(t) := x(t_0, x_0, t)$ satisfying:

1) $x(t)$ is continuous for all $t \in [t_0, T]$,
2) $x(t)$ is adapted to $\mathcal{F}_t$, and
3) for all $t \in [t_0, T]$

$$x(t) = x_0 + \int_{t_0}^{t} f(x,s)ds + \int_{t_0}^{t} g(x,s)\xi(s)ds. \tag{5}$$

The uniqueness of solution to (3) is in almost-sure sense. If for any $T > t_0$, system (3) has a unique solution on $[t_0, T]$, then it has a unique solution on the infinite interval $[t_0, \infty)$. In this case, we say system 1 is forward-complete, and $\{x(t) : t_0 \leq t < \infty\}$ is the unique global solution to system (3).

To analyze the existence and uniqueness of global solution to system (3), the following two preliminary assumptions are frequently used.

P1: Functions $f(x,t)$ and $g(x,t)$ are piecewise continuous in t, and Lipschitz in x, i.e., there exists a constant L such that $\forall x_1, x_2 \in \mathbb{R}^n, x_1 \neq x_2$,

$$|f(x_2,t) - f(x_1,t)| + \|g(x_2,t) - g(x_1,t)\| \leq L|x_2 - x_1|. \tag{6}$$

P2: There exists a constant $d_0 \geq 0$ such that

$$|f(0,t)| + \|g(0,t)\| < d_0. \tag{7}$$

The following conclusion named as Bellman-Gronwall Lemma can be found in [18].

Lemma 1: Let $\lambda \geq 0$ be a constant and $k(t)$ be a nonnegative piecewise continuous function of time t. If the function $y(t)$ satisfies the inequality

$$y(t) \leq \lambda + \int_{t_0}^{t} k(s)y(s)ds, \qquad \forall\, t \geq t_0 \geq 0$$

then

$$y(t) \leq \lambda e^{\int_{t_0}^{t} k(s)ds}, \qquad \forall\, t \geq t_0 \geq 0.$$

The following result can be found in [19].

Lemma 2: Let the function $y(t)$ be absolutely continuous for $t \geq t_0$ and let its derivative satisfy the inequality

$$\dot{y}(t) \leq k(t)y(t) + h(t)$$

for almost all $t \geq t_0$, where $k(t)$ and $h(t)$ are almost everywhere continuous functions integrable over every finite interval. Then for $t \geq t_0$

$$y(t) \leq y(t_0)e^{\int_{t_0}^{t} k(s)ds} + \int_{t_0}^{t} e^{\int_{s}^{t} k(u)du} h(s)ds.$$

We begin with presenting a result about existence and uniqueness of global solution to random affine systems.

Lemma 3: Under assumptions *A1*, *P1* and *P2*, system (3) has a unique solution $x(t)$ on $[t_0, \infty)$.

Proof: First we prove the result on the interval $[t_0, T]$, where $T \geq t_0$ is any finite instant. From $\sup_{t_0 \leq s \leq t} E|\xi(s)|^2 < \infty$, for all $t \geq t_0$, one can verify $|\xi(t)| < \infty$, a.s.

Uniqueness. Let $x(t)$ and $\bar{x}(t)$ be two solutions of system (3). By (5) and (6), we have

$$|x(t) - \bar{x}(t)| \leq \int_{t_0}^{t} |f(x,s) - f(\bar{x},s)|\, ds$$

$$+\int_{t_0}^{t} \|g(x,s)-g(\bar{x},s)\| \, |\xi(s)| \, ds$$
$$\leq \varepsilon + L\int_{t_0}^{t} |x-\bar{x}| \left(1+|\xi(s)|\right) ds$$

where $\varepsilon \geq 0$ is a parameter. According to Lemma 1

$$|x(t)-\bar{x}(t)| \leq \varepsilon e^{L\int_{t_0}^{T}(1+|\xi(s)|)ds}$$

which, together with the arbitrariness of ε and the $|\xi(t)| < \infty$, a.s., implies $x(t)=\bar{x}(t)$, a.s., for all $t \in [t_0, T]$.

Existence. For any $T \geq t_0$, applying the method of successive approximations to (5) on the interval $[t_0, T]$, we define a series of processes as

$$\begin{aligned} x^{(0)}(t) &\equiv x_0 \\ x^{(n+1)}(t) &= x_0 + \int_{t_0}^{t} f\left(x^{(n)}(s), s\right) ds \\ &\quad + \int_{t_0}^{t} g\left(x^{(n)}(s), s\right) \xi(s) ds, \forall n \geq 0. \end{aligned} \quad (8)$$

We get the estimates

$$\left|x^{(1)}(t)-x_0\right| \leq (L|x_0|+d_0)\int_{t_0}^{t} (1+|\xi(s)|)\, ds$$
$$\left|x^{(n+1)}(t)-x^{(n)}(t)\right|$$
$$\leq L\int_{t_0}^{t} \left|x^{(n)}(s)-x^{(n-1)}(s)\right| (1+|\xi(s)|)\, ds, \quad \text{a.s.}$$

and these imply the inequality

$$\left|x^{(n+1)}(t)-x^{(n)}(t)\right| \leq \left(|x_0|+\frac{d_0}{L}\right) \times \frac{\left[L\int_{t_0}^{t}(1+|\xi(s)|)\, ds\right]^n}{n!}, \quad \text{a.s.} \quad (9)$$

then the series

$$x^{(0)}(t) + \sum_{n=1}^{\infty} \left|x^{(n)}(t)-x^{(n-1)}(t)\right|$$

uniformly converges almost surely. Then there exists a process $x(t)$ such that

$$x(t) = \lim_{n\to\infty} x^{(n)}(t), \quad \text{a.s.}, \quad (10)$$

for all $t \in [t_0, T]$. Using this limitation to (8), one can verify that $x(t)$ satisfies (5). Obviously the sample of $x(t)$ is continuous in $t \in [t_0, T]$. Since $\xi(t)$ is adapted to $\mathcal{F}_t$, then $x^{(0)}(t)$ and $x^{(1)}(t)$ are adapted. Following recursive procedures, it can be inferred that $x^{(n)}(t)$ is adapted to $\mathcal{F}_t$, thereby $x(t)$ is adapted to $\mathcal{F}_t$, according to (10).

Finally, since $T \geq t_0$ is arbitrary, the result of this lemma holds on $[t_0, \infty)$. ∎

Some traditional notions about the space of states, such as region of attraction in [20], can not play the same roles as they are in the deterministic case, because it is difficult to define them in the almost-sure sense. Therefore, many methods expressed in the viewpoint of space in the deterministic case, have to be turned into expressions in the viewpoint of time. Introduce the first exit time from a region $U_k = \{x : |x| < k\}$ and its limit:

$$\sigma_k = \inf\{t \geq t_0 : |x(t)| \geq k\}, \sigma_\infty = \lim_{k\to\infty} \sigma_k \quad (11)$$

where $\inf \emptyset = \infty$ is assumed. An $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted continuous stochastic process $\{x(t) : t_0 \leq t < \sigma\infty\}$ is called a maximal solution of system (3), if $x(t_0) = x_0$ and for all $t \in [t_0, \infty)$ and $k > 0$

$$x(t\wedge\sigma_k) = x_0 + \int_{t_0}^{t\wedge\sigma_k} f(x,s)ds + \int_{t_0}^{t\wedge\sigma_k} g(x,s)\xi(s)ds, \quad \text{a.s.} \quad (12)$$

Here, $[t_0, \sigma_\infty)$ is called the maximal existence interval. If, furthermore, $\sigma_\infty < \infty$, a.s., then $\{x(t) : t_0 \leq t < \sigma_\infty\}$ is called a maximal local solution and σ_∞ is called the explosion time. A maximal solution $\{x(t) : t_0 \leq t < \sigma_\infty\}$ is said to be unique if any other maximal solution $\{\bar{x}(t) : t_0 \leq t < \bar{\sigma}_\infty\}$ is indistinguishable from it, namely, $\sigma_\infty = \bar{\sigma}_\infty$ and $x(t) = \bar{x}(t)$ for all $t_0 \leq t < \sigma_\infty$ with probability 1.

The strictness of global Lipschitz condition in preliminary assumption *P1* leads to poor applications of the model, which motivates us to research a milder one.

$\mathbf{P1}'$: For any $R > 0$, there exists a constant L_R possibly depending on R such that $\forall x_1, x_2 \in U_R, x_1 \neq x_2$

$$|f(x_1,t)-f(x_2,t)| + \|g(x_1,t)-g(x_2,t)\| \leq L_R|x_2-x_1|. \quad (13)$$

The existence-and-uniqueness of the maximal solution is given by the following lemma.

Lemma 4: Under assumptions *A1*, $\mathbf{P1}'$ and **P2**, system (3) has a unique solution in maximal interval $[t_0, \sigma_\infty)$.

Proof: For each $k \geq 1$, define the truncation functions

$$f_k(x,t) = \begin{cases} f(x,t), & \text{if } |x| \leq k \\ f(kx/|x|, t), & \text{if } |x| > k \end{cases}$$
$$g_k(x,t) = \begin{cases} g(x,t), & \text{if } |x| \leq k \\ g(kx/|x|, t), & \text{if } |x| > k \end{cases}$$

then f_k and g_k satisfy Lipschitz conditions. Hence by Lemma 3, there exists a unique solution $x_k(\cdot)$ to equation

$$\dot{x}_k(t) = f_k(x_k,t) + g_k(x_k,t)\xi(t), t \in [t_0, \infty) \quad (14)$$

where $x_k(t_0) = x_0$. It is easy to show that

$$x_k(t) = x_{k+1}(t), \forall t \in [t_0, \sigma_k]. \quad (15)$$

Define

$$x(t) = x_k(t), t \in [\sigma_{k-1}, \sigma_k), k \geq 1.$$

By (15), $x(t\wedge\sigma_k) = x_k(t\wedge\sigma_k)$. It follows from (14) that

$$\begin{aligned} x(t\wedge\sigma_k) &= x_0 + \int_{t_0}^{t\wedge\sigma_k} f_k(x(s),s)ds + \int_{t_0}^{t\wedge\sigma_k} f_k(x(s),s)\xi(s)ds \\ &= x_0 + \int_{t_0}^{t\wedge\sigma_k} f(x(s),s)ds + \int_{t_0}^{t\wedge\sigma_k} g(x(s),s)\xi(s)ds \end{aligned}$$

for all $t \in [t_0, \infty), k \geq 1$. If $\sigma_\infty < T < \infty$, then

$$\limsup_{t\to\sigma_\infty} |x(t)| \geq \limsup_{k\to\infty} |x(\sigma_k)| = \limsup_{k\to\infty} |x_k(\sigma_k)| = \infty.$$

Hence, $\{x(t) : t_0 \leq t < \sigma_\infty\}$ is a maximal solution. The uniqueness can be easily verified by that of x_k. ■

Roughly speaking, the global Lipschitz condition leads to the existence of a global solution, and local condition leads to that of a maximal solution. To obtain a global solution from local Lipschitz condition, we need additional information.

Lemma 5: For system (3) under assumptions **A1**, $\mathbf{P1}'$ and **P2**, if there exist a positive function $V(x(t), t) \in C$ and constants $c, d > 0$ such that for all $t \geq t_0$

$$\lim_{k\to\infty} \inf_{|x|>k} V(x,t) = \infty, \tag{16}$$

and

$$EV(x(t\wedge\sigma_k), t\wedge\sigma_k) \leq de^{ct}, \forall k > 0 \tag{17}$$

then system (3) has a unique solution $x(t)$ on $[t_0, \infty)$.

Proof: From Lemma 4, there exists a maximal solution $x(t)$ on $[t_0, \sigma_\infty)$. We need to show $\sigma_\infty = \infty$, a.s. Otherwise, we can find a pair of positive constants ε and T such that

$$P\{\sigma_\infty \leq T\} > 2\varepsilon.$$

Since $\sigma_\infty = \lim_{k\to\infty} \sigma_k$ almost surely, there exists a larger integer k_0 such that for any $k \geq k_0$,

$$P\{\sigma_k \leq T\} > \varepsilon. \tag{18}$$

Fixing $k \geq k_0$, for any $t_0 \leq t \leq T$, by (17), one has

$$EV\left(x(T\wedge\sigma_k), T\wedge\sigma_k\right) \leq de^{cT}$$

which means

$$E\left[I_{\sigma_k\leq T} V\left(x(\sigma_k), \sigma_k\right)\right] \leq de^{cT}. \tag{19}$$

On the other hand, if we define

$$h_k = \inf\left\{V(x,t) : |x| \geq k, t \in [t_0, T]\right\}$$

then $h_k \to \infty$ as $k \to \infty$ by (16). It follows from (18) and (19) that

$$de^{cT} \geq h_k P\{\sigma_k \leq T\} \geq \varepsilon h_k.$$

Letting $k \to \infty$ yields a contradiction, so we must have $\sigma_\infty = \infty$, a.s. ■

Remark 1: Based on locally Lipschitz conditions of f and g and a Lyapunov function V, the global solution was considered in [2], while additional assumptions such as $\sup_{x,t}\{\|g(x,t)\|\} < c$ (constant) and V being globally Lipschitz were imposed to system (3). ■

Definition 1: The stochastic process $\phi(t) : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ is called (strongly) bounded in probability if for any $\varepsilon > 0$ there exists an $r > 0$ such that

$$P\left\{\sup_{t\geq t_0} |\phi(t)| > r\right\} \leq \varepsilon.$$

The boundedness in probability here is stronger than the (weak) boundedness in probability proposed in [2, p. 13] considering the fact $P\{\sup_{t\geq t_0}|\phi(t)|>r\} \geq \sup_{t\geq t_0} P\{|\phi(t)|>r\}$.

Lemma 6: For system (3) under assumptions **A1**, $\mathbf{P1}'$ and **P2**, if there exist a positive-definite function $V(x,t) \in C$ and a constant $d \geq 0$ such that (16) and

$$EV\left(x(t\wedge\sigma_k), t\wedge\sigma_k\right) \leq d, \forall t \geq t_0, k > 0 \tag{20}$$

then system (3) has a unique solution $x(t)$ on $[t_0, \infty)$, which is bounded in probability.

Proof: The proof is the same as the proof of [21, Lemma 2]. ■

III. Noise-to-State Stability

Even in the deterministic case, the concept of stability can be given various meanings. The diversity is even greater in the presence of randomness. We shall not list here all the possible definitions, but we shall confine ourselves to those plausible in control field such as stability, asymptotic stability, exponential stability, ultimate boundedness and ISS (or NSS).

Regarding $\xi(t)$ as a stochastic input, notions of NSS will be presented in a background different from [10] where NSS was first proposed for SDEs.

A condition stricter than **A1** will be used to prove the existence-and-uniqueness of solution.

$\mathbf{A1}'$: Process $\xi(t)$ is $\mathcal{F}_t$-adapted and piecewise continuous, and there exists parameters $c_0, d_0 > 0$ such that

$$E|\xi(t)|^2 \leq d_0 e^{c_0 t}, \forall t \geq t_0. \tag{21}$$

In this section, we further consider system (3), i.e.,

$$\dot{x} = f(x,t) + g(x,t)\xi(t), x(t_0) = x_0 \tag{22}$$

where functions f and g satisfy assumptions $\mathbf{P1}'$ and **P2**, and $\xi(t)$ satisfies assumption $\mathbf{A1}'$.

Definition 2: System (22) is said to be noise-to-state stable in the m-th moment (NSS-m-M) if there exist a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class-$\mathcal{K}$ function $\gamma(\cdot)$ such that $\forall t \in [t_0, \infty)$

$$E|x(t)|^m \leq \beta\left(|x_0|, t-t_0\right) + \gamma\left(\sup_{t_0\leq s\leq t} E|\xi(s)|^2\right). \tag{23}$$

Definition 3: System (22) is said to be noise-to-state stable in probability (NSS-P) if for any $\varepsilon > 0$ there exist a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class-$\mathcal{K}$ function $\gamma(\cdot)$ such that for all $t \in [t_0, \infty)$ and $x_0 \in \mathbb{R}^n$

$$P\left\{|x(t)| \leq \beta\left(|x_0|, t-t_0\right) + \gamma\left(\sup_{t_0\leq s\leq t} E|\xi(s)|^2\right)\right\} \geq 1-\varepsilon. \tag{24}$$

Theorem 1: Under assumptions $\mathbf{A1}'$, $\mathbf{P1}'$ and **P2**, for system (22), assume that there exist a function $V \in C^1$ and constants $a_1, a_2, a, d > 0$ such that

$$a_1|x|^m \leq V(x) \leq a_2|x|^m \tag{25}$$

$$\frac{\partial V}{\partial x} f(x,t) + d\left|\frac{\partial V}{\partial x} g(x,t)\right|^2 \leq -a|x|^m. \tag{26}$$

Then there exists a unique global solution to system (22) and the system is NSS-m-M.

Proof: From (26), the derivative of V along system (22) satisfies that

$$\begin{aligned}\dot{V}\left(x(t)\right) &= \frac{\partial V}{\partial x} f(x,t) + \frac{\partial V}{\partial x} g(x,t)\xi(t)\\ &\leq \frac{\partial V}{\partial x} f(x,t) + d\left|\frac{\partial V}{\partial x} g(x,t)\right|^2 + \frac{1}{4d}|\xi(t)|^2\\ &\leq -\frac{a}{a_2} V(x) + \frac{1}{4d}|\xi(t)|^2. \end{aligned} \tag{27}$$

Taking integrals first in $[t_0, t\wedge\sigma_k)$ and then expectations on both sides of (27), we have

$$EV\left(x(t\wedge\sigma_k)\right)-V(x_0) \le -\frac{a}{a_2}E\int_{t_0}^{t\wedge\sigma_k}V(x)ds+\frac{1}{4d}E\int_{t_0}^{t}|\xi(s)|^2ds \quad (28)$$

which, together with (21), gives

$$EV(x(t\wedge\sigma_k))<V(x_0)+\frac{d_0}{4c_0d}e^{c_0t}\le\left(V(x_0)+\frac{d_0}{4c_0d}\right)e^{c_0t} \quad (29)$$

therefore, according to Lemma 5, the existence of solution on $[t_0,\infty)$ can be obtained, that is, $\sigma_\infty=\infty$ almost surely. It follows from $\sigma_\infty=\infty$, a.s., and (27) that

$$V\left(x(t)\right)<\infty, \dot V\left(x(t)\right)<\infty,$$

then from Fubini's theorem [22, Theorem 2.39], we have

$$\int_{t_1}^{t}E\dot V(x(s))ds=E\int_{t_1}^{t}\dot V(s)ds=EV(x(t))-EV(x(t_1)),$$

i.e.,

$$E\frac{dV\left(x(t)\right)}{dt}=\frac{dEV\left(x(t)\right)}{dt} \quad (30)$$

which means the exchangeability of expectation and derivative. By defining $v(t)=EV(x(t))$, according to Lemma 2, we have

$$v(t)\le|v(t_0)|e^{-\frac{a}{a_2}(t-t_0)}+\frac{a_2}{4ad}\sup_{t_0\le s\le t}E|\xi(s)|^2 \quad (31)$$

which, together with (25), means that

$$E|x(t)|^m\le\frac{a_2}{a_1}|x_0|^me^{-\frac{a}{a_2}(t-t_0)}+\frac{a_2}{4aa_1d}\sup_{t_0\le s\le t}E|\xi(s)|^2.$$

Thus, we complete the proof. ■

From the above proof, the following corollary is easily obtained.

Corollary 1: The result of Theorem 1 remains valid if we replace (26) with (27).

Theorem 2: Under assumptions $\mathbf{A1}'$, $\mathbf{P1}'$ and $\mathbf{P2}$, for system (22), assume that there exist a parameter $d>0$, a function $V\in C^1$, class-$\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$ and a class-$\mathcal{K}$ function α such that

$$\underline{\alpha}(|x|)\le V(x)\le\overline{\alpha}(|x|), \quad (32)$$

$$\frac{\partial V}{\partial x}f(x,t)+d\left|\frac{\partial V}{\partial x}g(x,t)\right|^2\le-\alpha(|x|). \quad (33)$$

Then there exists a unique global solution to system (22). If $\alpha(\bar\alpha^{-1}(\cdot))$ is a convex function, the system is NSS-P.

Proof: From (33), the derivative of V along system (22) satisfies that

$$\begin{aligned}\dot V(x(t))&=\frac{\partial V}{\partial x}f(x(t),t)+\frac{\partial V}{\partial x}g(x(t),t)\xi(t)\\&\le\frac{\partial V}{\partial x}f(x(t),t)+d\left|\frac{\partial V}{\partial x}g(x(t),t)\right|^2+\frac{1}{4d}|\xi(t)|^2\\&\le-\alpha(|x(t)|)+\frac{1}{4d}|\xi(t)|^2. \end{aligned}\quad (34)$$

By following the same line as the proof from (27) to (29), the existence of solution on $[t_0,\infty)$ can be obtained, that is, $\sigma_\infty=\infty$, a.s. It follows from (34) that

$$\dot V\left(x(t)\right)\le-\alpha\left(\bar\alpha^{-1}\left(V(x)\right)\right)+\frac{1}{4d}|\xi(t)|^2. \quad (35)$$

By defining $v(t)=EV(x(t))$, from (30) and the convexity of $\alpha(\bar\alpha^{-1}(s))$, we have

$$\begin{aligned}\dot v(t)&=E\dot V\left(x(t)\right)\le-E\alpha\left(\bar\alpha^{-1}\left(V(x(t))\right)\right)+\frac{1}{4d}E|\xi(t)|^2\\&\le-\alpha\left(\bar\alpha^{-1}\left(v(t)\right)\right)+\frac{1}{4d}E|\xi(t)|^2\end{aligned}$$

where Jensen's inequality is used to conclude $E\alpha(\bar\alpha^{-1}(V\times(x(t)))\ge\alpha(\bar\alpha^{-1}(EV(x(t)))$, therefore, from [23] and [24], there exist a class-$\mathcal{KL}$ function $\bar\beta$ and a class-$\mathcal{K}_\infty$ function $\bar\gamma$ such that

$$v(t)\le\bar\beta(|x_0|,t-t_0)+\bar\gamma\left(\frac{1}{d}\sup_{t_0\le s\le t}E|\xi(s)|^2\right). \quad (36)$$

According to Chebyshev's inequality, for any $\varepsilon>0$, it comes from (32) and (36) that

$$\begin{aligned}&P\{\underline{\alpha}(|x(t)|)>\frac{1}{\varepsilon}\bar\beta(|x_0|,t-t_0)+\frac{1}{\varepsilon}\bar\gamma(\frac{1}{d}\sup_{t_0\le s\le t}E|\xi(s)|^2)\}\\&\le P\{V(x(t))>\frac{1}{\varepsilon}\bar\beta(|x_0|,t-t_0)+\frac{1}{\varepsilon}\bar\gamma(\frac{1}{d}\sup_{t_0\le s\le t}E|\xi(s)|^2)\}\\&\le\frac{EV(x)\varepsilon}{\bar\beta(|x_0|,t-t_0)+\bar\gamma(\frac{1}{d}\sup_{t_0\le s\le t}E|\xi(s)|^2)}\le\varepsilon,\end{aligned}$$

thus, by taking $\beta(\cdot,\cdot)=\underline{\alpha}^{-1}((2/\varepsilon)\bar\beta(\cdot,\cdot))$ and $\gamma(\cdot)=\underline{\alpha}^{-1}((2/\varepsilon)\bar\gamma(\cdot))$, we have

$$P\{|x(t)|\le\beta(|x_0|,t-t_0)+\gamma(\frac{1}{d}\sup_{t_0\le s\le t}E|\xi(s)|^2)\}\ge1-\varepsilon$$

which completes the proof. ■

From the proof procedure of Theorem 2, we can obtain the following result easily.

Corollary 2: The result of Theorem 2 remains valid if we replace (33) with (34).

Remark 2: In Theorems 1-2 and their corollaries, parameter d can be neglected by taking 1 and there is no effect on the results. The importance of d can be found in the controller design in Section VII. ■

Remark 3: In [2], the effect of noise on state of system (22) was described by the stability of its truncated systems $\dot x=f(x,t), x(t_0)=x_0$ under small random perturbations. This was researched under the condition $|x(t_0)|+\sup_{x,t}\{\|g(x,t)\|\}<c$ and $V(x,t)$ being globally Lipschitz in x. Compared with [2, Theorem 1.6.2], these conditions are all removed from Theorems 1 and 2 in this paper, which gives conveniences of wide applications. ■

IV. Asymptotic Gain Properties of System

When the mean-square of disturbance is bounded by an exponential function of time, notions and criteria of NSS-m-M and NSS-P are proposed, while if the bound of mean-square of disturbance is a constant, some more specific properties than NSS are expected.

For further arguments, we need the following assumption.

$\mathbf{A1}''$: Process $\xi(t)$ is $\mathcal{F}_t$-adapted and piecewise continuous, and there exists a constant $K>0$ such that

$$\sup_{t\ge t_0}E|\xi(t)|^2<K. \quad (37)$$

In this section, assumptions $\mathbf{A1}''$, $\mathbf{P1}'$ and $\mathbf{P2}$ are imposed to system (22).

For nonlinear systems with bounded disturbance (in the deterministic case), estimation of ultimate bound of state was researched in [20], which motivates the following notions.

Definition 4: The state of system (22) has an asymptotic gain in the m-th moment (AG-m-M) if there exists a class-$\mathcal{K}$ function $\gamma(\cdot)$ such that, for any $x_0 \in R^n$

$$\lim_{t\to\infty} E|x(t)|^m \leq \gamma(K). \tag{38}$$

Definition 5: The state of system (22) has an asymptotic gain in probability (AG-P) if for any $\varepsilon > 0$ there exists a class-$\mathcal{K}$ function $\gamma(\cdot)$ such that, for any $x_0 \in R^n$

$$P\left\{\lim_{t\to\infty} |x(t)| \leq \gamma(K)\right\} \geq 1-\varepsilon. \tag{39}$$

Comparing to Definitions 2-3, the corresponding criteria of Definitions 4-5 can be obtained.

Theorem 3: Under assumptions $\mathbf{A1}''$, $\mathbf{P1}'$ and $\mathbf{P2}$, for system (22), if there exists a function $V \in C^1$ and constants $a_1, a_2, a, d > 0$ such that

$$a_1|x|^m \leq V(x) \leq a_2|x|^m \tag{40}$$

$$\frac{\partial V}{\partial x} f(x,t) + d\left|\frac{\partial V}{\partial x} g(x,t)\right|^2 \leq -a|x|^m \tag{41}$$

then system (22) is NSS-m-M and has a unique global solution, and the state of system has an AG-m-M.

Proof: According to Theorem 1, the existence-and-uniqueness of solution and NSS-m-M of system can be obtained, and moreover, for the given $d > 0$, we have

$$E|x(t)|^m \leq \beta\left(|x_0|, t-t_0\right) + \gamma\left(\frac{K}{d}\right). \tag{42}$$

By letting $t \to \infty$, (38) can be concluded, which completes the proof. ∎

Corollary 3: The result of Theorem 3 remains valid if we replace (41) with (27).

Theorem 4: Under assumptions $\mathbf{A1}''$, $\mathbf{P1}'$ and $\mathbf{P2}$, for system (22), if there exist a parameter $d > 0$, a function $V \in C^1$ and class-$\mathcal{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha}$ and a class-$\mathcal{K}$ function α such that

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \tag{43}$$

$$\frac{\partial V}{\partial x} f(x,t) + d\left|\frac{\partial V}{\partial x} g(x,t)\right|^2 \leq -\alpha(|x|) \tag{44}$$

then there exists a unique global solution to system (22). If $\alpha(\bar{\alpha}^{-1}(\cdot))$ is a convex function, the system is NSS-P, and the state of system has an AG-P.

Proof: According to Theorem 2, the existence-and-uniqueness of solution and NSS-P of system can be obtained, and moreover, for the given $d > 0$ and any $\varepsilon > 0$, we have

$$P\left\{|x(t)| \leq \beta\left(|x_0|, t-t_0\right) + \gamma\left(\frac{K}{d}\right)\right\} \geq 1-\varepsilon. \tag{45}$$

By letting $t \to \infty$, (39) can be concluded. ∎

Corollary 4: The result of Theorem 4 remains valid if we replace (44) with (34).

V. Global Asymptotic Stability

Some definitions of stability for stochastic systems (2) can be found in [2]–[8], [10], [11], which are now redefined for random systems in this section.

To guarantee $x(t) \equiv 0$ being the equilibrium, preliminary assumption $\mathbf{P2}$ to system (22) needs be replaced with the following assumption.

$\mathbf{P2}'$: Functions f and g vanish at the origin, i.e., $f(0,t) = 0$ and $g(0,t) = 0$, $\forall t \in [t_0, \infty)$.

In this section, assumptions $\mathbf{A1}''$, $\mathbf{P1}'$ and $\mathbf{P2}'$ are proposed for system (22).

Definition 6: For system (22), the equilibrium $x(t) \equiv 0$ is said to be globally stable in probability (GS-P) if for every $\varepsilon > 0$ there exists a class-$\mathcal{K}$ function $\gamma(\cdot)$ such that $\forall t \geq t_0, x_0 \in \mathbb{R}^n \backslash \{0\}$

$$P\left\{|x(t)| < \gamma\left(|x_0|\right)\right\} \geq 1-\varepsilon. \tag{46}$$

Definition 7: For system (22), the equilibrium $x(t) \equiv 0$ is said to be globally asymptotically stable in probability (GAS-P) if for every $\varepsilon > 0$ there exists a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that $\forall t \geq t_0, x_0 \in \mathbb{R}^n \backslash \{0\}$

$$P\left\{|x(t)| \leq \beta(|x_0|, t-t_0)\right\} \geq 1-\varepsilon. \tag{47}$$

Definition 8: For system (22), the equilibrium $x(t) \equiv 0$ is said to be globally stable in m-th moment (GS-m-M) if there exists a class-$\mathcal{K}$ function $\gamma(\cdot)$ such that $\forall t \geq t_0, x_0 \in \mathbb{R}^n \backslash \{0\}$

$$E\left|x(t)\right|^m < \gamma\left(|x_0|\right). \tag{48}$$

Definition 9: For system (22), the equilibrium $x(t) \equiv 0$ is said to be globally asymptotically stable in m-th moment (GAS-m-M) if there exists a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that $\forall t \geq t_0, x_0 \in \mathbb{R}^n \backslash \{0\}$

$$E\left|x(t)\right|^m < \beta\left(|x_0|, t-t_0\right). \tag{49}$$

Definition 10: For system (22), the equilibrium $x(t) \equiv 0$ is said to be exponentially stable in m-th moment (ES-m-M) if there exist parameters $k_1, k_2 > 0$ such that $\forall t \geq t_0, x_0 \in \mathbb{R}^n \backslash \{0\}$

$$E\left|x(t)\right|^m < k_1|x_0|^m e^{-k_2(t-t_0)}. \tag{50}$$

Criteria on GAS-P and ES-m-M of system are to be presented, and others can also be addressed and proved in similar ways.

A2: For a stochastic process $\xi(t)$, if for any $\varepsilon > 0, \delta > 0$, there exists a $T > t_0$ such that for all $t \geq T$

$$P\left\{\left|\frac{1}{t-t_0}\int_{t_0}^{t} |\xi(s)|^2 ds - E|\xi(t)|^2\right| \geq \delta\right\} \leq \varepsilon \tag{51}$$

we say that $|\xi|^2$ satisfies the weak law of large numbers.

Theorem 5: Under assumptions $\mathbf{A1}''$, $\mathbf{A2}$, $\mathbf{P1}'$ and $\mathbf{P2}'$, for system (22), assume that there exist a function $V \in C^1$ and class-$\mathcal{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha}$ and constants $c_1, c_2 > 0$ such that

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \tag{52}$$

$$\frac{\partial V}{\partial x} f(x,t) \leq -c_1 V(x(t)), \left|\frac{\partial V}{\partial x} g(x,t)\right| \leq c_2 V(x(t)). \tag{53}$$

If $c_1 > 2c_2\sqrt{K}$, then there exists a unique solution to system (22) on $[t_0, \infty)$, and its equilibrium is GAS-P.

Proof: From (53), the derivative of V along system (22) satisfies that

$$\dot{V}(x(t)) \leq \frac{\partial V}{\partial x} f(x,t) + \left(\frac{\partial V}{\partial x} g(x,t)\right)\xi(t) \leq (-c_1 + c_2|\xi(t)|)V(x(t)) \tag{54}$$

which, according to Lemma 2, implies that

$$V(x(t\wedge\sigma_k)) \leq V(x_0)e^{\int_{t_0}^{t}(-c_1+c_2|\xi(s)|)ds} = V(x_0)e^{-c_1(t-t_0)}e^{c_2\int_{t_0}^{t}|\xi(s)|ds} < \infty. \tag{55}$$

From (37), it is obvious $|\xi(t)| < \infty$, a.s., then from (52) and (55) one has $\sigma_\infty = \infty$, that is, there exists a unique solution to system (22) on $[t_0, \infty)$, according to Lemma 4. Letting $k \to \infty$ in (55), we have

$$V(x(t)) \leq V(x_0)e^{-c_1(t-t_0)}e^{c_2\int_{t_0}^{t}|\xi(s)|ds}. \tag{56}$$

1) Attraction: For each $\varepsilon > 0$ and $\delta \in (0, 3K)$, defining

$$A = \left\{\left|\frac{1}{t-t_0}\int_{t_0}^{t}|\xi(s)|^2 ds - E|\xi(t)|^2\right| \leq \delta\right\}$$

from (51), there exists a $T > 0$ such that for all $t \geq T$

$$P\{A\} \geq 1 - \varepsilon. \tag{57}$$

Combining the definition A with $\sup_{t\geq t_0} E|\xi(t)|^2 < K$ gives

$$\int_{t_0}^{t}|\xi(s)|^2 ds \leq t - t_0)(E|\xi(t)|^2 + \delta) \leq 4K(t-t_0), \quad \omega \in A, \quad t \geq T \tag{58}$$

then, we have

$$\int_{t_0}^{t}|\xi(s)|ds \leq c\sqrt{t-t_0}\left(\int_{t_0}^{t}|\xi(s)|^2 ds\right)^{\frac{1}{2}} \leq 2\sqrt{K}(t-t_0), \omega \in A, t \geq T. \tag{59}$$

Substituting (59) into (56) gives

$$V(x(t)) \leq V(x_0)e^{-(c_1-2c_2\sqrt{K})(t-t_0)}, \forall \omega \in A, t \geq T \tag{60}$$

which, along with (52) and (57), results in

$$P\left\{|x(t)| \leq \underline{\alpha}^{-1}(\bar{\alpha}(|x_0|)e^{-(c_1-2c_2\sqrt{K})(t-t_0)}\right\} \geq 1 - \varepsilon, \quad \forall t \geq T. \tag{61}$$

2) Stability: From $\sup_{t\geq t_0} E|\xi(t)|^2 < K$, for the given ε, there exists a $\delta_0 > 0$ such that

$$P\{|\xi(t)| > \delta_0\} \leq \frac{E|\xi(t)|^2}{\delta_0^2} < \frac{K}{\delta_0^2} = \varepsilon, \qquad \forall t \geq t_0$$

which, together with (56), leads to

$$P\{V(x(t)) \leq V(x_0)M\} \geq 1 - \varepsilon, \quad t \leq T \tag{62}$$

where $M = e^{(c_2\delta_0 - c_1)(T-t_0)}$. From (52) and (62), it is obtained that

$$P\left\{|x(t)| \leq \underline{\alpha}^{-1}(M\bar{\alpha}(|x_0|))\right\} \geq 1 - \varepsilon, \forall t \leq T. \tag{63}$$

Combining (61) with (63) gives

$$P\{|x(t)| \leq \alpha(|x_0|)\} \geq 1 - \varepsilon, \forall t \geq t_0. \tag{64}$$

where $\alpha(|x_0|) = \underline{\alpha}^{-1}(M\bar{\alpha}(|x_0|)) + \underline{\alpha}^{-1}(\bar{\alpha}(|x_0|))\bar{M}$ and $\bar{M} = e^{-(c_1-2c_2\sqrt{K})(T-t_0)}$.

3) Asymptotic Stability: By combining (61) with (64), it can be learned that for each $\varepsilon > 0$ there exists a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that

$$P\{|x(t)| \leq \beta(|x_0|, t-t_0)\} \geq 1 - \varepsilon, \tag{65}$$

thus we complete the proof. ■

Before further arguments, we present an estimation of stochastic processes.

A3: For a stationary process $\xi(t)$, there exists a function $\delta(\cdot)$ such that, for any given $\varepsilon > 0$, $t_1 \geq t_0$, we have

$$Ee^{\varepsilon\int_{t_0}^{t_1}|\xi(s)|ds} \leq e^{\delta(\varepsilon)(t_1-t_0)} \tag{66}$$

which will play a major role in the theory of asymptotical stability of random systems.

Theorem 6: For system (22) under assumptions **A1**$''$, **A3**, **P1**$'$ and **P2**, assume that there exist a function $V \in C^1$ and constants $a_1, a_2, c_1, c_2 > 0$ such that

$$a_1|x|^m \leq V(x) \leq a_2|x|^m \tag{67}$$

$$\frac{\partial V}{\partial x} f(x,t) \leq -c_1 V(x), \left|\frac{\partial V}{\partial x} g(x,t)\right| \leq c_2 V(x). \tag{68}$$

If

$$c_1 > \delta(c_2), \tag{69}$$

then there exists a unique global solution to system (22), and its equilibrium is ES-m-M.

Proof: From (68), the derivative of V along system (22) satisfies that, for all $k > 0$ and $t_0 \leq t < \sigma_k$

$$\dot{V}(x(t)) \leq \frac{\partial V}{\partial x} f(x(t),t) + \left(\frac{\partial V}{\partial x} g(x(x),t)\right)\xi(t) \leq (-c_1 + c_2|\xi(t)|)V(x(t)), \quad \text{a.s.} \tag{70}$$

Following the same line as the proof from (54) to (55), the existence-and-uniqueness of solution to system (22) on $[t_0, \infty)$ can be verified. According to Lemma 2, it comes from (67) and (70) that

$$a_1|x(t\wedge\sigma_k)|^m \leq V(x(t\wedge\sigma_k)) \leq V(x_0)e^{\int_{t_0}^{t}(-c_1+c_2|\xi(s)|)ds} \leq a_2|x_0|^m e^{-c_1(t-t_0)}e^{c_2\int_{t_0}^{t}|\xi(s)|ds} \tag{71}$$

then, by letting $k \to \infty$, we have

$$E|x(t)|^m \leq \frac{a_2}{a_1}|x_0|^m e^{-c_1(t-t_0)}Ee^{c_2\int_{t_0}^{t}|\xi(s)|ds}. \tag{72}$$

According to (66), we have

$$Ee^{c_2\int_{t_0}^{t}|\xi(s)|ds} \leq e^{\delta(c_2)(t-t_0)}$$

which, combining with (72), gives

$$E|x(t)|^m \leq \frac{a_2}{a_1}|x_0|^m e^{-(c_1-\delta(c_2))(t-t_0)}. \tag{73}$$

By noting condition (69), one completes the proof. ■

Remark 4: In [2], the asymptotic stability of system (22) was analyzed by using a Lyapunov function $V(x,t)$ satisfying global Lipschitz condition: there exists a constant

$$B = \sup_{t>0, x_i \in \mathbb{R}^n, x_1 \neq x_2} \frac{|V(x_2,t) - V(x_1,t)|}{|x_2 - x_1|}.$$

Other conditions such as $\sup_{x,t}\{g(x,t)\} < c_2 V$ or $V(x,t) > c_3|x|$ were also used in different situations. Comparing Theorems 1.5.1 and 1.5.2 in [2] with Theorems 5 and 6, these conditions are all removed or replaced by milder ones in this paper, which is popular to the the practical applications. ■

Remark 5: In [16], stability and asymptotic stability were proved by using Lyapunov function satisfying $V(x,t) \geq a|x|$ for a constant $a > 0$. The main stability conditions with respect to (22) are given in terms of a Lyapunov function $V(x,t) \geq 0$ such that $E\dot{V}(x,t) < 0$. However, in order to calculate the expectation $E\dot{V}(x,t)$ one must solve the system (22) with a suitable initial condition, and this limits the practical use of the criterion. ■

Remark 6: For different purposes, assumptions $\mathbf{P1}$, $\mathbf{P1}'$, $\mathbf{P2}$ and $\mathbf{P2}'$ are proposed to functions f and g of system, and assumptions $\mathbf{A1}$, $\mathbf{A1}'$, $\mathbf{A1}''$, $\mathbf{A2}$ and $\mathbf{A3}$ are imposed to stochastic process $\xi(t)$. It should be noted that some relations among these conditions exist:

$$\mathbf{A1}'' \Rightarrow \mathbf{A1}' \Rightarrow \mathbf{A1}, \mathbf{P1}' \Rightarrow \mathbf{P1}, \mathbf{P2}' \Rightarrow \mathbf{P2},$$

which can be easily verified. ■

VI. Estimations and Simulations of Some Stochastic Processes

For system (3), the existence-and-uniqueness of solution is analyzed under the assumption $\mathbf{A1}$: $\sup_{t\geq t_0} E|\xi(t)|^2 < \infty$. For system (22), conclusions about noise-to-state stability are obtained under the assumption $\mathbf{A1}'$: $E|\xi(t)|^2 < d_0 e^{c_0 t}$. The asymptotic stabilities require more specific statistic properties such as $\mathbf{A1}''$, $\mathbf{A2}$ and $\mathbf{A3}$. The reasonability of these requirements in previous sections on the stochastic processes $\xi(t)$ will be investigated in this section.

A. Second-Order Process

Suppose that $\xi(t)(t \in [t_0,\infty))$ is a stochastic process with state space in R^l, defined on a probability space $(\Omega, \mathcal{F}, P)$. The process $\xi(t)$ is a second-order process if $E|\xi(t)|^2 < \infty$ for every $t \in [t_0,\infty)$. From basic properties of higher moments, this condition implies that $E|\xi(t)| < \infty$ and also that $E(|\xi^T(s)\xi(t)|) < \infty$ for every $s,t \in [t_0,\infty)$.

For a second-order process $\xi(t)$, define expectation, variance and covariance

$$E\xi(t) = m(t)$$
$$E|\xi(t) - m(t)|^2 = D(t)$$
$$E(\xi(t) - m(t))^T(\xi(s) - m(s)) = H(s,t)$$

respectively. By the second-order assumption in the last paragraph, these are well defined. In general of course, these functions do not determine the finite dimensional distributions of $\xi(t)$, but are nonetheless important. In some applications, the mean and covariance functions may be known, at least approximately, when the finite dimensional distributions are not.

It is obvious that a stochastic process $\xi(t)$ is second-order if and only if it satisfies assumption $\mathbf{A1}$.

B. Widely Periodic Process

A second-order process $\xi(t)$ is said to be widely periodic with period θ if there exists a constant $\theta > 0$ such that its moments satisfy

$$m(t+\theta) = m(t), \quad D(t+\theta) = D(t)$$

and

$$H(s+\theta, t+\theta) = H(s,t), \qquad \forall\, s,t \geq t_0.$$

Assume $\xi(t)$ is piecewise continuous, which implies the same property of $m(t)$ and $D(t)$, thus $E|\xi(t)|^2 = D(t) + |m(t)|^2$ is bounded in $[t_0, t_0+\theta]$, so it is bounded in $[t_0,\infty)$. This leads to a claim that for a widely periodic process $\xi(t)$ which is piecewise continuous there exists a constant $K > 0$ such that $\sup_{t\geq t_0} E|\xi(t)|^2 < K$, that is, assumption $\mathbf{A1}''$ holds.

C. Widely Stationary Process

One popular kind of second-order processes in practice is the widely stationary one, i.e., a second-order process $\xi(t)$ whose moments satisfy

$$m(t) = m, D(t) = D, H(s,t) = H(t-s), \forall s,t \geq t_0.$$

Since $E|\xi(t)|^2 = m^2 + D$, then $\sup_{t\geq t_0} E|\xi(t)|^2 \leq m^2 + D$.

It is obvious that a widely stationary process is widely periodic with arbitrary period and thus satisfies assumption $\mathbf{A1}''$.

D. Strictly Stationary Process

We say a process $\xi(t)$ defined on $(\Omega, \mathcal{F}, P)$ is strictly stationary, if for any $t_1, \cdots, t_n, t \geq t_0$, the distribution of $\xi(t_1 + t), \cdots, \xi(t_n + t)$ is independent of t.

Since a strictly stationary process isn't necessarily a second-order process, then, strictly stationary process and widely stationary one can't cover each other.

For a second-order process $\xi(t)$, if it is strictly stationary, then it must be widely stationary, and thus it satisfies assumption $\mathbf{A1}''$.

As a counterexample, white noise is strictly stationary, but it is not a second-order process. It does not satisfies assumptions $\mathbf{A1}$, $\mathbf{A1}'$ and $\mathbf{A1}''$.

E. Mean-Ergodicity and Variance-Ergodicity of Widely Stationary Process

Given a widely stationary process $\xi(t)$, let us estimate its mean $m = E\xi(t)$. For this end, we define its mean in time

$$m_T = \frac{1}{T - t_0} \int_{t_0}^{T} \xi(t) dt.$$

Clearly, m_T is a random variable with the mean m. If its variance $E|m_T - m|^2 \to 0$ as $T \to \infty$, then we say that the process $\xi(t)$ is mean-ergodic. According to [25], $\xi(t)$ is mean-ergodic if and only if $\lim_{T\to\infty}(1/T - t_0)\int_{t_0}^{T} H(s)ds = 0$.

Turn to estimate $D = E|\xi(t) - m|^2$. Under assumption of $m = 0$, to estimate the mean of $|\xi(t)|^2$, let us introduce

$$D_T = \frac{1}{T - t_0} \int_{t_0}^{T} |\xi(t)|^2 dt.$$

If its variance $E|D_T - D|^2 \to 0$ as $T \to \infty$, then we say that the process $\xi(t)$ is variance-ergodic, i.e., $|\xi(t)|^2$ obeys the mean-square law of large number. According to [25], $\xi(t)$ is variance-ergodic if and only if $\lim_{T\to\infty}(1/T - t_0)\int_{t_0}^{T} E \times (|\xi(t+s)|^2|\xi(t)|^2)ds = D^2$.

In this case, $|\xi(t)|^2$ also obeys the weak law of large number (51) and thus assumption **A2** is satisfied.

F. Ergodicity of Strictly Stationary Process

For a strictly stationary process $\xi(t)$. Define $\mathcal{F}_{\xi_t}$ as the σ-field generated by $\xi(t)$. A strictly stationary process $\xi(t)$ is called ergodic if for every set $A \in \mathcal{F}_{\xi_t}$ there holds $P(A) = 0$ or 1. According to [26], for an ergodic process $\xi(t)$, if there exists a measurable function f such that $E|f(\xi(t))| < \infty$ for any $t > t_0$, we have

$$\lim_{t\to\infty} \frac{1}{t - t_0} \int_{t_0}^{t} f(\xi(s))ds = Ef(\xi(t)), \quad \text{a.s.}$$

i.e., $f(\xi(t))$ satisfies the strong law of large number. This implies that $f(\xi(t))$ also satisfies the weak law of large number, then, letting $f(\xi(t)) = |\xi(t)|^2$, assumption **A2** is satisfied.

G. Stationary Gaussian Process

Suppose that $\xi(t) \in R^l(t \in [t_0, \infty))$ is a stochastic process. Then $\xi(t)$ is said to be a Gaussian process if all of the finite dimensional distributions are normal.

According to (1.7.15) in [2], if a Gaussian process $\xi(t)$ satisfies

$$E|\xi(t)|^2 \le c_1, \int_{t_0}^{\infty} \|H(s,t)\|dt \le c_2$$

for some $c_i > 0$ and all $t \ge t_0$, then the following estimate holds for all $\varepsilon > 0$ and $t_1 > t_0$

$$Ee^{\varepsilon\int_{t_0}^{t_1}|\xi(s)|ds} \le Ee^{\varepsilon\int_{t_0}^{t_1}(|E\xi(s)|+|\xi(s)-E\xi(s)|)ds} \le e^{\varepsilon\sqrt{c_1}(t_1-t_0)}e^{\varepsilon(\sqrt{c_1}+\frac{\varepsilon c_2}{2})(t_1-t_0)} = e^{\varepsilon(2\sqrt{c_1}+\frac{\varepsilon c_2}{2})(t_1-t_0)}$$

thus assumption **A3** is satisfied.

H. Producing a Stationary Process Using Matlab

It is interesting to invesqtigate how to produce stationary processes by using Matlab.

Consider a stochastic process $\xi(t) \in \mathbb{R}, t \in T$, defined by

$$\xi(t) = a\cos(\lambda t + U) \tag{74}$$

where a and λ are real constants and U is a random variable uniformly distributed on the interval $[0, 2\pi]$. It is not only a strictly stationary process but also a widely one, and $E\xi^2(t) < a^2$. Uniform distribution can be produced by Matlab software.

Consider another stochastic process $\xi(t) \in \mathbb{R}$ produced By

$$q\dot{\xi}(t) = -\xi(t) + w(t), \xi(0) = 0, \tag{75}$$

where $q > 0$ is a parameter and $w(t) \in \mathbb{R}$ is a zero-mean white noise whose spectral function equals a constant $A > 0$. According to P.135 of [27], $\xi(t)$ is a zero-mean widely stationary process with mean-square value $A/2q$.

Consider (75) where $w(t) \in \mathbb{R}$ is a zero-mean bandlimited white noise whose power spectrum is

$$F_w(j\lambda) = \begin{cases} A, & |\lambda| \le \lambda_c \\ 0, & \text{otherwise} \end{cases}$$

where $A > 0$ is noise power and $\lambda_c > 0$ is bandwidth. According to P.135 of [27], $\xi(t)$ is a zero-mean widely stationary process whose power spectrum and variance equal to

$$F_\xi(j\lambda) = \begin{cases} \frac{A}{1+q^2\lambda^2}, & |\lambda| \le \lambda_c \\ 0, & \text{otherwise} \end{cases}$$

$$E|\xi|^2 = \frac{1}{2\pi}\int_{-\lambda_c}^{\lambda_c} \frac{A}{1+q^2\lambda^2}d\lambda = \frac{A}{\pi q}\tan^{-1}(\lambda_c q)$$

respectively. It is the bandlimited white noise instead of pure white noise that is physically plausible and can be generated by a block in Simulink software. In bandlimited-white-noise block, there are two dialog box parameters: noise power A and correlation time $t_c \approx (1/100)(2\pi/\lambda_c)$.

VII. Some Applications in Control Problems

The reasonability of Lyapunov criteria can be verified by applying them in controller design.

Consider a random nonlinear control system

$$\begin{aligned} \dot{x} &= f(x,u,t) + g(x,t)\xi(t), x(t_0) = x_0 \\ y &= h(x) \end{aligned} \tag{76}$$

where $\xi(t)$ satisfies assumption $\mathbf{A1}''$, i.e., $\sup_{t\ge t_0} E|\xi(t)|^2 \le K$.

Many control objectives can be achieved by designing controllers in the form

$$u = u(x,z,t), \dot{z} = \chi(x,z)$$

where z is a dynamic.

Functions $f(x,u(x,z,t),t)$, $g(x,u(x,z,t),t)$, $h(x)$ and $\chi(x,z)$ satisfy locally Lipschitz condition and local boundedness.

Firstly, for regulation, we give the following example.

Example 1: Consider how to design a controller to regulate the state of system

$$\dot{x} = u + \xi(t), x(0) = x_0, \tag{77}$$

where $x \in \mathbb{R}$ and stochastic process $\xi \in \mathbb{R}$ satisfies $\sup_{t\ge t_0} E|\xi(t)|^2 < K$ (constant), to a set-point x_s.

Select

$$u = -c(x - x_s),$$

where $c > 0$ is a design parameter, and choose a Lyapunov function

$$V = \frac{1}{2}(x - x_s)^2$$

then, we have

$$\dot{V} \le (x - x_s)(-c(x - x_s) + \xi(t)) \le -cV + \frac{|\xi(t)|^2}{2c}$$

which, according to Lemma 2, means that

$$E|x - x_s|^2 \le |x_0 - x_s|^2 e^{-c(t-t_0)} + \frac{K}{c^2}.$$

Therefore, according to Theorem 3, system (77) is NSS-2-M, and its AG-2-M is $1/c^2$, which can be made arbitrarily small by tuning design parameter c large enough. ■

Secondly, pay attention to the tracking of random nonlinear system.

Example 2: Consider a strict-feedback system with stochastic disturbance as follows:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + x_2 + h_1(x_1)\xi_1 \\ \dot{x}_2 &= f_2(x_1, x_2) + u + h_2(x_1, x_2)\xi_2 \\ y &= x_1 \end{aligned} \tag{78}$$

where $x = (x_1, x_2)^T \in \mathbb{R}^2$ is state, $u \in \mathbb{R}$ is control, $y \in \mathbb{R}$ is the output, functions f_i, g_i, h_i are locally lipschitz, and the adapted stochastic process $\xi = (\xi_1, \xi_2)^T \in \mathbb{R}^2$ is piecewise continuous and satisfies $\sup_{t \ge t_0} E|\xi(t)|^2 < K$.

The control object is to design a smooth controller $u = u(x)$ such that the output of the closed-loop system can track a known reference y_r as close as possible, where $y_r, \dot{y}_r$ and $\ddot{y}_r$ are bounded.

The integral backstepping controller can be designed step by step.

Step 1: Introduce a transform

$$z_1 = x_1 - y_r \quad z_2 = x_2 - \dot{y}_r - \alpha_1$$

where α_1 is a function to be designed, and choose a Lyapunov-like function

$$V_1 = \frac{1}{2} z_1^2.$$

The derivative of V_1 satisfies

$$\begin{aligned} \dot{V}_1 &= z_1(f_1(x_1) + x_2 + h_1(x_1)\xi_1 - \dot{y}_r) \\ &\le z_1\left(f_1(x_1) + \frac{d_1}{4} z_1 h_1^2(x_1) + \alpha_1\right) + z_1 z_2 + \frac{1}{d_1}\xi_1^2 \end{aligned} \tag{79}$$

where Young's inequality is used. By selecting

$$\alpha_1 = -c_1 z_1 - f_1(x_1) - \frac{d_1}{4} z_1 h_1^2(x_1)$$

it follows from (79) that

$$\dot{V}_1 \le z_1 z_2 - c_1 z_1^2 + \frac{1}{d_1}\xi_1^2. \tag{80}$$

Step 2: The second Lyapunov-like function is

$$V_2 = V_1 + \frac{1}{2} z_2^2.$$

The derivative of V_2 satisfies

$$\begin{aligned} \dot{V}_2 = \dot{V}_1 &+ z_2(f_2(x) + u + h_2(x)\xi_2 - \ddot{y}_r \\ &- \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r - \frac{\partial \alpha_1}{\partial x_1}(f_1(x_1) + x_2) - \frac{\partial \alpha_1}{\partial x_1} h_1(x_1)\xi_1). \end{aligned} \tag{81}$$

According to Young's inequality, one has

$$\begin{aligned} z_2 h_2(x)\xi_2 &\le \frac{d_2}{4} h_2^2(x) z_2^2 + \frac{1}{d_2}\xi_2^2 \\ z_2 \frac{\partial \alpha_1}{\partial x_1} h_1(x_1)\xi_1 &\le \frac{d_3}{4}\left[\frac{\partial \alpha_1}{\partial x_1} h_1(x_1)\right]^2 z_2^2 + \frac{1}{d_3}\xi_1^2. \end{aligned} \tag{82}$$

Substituting (82) into (81) results in

$$\begin{aligned} \dot{V}_2 \le \dot{V}_1 &+ z_2(f_2(x) + \frac{d_2}{4} z_2 h_2^2(x) - \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r \\ &- \frac{\partial \alpha_1}{\partial x_1}\left(f_1(x_1) + x_2) + u - \ddot{y}_r + \frac{d_3}{4}\left[\frac{\partial \alpha_1}{\partial x_1} h_1(x_1)\right]^2 z_2\right) \\ &+ \frac{1}{d_2}\xi_2^2 + \frac{1}{d_3}\xi_1^2 \end{aligned} \tag{83}$$

which, together with (80), leads to

$$\begin{aligned} \dot{V}_2 \le z_2(f_2(x) &+ \frac{d_2}{4} z_2 h_2^2(x) + z_1 - \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r \\ &- \frac{\partial \alpha_1}{\partial x_1}\left(f_1(x_1) + x_2) + u - \ddot{y}_r + \frac{d_3}{4}\left[\frac{\partial \alpha_1}{\partial x_1} h_1(x_1)\right]^2 z_2\right) \\ &- c_1 z_1^2 + \frac{1}{d_1}\xi_1^2 + \frac{1}{d_3}\xi_1^2 + \frac{1}{d_2}\xi_2^2. \end{aligned} \tag{84}$$

Choose

$$\begin{aligned} u = -c_2 z_2 - f_2(x) - z_1 &+ \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r - \frac{d_2}{4} z_2 h_2^2(x) \\ &+ \frac{\partial \alpha_1}{\partial x_1}(f_1(x_1) + x_2) + \ddot{y}_r - \frac{d_3}{4}\left[\frac{\partial \alpha_1}{\partial x_1} h_1(x_1)\right]^2 z_2 \end{aligned} \tag{85}$$

such that

$$\dot{V}_2 \le -c_1 z_1^2 - c_2 z_2^2 + \left(\frac{1}{d_1} + \frac{1}{d_3}\right)\xi_1^2 + \frac{1}{d_2}\xi_2^2$$

which satisfies

$$\dot{V} \le -cV + \frac{1}{d}|\xi|^2 \tag{86}$$

where $c = 2\min\{c_1, c_2\}$ and $1/d = \max\{(1/d_1) + (1/d_3), (1/d_2)\}$. From (86) and $\sup_{t \ge t_0} E|\xi(t)|^2 < K$, the tracking error satisfies

$$E|y - y_r|^2 \le 2EV(t) \le 2V(x_0)e^{-c(t-t_0)} + \frac{2K}{cd} \tag{87}$$

whose proof can be refereed to (31).

According to Theorem 4, from (86), the closed-loop system is NSS-P, and the state of the closed-loop system has an AG-P, therefore, all the signals in the closed-loop system are bounded in probability. From (86), the tracking error can be made arbitrarily small by tuning $d_i (i = 1, 2, 3)$ large enough.

To perform a simulation to system (78), the reference signal $y_r = \sin t$, functions $f_i, h_i (i = 1, 2)$ are chosen as

$$f_1(x_1) = x_1^2 + 1, h_1(x_1) = -x_1, f_2 = x_1^2 - x_2^2, h_2 = x_1 x_2$$

the initial values $x_1(0) = 0.4$, $x_2(0) = -1$, the design parameters $c_1 = c_2 = 1$, $d_1 = 50$, $d_2 = d_3 = 2$, and the disturbances ξ_1 and ξ_2 are all produced by (75) where coefficient $q = 1$, noise power $A = 0.1$ and sample time $t_c = 0.1$. From Fig. 1, it can be learned that the tracking error is very small and all the other signals including x_2 and control u are bounded,

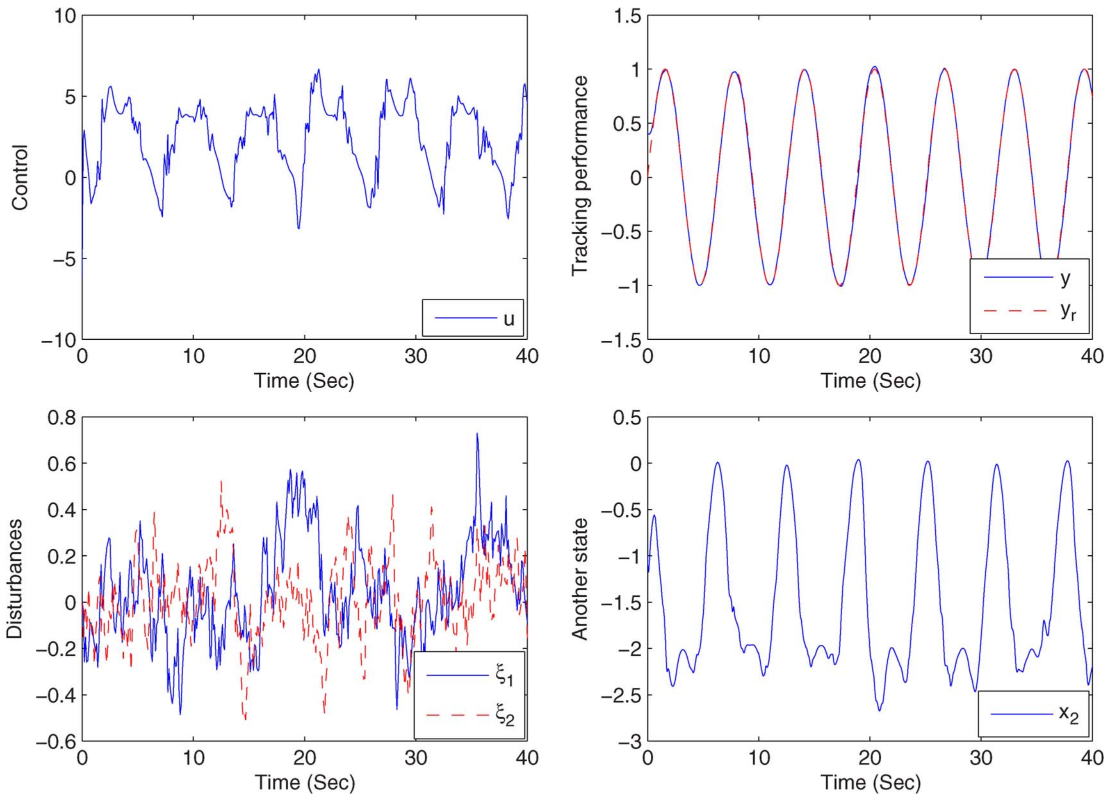

Fig. 1. Responses of closed-loop system.

which demonstrates the efficiency of the controller and the reasonability of stability analysis. ■

Finally, an example of asymptotic stabilization is considered.

Example 3: A linear random system is given as follows:

$$\dot{x} = Ax + Bu + Cx\xi, x(0) = x_0 \quad (88)$$

where $x \in \mathbb{R}^n$ and $\xi(t) \in \mathbb{R}$ satisfies $\sup_{t\geq t_0} E|\xi(t)|^2 < K$ (constant) and $|\xi(t)|^2$ satisfies the weak law of large numbers.

Choose a Lyapunov function

$$V = x^T Px.$$

Consider a controller $u = -cx$.

If there exists a parameter c such that the following LMI has a positive definite solution:

$$\begin{aligned} P\bar{A} + \bar{A}^T P + c_1 P < 0 \\ PC + C^T P - c_2 P < 0 \end{aligned} \quad (89)$$

where $\bar{A} = A - Bc$ and $c_1 > 2c_2\sqrt{K}$, that is

$$\frac{\partial V}{\partial x} f(x,t) \leq -c_1 V(x(t)), \left|\frac{\partial V}{\partial x} g(x,t)\right| \leq c_2 V(x(t)). \quad (90)$$

Therefore, $x = 0$ is GAS-P, according to Theorem 5. ■

VIII. Comparison With Stochastic Differential Equation

Great achievements in stability analysis have been obtained for SDE described by

$$dx = f(x,t,u)dt + g(x,t)dW_t \quad (91)$$

where W_t is a Wiener process.

In the deterministic case, many results of asymptotic stabilization can also be used in asymptotic tracking problems by performing a coordination transformation. It is not the same case for stochastic control systems (91). About the tracking problems, no asymptotic stability results can be found in the existing references.

Example 4: By taking $u = -cx$ with a parameter $c > 1$, the System

$$dx = udt + xdW_t \quad (92)$$

can be stabilized to $x \equiv 0$. In fact, by using Lyapunov function $V = (1/2)x^2$, one can obtain $\mathcal{L}V \leq -(2c-1)V$, which implies the stability of equilibrium. ■

Example 5: For system (92), we can not find a controller to asymptotically regulate the state to a set-point $x = x_e \neq 0$ in probability.

By selecting $u = -c(x - x_e)$ and Lyapunov function $V = (1/4)(x - x_e)^4$ such that

$$\begin{aligned} \mathcal{L}V &= -c(x-x_e)^4 + \frac{3}{2}x^2(x-x_e)^2 \\ &\leq -c(x-x_e)^4 + 3(x-x_e)^4 + 3(x-x_e)^2 x_e^2 \\ &\leq -c(x-x_e)^4 + 3(x-x_e)^4 + \frac{9}{4\varepsilon}(x-x_e)^4 + \varepsilon x_e^2 \\ &\leq -(c - 3 - \frac{9}{4\varepsilon})(x-x_e)^4 + \varepsilon x_e^2 \end{aligned}$$

where $\varepsilon > 0$ is any parameter, which leads to

$$E|x - x_e|^4 \leq |x_0|^4 e^{-(c-3-\frac{9}{4\varepsilon})(t-t_0)} + \frac{4\varepsilon x_e^2}{c - 3 - \frac{9}{4\varepsilon}}.$$

Thus c should be chosen such that $c > 3 + (9/4\varepsilon)$. The regulation error of this example depends on the set-point x_e, which implies that the regulation task becomes more difficult with x_e being farther away from the origin. For system (77), the regulation error is independent of x_e. ■

Example 6: For system (92), we cannot find a controller to let the state track asymptotically a reference signal x_r in probability. Assume that $|x_r| \leq d_r, |\dot{x}_r| \leq \bar{d}_r$.

By selecting $u = -c(x - x_r) + \dot{x}_r$ and Lyapunov function $V = (1/4)(x - x_r)^4$ such that

$$\begin{aligned}\mathcal{L}V &= (x - x_r)^3 u - (x - x_r)^3 \dot{x}_r + \frac{3}{2}x^2(x - x_r)^2 \\ &= -c(x - x_r)^4 + \frac{3}{2}x^2(x - x_r)^2 \\ &\leq -(c - 3 - \frac{9}{4\varepsilon})(x - x_r)^4 + \varepsilon x_r^2\end{aligned}$$

which leads to

$$E|x - x_r|^4 \leq |x_0|^4 e^{-(c-3-\frac{9}{4\varepsilon})(t-t_0)} + \frac{4\varepsilon d_r^2}{c - 3 - \frac{9}{4\varepsilon}}.$$

Similarly, the tracking error depends on the bound of x_r. For system (78), the tracking error is independent of reference signal. ■

Remark 7: RDE model (see (3)) has no so significant impacts on the physics, mathematics and social science as SDE model (see (91)), but has at least the following advantages.

1) The error of tracking (or regelation) does not depend on the reference signal, which is more reasonable in practice.
2) Many analysis tools proposed for ordinary differential equation can be applied in RDE models. ■

IX. Conclusion

The existing results of Lyapunov stability for RDEs were given under some strict assumptions (see, [2], [16], [17]). In this paper, we try to replace these requirements with some milder ones, and develop a theoretical framework of Lyapunov stability, parallelling to that of SDEs. One outstanding advantage of controllers designed for random nonlinear systems is that the tracking errors (or regulation errors) are independent of the magnitude of the reference signals. So far, the framework is uncompleted, thus it needs develop in many directions such as optimal control, observer-based control, H_2/H_∞ control, inverse Lyapunov function design, dissipativity, and controller design for mechanical and power systems.

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Zhaojing Wu (M'09) was born in Qufu, Shandong, China, in 1970. He received the M.S. and Ph.D. degrees from Qufu Normal University and Northeastern University, Shenyang, China, in 2003 and 2005, respectively.

He is currently a Professor in the School of Mathematics and Information Science, Yantai University, Shandong, China. His research interests include nonlinear control, adaptive control, stochastic stability analysis, stochastic dissipative systems, stochastic Hamiltonian systems, and stochastic switched systems.

Dr. Wu was the outstanding reviewer for IEEE Transactions on Automatic Control in 2012.