

Reconstructing the state of a Boolean control network via state feedback

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Abstract—The goal of this paper is to investigate under what conditions a Boolean Control Network admits a state feedback control law that makes the resulting Boolean Network reconstructable. Starting from an algebraic representation of the Boolean Control Network, we first propose a result that allows to significantly reduce the problem size, and hence to mitigate the curse of dimensionality that typically arises when dealing with logical systems of very large size. Subsequently, we provide a necessary and sufficient condition for the problem solvability that relies on the algebra of noncommutative polynomials. Finally, when such a condition holds, we present a procedure to design a possible state feedback controller that achieves the desired result.

I. INTRODUCTION

The last 15 years have witnessed a surge of interest in Boolean Control Networks (BCNs), motivated by the large number of application areas where such networks represent effective modelling tools. This is the case when it is important to highlight the logical relationships among the describing variables, rather than predicting their numerical values. In addition, functional modelling can be a preliminary modelling phase, or the only meaningful description one may search for, based on the limited information available. BCNs have been successfully applied to different fields, such as biology [25], [28], smart homes [15], multi-agent systems and consensus problems [13], [21], or game theory [4], [26]. However, the application area where they proved to be more effective is represented by gene regulatory networks [16], [24]. Genes behave as binary devices whose status can be active or inactive, and their time evolutions are mutually related by logical relationships, that formalise activation/inhibition processes.

The algebraic approach to Boolean Control Networks proposed by D. Cheng and co-authors [2], [5], [6], [7] has offered a very successful tool to formalise and solve control problems in the context of logical networks. Indeed, the semitensor product of matrices allows to represent BCNs as state space models whose describing variables are canonical vectors. In this way, classical concepts and methods developed for linear time invariant state-space models have been tailored to BCNs, thus leading to matrix based characterizations for a number of properties and control problems, such as stability, stabilizability, controllability [8], [10], [18], [19], disturbance decoupling [3], [23], observability and reconstructability [9], [27], [29], fault detection [12], [31], and optimal control [11], [17], [32].

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As in classical Systems Theory, reconstructability is the property of a system to reveal its current status provided that the input acting on the system (if any) and the corresponding output evolution have been measured over a sufficiently large time window. This property, which is a weaker requirement with respect to observability, often represents a mandatory feature for a logical network and provides a clear indication of how well the output measurements have been chosen. Indeed, for the correct functioning of a BCN (of a BN) it is of fundamental importance to be able to deduce the internal logical status from the input and output (the output) measurements. This is quite intuitive if one thinks of fault detection problems, since output measurements are typically used to detect and identify system failures. Indeed, in [12] the connection between reconstructability and the solvability of fault detection problems has been explored. Furthermore, in biological and medical applications, being able to reconstruct the internal state of the system from its external behaviour means to be able to perform a correct diagnosis by making use of the testing input and the corresponding output measurements. In particular, when a BN models a gene regulatory network, the possibility of reconstructing the state of the BN from the output measurements means to be able to understand if the gene regulatory network is in a healthy situation or not. For instance, detecting the level of an oxidative stress response from output measurements allows to prevent major side effects as cancer, cardiovascular disease, chronic inflammation, and neurodegenerative disorders [25]. The goal of this paper is to investigate under what conditions, given a BCN, it is possible to design a state feedback control law such that the resulting BN is reconstructable. To this end, we first propose, in Section III, a result that allows to significantly reduce the problem size, and hence to mitigate the curse of dimensionality that typically arises when dealing with logical systems of very large size. Subsequently, in Section IV, we provide a necessary and sufficient condition for the problem solvability that relies on the algebra of noncommutative polynomials. Finally, in Section V, when such a condition holds, we present a procedure to design a state feedback controller that achieves the desired result.

Note that there is no contradiction in designing a state feedback law, which suggests that the state is accessible, with the goal of making the resulting system reconstructable (which allows to rely on the output measurements to deduce its state). First of all, the system state may be accessible during the design phase, but it may be advisable that the end-users have no access to it, and yet they can deduce its evolution from external measurements. This is the case for several electronic devices and in particular biomedical

wearable devices. Secondly, as clearly discussed in the sequel, when dealing with BCNs a state feedback law can be implemented by simply designing offline the controlled BN, without resorting to real time measurements of the state variables at each time instant. Moreover, the proposed feedback control law does not aim at reducing the degrees of freedom of the BCN, by converting it into a BN, in order to guarantee reconstructability. Rather, the idea is to apply the feedback law on a finite time interval to reconstruct the system state (a sort of “preliminary off-line procedure”) and then leave the BCN evolve according to its standard operating conditions, meanwhile keeping track of the internal state evolution.

Notation. \mathbb{Z}_+ denotes the set of nonnegative integers. Given two integers $k, n \in \mathbb{Z}_+$, with $k \leq n$, we denote by $[k, n]$ the set of integers $\{k, k+1, \dots, n\}$. We consider Boolean vectors and matrices, taking values in $\mathcal{B} := \{0, 1\}$, with the usual Boolean operations.

δ_k^i denotes the i th *canonical vector* of size k (namely the i th column of the identity matrix I_k), \mathcal{L}_k the set of all k -dimensional canonical vectors, and $\mathcal{L}_{k \times n} \subset \mathcal{B}^{k \times n}$ the set of all $k \times n$ *logical matrices*, i.e., matrices whose columns are canonical vectors of size k . Any logical matrix $L \in \mathcal{L}_{k \times n}$ can be represented as $L = [\delta_k^{i_1} \ \delta_k^{i_2} \ \dots \ \delta_k^{i_n}]$, for suitable indices $i_1, i_2, \dots, i_n \in [1, k]$. The k -dimensional vector with all entries equal to 1 is denoted by $\mathbf{1}_k$. We denote by $\text{blkdg}\{F_1, F_2, \dots, F_k\}$ the square block diagonal matrix having the square matrices F_1, F_2, \dots, F_k as diagonal blocks.

Given a Boolean matrix $L \in \mathcal{B}^{k \times k}$ (in particular, a logic matrix $L \in \mathcal{L}_{k \times k}$), its (ℓ, j) th entry is denoted by $[L]_{\ell j}$, its i th column by $\text{col}_i(L)$, and its *nonzero pattern* by

$$\overline{ZP}(L) := \{(i, j) \in [1, k] \times [1, k] : [L]_{ij} = 1\}.$$

A Boolean matrix $L \in \mathcal{B}^{k \times k}$ is *irreducible* [14] if either $k = 1$ or $k > 1$ and no $k \times k$ permutation matrix P can be found such that

$$P^\top L P = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix},$$

where L_{11} and L_{22} are square matrices.

We associate [1] with the matrix L a *directed graph (digraph)* $\mathcal{D}(L) = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} = \{1, \dots, k\}$ the set of vertices and \mathcal{E} the set of arcs (edges). There is an arc (j, ℓ) from j to ℓ if and only if $(\ell, j) \in \overline{ZP}(L)$. A sequence $j_1 \rightarrow j_2 \rightarrow \dots \rightarrow j_r \rightarrow j_{r+1}$ in $\mathcal{D}(L)$ is a *path* of length r from j_1 to j_{r+1} provided that $(j_1, j_2), \dots, (j_r, j_{r+1})$ are arcs of $\mathcal{D}(L)$. A closed path is a *cycle*. In particular, a cycle γ with no repeated vertices is called *elementary*, and its length $|\gamma|$ coincides with the number of its (distinct) vertices.

Two distinct vertices h and k are said to *communicate* if there is a path from h to k and conversely. Each vertex is assumed to communicate with itself. The concept of communicating vertices allows to partition the set of vertices \mathcal{V} into communication classes. A *communication class* is a maximal set of vertices that communicate with each other. Communication classes with no outgoing arcs are called *final*. A digraph is *strongly connected* if it consists of a single

communication class. $\mathcal{D}(L)$ is strongly connected if and only if L is irreducible.

There is a bijective correspondence between Boolean variables $X \in \mathcal{B}$ and vectors $\mathbf{x} \in \mathcal{L}_2$, defined by the relationship $\mathbf{x} = [X \ \bar{X}]^\top$, where \bar{X} is the negation of X . We introduce the (*left*) *semi-tensor product* \ltimes between matrices (in particular, vectors) as follows [7], [18], [20]: given $L_1 \in \mathcal{B}^{r_1 \times c_1}$ and $L_2 \in \mathcal{B}^{r_2 \times c_2}$, we set

$$L_1 \ltimes L_2 := (L_1 \otimes I_{T/c_1})(L_2 \otimes I_{T/r_2}), \quad T := \text{l.c.m.}\{c_1, r_2\},$$

where l.c.m. denotes the least common multiple. The semi-tensor product represents an extension of the standard matrix product, by this meaning that if $c_1 = r_2$, then $L_1 \ltimes L_2 = L_1 L_2$. Note that if $\mathbf{x}_1 \in \mathcal{L}_{r_1}$ and $\mathbf{x}_2 \in \mathcal{L}_{r_2}$, then $\mathbf{x}_1 \ltimes \mathbf{x}_2 \in \mathcal{L}_{r_1 r_2}$. For the various properties of the semi-tensor product we refer to [7]. By resorting to the semi-tensor product, we can extend the previous correspondence to a bijective correspondence between \mathcal{B}^n and \mathcal{L}_{2^n} , by mapping the vector $X = [X_1 \ X_2 \ \dots \ X_n]^\top \in \mathcal{B}^n$ into

$$\mathbf{x} := \begin{bmatrix} X_1 \\ \bar{X}_1 \end{bmatrix} \ltimes \begin{bmatrix} X_2 \\ \bar{X}_2 \end{bmatrix} \ltimes \dots \ltimes \begin{bmatrix} X_n \\ \bar{X}_n \end{bmatrix}.$$

We let Φ denote the *power-reducing matrix*, i.e., the matrix of suitable size such that $\Phi \mathbf{x}(t) = \mathbf{x}(t) \ltimes \mathbf{x}(t)$.

Finally, we need some definitions borrowed from the algebra of noncommutative polynomials [22]. Given the *alphabet* $\Xi = \{\xi_1, \xi_2, \dots, \xi_P\}$, we denote by Ξ^* the set of all *words* $w = \xi_{i_k} \xi_{i_{k-1}} \dots \xi_{i_1}$, $k \in \mathbb{Z}_+$, $\xi_{i_h} \in \Xi$ (including the empty word $\varepsilon = \emptyset$). The integer k is called the *length* of w and is denoted by $|w|$. Note that $|\varepsilon| = 0$.

If $\tilde{w} = \xi_{j_m} \xi_{j_{m-1}} \dots \xi_{j_1}$ is another element of Ξ^* , the product $w\tilde{w}$ is defined by concatenation as $\xi_{i_k} \xi_{i_{k-1}} \dots \xi_{i_1} \xi_{j_m} \xi_{j_{m-1}} \dots \xi_{j_1}$. This induces in Ξ^* the monoid structure, with ε as unit element. Clearly, $|w\tilde{w}| = |w| + |\tilde{w}|$. $\mathcal{B}\langle \xi_1, \xi_2, \dots, \xi_P \rangle$ is the set of polynomials with Boolean coefficients, in the noncommutative variables $\xi_1, \xi_2, \dots, \xi_P$. Given a word $w \in \Xi^*$ of length k , say $w = \xi_{i_k} \xi_{i_{k-1}} \dots \xi_{i_1}$, the *cyclic permutation of one step* of w , σw , is $\sigma w = \xi_{i_1} \xi_{i_k} \xi_{i_{k-1}} \dots \xi_{i_2}$. The extension to the concept of *cyclic permutation of d steps*, with $d \in \mathbb{Z}_+$, is immediate.

II. PROBLEM STATEMENT AND PRELIMINARY ANALYSIS

By a *Boolean control network* we mean a state-space model described as follows

$$X(t+1) = f(X(t), U(t)), \quad t \in \mathbb{Z}_+, \quad (1)$$

$$Y(t) = h(X(t)), \quad (2)$$

where $X(t)$, $U(t)$ and $Y(t)$ are the n -dimensional state variable, the m -dimensional input and the p -dimensional output at time t , taking values in \mathcal{B}^n , \mathcal{B}^m , and \mathcal{B}^p , respectively. The logic function f maps pairs of Boolean vectors into a Boolean vector, that is $f : \mathcal{B}^n \times \mathcal{B}^m \rightarrow \mathcal{B}^n$. Similarly, h is a logic function from \mathcal{B}^n to \mathcal{B}^p . Once we represent state, input and output Boolean variables by means of canonical vectors belonging to \mathcal{L}_N , $N := 2^n$, \mathcal{L}_M , $M := 2^m$, and \mathcal{L}_P , $P := 2^p$, respectively, by making use of the semi-tensor

product \times we can represent the BCN (1)-(2) by means of its algebraic description [7]:

$$\mathbf{x}(t+1) = L \times \mathbf{u}(t) \times \mathbf{x}(t), \quad t \in \mathbb{Z}_+, \quad (3)$$

$$\mathbf{y}(t) = H\mathbf{x}(t), \quad (4)$$

where $\mathbf{x}(t) \in \mathcal{L}_N$, $\mathbf{u}(t) \in \mathcal{L}_M$ and $\mathbf{y}(t) \in \mathcal{L}_P$. The matrix $L \in \mathcal{L}_{N \times NM}$, whose columns are canonical vectors of size N , can be split into M square blocks of size N :

$$L = [L_1 \quad L_2 \quad \dots \quad L_M].$$

Each matrix $L_i \in \mathcal{L}_{N \times N}$ describes the behavior of the Boolean network (BN) (the i th subsystem of the BCN)

$$\mathbf{x}(t+1) = L_i \mathbf{x}(t), \quad t \in \mathbb{Z}_+, \quad (5)$$

one gets when $\mathbf{u}(t) = \delta_M^i$, for every $t \in \mathbb{Z}_+$. Note that everything we will say in the following applies to BCNs (3)-(4) for which N, M and P are arbitrary nonnegative integers and not necessarily powers of 2.

The problem we address in this paper is the following one:

Feedback Reconstructability (FR) Problem: *Given a BCN (3)-(4), determine (if possible) a state feedback law*

$$\mathbf{u}(t) = K\mathbf{x}(t), \quad (6)$$

such that the resulting BN:

$$\mathbf{x}(t+1) = \tilde{L}\mathbf{x}(t), \quad t \in \mathbb{Z}_+, \quad (7)$$

$$\mathbf{y}(t) = H\mathbf{x}(t), \quad (8)$$

where (see Notation for the symbol Φ)

$$\tilde{L} := L \times K \times \Phi \in \mathcal{L}_{N \times N},$$

is reconstructable [9], [30], by this meaning that $T \in \mathbb{Z}_+$ can be found such that the knowledge of the output trajectory $\mathbf{y}(t), t \in [0, T]$, allows to uniquely determine $\mathbf{x}(T)$ (and hence $\mathbf{x}(t)$ for every $t \geq T$).

We preliminary notice that the effect of a state-feedback law as in (6) is to create a BN by associating each single state variable $\mathbf{x} = \delta_N^j$ with one specific input value $\mathbf{u} = \delta_M^i$. As a consequence, the state-feedback law in (3) maps the state of the resulting BN from $\mathbf{x} = \delta_N^j$ to $L_i \delta_N^j$. Therefore, the practical effect of a state-feedback is to transform a BCN into a BN, by choosing for each state $\mathbf{x} = \delta_N^j$ its successor in the set $\{L_i \delta_N^j, i \in [1, M]\}$. So, in a sense, a state-feedback does not introduce new dynamics, just selects for each state a specific state transition among those already available.

If we introduce the Boolean matrix

$$L_{tot} := L_1 \vee L_2 \vee \dots \vee L_M, \quad (9)$$

it is immediate to realise that the matrices $\tilde{L} \in \mathcal{L}_{N \times N}$ that can be obtained via state-feedback are those and only those such that for each index $j \in [1, N]$ the j th column of \tilde{L} , $\text{col}_j(\tilde{L})$, belongs to $\{\text{col}_j(L_i), i \in [1, M]\}$, namely it coincides with δ_N^ℓ for some $\ell \in \{i \in [1, M] : [L_{tot}]_{ij} = 1\}$. So, we have proved the following result.

Proposition 1. *Given a BCN described as in (3)-(4), there exists a state feedback matrix $K \in \mathcal{L}_{M \times N}$ such that the resulting BN is described by the state matrix $\tilde{L} \in \mathcal{L}_{N \times N}$ if and only if \tilde{L} satisfies $\overline{ZP}(\tilde{L}) \subseteq \overline{ZP}(L_{tot})$. If so, the matrix K can be determined as follows*

$$K = [\delta_M^{i_1} \quad \delta_M^{i_2} \quad \dots \quad \delta_M^{i_N}],$$

where $i_k \in [1, M]$ is any index such that $\text{col}_k(L_{i_k}) = \text{col}_k(\tilde{L}), k \in [1, N]$.

This analysis of the concept of state feedback for BCNs can be alternatively expressed in graph terms. Denote by $\mathcal{D}(L_{tot}) = ([1, N], \mathcal{E}_{tot})$ the digraph associated with L_{tot} . The effect of state-feedback coincides with that of selecting for each node only one of its outgoing arcs, and this leads to $\mathcal{D}(\tilde{L})$, that uniquely identifies the logic matrix \tilde{L} .

Another remark that will help us solving the problem pertains the characterization of the reconstructability of a BN. To formalise it, we introduce some notation. Every periodic state trajectory (of period k) of the BN (7)-(8), i.e.,

$$\begin{cases} \mathbf{x}(t) = \delta_N^{i_{t+1}}, & t \in [0, k-1]; \\ \mathbf{x}(t) = \mathbf{x}(t-k), & t \in [k, +\infty), \end{cases}$$

can be uniquely represented by the ordered k tuple $(\delta_N^{i_1}, \delta_N^{i_2}, \dots, \delta_N^{i_k})$. A similar notation can be adopted for periodic output trajectories. The correspondence between periodic trajectories and k tuples representing them is bijective.

Proposition 2. (Theorem 2 in [9]) *The BN (7)-(8) is reconstructable if and only if distinct periodic state trajectories $(\delta_N^{i_1}, \delta_N^{i_2}, \dots, \delta_N^{i_k})$ (equivalently, distinct cycles (i_1, i_2, \dots, i_k) in $\mathcal{D}(\tilde{L})$) induce distinct periodic output sequences $(H\delta_N^{i_1}, H\delta_N^{i_2}, \dots, H\delta_N^{i_k})^1$.*

III. PROBLEM SIMPLIFICATION

When N is large, solving the FR problem for the pair (L, H) can be quite demanding. We want to show, however, that under certain conditions (that typically arise when dealing with BCNs representing gene regulatory networks, whose associated digraphs are quite sparse) the FR problem may be significantly reduced in size. In fact, if we partition the digraph $\mathcal{D}(L_{tot})$ into disjoint communication classes, then the problem of determining a state-feedback (equivalently a logical matrix \tilde{L} with $\overline{ZP}(\tilde{L}) \subseteq \overline{ZP}(L_{tot})$) such that the resulting BN is reconstructable can be solved by considering only the states that belong to the final classes, namely classes devoid of outgoing arcs.

To this end, it is convenient to introduce the following not restrictive assumption, which is based on the fact that every positive matrix can be brought by means of a suitable permutation (namely by means of a suitable relabelling of

¹This implies, in particular, that every periodic state trajectory of minimal period k induces a periodic output trajectory of minimal period k .

its row and column indices) to Frobenius (block triangular) normal form with irreducible diagonal blocks² [14].

Assumption 1: The matrix L_{tot} is block-partitioned as:

$$L_{tot} = \begin{bmatrix} T & 0 \\ Q & F \end{bmatrix}, \quad (10)$$

where:

(i) $T \in \mathcal{B}^{\tau \times \tau}$ is a lower block triangular matrix with irreducible diagonal blocks,

(ii) $F \in \mathcal{B}^{(N-\tau) \times (N-\tau)}$ is a block diagonal matrix with irreducible nonzero diagonal blocks,

(iii) for every $j \in [1, \tau]$ there exist $h \in \mathbb{Z}_+$ and $i \in [\tau + 1, N]$ such that $[L_{tot}^h]_{ij} \neq 0$.

Therefore, the matrix H is accordingly block-partitioned as

$$H = \begin{bmatrix} H_T & H_F \end{bmatrix}, \quad (11)$$

where $H_F \in \mathcal{L}_{P \times (N-\tau)}$.

Theorem 1. Given a BCN (3)-(4), whose matrices L_{tot} and H satisfy Assumption 1, the FR problem is solvable, namely there exists $K \in \mathcal{L}_{M \times N}$ such that the resulting BN (7)-(8) is reconstructable, if and only if the FR problem is solvable for the reduced BCN of size $N - \tau$

$$\mathbf{z}(t+1) = A \times \mathbf{u}(t) \times \mathbf{z}(t), \quad t \in \mathbb{Z}_+, \quad (12)$$

$$\mathbf{y}(t) = H_F \mathbf{z}(t) \quad (13)$$

where $A = [A_1 \ A_2 \ \dots \ A_M]$, with

$$A_i := [0 \ I_{N-\tau}] L_i \begin{bmatrix} 0 \\ I_{N-\tau} \end{bmatrix}.$$

Proof. Necessity is obvious. As far as sufficiency is concerned, we first observe that, under Assumption 1, the nodes of $\mathcal{D}(L_{tot})$ belonging to the final communication classes are those and those only labelled by $[\tau + 1, N]$. On the other hand, for each of the nodes labelled by $\mathcal{T} := [1, \tau]$, there is a path from such node to a node in $\mathcal{F} := [\tau + 1, N]$.

For each $i \in \mathcal{T}$, we let ν_i be the minimum length of any such path, which corresponds to the minimum power k of L_{tot} such that in

$$L_{tot}^k = \begin{bmatrix} T^k & 0 \\ Q^{(k)} & F^k \end{bmatrix} \quad (14)$$

the i -th column of the block $Q^{(k)}$ is nonzero.

Partition now \mathcal{T} into the following subsets

$$\mathcal{T}_r := \{i \in \mathcal{T} : \nu_i = r\}, \quad (15)$$

and denote by \bar{r} the largest value of r such that $\mathcal{T}_r \neq \emptyset$. Consequently, $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \dots \cup \mathcal{T}_{\bar{r}} = \mathcal{T}$.

The following procedure shows that it is always possible to

²The Frobenius normal form is unique up to a permutation, consequently also the form (10) is uniquely determined up to a permutation. This implies that for any positive matrix P if we denote by

$$\begin{bmatrix} T_1 & 0 \\ Q_1 & F_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} T_2 & 0 \\ Q_2 & F_2 \end{bmatrix}$$

two block triangular matrices satisfying (i)-(iii) to which P can be reduced by means of permutations, then $\dim T_1 = \dim T_2$ and there exist permutation matrices Π and $\tilde{\Pi}$ such that $T_2 = \Pi^T T_1 \Pi$ and $F_2 = \tilde{\Pi}^T F_1 \tilde{\Pi}$.

determine a state feedback law that drives any state δ_N^i , with $i \in \mathcal{T}$, into some state δ_N^j , $j \in \mathcal{F}$, in at most \bar{r} steps.

(1) For every node $i \in \mathcal{T}_1$, select an edge from i to some vertex $j \in \mathcal{F}$. This amounts to replacing in L_{tot} the i th column with the canonical vector δ_N^j (with $j \in \overline{ZP}(L_{tot} \delta_N^i) \cap \mathcal{F}$).

(2) Next, for every $i \in \mathcal{T}_2$, select an edge from i to some vertex $j \in \mathcal{T}_1$. Again, this amounts to replacing in L_{tot} the i th column with the canonical vector δ_N^j (with $j \in \overline{ZP}(L_{tot} \delta_N^i) \cap \mathcal{T}_1$).

(3) Similarly, for every $i \in \mathcal{T}_3$, select an edge from i to some vertex $j \in \mathcal{T}_2$ and so on.

In this way we have replaced the first τ columns of L_{tot} with canonical vectors in such a way that in the matrix thus obtained

$$\tilde{L}_{tot} = \begin{bmatrix} N & 0 \\ \tilde{Q} & F \end{bmatrix},$$

the block N is nilpotent and $\overline{ZP} \left(\begin{bmatrix} N \\ \tilde{Q} \end{bmatrix} \right) \subseteq \overline{ZP} \left(\begin{bmatrix} T \\ Q \end{bmatrix} \right)$. It is now clear that for every logical matrix $\tilde{F} \in \mathcal{L}_{(N-\tau) \times (N-\tau)}$ such that $\overline{ZP}(\tilde{F}) \subseteq \overline{ZP}(F)$, we have that in the directed graph of the logical matrix

$$\tilde{L} = \begin{bmatrix} N & 0 \\ \tilde{Q} & \tilde{F} \end{bmatrix},$$

all possible cycles involve only vertices of \mathcal{F} . Consequently, by Proposition 2, the pair (\tilde{L}, H) is reconstructable if the pair (\tilde{F}, H_F) is reconstructable. But this means that the FR problem is solvable for the original BCN if it is solvable for the BCN that corresponds to the last $N - \tau$ entries/nodes of the original BCN. ■

Example 1. Assume that $N = 9$ and

$$L_{tot} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & | & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & | & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & | & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} T & 0 \\ Q & F \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & | & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & | & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & | & 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} H_T & H_F \end{bmatrix}.$$

Note that T is lower block triangular with two irreducible diagonal blocks of sizes 2 and 3, respectively, while F is block diagonal with two irreducible diagonal blocks, both of size 2. So, the pair (L_{tot}, H) satisfies Assumption 1. The digraph associated with the previous matrix pair is given in Figure 1 a), where vertex i corresponds to the state δ_9^i and we have represented in blue the states δ_9^i such that $H\delta_9^i = \delta_3^1$, in green the states δ_9^i such that $H\delta_9^i = \delta_3^2$ and in orange the states δ_9^i such that $H\delta_9^i = \delta_3^3$. It is easy to see that there are 4 communication classes, two transient ones: $\mathcal{T}_1 = \{1, 2\}$, $\mathcal{T}_2 = \{3, 4, 5\}$ and two final ones $\mathcal{F}_1 = \{6, 7\}$ and $\mathcal{F}_2 = \{8, 9\}$.

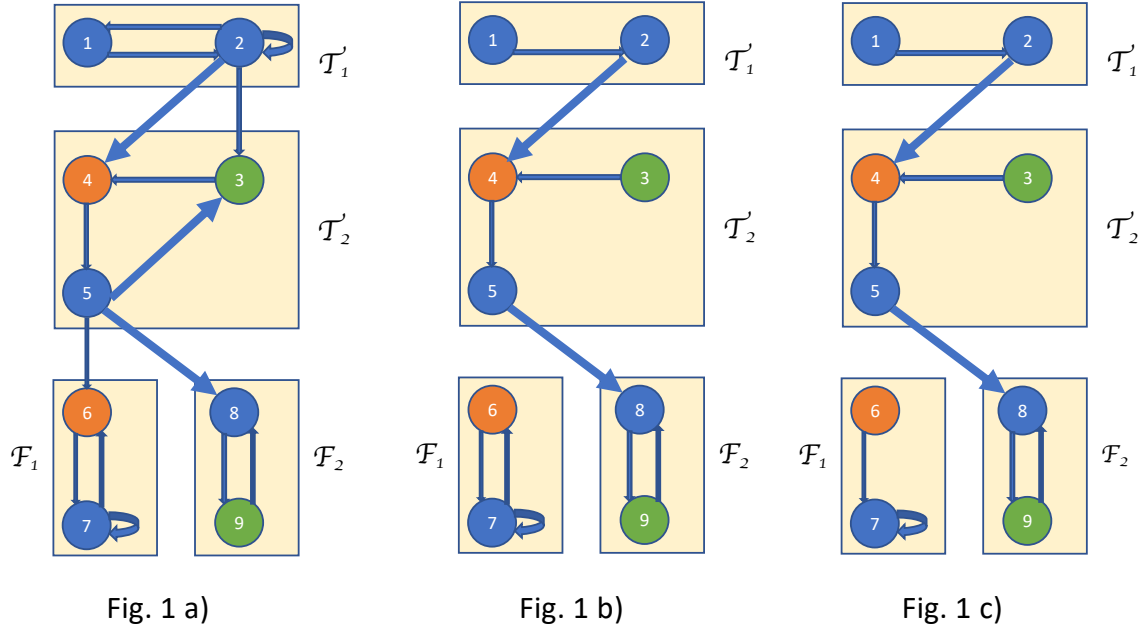


Fig. 1. Three digraphs associated with the three steps in Example 1: a) The first figure represents $\mathcal{D}(L_{tot})$, b) The second figure represents $\mathcal{D}(\tilde{L}_{tot})$, c) The third figure represents $\mathcal{D}(\tilde{L})$.

By following the procedure presented in the proof of Theorem 1, we first obtain the matrix \tilde{L}_{tot} , that corresponds to a subgraph of $\mathcal{D}(L_{tot})$ where all vertices in $\mathcal{T}_1 \cup \mathcal{T}_2$ do not belong to cycles, have a single outgoing arc, and they can reach at least one vertex in $\mathcal{F}_1 \cup \mathcal{F}_2$:

$$\tilde{L}_{tot} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & | & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & | & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} N & 0 \\ \tilde{Q} & F \end{bmatrix},$$

where N is nilpotent. The digraph associated to the previous matrix is given in Figure 1 b).

Finally, for every node in $\mathcal{F}_1 \cup \mathcal{F}_2$ we select a single outgoing edge, in order to ensure that no distinct cycles in $\mathcal{F}_1 \cup \mathcal{F}_2$ correspond to the same periodic output trajectory. A possible solution that makes the pair (\tilde{L}, H) reconstructable (see Figure 1 c)) is

$$\tilde{L} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & | & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & | & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 1 & 0 \end{bmatrix}.$$

IV. CHECKING PROBLEM SOLVABILITY BY MEANS OF POLYNOMIAL MATRICES IN NONCOMMUTATIVE VARIABLES

In the previous section we have proved that in order to solve the FR problem we can always reduce ourselves to the case of a BCN (3)-(4), whose matrix L_{tot} is block diagonal with irreducible diagonal blocks. In order to provide a necessary and sufficient condition for the problem solvability for BCNs satisfying this assumption, we first consider the case when L_{tot} is an irreducible matrix and we assume, without loss of generality, that the state to output matrix H takes the form

$$H = [\delta_P^1 \mathbf{1}_{n_1}^\top \quad \delta_P^2 \mathbf{1}_{n_2}^\top \quad \dots \quad \delta_P^P \mathbf{1}_{n_P}^\top]. \quad (16)$$

This means that the first n_1 states generate the output δ_P^1 , the second n_2 states generate the output δ_P^2 and so on. We can always reduce ourselves to this situation by means of a suitable permutation of the state variables (and possibly deleting outputs that do not correspond to any state). Clearly, $n_1 + n_2 + \dots + n_P = N$ and the matrices L_i and hence L_{tot} need to be accordingly permuted.

We now introduce the polynomial matrix in the noncommutative variables $\xi_1, \xi_2, \dots, \xi_P$ with coefficients in \mathcal{B} :

$$M(\xi_1, \dots, \xi_P) := L_{tot} \text{blkd}g\{\xi_1 I_{n_1}, \xi_2 I_{n_2}, \dots, \xi_P I_{n_P}\}. \quad (17)$$

This amounts to multiplying all entries in the first n_1 columns of L_{tot} by ξ_1 , the subsequent n_2 columns of L_{tot} by ξ_2 , and so on.

If we consider the generic k -th power of the matrix M , each (i, j) -th entry of M can be expressed as the sum of words, generically denoted by $w_{ij}^{(\ell)}(\xi_1, \xi_2, \dots, \xi_P) \in \Xi^*$, each of them of length k , i.e.,

$$[M^k(\xi_1, \xi_2, \dots, \xi_P)]_{ij} = \sum_{\ell} w_{ij}^{(\ell)}(\xi_1, \xi_2, \dots, \xi_P),$$

with $|w_{ij}^{(\ell)}(\xi_1, \xi_2, \dots, \xi_P)| = k$ for every ℓ . The presence of the specific word $w_{ij}^{(\ell)}(\xi_1, \xi_2, \dots, \xi_P) = \xi_{i_k} \xi_{i_{k-1}} \dots \xi_{i_1}$ in $[M^k(\xi_1, \xi_2, \dots, \xi_P)]_{ij}$ indicates that there exists a path in $\mathcal{D}(L_{tot})$ from the vertex j to the vertex i of length k , (equivalently a sequence of k state transitions from δ_N^j to δ_N^i), that generates, in the order, the output values $\delta_P^1, \delta_P^2, \dots, \delta_P^k$.

The following proposition provides a necessary and sufficient condition for the FR problem solution, when L_{tot} is an irreducible Boolean matrix.

Proposition 3. *Given a BCN (3)-(4), assume that the matrix L_{tot} , defined as in (9), is irreducible, and let $M(\xi, \xi_2, \dots, \xi_P)$ be the associated matrix in (17).*

The FR problem is solvable, namely there exists $K \in \mathcal{L}_{M \times N}$ such that the resulting BN (7)-(8) is reconstructable, if and only if there exist $\bar{\kappa}$ and i both in $[1, N]$ such that $[M^{\bar{\kappa}}(\xi_1, \xi_2, \dots, \xi_P)]_{ii} \neq 0$ and if $\bar{\kappa} \geq 2$ then one of the words composing $[M^{\bar{\kappa}}(\xi_1, \xi_2, \dots, \xi_P)]_{ii}$, say $w_{ii}^{(\ell)}(\xi_1, \xi_2, \dots, \xi_P) = \xi_{i_{\bar{\kappa}}} \xi_{i_{\bar{\kappa}-1}} \dots \xi_{i_1}$, satisfies

$$\sigma^d w_{ii}^{(\ell)}(\xi_1, \dots, \xi_P) = w_{ii}^{(\ell)}(\xi_1, \dots, \xi_P) \Rightarrow d \text{ multiple of } \bar{\kappa}. \quad (18)$$

Proof. [Sufficiency] Suppose that one of the terms of $[M^{\bar{\kappa}}(\xi_1, \xi_2, \dots, \xi_P)]_{ii}$ is $w_{ii}^{(\ell)}(\xi_1, \xi_2, \dots, \xi_P) = \xi_{i_{\bar{\kappa}}} \xi_{i_{\bar{\kappa}-1}} \dots \xi_{i_1}$.

This implies that in $\mathcal{D}(L_{tot})$ there is a cycle of length $\bar{\kappa}$ (if $\bar{\kappa} = 1$ a self-loop), say $(j_1, j_2, \dots, j_{\bar{\kappa}})$, by this meaning that $(j_h, j_{h+1}), h \in [1, \bar{\kappa} - 1]$, and $(j_{\bar{\kappa}}, j_1)$ are all arcs in $\mathcal{D}(L_{tot})$. Moreover, the cycle corresponds to the periodic state trajectory, with period $\bar{\kappa}$ (a constant trajectory if $\bar{\kappa} = 1$), $(\delta_N^{j_1}, \delta_N^{j_2}, \dots, \delta_N^{j_{\bar{\kappa}}})$, and the corresponding output trajectory $(\delta_P^{j_1}, \delta_P^{j_2}, \dots, \delta_P^{j_{\bar{\kappa}}})$ is, in turn, periodic with minimal period $\bar{\kappa}$ (by the assumption (18)).

We now proceed as follows: we preserve all arcs $(j_h, j_{h+1}), h \in [1, \bar{\kappa} - 1]$, and $(j_{\bar{\kappa}}, j_1)$ (equivalently, the corresponding entries in L_{tot}) and delete all the other outgoing arcs from the vertices in $\mathcal{D}_0 := \{j_1, j_2, \dots, j_{\bar{\kappa}}\}$. Since $\mathcal{D}(L_{tot})$ is irreducible, for every $i \in [1, N] \setminus \mathcal{D}_0$ there exists a path from i to one of the vertices in \mathcal{D}_0 . We let \mathcal{D}_1 be the set of vertices $i \in [1, N] \setminus \mathcal{D}_0$ whose distance from \mathcal{D}_0 is 1. For each such node we preserve a single arc (i, j) , with $j \in \mathcal{D}_0$. We then consider all vertices $i \in [1, N] \setminus (\mathcal{D}_0 \cup \mathcal{D}_1)$ whose distance from \mathcal{D}_1 is 1 and denote such set by \mathcal{D}_2 . For each such node we preserve a single arc (i, j) , with $j \in \mathcal{D}_1$. By proceeding in this way we construct a subgraph of $\mathcal{D}(L_{tot})$ with vertices $[1, N]$ and a single outgoing arc for each vertex. Moreover, the cycle $(j_1, j_2, \dots, j_{\bar{\kappa}})$ belongs to such subgraph and it can be reached from every other node. If we denote by \tilde{L} the adjacency matrix of such subgraph, \tilde{L} is logical, the only attractor of the BN with matrix \tilde{L} is the cycle $(j_1, j_2, \dots, j_{\bar{\kappa}})$. By assumption (18) and Proposition 2, the pair (\tilde{L}, H) is reconstructable.

[Necessity] Suppose that there exists a logical matrix $\tilde{L} \in \mathcal{L}_{N \times N}$, with $\overline{ZP}(\tilde{L}) \subseteq \overline{ZP}(L_{tot})$, such that (\tilde{L}, H) is reconstructable. Then there exists a cycle of some length, say $\bar{\kappa} \in [1, N]$, $(j_1, j_2, \dots, j_{\bar{\kappa}})$ in $\mathcal{D}(\tilde{L}) \subseteq \mathcal{D}(L_{tot})$, and if $\bar{\kappa} \geq 2$ then the corresponding output sequence $(H\delta_N^{j_1}, H\delta_N^{j_2}, \dots, H\delta_N^{j_{\bar{\kappa}}}) =$

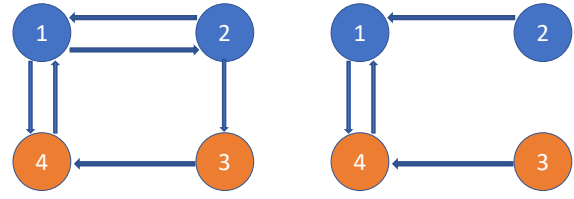


Fig. 2 a)

Fig. 2 b)

Fig. 2. $\mathcal{D}(L_{tot})$ (on the left) and $\mathcal{D}(\tilde{L})$ (on the right) for Example 2.

$(\delta_P^{i_1}, \delta_P^{i_2}, \dots, \delta_P^{i_{\bar{\kappa}}})$ is periodic with minimal period $\bar{\kappa}$. But this means that there exists $i \in [1, N]$ such that one of the terms of $[M^{\bar{\kappa}}(\xi_1, \xi_2, \dots, \xi_P)]_{ii}$ is $w_{ii}^{(\ell)}(\xi_1, \xi_2, \dots, \xi_P) = \xi_{i_{\bar{\kappa}}} \xi_{i_{\bar{\kappa}-1}} \dots \xi_{i_1}$, and when $\bar{\kappa} \geq 2$ the minimality of the period $\bar{\kappa}$ implies that condition (18) necessarily holds. ■

Example 2. *Consider the matrix pair*

$$L_{tot} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

that corresponds to the digraph of Fig. 2, where we have represented in blue the states δ_4^i such that $H\delta_4^i = \delta_2^1$, and in orange the states δ_4^i such that $H\delta_4^i = \delta_2^2$. Note that L_{tot} is irreducible, while $n_1 = n_2 = 2$. We associate with (L_{tot}, H) the matrix

$$M(\xi_1, \xi_2) = \begin{bmatrix} 0 & \xi_1 & 0 & \xi_2 \\ \xi_1 & 0 & 0 & 0 \\ 0 & \xi_1 & 0 & 0 \\ \xi_1 & 0 & \xi_2 & 0 \end{bmatrix}.$$

Clearly, all diagonal entries of M are zero, while

$$M^2(\xi_1, \xi_2) = \begin{bmatrix} \xi_1 \xi_1 + \xi_2 \xi_1 & 0 & \xi_2 \xi_2 & 0 \\ 0 & \xi_1 \xi_1 & 0 & \xi_1 \xi_2 \\ \xi_1 \xi_1 & 0 & 0 & 0 \\ 0 & \xi_2 \xi_1 + \xi_1 \xi_1 & 0 & \xi_1 \xi_2 \end{bmatrix}.$$

So, it is easy to see that, for instance $[M^2]_{44} = \xi_1 \xi_2$ (and $[M^2]_{11}$ includes the word $\xi_2 \xi_1$). Therefore we can keep, the cycle of length 2 in $\mathcal{D}(L_{tot})$ (see Figure 2) involving the vertices³ $\{1, 4\}$ and retain, for instance, the edges $(3, 4)$ and $(2, 1)$ to obtain a directed graph $\mathcal{D}(\tilde{L})$ (see Figure 3) where there is a single cycle of length 2 whose nodes correspond to different outputs, so that (\tilde{L}, H) is reconstructable, where

$$\tilde{L} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

The previous result can be easily extended to the case of a matrix L_{tot} that consists of k irreducible diagonal blocks, namely takes the form:

$$L_{tot} = \text{blkdiag}\{F_1, F_2, \dots, F_k\}, \quad (19)$$

³The general procedure to deduce \tilde{L} from L_{tot} will be described later in this section.

where each $F_i \in \mathcal{B}^{n_i \times n_i}$ is an irreducible Boolean matrix. Accordingly, the matrix H can be partitioned so that $H = [H_1 \ H_2 \ \dots \ H_k]$, where each block H_i has a number of columns equal to n_i . Subsequently, we can permute the columns of each block H_i (and hence the corresponding entries in F_i) based on the corresponding output values, so that H_i has the structure (16), namely

$$H_i = \begin{bmatrix} \delta_P^1 \mathbf{1}_{n_1}^\top & \delta_P^2 \mathbf{1}_{n_2}^\top & \dots & \delta_P^P \mathbf{1}_{n_P}^\top \end{bmatrix}.$$

If we now define the matrix in the noncommutative variables

$$M(\xi_1, \xi_2, \dots, \xi_P) = \text{blkdg}\{M^{(1)}, \dots, M^{(k)}\} := L_{tot}\Omega, \quad (20)$$

where

$$\begin{aligned} \Omega &:= \text{blkdg}\{\Omega^{(1)}, \Omega^{(2)}, \dots, \Omega^{(k)}\}, \\ \Omega^{(i)} &:= \text{blkdg}\{\xi_1 I_{n_1}, \xi_2 I_{n_2}, \dots, \xi_P I_{n_P}\}, \quad i \in [1, k], \end{aligned}$$

then we easily extend the result of Proposition 3 as follows.

Theorem 2. *Given a BCN (3)-(4), assume that the matrix L_{tot} , defined as in (9), is described as in (19), with $F_i, i \in [1, k]$, irreducible Boolean matrices, and let $M(\xi_1, \dots, \xi_P)$ be the associated matrix in (20).*

The FR problem is solvable if and only if for each $i \in [1, k]$ there exist $\bar{\kappa}_i$ and j_i in $[1, n_i]$ such that by choosing one of the words composing $[M^{\bar{\kappa}_i}(\xi_1, \xi_2, \dots, \xi_P)]_{j_i j_i}$, say $w_i(\xi_1, \xi_2, \dots, \xi_P)$ with $|w_i| = \bar{\kappa}_i$, we obtain a set of k words $\{w_1, w_2, \dots, w_k\}$ for which the following property holds: for every $\ell, m \in [1, k]$ and $d \in \mathbb{Z}_+$

$$\sigma^d w_\ell(\xi_1, \dots, \xi_P) = w_m(\xi_1, \dots, \xi_P) \Rightarrow \begin{cases} d \text{ is a multiple of } \bar{\kappa}_\ell \\ \text{and } \ell = m. \end{cases}$$

Theorems 1 and 2 together provide a general answer to the FR problem. Indeed, by Theorem 1 the problem is solvable if and only if it is solvable for the pair (A, H_F) , where the matrices A and H_F are submatrices of L and H obtained as described in Assumption 1 and in the statement of Theorem 1. While the problem can be solved for the pair (A, H_F) if and only if the conditions given in Theorem 2 hold.

V. DETERMINING A MATRIX K THAT SOLVES THE FR PROBLEM

Theorem 2 gives necessary and sufficient conditions for the problem solvability, but it does not provide an explicit procedure to construct a feedback matrix K that solves the problem. We want to investigate how the previous approach, based on polynomial matrices in noncommutative variables, can be used to obtain such a matrix K .

We assume that the matrix L_{tot} is irreducible, and hence refer to the special case of Theorem 2 addressed in Proposition 3. The general situation corresponding to Theorem 2 can be addressed according to the same logic, but it is just more complicated.

Define $\bar{\kappa} := \min\{h \in [1, N]: \exists i \in [1, N] \text{ such that } [M^h(\xi_1, \xi_2, \dots, \xi_P)]_{ii} \neq 0 \text{ includes a word } w \text{ s.t. } \sigma^d w \neq w, \forall d \in [1, h-1]\}$.

Case 1: If $\bar{\kappa} = 1$, i.e., $M(\xi_1, \xi_2, \dots, \xi_P)$ has a nonzero diagonal entry, say $[M(\xi_1, \xi_2, \dots, \xi_P)]_{ii}$, this means that there exists $j \in [1, M]$ such that $\delta_N^i = (L \delta_M^j) \delta_N^i = L_j \delta_N^i$ (i.e., δ_N^i is an equilibrium point of the j th subsystem of the BCN). Equivalently, there is a self-loop in $\mathcal{D}(L_{tot})$. In this case, we have a standard problem of state-feedback stabilisation to an equilibrium point, and a matrix K that solves this problem can be obtained, e.g., with the technique described in [10].

Case 2: If $\bar{\kappa} \geq 2$, let $w = \xi_{i_{\bar{\kappa}}} \dots \xi_{i_2} \xi_{i_1}$ be a word satisfying the assumptions of Proposition 3. This implies that (1) the set $\mathcal{W} := \{w, \sigma w, \dots, \sigma^{\bar{\kappa}-1} w\}$ consists of distinct words and (2) w represents the output sequence corresponding to an elementary cycle of length $\bar{\kappa}$ in $\mathcal{D}(L_{tot})$, meaning that there are $\bar{\kappa}$ distinct nodes belonging to that cycle.

Case 2a: If for each $d \in [1, \bar{\kappa}]$ there exists a single diagonal entry of $M^{\bar{\kappa}}(\xi_1, \xi_2, \dots, \xi_P)$ including the word $\sigma^{d-1} w$, say the j_d th entry, then we have found the nodes of the cycle and the exact order in which they appear. Indeed, the bijective correspondence between words of the set \mathcal{W} and diagonal entries of $M^{\bar{\kappa}}(\xi_1, \xi_2, \dots, \xi_P)$ allows to say that the cycle in $\mathcal{D}(L_{tot})$ corresponding to the word w is $(j_1, j_2, \dots, j_{\bar{\kappa}})$.

Case 2b: If Case 2a does not hold, this means that there is more than one elementary cycle of length $\bar{\kappa}$ in $\mathcal{D}(L_{tot})$ corresponding to the same periodic output trajectory of minimal period $\bar{\kappa}$ represented by the word w . To identify one of them, one can start from any state δ_N^i , where i is such that $[M^{\bar{\kappa}}(\xi_1, \xi_2, \dots, \xi_P)]_{ii}$ includes the word $w = \xi_{i_{\bar{\kappa}}} \dots \xi_{i_2} \xi_{i_1}$.

Set $j_1 := i$. Clearly j_1 is a node of the cycle and it corresponds to the output value $\delta_P^{i_1}$.

Set, now,

$$\begin{aligned} \mathcal{V}_2 &:= \{j \in [1, N]: \delta_P^{i_2} = H \delta_N^j \wedge (j_1, j) \in \mathcal{E}\}, \\ \mathcal{E}_{12} &:= \{(j_1, j) \in \mathcal{E}: j \in \mathcal{V}_2\}. \end{aligned}$$

In this way we have determined all the states that can be reached from vertex $j_1 = i$ in one step and that generate the output $y = \delta_P^{i_2}$, conformably with the structure of the word w . Subsequently, set

$$\begin{aligned} \mathcal{V}_3 &:= \{j \in [1, N]: \delta_P^{i_3} = H \delta_N^j \wedge \exists h \in \mathcal{V}_2 \text{ s.t. } (h, j) \in \mathcal{E}\}, \\ \mathcal{E}_{23} &:= \{(h, j) \in \mathcal{E}: h \in \mathcal{V}_2, j \in \mathcal{V}_3\}. \end{aligned}$$

By proceeding in this way, we define all vertex sets \mathcal{V}_h and all edge sets $\mathcal{E}_{h, h+1}$, $h = 1, 2, \dots, \bar{\kappa} + 1$. Clearly, $j_1 = i \in \mathcal{V}_{\bar{\kappa}+1}$, and we can determine backward both the nodes in the cycle and the edges that connect them⁴. Indeed, as $i \in \mathcal{V}_{\bar{\kappa}+1}$ then there exists $j_{\bar{\kappa}} \in \mathcal{V}_{\bar{\kappa}}$ such that $\delta_P^{i_{\bar{\kappa}}} = H \delta_N^{j_{\bar{\kappa}}}$ and $(j_{\bar{\kappa}}, j_1) \in \mathcal{E}$. Similarly, there exists $j_{\bar{\kappa}-1} \in \mathcal{V}_{\bar{\kappa}-1}$ such that $\delta_P^{i_{\bar{\kappa}-1}} = H \delta_N^{j_{\bar{\kappa}-1}}$ and $(j_{\bar{\kappa}-1}, j_{\bar{\kappa}}) \in \mathcal{E}$, and so on till we find $j_2 \in \mathcal{V}_2$ such that $\delta_P^{i_2} = H \delta_N^{j_2}$ and $(j_1, j_2) \in \mathcal{E}$. Note that the solution is not necessarily unique.

Both in Case 2a and in Case 2b, once a cycle has been identified, the feedback matrix K can be designed by following the procedure to stabilise a BCN to a given limit cycle described, for instance, in [10].

⁴In case in each set $\mathcal{E}_{h, h+1}$ one can memorize for each edge (h, j) also the value of the input corresponding to the state transition from δ_N^h to δ_N^j . This would make the construction of the matrix K more efficient.

Remark 1. Note that if instead of the state feedback $\mathbf{u}(t) = K\mathbf{x}(t)$ we adopt a state feedback of the following type

$$\mathbf{u}(t) = K \times \mathbf{v}(t) \times \mathbf{x}(t),$$

where $\mathbf{v}(t)$ is an independent input, we can move from the original BCN to a new BCN that preserves all the properties of the original one, but for a special (constant) choice of the input value $\mathbf{v}(t)$ becomes a reconstructable BN. To this end it is sufficient to impose that \mathbf{v} assumes $M + 1$ values and the controlled BCN takes the following form:

$$\begin{aligned} \mathbf{x}(t+1) &= L_K \times \mathbf{v}(t) \times \mathbf{x}(t), \\ L_K &= [L_1 \quad L_2 \quad \dots \quad L_M \quad \tilde{L}]. \end{aligned}$$

Therefore when $\mathbf{v}(t) \in \{\delta_{M+1}^i, i \in [1, M]\}$ the controlled BCN behaves as the original one, while when $\mathbf{v}(t) = \delta_{M+1}^{M+1}$ the BCN becomes a reconstructable BN. In this way, one can first “offline” identify the current BCN state (by assuming $\mathbf{v}(t) = \delta_{M+1}^{M+1}$ for $t \in [0, T]$, and T sufficiently large) and then let the BCN operate as the original one.

A possible state-feedback matrix that achieves this goal takes the form $K = [K_1 \quad K_2 \quad \dots \quad K_M \quad \tilde{K}]$, where $K_i = \delta_{M+1}^i \mathbf{1}_N^\top$, $i \in [1, M]$, while $\tilde{K} \in \mathcal{L}_{M \times N}$ makes the BN

$$\mathbf{x}(t+1) = (L \times \tilde{K} \times \Phi)\mathbf{x}(t) = \tilde{L}\mathbf{x}(t)$$

reconstructable from the output $\mathbf{y}(t) = H\mathbf{x}(t)$.

VI. CONCLUSIONS

In this paper we have addressed the Feedback Reconstructability problem for BCNs. We have first simplified the problem, showing that the solvability analysis can be performed on a BCN of smaller (oftentimes, much smaller) size than the original one. Subsequently we have provided necessary and sufficient conditions for the problem solvability. Finally, we have proposed a method to derive possible state feedback matrices that solve the problem.

It is interesting to remark that the key idea underlying the problem solution is to reduce the FR problem to a stabilisation problem either to an equilibrium point or to a limit cycle (or, of course, more equilibrium points and limit cycles, if we have different disjoint components), like those addressed in [10]. However, while typical stabilisation problems are formalised by referring to specific limit cycles (in particular, equilibria), a priori assigned, in this case every choice of the limit cycles is possible provided that each of them corresponds to a periodic output trajectory that is distinguishable from all the others. In other words, there must be a bijective correspondence between periodic state trajectories and periodic output trajectories that feedback “preserves” in the resulting BN.

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