Risk-Aware Stability, Ultimate Boundedness, and Positive Invariance

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***Abstract*—This article introduces the notions of stability, ultimate boundedness, and positive invariance for stochastic systems in view of risk. More specifically, those notions are defined in terms of the worst-case conditional value-at-risk (CVaR), which quantifies the worst-case conditional expectation of losses exceeding a certain threshold over a set of possible uncertainties. Those notions allow us to focus our attention on the tail behavior of stochastic systems in the analysis of dynamical systems and the design of controllers. Furthermore, some event-triggered control strategies that guarantee ultimate boundedness and positive invariance with specified bounds are derived using the obtained results and illustrated using numerical examples.**

***Index Terms*—Conditional value-at-risk, event-triggered control, positive invariance, stability, stochastic systems, ultimate boundedness.**

I. Introduction

Whether it is a financial portfolio or an engineering system, we often need to accept a certain level of risk to balance the pros and cons of spending costs in the decision-making processes under uncertainties. This is particularly true when the uncertainty distributions have an unbounded support because a guarantee of 100% ideal satisfaction is impossible or requires an infinite amount of cost. Therefore, risk has been studied for a long time, not only in the financial industry [1], [2], [3], [4], but also in the wide areas of engineering [5], [6], [7] using many different approaches.

The notions of stability, ultimate boundedness, and positive invariance are fundamental in the analysis of dynamical systems and in the design of controllers [8]. To deal with stochastic uncertainties in dynamical systems [9], the concept of probabilistic stability was introduced in [10]. Later, the concepts of probabilistic set invariance and ultimate boundedness were introduced for discrete-time linear systems in [11] and extended to continuous-time linear systems in [12]. Those probabilistic notions are defined using chance constraints. Another popular approach to dealing with stochastic uncertainties is to consider mean square and p-stability [13]. In this direction, ultimate boundedness [14] and positive invariance [15], [16] have been also investigated. However, the use of those notions may result in a significant loss, especially when the uncertainty distribution has a fat tail. This is because they do not characterize the tail risk—the risk that has a low probability of occurring, but if it does occur, it will result in a large loss.

To take into account the tail risk in the controller design, this article introduces definitions of stability, ultimate boundedness, and positive invariance in terms of the worst-case conditional value-at-risk (CVaR) for stochastic systems. This allows us to design controllers that guarantee that the expected value of the constraint-violating cases is small. CVaR is a relatively new risk measure that is defined as the conditional expectation of losses exceeding a certain threshold [17]. It is known that the (worst-case) CVaR is a coherent risk measure [4] and enjoys nice mathematical properties. In the recent controls community, CVaR has been used such as for optimal control [18], [19], [20], collision avoidance [21], and risk-perception-aware control under dynamic spatial risks in [22]. However, the computation of CVaR requires knowledge of the probability distribution of uncertainties, which is not always available. The worst-case CVaR, on the other hand, is the supremum of CVaR over a set of possible disturbances [4], [23]. Thus, the computation of the worst-case CVaR does not require knowledge of the probability distribution of uncertainties. For this reason, as well as its computational tractability, the worst-case CVaR is well-suited for risk-aware control design in practice. The worst-case CVaR has been used such as for the shortest path problem [24] and the multistage stochastic constrained control problem [25] as well as the author's earlier work about linear quadratic control [26].

The rest of this article is organized as follows. After introducing notation, definitions, and properties of the worst-case CVaR in Section II, Section III presents the system model we consider. Sections IV–VI discuss the properties of stability, ultimate boundedness, and positive invariance using the worst-case CVaR, respectively. Based on the results in those sections, approaches to risk-aware event-triggered control are discussed in Section VII, which is followed by numerical examples in Section VIII, and finally, Section IX concludes this article.

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II. Preliminaries

A. Notation

The sets of real numbers, real vectors of length n, and real matrices of size $n \times m$ are denoted by $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$, respectively. The sets of nonnegative numbers, nonnegative integers, and positive integers are denoted by $\mathbb{R}_{\geq 0}$, $\mathbb{Z}_{\geq 0}$, and $\mathbb{Z}_{>0}$, respectively. For $M \in \mathbb{R}^{n\times n}$, $\rho(M)$ denotes the spectral radius of M and $M \succ 0$ indicates M is positive definite. $M^\top$ denotes the transpose of a real matrix M and $\mathrm{Tr}(M)$ denotes the trace of M. I_n denotes the identity matrix of size n. The Kronecker product of two matrices X and Y is denoted as $X \otimes Y$. For $x \in \mathbb{R}$, $x^+ = \max\{x, 0\}$. For a vector $v \in \mathbb{R}^n$, $\|v\|$ denotes the Euclidean norm. For a matrix M, $\|M\|$ denotes the maximum singular value norm. Recall that a function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{K}$ function if it is continuous, strictly increasing and $\gamma(0) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{KL}$ function if for each fixed $t \geq 0$, $\beta(s, t) \in \mathcal{K}$ with respect to s and for each fixed $s \geq 0$, $\beta(s, t)$ is decreasing with respect to t and $\lim_{t\to\infty} \beta(s, t) = 0$.

B. Conditional Value-at-Risk (CVaR)

Let $\mu \in \mathbb{R}^n$ be the mean and $\Sigma \in \mathbb{R}^{n\times n}$ be the covariance matrix of the random vector $\xi \in \mathbb{R}^n$ under the true distribution $\mathbb{P}$, which is the probability law of ξ. Thus, it is implicitly assumed that the random

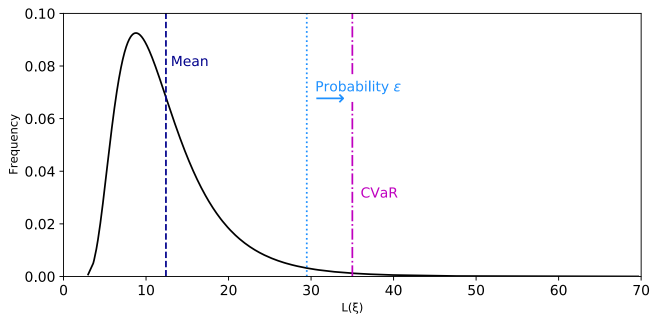


Fig. 1. Illustration of mean and CVaR.

vector ξ has finite second-order moments. Let $\mathcal{P}$ denote the set of all probability distributions on $\mathbb{R}^n$ that have the same first- and second-order moments as $\mathbb{P}$, i.e.,

$$\mathcal{P} = \left\{ \mathbb{P} : \mathbb{E}_{\mathbb{P}} \left[\begin{bmatrix} \xi_i \\ 1 \end{bmatrix} \begin{bmatrix} \xi_j \\ 1 \end{bmatrix}^\top \right] = \begin{bmatrix} \Sigma\delta_{ij} & 0 \\ 0^\top & 1 \end{bmatrix} \forall i,j \right\}.$$

Here, δ_{ij} denotes the Kronecker delta and $\mathbb{E}_{\mathbb{P}}[\cdot]$ denotes the expectation with respect to $\mathbb{P}$. The true underlying probability measure $\mathbb{P}$ is not known exactly, but it is known that $\mathbb{P} \in \mathcal{P}$.

Definition 2.1 (Conditional Value-at-Risk (CVaR) [17], [27]): For a given measurable loss function $L : \mathbb{R}^n \to \mathbb{R}$, a probability distribution $\mathbb{P}$ on $\mathbb{R}^n$ and a level $\varepsilon \in (0,1)$, the CVaR at ε with respect to $\mathbb{P}$ is defined as

$$\mathbb{P}\text{-CVaR}_\varepsilon[L(\xi)] = \inf_{\beta\in\mathbb{R}} \left\{ \beta + \frac{1}{\varepsilon}\mathbb{E}_{\mathbb{P}}[(L(\xi)-\beta)^+] \right\}.$$

CVaR is the conditional expectation of loss above the $(1-\varepsilon)$-quantile of the loss function [27] and quantifies the tail risk (see Fig. 1).

The worst-case CVaR is the supremum of CVaR over a given set of probability distributions as defined as follows.

Definition 2.2 (Worst-case CVaR [27]): The worst-case CVaR over $\mathcal{P}$ is given by

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[L(\xi)] = \inf_{\beta\in\mathbb{R}} \left\{ \beta + \frac{1}{\varepsilon} \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}[(L(\xi)-\beta)^+] \right\}.$$

Here, the exchange between the supremum and infimum is justified by the stochastic saddle point theorem [28].

If $L(\xi)$ is quadratic with respect to ξ, the worst-case CVaR can be computed by a semidefinite program [27], [29]. Furthermore, if the mean of the random vector ξ is zero, the following easy-to-compute bounds are obtained.

Lemma 2.3 (Bounds for worst-case CVaR [26]): Suppose $\mu = 0$ and $L(\xi) = \|A\xi + b\|^2 + c$ with some $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then

$$c + b^\top b + \frac{1}{\varepsilon}\left(\text{Tr}(\Sigma A^\top A)\right) \leq \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[L(\xi)] \leq c + \frac{1}{\varepsilon}\left(\text{Tr}(\Sigma A^\top A) + b^\top b\right).$$

If $b = 0$ and $c = 0$, then it follows that

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[L(\xi)] = \frac{1}{\varepsilon}\text{Tr}(\Sigma A^\top A).$$

To deal with dynamical systems, the following is a simple, but useful result.

Lemma 2.4: Suppose $\xi_1, \xi_2, \ldots, \xi_m \in \mathbb{R}^n$ are independent and identically distributed random vectors under the true distribution $\mathbb{P} \in \mathcal{P}$. Let $\bar{\xi} = [\xi_1^\top, \xi_2^\top, \ldots, \xi_m^\top]^\top$. Then, $\bar{\xi} \in \mathbb{R}^{nm}$ is a random vector with the mean zero and covariance $I_m \otimes \Sigma$. Thus, the true underlying probability measure $\mathbb{P}$ for ξ satisfies $\mathbb{P} \in \mathcal{P}_{\text{aug}}$, where

$$\mathcal{P}_{\text{aug}} = \left\{ \mathbb{P} : \mathbb{E}_{\mathbb{P}} \left[\begin{bmatrix} \bar{\xi}_i \\ 1 \end{bmatrix} \begin{bmatrix} \bar{\xi}_j \\ 1 \end{bmatrix}^\top \right] = \begin{bmatrix} (I_m \otimes \Sigma)\delta_{ij} & 0 \\ 0 & 1 \end{bmatrix} \forall i,j \right\}.$$

Moreover, for a matrix $A \in \mathbb{R}^{\ell \times nm}$, it holds that

$$\sup_{\mathbb{P}\in\mathcal{P}_{\text{aug}}} \mathbb{P}\text{-CVaR}_\varepsilon[\|A\xi\|^2] = \frac{1}{\varepsilon}\text{Tr}((I_m \otimes \Sigma)A^\top A).$$

Proof: Follows from the assumption that $\xi_1, \xi_2, \ldots, \xi_m \in \mathbb{R}^n$ are independent and identically distributed. The second part follows from Lemma 2.3. ■

One reason that the (worst-case) CVaR is popular for risk assessment is its mathematically attractive properties of coherency.

Proposition 2.5 (Coherence properties [4], [30]): The worst-case CVaR is a coherent risk measure, i.e., it satisfies the following properties: Let $L_1 = L_1(\xi)$ and $L_2 = L_2(\xi)$ be two measurable loss functions.

1) Subadditivity: For all L_1 and L_2

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[L_1 + L_2] \leq \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[L_1] + \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[L_2].$$

2) Positive homogeneity: For a positive constant $a > 0$

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[aL_1] = a \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[L_1].$$

3) Monotonicity: If $L_1 \leq L_2$ almost surely

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[L_1] \leq \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[L_2].$$

4) Translation invariance: For a constant c.

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[L_1 + c] = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[L_1] + c.$$

III. System Model

This section introduces a model of a linear system with stochastic disturbances.

Consider the discrete-time linear stochastic system

$$x_{t+1} = Ax_t + Dv_t + Ew_t \tag{1}$$

where $x_t \in \mathbb{R}^{n_x}$ is the state, $v_t \in \mathbb{R}^{n_v}$ is the input and $w_t \in \mathbb{R}^{n_w}$ is the disturbance, respectively, at discrete time instant $t \in \mathbb{Z}_{\geq 0}$. $A \in \mathbb{R}^{n_x \times n_x}$, $D \in \mathbb{R}^{n_x \times n_v}$, and $E \in \mathbb{R}^{n_x \times n_w}$ are constant matrices. It is assumed that the initial condition $x_0 \in \mathbb{R}^{n_x}$ is given, and that w_t are independent and identically distributed random vectors with the mean zero and covariance $\Sigma_w \succ 0$ for all $t \in \mathbb{Z}_{\geq 0}$. The true underlying probability measure $\mathbb{P}$ for w_t is not known exactly, but it is known that $\mathbb{P} \in \mathcal{P}$, where

$$\mathcal{P} = \left\{ \mathbb{P} : \mathbb{E}_{\mathbb{P}} \left[\begin{bmatrix} w_i \\ 1 \end{bmatrix} \begin{bmatrix} w_j \\ 1 \end{bmatrix}^\top \right] = \begin{bmatrix} \Sigma_w\delta_{ij} & 0 \\ 0^\top & 1 \end{bmatrix} \forall i,j \right\}. \tag{2}$$

It is also assumed $v_t \in \mathcal{V}$, where

$$\mathcal{V} = \{v \in \mathbb{R}^{n_v} : \|v\| \leq d\} \tag{3}$$

for a given d.

For $t \in \mathbb{Z}_{>0}$, the state evolution of (1) can be expressed by

$$x_t = F_t x_0 + G_t \bar{v}_t + H_t \bar{w}_t \tag{4}$$

using

$$\begin{aligned} F_t &= A^t \\ G_t &= \begin{bmatrix} A^{t-1}D & A^{t-2}D & \cdots & D \end{bmatrix} \\ H_t &= \begin{bmatrix} A^{t-1}E & A^{t-2}E & \cdots & E \end{bmatrix} \\ \bar{w}_t &= [w_0^\top,\ w_1^\top, \ldots, w_{t-1}^\top]^\top \\ \bar{v}_t &= [v_0^\top,\ v_1^\top, \ldots, v_{t-1}^\top]^\top. \end{aligned} \tag{5}$$

With this notation, from Lemma 2.4, $\bar{w}_t \in \mathbb{R}^{n_w t}$ is a random vector with the mean zero and covariance $I_t \otimes \Sigma_w$. Thus, the true underlying probability measure $\mathbb{P}$ for $\bar{w}_t$ satisfies $\mathbb{P} \in \mathcal{P}_t$, where

$$\mathcal{P}_t = \left\{ \mathbb{P} : \mathbb{E}_{\mathbb{P}} \left[\begin{bmatrix} \bar{w}_{t,i} \\ 1 \end{bmatrix} \begin{bmatrix} \bar{w}_{t,j} \\ 1 \end{bmatrix}^\top \right] = \begin{bmatrix} (I_t \otimes \Sigma_w)\delta_{ij} & 0 \\ 0 & 1 \end{bmatrix} \forall i, j \right\}. \tag{6}$$

IV. Stability

This section introduces the stability notions in terms of the worst-case CVaR.

The input-to-state stability using the worst-case CVaR is defined as follows.

Definition 4.1 (Practical input-to-state stability): The system (1) is practically worst-case CVaR input-to-state stable if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$, and $c \geq 0$ such that

$$\begin{aligned} &\sup_{\mathbb{P} \in \mathcal{P}_t} \mathbb{P}\text{-CVaR}_\varepsilon [\|x_t\|^2] \\ &\leq \beta(\|x_0\|^2, t) + \gamma \left(\max_{\tau \in [0, t-1]} (\|v_\tau\|^2) \right) + c \ \forall t \in \mathbb{Z}_{\geq 0}. \end{aligned} \tag{7}$$

The following lemma provides conditions for a system to be practically worst-case CVaR input-to-state stable.

Lemma 4.2 (Conditions for practical input-to-state stability): The system (1) is practically worst-case CVaR input-to-state stable if and only if $\rho(A) < 1$.

Proof: The necessity is clear. For sufficiency, we first note that under the condition, there exist $\mu > 0$ and $\lambda \in [0, 1)$ such that

$$\|A^t\| \leq \mu \lambda^t. \tag{8}$$

Define $\bar{v}_{t-1} = \max_{\tau \in [0, t-1]} (\|v_\tau\|) (\leq d)$, then

$$\begin{aligned} \|G_t \bar{v}_t\|^2 &= \left\| \sum_{k=0}^{t-1} A^k D v_{t-1-k} \right\|^2 \\ &\leq \left\| \sum_{k=0}^{t-1} \|A^k\| \|D\| \|v_{t-1-k}\| \right\|^2 \\ &\leq \left\| \sum_{k=0}^{t-1} \mu \lambda^k \|D\| \max_{\tau \in [0, t-1]} (\|v_\tau\|) \right\|^2 \leq \left(\frac{\mu \|D\|}{1 - \lambda} \right)^2 \bar{v}_{t-1}^2. \end{aligned} \tag{9}$$

Because for two vectors $x, y \in \mathbb{R}^n$, it holds that

$$2x^\top y \leq \alpha^2 \|x\|^2 + \frac{1}{\alpha^2} \|y\|^2 \ \forall \alpha > 0$$

it follows that

$$\begin{aligned} \|x_t\|^2 &= \|F_t x_0 + G_t \bar{v}_t + H_t \bar{w}_t\|^2 \\ &\leq (1 + \alpha_1^2) \|F_t x_0\|^2 + \left(1 + \frac{1}{\alpha_1^2}\right) (1 + \alpha_2^2) \|G_t \bar{v}_t\|^2 \\ &\quad + \left(1 + \frac{1}{\alpha_1^2}\right) \left(1 + \frac{1}{\alpha_2^2}\right) \|H_t \bar{w}_t\|^2 \\ &\leq (1 + \alpha_1^2) \mu^2 \lambda^{2t} \|x_0\|^2 + \left(1 + \frac{1}{\alpha_1^2}\right) (1 + \alpha_2^2) \left(\frac{\mu \|D\|}{1 - \lambda} \right)^2 \bar{v}_{t-1}^2 \\ &\quad + \left(1 + \frac{1}{\alpha_1^2}\right) \left(1 + \frac{1}{\alpha_2^2}\right) \|H_t \bar{w}_t\|^2 \end{aligned} \tag{10}$$

for any $\alpha_1, \alpha_2 > 0$. Using Proposition 2.5 along with Lemma 2.4, it follows that

$$\begin{aligned} &\sup_{\mathbb{P} \in \mathcal{P}_t} \mathbb{P}\text{-CVaR}_\varepsilon [\|x_t\|^2] \\ &\leq (1 + \alpha_1^2) \mu^2 \lambda^{2t} \|x_0\|^2 + \left(1 + \frac{1}{\alpha_1^2}\right) (1 + \alpha_2^2) \left(\frac{\mu \|D\|}{1 - \lambda} \right)^2 \bar{v}_{t-1}^2 \\ &\quad + \left(1 + \frac{1}{\alpha_1^2}\right) \left(1 + \frac{1}{\alpha_2^2}\right) \frac{1}{\varepsilon} \mathrm{Tr}(P). \end{aligned} \tag{11}$$

Here, we used

$$\begin{aligned} \mathrm{Tr}((I_t \otimes \Sigma_w) H_t^\top H_t) &= \mathrm{Tr} \left(\sum_{k=0}^{t-1} (A^k) E \Sigma_w E^\top (A^\top)^k \right) \\ &\leq \mathrm{Tr} \left(\sum_{k=0}^{\infty} (A^k) E \Sigma_w E^\top (A^\top)^k \right) \\ &= \mathrm{Tr}(P) \end{aligned} \tag{12}$$

where $P \succ 0$ is the solution to the Lyapunov equation

$$APA^\top - P + E \Sigma_w E^\top = 0. \tag{13}$$

Thus, choosing

$$\begin{aligned} \beta(s, t) &= (1 + \alpha_1^2) \mu^2 \lambda^{2t} s \\ \gamma(\bar{v}_{t-1}) &= \left(1 + \frac{1}{\alpha_1^2}\right) (1 + \alpha_2^2) \left(\frac{\mu \|D\|}{1 - \lambda} \right)^2 \bar{v}_{t-1}^2 \\ c &= \left(1 + \frac{1}{\alpha_1^2}\right) \left(1 + \frac{1}{\alpha_2^2}\right) \frac{1}{\varepsilon} \mathrm{Tr}(P) \end{aligned} \tag{14}$$

in (11) verifies Definition 4.1. ■

If there is no input, we can simplify the definition and lemma as follows.

Definition 4.3 (Practical asymptotic stability): The system (1) with $D = 0$ is practically asymptotically worst-case CVaR stable if there exist $\beta \in \mathcal{KL}$ and a constant $c \geq 0$ such that

$$\sup_{\mathbb{P} \in \mathcal{P}_t} \mathbb{P}\text{-CVaR}_\varepsilon [\|x_t\|^2] \leq \beta(\|x_0\|^2, t) + c \ \forall t \in \mathbb{Z}_{\geq 0}. \tag{15}$$

Corollary 4.4 (Conditions for practical asymptotic stability): The system (1) with $D = 0$ is practically asymptotically worst-case CVaR stable if and only if $\rho(A) < 1$.

Proof: Similarly to the proof of Lemma 4.2, using

$$\beta(s, t) = (1 + \alpha^2) \mu^2 \lambda^{2t} s, \ c = \left(1 + \frac{1}{\alpha^2}\right) \frac{1}{\varepsilon} \mathrm{Tr}(P). \tag{16}$$

in (11) verifies Definition 4.3. ■

The results in this section indicate that the standard stability condition $\rho(A) < 1$ also guarantees that the expected value of the tail of the squared norm of the states does not grow as long as the inputs are bounded. The bounds obtained here are not tight, however, tight bounds can be obtained for computing the worst-case CVaR of $\|x_t\|^2 = \|F_t x_0 + G_t \bar{v}_t + H_t \bar{w}_t\|^2$ at each t using a semidefinite program as shown in [27] and [29].

Remark 4.5: An earlier version of this article [31] inspired the development of risk-aware stability theory using other measures and discussions of the relations between them [32].

V. Ultimate Boundedness

Here, we introduce a notion of worst-case CVaR ultimate bound, which is an extension of the probabilistic ultimate bound [11]. Ultimate bounds can be found for practically worst-case CVaR input-to-state stable systems, i.e., for systems with $\rho(A) < 1$.

Definition 5.1 (Ultimate bound): A compact set $\Omega \in \mathbb{R}^{n_x}$ is a worst-case CVaR ultimate bound set for the system (1) if for every initial state x_0, there exists $T = T(x_0) > 0$ such that for any $\eta > 0$, $\sup_{\mathbb{P}\in\mathcal{P}_t} \mathbb{P}\text{-CVaR}_\varepsilon[x \in \Omega] \le \eta$ for all $t \ge T$. In particular, a positive scalar r is said to be a worst-case CVaR ultimate bound if $\Omega = \{x \in \mathbb{R}^{n_x} : \|x_t\|^2 \le r^2\}$.

Without stochastic disturbance, this definition agrees with the standard ultimate bound definition that r is an ultimate bound if for every initial state x_0, there exists $T = T(x_0) > 0$ such that $\|x_t\|^2 \le r^2 + \eta$ for all $t \ge T$.

A worst-case CVaR ultimate bound can be found as follows.

Theorem 5.2 (Ultimate bound): Consider the system (1) with $\rho(A) < 1$

$$r = \frac{\mu\|D\|}{1-\lambda} d + \sqrt{\frac{1}{\varepsilon}\text{Tr}(P)} \tag{17}$$

with d defined in (3), μ and λ that satisfy (8) and $P \succ 0$ that satisfies the Lyapunov (13) is a worst-case CVaR ultimate bound for the system (1).

Proof: By using $\bar{v}_{t-1} \le d$ and choosing

$$\alpha_1^2 = \frac{r^2}{\eta - c},\ \alpha_2^2 = \frac{1-\lambda}{\mu\|D\|d}\sqrt{\frac{1}{\varepsilon}\text{Tr}(P)} \tag{18}$$

for some $c \in (0, \eta)$ in (11), it follows that

$$\begin{aligned}
&\sup_{\mathbb{P}\in\mathcal{P}_t} \mathbb{P}\text{-CVaR}_\varepsilon[\|x_t\|^2] \\
&\le (1+\alpha_1^2)\mu^2\lambda^{2t}\|x_0\|^2 + \left(1 + \frac{1}{\alpha_1^2}\right)\left(\frac{\mu\|D\|}{1-\lambda}d + \sqrt{\frac{1}{\varepsilon}\text{Tr}(P)}\right)^2 \\
&= \left(1 + \frac{r^2}{\eta - c}\right)\mu^2\lambda^{2t}\|x_0\|^2 + \left(1 + \frac{\eta - c}{r^2}\right) r^2 \\
&= \left(1 + \frac{r^2}{\eta - c}\right)\mu^2\lambda^{2t}\|x_0\|^2 + r^2 + \eta - c.
\end{aligned} \tag{19}$$

Because the first term approaches to 0 as t goes to infinity, there exists T such that $\sup_{\mathbb{P}\in\mathcal{P}_t} \mathbb{P}\text{-CVaR}_\varepsilon[\|x_t\|^2] \le r^2 + \eta$ for all $t \ge T$. ∎

The result can be simplified in the case of $D = 0$ as follows.

Corollary 5.3 (Ultimate bound): Consider the system (1) with $\rho(A) < 1$ and $D = 0$.

$$r = \sqrt{\frac{1}{\varepsilon}\text{Tr}(P)} \tag{20}$$

is a worst-case CVaR ultimate bound for (1) without inputs.

Proof: Substitute $D = 0$ in the proof of Theorem 5.2. ∎

Corollary 5.3 indicates that the ultimate bound is proportional to the square root of the inverse of the risk level ε. In other words, if we are considering the worst 10% cases and would like to tighten it to 2.5%, then the ultimate bound must be increased by a factor of 2.

VI. Positive Invariance

The notion of positive invariance is yet another important concept, which is intimately related to ultimate boundedness.

Definition 6.1 (Robust positively invariant set): Let $f : \mathbb{R}^{n_x} \to \mathbb{R}$ be a continuous function. A domain $\mathcal{D} = \{x \in \mathbb{R}^{n_x} : f(x) \le 0\}$ is a worst-case CVaR robust positively invariant set for the system (1) if for any state $x_t \in \mathcal{D}$, $\sup_{\mathbb{P}\in\mathcal{P}_k} \mathbb{P}\text{-CVaR}_\varepsilon[f(x_{t+k})] \le 0$ for all $k \in \mathbb{Z}_{>0}$.

A worst-case CVaR robust positively invariant set can be found as follows.

Theorem 6.2 (Robust positively invariant set): Consider the system (1) with $\|A\| < 1$. For any $\eta > 0$, the set

$$\mathcal{D} = \{x \in \mathbb{R}^{n_x} : \|x\|^2 \le r^2 + \eta\} \tag{21}$$

where

$$r = \frac{1}{1 - \|A\|}\left(\|D\|d + \sqrt{\frac{1}{\varepsilon}\text{Tr}(\Sigma_w E^\top E)}\right) \tag{22}$$

is a worst-case CVaR robust positively invariant set for (1). Note d is defined in (3).

Proof: First, we show that $x_t \in \mathcal{D}$ implies $\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[\|x_{t+1}\|^2] \le r^2 + \eta$. Similarly to (10), we have

$$\begin{aligned}
\|x_{t+1}\|^2 &= \|Ax_t + Dv_t + Ew_t\|^2 \\
&\le (1+\alpha_1^2)\|Ax_t\|^2 + \left(1 + \frac{1}{\alpha_1^2}\right)(1+\alpha_2^2)\|D\|^2 d^2 \\
&\quad + \left(1 + \frac{1}{\alpha_1^2}\right)\left(1 + \frac{1}{\alpha_2^2}\right)\|Ew_t\|^2.
\end{aligned} \tag{23}$$

Choose $\alpha_1 > 0$ that satisfies

$$(1+\alpha_1^2)\|A\| = 1 \tag{24}$$

and $\alpha_2 > 0$ that satisfies

$$\alpha_2^2\|D\|d = \sqrt{\frac{1}{\varepsilon}\text{Tr}(\Sigma_w E^\top E)}. \tag{25}$$

Using Proposition 2.5 as well as the norm submultiplicativity and Lemma 2.3, it follows that

$$\begin{aligned}
&\sup_{\mathbb{P}\in\mathcal{P}_{t+1}} \mathbb{P}\text{-CVaR}_\varepsilon[\|x_{t+1}\|^2] \\
&\le \|A\|\|x_t\|^2 + \frac{1}{1-\|A\|}\left(\|D\|d + \sqrt{\frac{1}{\varepsilon}\text{Tr}(\Sigma_w E^\top E)}\right)^2.
\end{aligned} \tag{26}$$

Hence, if $x_t \in \mathcal{D}$, using the condition (22), it follows that

$$\begin{aligned}
&\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[\|x_{t+1}\|^2] \\
&\le \|A\|(r^2+\eta) + \frac{1}{1-\|A\|}(1-\|A\|)^2 r^2 \\
&\le r^2 + \eta.
\end{aligned} \tag{27}$$

Next, we show $\sup_{\mathbb{P}\in\mathcal{P}_k} \mathbb{P}\text{-CVaR}_\varepsilon[\|x_{t+k}\|^2] \le r^2 + \eta$ for some $k > 0$ implies $\sup_{\mathbb{P}\in\mathcal{P}_{k+1}} \mathbb{P}\text{-CVaR}_\varepsilon[\|x_{t+k+1}\|^2] \le r^2 + \eta$. For any

$\alpha_1, \alpha_2 > 0$

$$\|x_{t+k+1}\|^2 = \|Ax_{t+k} + Dv_{t+k} + Ew_{t+k}\|^2$$
$$\leq (1+\alpha_1^2)\|Ax_{t+k}\|^2 + \left(1+\frac{1}{\alpha_1^2}\right)(1+\alpha_2^2)\|D\|^2 d^2$$
$$+ \left(1+\frac{1}{\alpha_1^2}\right)\left(1+\frac{1}{\alpha_2^2}\right)\|Ew_{t+k}\|^2. \quad (28)$$

Choose $\alpha_1 > 0$ and $\alpha_2 > 0$ that satisfy (24) and (25), respectively. Similarly to before, we have

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[\|x_{t+k+1}\|^2]$$
$$\leq \|A\| \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[\|x_{t+k}\|^2] + \frac{1}{1-\|A\|}(1-\|A\|)^2 r^2$$
$$\leq r^2 + \eta. \quad (29)$$

This completes the proof. ∎

The result can be simplified in the case of $D = 0$ as follows.

Definition 6.3 (Positively invariant set): Let $f : \mathbb{R}^{n_x} \to \mathbb{R}$ be a continuous function. A domain $\mathcal{D} = \{x \in \mathbb{R}^{n_x} : f(x) \leq 0\}$ is a worst-case CVaR positively invariant set for the system (1) with $D = 0$ if for any state $x_t \in \mathcal{D}\sup_{\mathbb{P}\in\mathcal{P}_k} \mathbb{P}\text{-CVaR}_\varepsilon[f(x_{t+k})] \leq 0$ for all $k \in \mathbb{Z}_{>0}$.

A worst-case CVaR positively invariant set can be found as follows.

Corollary 6.4 (A positively invariant set): Consider the system (1) with $\|A\| < 1$ and $D = 0$. For any $\eta > 0$, the set $\mathcal{D}$ in (21), where

$$r = \frac{1}{1-\|A\|}\sqrt{\frac{1}{\varepsilon}\mathrm{Tr}(\Sigma_w E^\top E)} \quad (30)$$

is a worst-case CVaR positively invariant set for (1) with $D = 0$.

Proof: Substitute $D = 0$ in the proof of Theorem 6.2. ∎

In this section, we considered the systems with $\|A\| < 1$ instead of systems with $\rho(A) < 1$. This is because unlike stability and ultimate boundedness, the notion of positive invariance is the relation between the current state and the state at the next time, instead of the relation between the initial state and the state at some time sufficiently later. Therefore, Gelfand's formula cannot be utilized. Yet, Corollary 6.4 indicates that the parameter r of the positively invariant set is proportional to the square root of the inverse of the risk level ε as for the ultimate bound.

VII. Event-Triggered Control

This section presents an application of the results of Sections V and VI to developing event-triggered controllers, which reduce the control input updates while meeting the required risk-aware properties of the system.

A. System Description

Consider the discrete-time linear control system subject to stochastic disturbance

$$x_{t+1} = Ax_t + Bu_t + Ew_t. \quad (31)$$

This is the same as the system (1) except for the control input $u_t \in \mathbb{R}^{n_u}$. Here, we assume that a linear state feedback control $u_t = Kx_t$ has been designed.

To design trigger conditions, let us introduce the following notation: Let the triggering time sequence $\{t_k\}_{k\in\mathbb{Z}_{\geq 0}}$, and define the state used for the control input by

$$\hat{x}_t = x_{t_k} \ \forall t \in [t_k, t_{k+1}) \quad (32)$$

and the state error by

$$e_t = \hat{x}_t - x_t \ \forall t \in [t_k, t_{k+1}). \quad (33)$$

Then, the control law can be written as

$$u_t = K\hat{x}_t \ \forall t \in [t_k, t_{k+1}) \quad (34)$$

and the system (1) can be written as

$$x_{t+1} = (A+BK)x_t + BKe_t + Ew_t. \quad (35)$$

The rest of this section considers the event-triggering mechanism in the form of

$$t_{k+1} = \min\{t > t_k : \phi(x_t, \hat{x}_t) > \sigma\},\ t_0 = 0 \quad (36)$$

where the triggering function ϕ and the triggering threshold σ are to be designed. Note that such an event-trigger condition guarantees that $\phi(x_t, \hat{x}_t) \leq \sigma$ for all $t \in \mathbb{Z}_{\geq 0}$.

We consider static event-triggered control strategies that use a constant error threshold σ.

B. Ultimately Bounded Control

Event-triggered control strategies that guarantee ultimate boundedness are followed from Theorem 5.2.

Corollary 7.1: Suppose K is designed such that $\rho(A+BK) < 1$ and an ultimate bound

$$r > \sqrt{\frac{1}{\varepsilon}\mathrm{Tr}(\tilde{P})} \quad (37)$$

where $\tilde{P} \succ 0$ is the solution to the Lyapunov equation

$$(A+BK)\tilde{P}(A+BK)^\top - \tilde{P} + E\Sigma_w E^\top = 0 \quad (38)$$

is chosen. Then, the use of the event-triggered condition

$$\phi(x_t, \hat{x}_t) = \|\hat{x}_t - x_t\| = \|e_t\| > \sigma_1 \quad (39)$$

in (36) guarantees ultimate boundedness with r if σ_1 satisfies

$$\sigma_1 \leq \frac{1-\lambda}{\mu\|BK\|}\left(r - \sqrt{\frac{1}{\varepsilon}\mathrm{Tr}(\tilde{P})}\right) \quad (40)$$

where $\mu > 0$ and $\lambda \in [0, 1)$ satisfy

$$\|(A+BK)^t\| \leq \mu\lambda^t. \quad (41)$$

Note that the existence of such μ and λ are guaranteed under the condition $\rho(A+BK) < 1$.

Proof: By replacing A, D, and v_t in (1) and d in (3) by $A+BK$, BK, and e_t in (31) and σ_1 in (36), respectively, the result follows from Theorem 5.2. ∎

Corollary 7.1 provides a way to design an event-triggered condition. One can also find an ultimate bound for a given event-triggered condition; given $\sigma_1 = \sigma_1' > 0$ in (39), an ultimate bound is given by

$$r = \frac{\mu\|BK\|}{1-\lambda}\sigma_1' + \sqrt{\frac{1}{\varepsilon}\mathrm{Tr}(\tilde{P})}. \quad (42)$$

This should be a more direct corollary of Theorem 5.2.

A similar result can be obtained using the error threshold on the control input error.

Corollary 7.2: Suppose $r > 0$ is chosen to satisfy the condition (37). The use of the static event-triggered condition

$$\phi(x_t, \hat{x}_t) = \|K(\hat{x}_t - x_t)\| = \|Ke_t\| > \sigma_2 \quad (43)$$

with

$$\sigma_2 \leq \frac{1-\lambda}{\mu\|B\|}\left(r - \sqrt{\frac{1}{\varepsilon}\mathrm{Tr}(\tilde{P})}\right) \tag{44}$$

in (36) guarantees ultimate boundedness with r.

Proof: Similarly to Corollary 7.1, replace A, D, and v_t in (1) and d in (3) by $A + BK$, B, and Ke_t in (31) and σ_2 in (36), respectively, the result follows from Theorem 5.2. ■

C. Positively Invariant Control

Event-triggered control strategies that guarantee positive invariance are followed from Theorem 6.2.

Corollary 7.3: Suppose

$$r > \frac{1}{1-\|A+BK\|}\sqrt{\frac{1}{\varepsilon}\mathrm{Tr}(\Sigma_w E^\top E)} \tag{45}$$

is chosen. Then, the use of the event-triggered condition

$$\phi(x_t, \hat{x}_t) = \|\hat{x}_t - x_t\| = \|e_t\| > \sigma_3 \tag{46}$$

in (36) guarantees positive invariance with (21) if σ_3 satisfies

$$\sigma_3 \leq \frac{1}{\|BK\|}\left((1-\|A+BK\|)r - \sqrt{\frac{1}{\varepsilon}\mathrm{Tr}(\Sigma_w E^\top E)}\right). \tag{47}$$

Proof: Follows from Theorem 6.2. ■

A similar result can be obtained using the error threshold on the control input error:

Corollary 7.4: Suppose $r > 0$ is chosen to satisfy the condition (45). The use of the static event-triggered function

$$\phi(x_t, \hat{x}_t) = \|K(\hat{x}_t - x_t)\| = \|Ke_t\| > \sigma_4\|x_t\| \tag{48}$$

with

$$\sigma_4 \leq \frac{1}{\|B\|}\left((1-\|A+BK\|)r - \sqrt{\frac{1}{\varepsilon}\mathrm{Tr}(\Sigma_w E^\top E)}\right). \tag{49}$$

in (36) guarantees positive invariance with (21).

Proof: Follows from Theorem 6.2. ■

VIII. Numerical Examples

This section illustrates the proposed notions and the performances of the risk-aware event-triggered controllers.

A. Relations Between ε, Σ_w and r

Here, we illustrate the meanings of Theorems 5.2 and 6.2 by investigating how r in (17) and (22) of the ultimate bound and positive invariant set are affected as ε (risk level) or Σ_w (covariance) varies for the system (1) with

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0 & 0.5 \end{bmatrix},\ D = \begin{bmatrix} 0 & 0.7 \\ 0.2 & 0.5 \end{bmatrix}$$

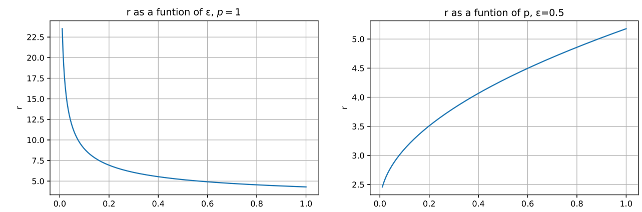
$$E = \begin{bmatrix} 1 & 2 \\ 0.5 & -0.5 \end{bmatrix},\ \Sigma_w = p\begin{bmatrix} 0.5 & 0 \\ 0 & 0.25 \end{bmatrix}$$

$$d = 0.3. \tag{50}$$

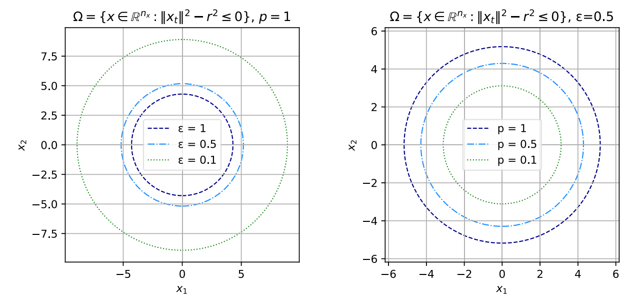
The covariance matrix Σ_w is varied by varying the value of p.

Fig. 2 illustrates (17), where Fig. 2(a) shows the ultimate bounds for different parameter values of ε and p, and Fig. 2(b) displays the plot of the boundaries of the worst-case CVaR ultimate bound sets. Fig. 3 represents the illustration of (22), with Fig. 3(a) presenting the bounds for positively invariant sets for different parameter values of ε and p, and Fig. 3(b) depicting the boundaries of worst-case CVaR robust positively invariant sets.

Those figures show similar patterns. As ε approaches 0, r values increase toward infinity, and as ε approaches 1, they decrease. Thus,

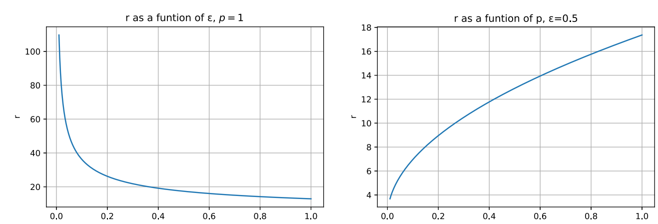


(a)

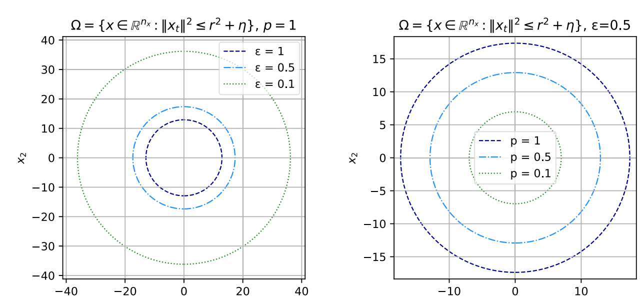


(b)

Fig. 2. Illustration of (17). (a) Ultimate bounds for different parameter values of ε (risk level) and p (covariance parameter). (b) Boundaries of worst-case CVaR ultimate bound sets $\Omega = \{x \in \mathbb{R}^{n_x} : \|x_t\|^2 - r^2 \leq 0\}$ for different parameter values of ε (risk level) and p (covariance parameter).



(a)



(b)

Fig. 3. Illustration of (22). (a) Bounds for positively invariant sets for different parameter values of ε (risk level) and p (covariance parameter). (b) Boundaries of worst-case CVaR robust positively invariant sets $\Omega = \{x \in \mathbb{R}^{n_x} : \|x_t\|^2 \leq r^2 + \eta\}$ for different parameter values of ε (risk level) and p (covariance parameter) with $\eta = 0.1$.

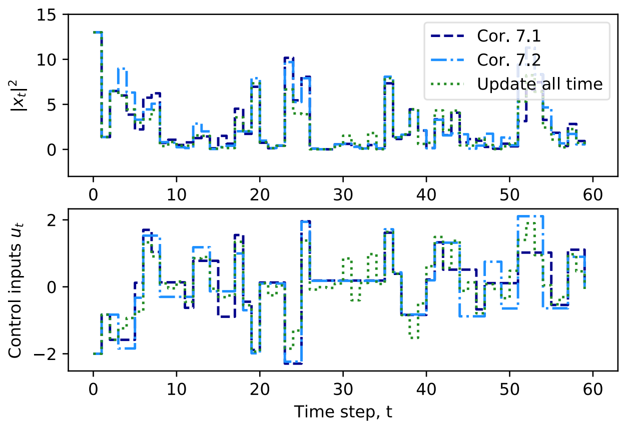


Fig. 4. Event-triggered controller performances and control inputs (ultimate boundedness).

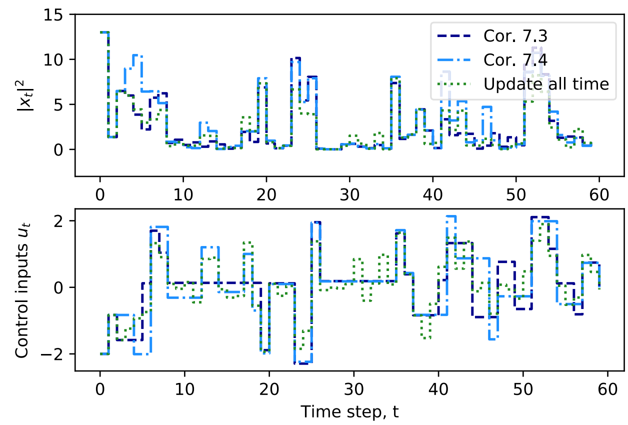


Fig. 5. Event-triggered controller performances and control inputs (positive invariance).

smaller ε results in larger worst-case CVaR ultimate bound sets and robust positively invariant sets. On the other hand, smaller p for the covariance Σ_w leads to smaller r values, resulting in smaller worst-case CVaR ultimate bound sets and robust positively invariant sets.

The figures emphasize the importance of considering varying risk levels and covariance in order to understand the behavior of the tail risks. For instance, consider Fig. 2(a); when estimating the value of r using the expected value (which corresponds to $\varepsilon = 1$), the r value is roughly 4.3. However, if the tail with $\varepsilon = 0.2$ is taken into account, the value of r becomes approximately 6.9. Consequently, if a controller is designed (e.g., A symbolizes a component of a system along with the controller) with the objective of achieving an ultimate bound of less than 5 based on the expected value, the tail will not meet this target.

B. Risk-Aware Event-Triggered Controllers

This subsection provides numerical examples of risk-aware event-triggered controllers developed earlier in this section. The performances are compared with nonevent-triggered controllers to see the effect of update reductions using the event-triggered conditions.

Consider the system (31) with

$$A = \begin{bmatrix} 1.2 & 0.3 \\ 0 & 0.5 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \; E = \begin{bmatrix} 1 & 2 \\ 0.5 & -0.5 \end{bmatrix}$$
$$x_0 = [2 \quad 3]^\top \tag{51}$$

subject to the zero-mean Gaussian disturbance with the covariance

$$\Sigma_w = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.25 \end{bmatrix} \tag{52}$$

and the state feedback gain

$$K = [-0.7 \quad -0.2]. \tag{53}$$

Choose $\varepsilon = 0.3$ to compute the worst-case CVaR. Namely, we focus our attention on the tail behavior of the worst 30%.

To see the performances of the controllers with Corollaries 7.1 and 7.2, choose $\mu = 1$, $\lambda = \|A + BK\|$, $r = 6 > \sqrt{\frac{1}{\varepsilon}\mathrm{Tr}(P)} = 2.94$ and $\sigma_1 = 1.36$, $\sigma_2 = 0.99$, which satisfy (40) and (44) with equalities, respectively. With those parameters, the event-triggered control performances and control inputs as well as those of a standard state feedback controller that updates the control input all the time are shown in Fig. 4. It is observed that the numbers of the control input updates were reduced to 27 and 25 during 60 time steps, respectively, while $\|x_t\|^2$ is always smaller than $15 < 6^2 = 36$, thus ultimate boundedness is achieved.

To see the performances of the controllers with Corollaries 7.3 and 7.4, choose $r = 10 > \sqrt{\frac{1}{(1-\|A+BK\|)^2}\frac{1}{\varepsilon}\mathrm{Tr}(\Sigma_w E^\top E)} = 6.54$ and $\sigma_3 = 1.52$, $\sigma_4 = 1.11$, which satisfy (47) and (49) with equalities, respectively. With those parameters, the event-triggered control performances and control inputs as well as those of a periodic controller are shown in Fig. 5. The numbers of control input updates were 24 and 23 during 60 time steps, respectively.

In all cases, it is observed that the number of control inputs achieved a 50% reduction while achieving the objectives. We see that the norms of the system states for the event-triggered controllers are larger than that of the non event-triggered controllers, as expected, due to the update reduction. The reductions and state norms are heavily dependent on the threshold σ. As we decrease the size of ε, we focus more on the tail behaviors thus reducing σ in (36) and increasing the frequency of the updates. On the other hand, large r and ε, and a small $\|A + BK\|$ increase the size of σ, thus reducing the number of updates.

IX. Conclusion

This article introduced the concepts of stability, ultimate boundedness, and positive invariance for stochastic systems using the worst-case CVaR to quantify the tail behavior of the stochastic systems. The benefit of the introduced notions is that they allow us to take into account the quantified tail risks in the controller design. In the controller design, the risk level $\varepsilon \in (0, 1)$ is a design parameter; which determines how much of the worst cases we focus on. If ε is close to 1, we are basically considering the mean performance, and if ε is close to 0, we are focusing on the performances of the worst cases. This article clarified how ε and the bounds r of the ultimate bound and the positive invariance are related.

It should be acknowledged that the use of the worst-case CVaR may lead to conservative results due to focusing on the distribution with the worst possible tails. However, as for many other robust control approaches, this conservatism is necessary for risk-aware control, where the goal is to design controllers that guarantee a certain level of performance even in the worst-case scenarios. Moreover, although inflating the terms that account for the covariance of the disturbance (e.g., $\mathrm{Tr}(\tilde{P})$ and $\mathrm{Tr}(\Sigma_w E^\top E)$) by $1/\varepsilon$ could appear crude, introducing this $1/\varepsilon$ factor is vital for addressing varying risk levels. This article clarified the relationship between the acceptable risk level, covariance, and its effects on r, which are used in the design of the controller, offering practitioners a straightforward and easily implementable method to deal with tail risks.

The introduced notions based on the worst-case CVaR are fundamental and applicable to systems where conventional stability, ultimate boundedness, and positive invariance have been considered such as model predictive controls and beneficial where the worst-case behaviors should be taken into account.

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