On the Role of Convexity/Concavity in Vector Fields, Flows, and Stability/Stabilizability

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***Abstract*—It is shown that strong convexity/concavity of a component of the vector field, as a function of the state variables, induces the same property on the corresponding component of the *flow*, as a function of the initial condition. Such an inherited property is then instrumental, for instance, for establishing several instability theorems, the proofs of which rely precisely on consequences of convexity/concavity of the *flow* with respect to the initial condition. Furthermore, the property of convexity/concavity permits the construction of a *canonical* Chetaev function to certify instability without explicitly resorting to the computation of the flow. Finally, necessary conditions for continuous stabilizability are derived, hence putting the properties of convexity/concavity of the vector field in relation to the well-known Brockett's theorem.**

***Index Terms*—Convex/concave functions, nonlinear systems, stability of equilibrium points.**

I. Introduction

IMPOSING the property of (asymptotic) stability to an equilibrium point or to a motion is probably the primary and unavoidable objective of any control system. It is therefore not surprising that, since the seminal paper of Lyapunov [1] (see also [2]), stability theory has received continual and increasing attention in the literature. In the last century several attempts have been pursued to generalize and extend the claims of [1] in different directions [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13] and for wider classes of linear and nonlinear systems [14], [15], [16], [17].

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Lyapunov's direct method allows concluding stability properties of a certain equilibrium point for linear and nonlinear systems while avoiding the explicit computation of the solution of the underlying ordinary differential equation. Thus, intensive research efforts have been devoted to the systematic construction of the so-called *Lyapunov functions* [18], [19], [20]. Nonetheless, it is worth observing that the availability of *easily verifiable* conditions to imply a certain property does not prevent simultaneous advancements in a deeper understanding of the nature of the property. In the case of stability theory, for instance, overcoming the dichotomy between systematic approaches devoted to study *"how to check"* that a certain property holds (constructing Lyapunov functions) and the mechanism *"why"* the property arises (establishing bounds on trajectories) has stimulated a significant enhancement of the understanding of the relevant underlying phenomena, i.e., alternative points of view on the study of the topic may benefit one from the other. As a few examples illustrating the effect of the intertwining of the two approaches, note that arguments based on the original definition of the notion of stability, in terms of bounds on the ensuing trajectories, led to a further elaboration on the definition of the property in [3] (related essentially to the concept of uniformity in time), which is interestingly implied by the same sufficient conditions as for the original definition in [1]. On the contrary, once the intuition on the implications of the property is solid, it may be possible to envision sufficient conditions that are built on completely different premises with respect to existing techniques. This has been, e.g., pursued in [12] by introducing conditions that are *dual* to the celebrated second Lyapunov's method, although yielding identical conclusions on the nature of the time-evolution of the solutions to a certain dynamic system.

In addition to the direct method discussed above, several *converse* results have been established [21], [22], [23], [24], [25], [26], [27]. These Converse Lyapunov Theorems imply the existence of a function with certain properties in the presence of asymptotically stable equilibrium points of the underlying dynamics.

By somewhat mirroring the point of view under which the issue of stability is approached, a few *instability* theorems have been proposed in the literature hitherto, the most popular among them being probably that stated by Chetaev [28]. While the literature concerning instability theorems is considerably more limited than the stability counterpart, alternative conditions for instability have been proposed in [29], based on the divergence of the underlying vector field, in [30], in which necessary conditions for instability are discussed, and in [31], in which the concept of stability restricted to the positive orthant is considered.

Finally, by building on the intuition that sufficient conditions for instability constitute in fact obstructions for imposing the property of asymptotic stability via the selection of a feedback control action, in [32], [33], and [34] the authors provided necessary conditions for smooth stabilizability, including the celebrated Brockett's Theorem discussed in [32].

The main contribution of this article is twofold. First, it is shown that the properties of strong convexity or concavity of a component of the vector field with respect to the state variable imply that the corresponding *flow*, interpreted as a function of the initial condition, exhibits in turn properties of convexity or concavity, respectively. In the second part of the article, instead, it is proved that the above implication is further instrumental to state several instability theorems based on convexity/concavity of the underlying flow with respect to the initial condition, even whenever the latter properties hold only locally in time. Furthermore, it is shown that convexity/concavity of a component of the vector field permits the construction of a *canonical* Chetaev function, which allows us to systematically verify instability of an equilibrium point, hence somewhat complementing the discussion about the abstract (trajectory-based) nature of the property of instability arising in the presence of convex/concave vector fields. This is inspired by the spirit of reconciling the dichotomy mentioned above in the case of stability. Finally, similar ideas are extended to the setting of controlled systems by providing necessary conditions for continuous stabilizability of a certain equilibrium point by means of smooth feedback. These statements, in turn, establish a connection between the well-known Brockett's Theorem and the properties of convexity and concavity of a component of the vector field.

The rest of the article is organized as follows. A few preliminary definitions and results are briefly recalled for completeness in Section II. In Section III, we establish an implication between the property of convexity or concavity of a component of the underlying vector field with respect to the state and an identical property of the corresponding flow with respect to the initial condition. Then, the above implication is instrumental for stating several instability theorems for nonlinear systems. These are discussed in detail in Section IV. Finally, similar considerations are extended to the case of control system in Section V, before conclusions are drawn in Section VI.

Notation: The set $\mathcal{C}^\kappa(\mathbb{R}^n)$, $\kappa \geqslant 0$, denotes the space of functions defined on $\mathbb{R}^n$ that admit continuous derivatives of order κ. Given a function $h \in \mathbb{R} \to \mathbb{R}$ in $\mathcal{C}^2(\mathbb{R})$, $\frac{dh}{dx}$ and $\frac{d^2h}{dx^2}$ denote the first and second derivatives, respectively. Given a multivariable function $g : \mathbb{R}^n \to \mathbb{R}$ in $\mathcal{C}^2(\mathbb{R}^n)$, $\frac{\partial g}{\partial x}(x) \in \mathbb{R}^{1\times n}$ denotes the gradient of g at x, whereas $\frac{\partial^2 g}{\partial x^2}(x) \in \mathbb{R}^{n\times n}$ denotes the Hessian matrix of g at x. For a vector-valued function $f : \mathbb{R}^n \to \mathbb{R}^n$, $\frac{\partial f}{\partial x}$ denotes its Jacobian matrix. Let $\mathcal{U}$ denote a subset of $\mathbb{R}^n$. Then $\partial\mathcal{U}$ and $\mathcal{U}^\circ$ denote the boundary and the interior, respectively, of $\mathcal{U}$. The notation $B_\varepsilon(x_0)$ describes the set $B_\varepsilon(x_0) := \{x \in \mathbb{R}^n : \|x - x_0\| < \varepsilon\}$.

II. Definitions and Preliminaries

Consider a nonlinear, autonomous, dynamical system described by the equation

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \tag{1}$$

with $x(t) \in \mathbb{R}^n$ denoting the state of the system and $x_0 \in \mathbb{R}^n$ prescribing the initial condition. The vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, $f \in \mathcal{C}^2(\mathbb{R}^n)$ is such that $f(0) = 0$, i.e., without loss of generality it is assumed that the origin is an equilibrium point of (1). The solution of the ordinary differential (1), parameterized with respect to time t and the initial condition x_0, is denoted by the *flow* $\varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $\varphi_t(x_0) = \varphi(t, x_0)$, which satisfies the equation

$$\frac{\partial}{\partial t}\varphi(t, x_0) = f\left(\varphi(t, x_0)\right) \tag{2}$$

for any $t \in \mathcal{I}_{x_0} \subseteq \mathbb{R}$, together with the boundary condition $\varphi(0, x_0) = x_0$, for all $x_0 \in \mathbb{R}^n$. The set $\mathcal{I}_{x_0}$ denotes the *maximal interval of existence* of the solution of the differential (2) passing through $x_0 \in \mathbb{R}^n$ at time $t = 0$. It is worth observing that the function $x_0 \mapsto \varphi_t(x_0)$ inherits regularity properties identical to the underlying f whenever it exists, hence it is at least twice continuously differentiable, see, e.g., [35]. Furthermore, as well known, the *flow* φ_t possesses also an interesting *fixed-point* characterization, namely

$$\varphi_t(x) = x + \int_0^t f\left(\varphi_s(x)\right) ds. \tag{3}$$

Note that, by slightly abusing the notation, the variable $x \in \mathbb{R}^n$ in (3) (as well as throughout the rest of the manuscript, whenever it does not create confusion) defines the initial condition of (1), rather than the state variable of (1).

The main objective of this article consists in characterizing the properties of convexity and concavity of the flow with respect to the initial condition $x \in \mathbb{R}^n$ and with respect to time $t \in \mathcal{I}_x$. Moreover, these properties are further put in relation with the property of stability (or instability) of equilibrium points. Toward this end, the notion of convex (multivariable) function is recalled in the following definition.

Definition 1: Consider a continuous function $g : \mathbb{R}^n \to \mathbb{R}$. The function is said to be *convex* in a set $\Omega \subseteq \mathbb{R}^n$ if its epigraph $\{(x, c) \in \Omega \times \mathbb{R} : c \geqslant g(x)\}$ is a convex set. A function g is *concave* if $-g$ is convex. ◦

More practical characterizations of convexity and of concavity have been provided. In fact, a continuously differentiable function $g : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if

$$g(x_2) \geqslant g(x_1) + \frac{\partial g}{\partial x}(x_1)\,(x_2 - x_1) \tag{4}$$

for all $x_i \in \mathbb{R}^n$, $i = 1, 2$.

Definition 2: Consider a continuous function $g : \mathbb{R}^n \to \mathbb{R}$. The function is said to be *strongly convex* in a set $\Omega \subseteq \mathbb{R}^n$ if there exists $\alpha > 0$ such that $g(x) - \frac{\alpha}{2}\|x\|^2$ is convex or, equivalently,

$$g(x_2) \geqslant g(x_1) + \frac{\partial g}{\partial x}(x_1)\,(x_2 - x_1) + \frac{\alpha}{2}\|x_2 - x_1\|^2 \tag{5}$$

for all $x_i \in \Omega$, $i = 1, 2$. ◦

Although the following technical lemma is stated for simplicity in the case of convex functions, the claims may be extended to the case of concave functions.

Lemma 1: Consider two strongly convex functions $g_i : \mathbb{R} \to \mathbb{R}$. Suppose that g_2 is nondecreasing. Then, the composition $g_2 \circ g_1$ is strongly convex. ◦

Proof: To begin with, recall that a function $g_i : \mathbb{R} \to \mathbb{R}$ is strongly convex if and only if there exists $\mu_i > 0$ such that

$$g_i\left(\alpha x_1 + (1-\alpha)x_2\right) \leqslant \alpha g_i(x_1) + (1-\alpha)g_i(x_2) - \frac{\alpha(1-\alpha)}{2}\mu_i\|x_1 - x_2\|^2 \quad (6)$$

for all $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}$ and $\alpha \in [0,1]$. Thus, one has that

$$\begin{aligned} g_2 \circ g_1\left(\alpha x_1 + (1-\alpha)x_2\right) &= g_2\left(g_1\left(\alpha x_1 + (1-\alpha)x_2\right)\right) \\ &\leqslant g_2\left(\alpha g_1(x_1) + (1-\alpha)g_1(x_2)\right) \\ &\leqslant \alpha g_2\left(g_1(x_1)\right) + (1-\alpha)g_2\left(g_1(x_2)\right) \\ &\quad - \frac{\alpha(1-\alpha)}{2}\mu_2\|x_1 - x_2\|^2 \end{aligned} \quad (7)$$

where the first inequality follows from the nondecreasing property of g_2 and strong convexity of g_1, which ensures that $g_1(\alpha x_1 + (1-\alpha)x_2) \leqslant \alpha g_1(x_1) + (1-\alpha)g_1(x_2)$. The last inequality follows from strong convexity of g_2 and implies strong convexity of the function $g_2 \circ g_1$ by inspecting the definition in (6). □

Finally, whenever g is a twice continuously differentiable function mapping $\mathbb{R}^n$ to $\mathbb{R}$, then g is convex if and only if the Hessian matrix is positive semidefinite, while g is strongly convex if and only if $(\partial^2 g(x)/\partial x^2) \geqslant \alpha I > 0$, for all x and some $\alpha > 0$.

III. Convexity and Concavity of Flows With Respect to Initial Conditions and Time

The main objective of this section consists in characterizing *convexity* and *concavity* properties of the flow φ_t with respect to time and to the initial condition $x \in \mathbb{R}^n$. Toward this end, and to provide a concise statement of the result, let the functions $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, n$ be such that

$$f(x) := \begin{bmatrix} f_1\left(x_1, \ldots, x_n\right) \\ f_2\left(x_1, \ldots, x_n\right) \\ \vdots \\ f_n\left(x_1, \ldots, x_n\right) \end{bmatrix} \quad (8)$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ defined from (1). An identical partitioning is assumed to hold also for the components of the resulting flow $\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$, namely $\varphi_t(x) = [\varphi_t^1(x), \ldots, \varphi_t^n(x)]^\top$.

Theorem 1: Consider the system (1). Let Ω be any arbitrary compact subset of $\mathbb{R}^n$. Suppose that the function $f_i : \mathbb{R}^n \to \mathbb{R}$, for some $i \in \{1, \ldots, n\}$, is strongly convex (concave) for all $x \in \Omega$. Then, the ith component of the flow φ_t^i is a strongly convex (concave) function of $x \in \Omega$, for all $t \in [t_\ell, t_u) \subset \mathcal{I}_x$, for any $t_\ell > 0$, and for some $t_u > 0$ such that $\varphi_t(x) \in \Omega$. ◦

Proof: Note that the claims are shown explicitly in the case of strongly convex vector fields, while the proof for the concave case can be immediately obtained by relying on "symmetric" arguments.[1] To show the claim, note first that the fixed-point characterization (3) of the flow implies that, for any pair $(x, x') \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\begin{aligned} \delta\varphi_t^i\left(x, x'\right) &:= \varphi_t^i(x) - \varphi_t^i(x') + \frac{\partial \varphi_t^i}{\partial x}(x)(x' - x) \\ &= x_i + \int_0^t f_i\left(\varphi_s(x)\right) ds - x_i' - \int_0^t f_i\left(\varphi_s\left(x'\right)\right) ds \\ &\quad + \left(e_i^\top + \int_0^t \frac{\partial f_i}{\partial x}\left(\varphi_s(x)\right) \frac{\partial \varphi_s}{\partial x}(x) ds\right)(x' - x) \\ &= \int_0^t \Big(f_i\left(\varphi_s(x)\right) - f_i\left(\varphi_s\left(x'\right)\right) \\ &\quad + \frac{\partial f_i}{\partial x}\left(\varphi_s(x)\right) \frac{\partial \varphi_s}{\partial x}(x)\left(x' - x\right)\Big) ds \end{aligned} \quad (9)$$

with $e_i \in \mathbb{R}^n$ denoting the ith element of the *canonical basis* of $\mathbb{R}^n$ and where the first equality is immediately obtained by the property of the flow in (3) and by observing that (see also (27) below and [35, Sec. 3.3])

$$\frac{\partial \varphi_t^i}{\partial x}(x) = e_i^\top + \int_0^t \frac{\partial f_i}{\partial x}\left(\varphi_s(x)\right) \frac{\partial \varphi_s}{\partial x}(x) ds. \quad (10)$$

The second-order (exact) Taylor expansion of the function $x' \mapsto f_i(\varphi_s(x'))$ around the point $x \in \mathbb{R}^n$ is provided by

$$\begin{aligned} f_i(\varphi_s(x')) = {} & f_i(\varphi_s(x)) + \frac{\partial f_i}{\partial x}(\varphi_s(x)) \frac{\partial \varphi_s}{\partial x}(x)(x' - x) \\ & + \frac{1}{2}(x' - x)^\top \left[\frac{\partial \varphi_s}{\partial x}(z)^\top \frac{\partial^2 f_i}{\partial x^2}(\varphi_s(z)) \frac{\partial \varphi_s}{\partial x}(z) \right. \\ & \left. + \frac{\partial}{\partial x}\left(\frac{\partial \varphi_s}{\partial x}(z)^\top \frac{\partial f_i}{\partial x}(\varphi_s(\lambda))^\top\right)\Bigg|_{\lambda = z}\right](x' - x) \end{aligned} \quad (11)$$

for a certain $z \in \mathbb{R}^n$ that is convex combination of x and x'. By inserting (11) into (9) it follows that

$$\begin{aligned} \delta\varphi_t^i(x, x') = {} & -\frac{1}{2}(x' - x)^\top \int_0^t \left[\frac{\partial \varphi_s}{\partial x}(z)^\top \frac{\partial^2 f_i}{\partial x^2}(\varphi_s(z)) \frac{\partial \varphi_s}{\partial x}(z) \right. \\ & \left. + \frac{\partial}{\partial x}\left(\frac{\partial \varphi_s}{\partial x}(z)^\top \frac{\partial f_i}{\partial x}(\varphi_s(\lambda))^\top\right)\Bigg|_{\lambda = z}\right] ds (x' - x) \\ =: {} & -\frac{1}{2}\left(x' - x\right)^\top \int_0^t \Psi(z, s) ds \; \left(x' - x\right). \end{aligned} \quad (12)$$

Since, by assumption, f_i is strongly convex, there exists $\alpha > 0$ such that $(\partial^2 f_i(x)/\partial x^2) \geqslant \alpha I > 0$ for all $x \in \Omega$. Furthermore, since $\varphi_0(x) = x$ for all $x \in \Omega$ one has that

$$\frac{\partial \varphi_0}{\partial x}(x) = I \quad (13)$$

while the components of the second derivative of the flow, appearing in the last term of Ψ in (12) are equal to zero, i.e.,

$$\frac{\partial^2 \varphi_0^j}{\partial x_k \partial x_\ell}(x) = 0 \quad (14)$$

[1]The same approach of limiting the proofs to the convex case is employed throughout the manuscript.

for $j = 1,\ldots,n$, $k = 1,\ldots,n$, and $\ell = 1,\ldots,n$. Thus, by continuity of the involved functions with respect to time and initial conditions and by compactness of the set Ω, there exist $\bar{\alpha} > 0$ and $\bar{t} > 0$ such that $\Psi(z,s) > \bar{\alpha} I$ for any $z \in \Omega$ and for all $s \in [0,\bar{t})$. Therefore, (12) implies that

$$\delta\varphi_t^i(x,x') < -\frac{1}{2}\|x'-x\|^2\bar{\alpha}t \leqslant -\frac{1}{2}\|x'-x\|^2\alpha_\ell \qquad (15)$$

with (constant) $\alpha_\ell := \bar{\alpha}t_\ell > 0$, for any $t_\ell \in (0,\bar{t})$ and for any pair $(x,x') \in \Omega\times\Omega$, which in turn ensures strong convexity of $\varphi_t^i(x)$ for all $t \in [t_\ell,\bar{t})$. □

The conclusions of Theorem 1 are now specialized to the case of scalar systems, namely (1) with $n = 1$. In this case, the claims can be immediately extended to hold globally with respect to t in $\mathcal{I}_x$, while also providing a converse implication.

Corollary 1: Consider the system (1) with $n = 1$ and the associated function $x \mapsto \varphi_t(x)$ for fixed $t \geqslant 0$. Let Ω be any arbitrary compact subset of $\mathbb{R}$.

i) Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is strongly convex (concave) in Ω. Then, the flow φ_t is a *strongly convex (concave)* function of $x \in \Omega$ for all $t > 0$ such that $\varphi_t(x) \in \Omega$.

ii) Suppose that the flow $\varphi_t(x)$ is a *convex (concave)* function of x for all $t \in \mathcal{I}_x$ such that $\varphi_t(x) \in \Omega$. Then, the function $f : \mathbb{R} \to \mathbb{R}$ is convex (concave) for all $x \in \Omega$. ◦

Proof: i) Strong convexity (concavity) of $\varphi_t(x)$ for all $x \in \Omega$ and all $t \in (t_\ell, t_u)$ has been shown in the proof of Theorem 1. Fix now $\tau \in (t_\ell,t_u)$ and $\tau' \in (t_\ell,t_u)$ with the property that $\tau + \tau' > t_u$. Hence, φ_τ and $\varphi_{\tau'}$ are strongly convex functions of x. It then follows, by Lemma 1 and by recalling that the flow φ_t of (1) with $n = 1$ is nondecreasing with respect to x, that the composition $\varphi_\tau \circ \varphi_{\tau'}(x) = \varphi_\tau(\varphi_{\tau'}(x))$ is strongly convex (concave). Therefore, by relying on the *semigroup property* of the flow, one has that the flow $\varphi_{\tau+\tau'}(x) = \varphi_\tau(\varphi_{\tau'}(x))$ is strongly convex, with $\tau + \tau' > t_u$. By iterating the same argument it follows that φ_t is strongly convex for all $t \in \mathcal{I}_x$ such that $\varphi_t(x)$ remains in Ω.

ii) The second claim is proved by contradiction. To this end, suppose that φ_t is a convex function of x, whereas the hypothesis is contradicted. Since the latter entails that, $\forall x \in \Omega$, $(d^2 f(x)/dx^2) \geqslant 0$, negating the hypothesis implies the existence of $\tilde{x} \in \Omega$ such that $(d^2 f(\tilde{x})/dx^2)$ is strictly negative. As a consequence, by continuity of the second-order derivative there exist $\delta > 0$ and a nonempty open interval $(\underline{x},\bar{x}) \subset \mathbb{R}$ such that $(d^2 f(\tilde{x})/dx^2) < -\delta$ for all $x \in (\underline{x},\bar{x})$. Since $\varphi_t(x)$ is convex, the inequality (15) must hold. However, for any x' such that the *intermediate point* z belongs to $(\underline{x},\bar{x})$ it follows that for any $0 < \varepsilon_1 < \delta$ and $\varepsilon_2 > 0$ there exists $\bar{s}$ such that *(i)* $(d^2 f(\varphi_s(z))/dx^2) < -\varepsilon_1$, *(ii)* $(d\varphi_s(z)/dx) > 1 - \varepsilon_2$, *(iii)* $(df(\varphi_s(z))/dx)(d^2\varphi_s(z)/dx^2) < \varepsilon_2$ for all $s \in (0,\bar{s})$. Items *(ii)* and *(iii)* hold by continuity of the involved functions and by recalling that

$$\left.\frac{d\varphi_s(z)}{dx}\right|_{s=0} = 1\,, \qquad \frac{df(\varphi_s(z))}{dx}\left.\frac{d^2\varphi_s(z)}{dx^2}\right|_{s=0} = 0\,.$$

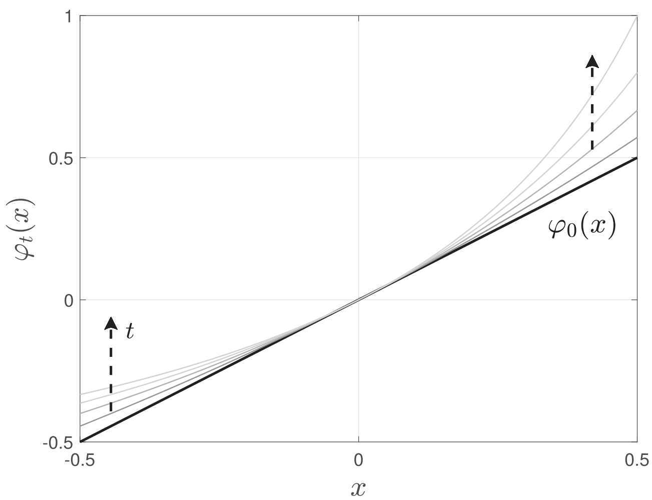

Fig. 1. Graph of the flow $x \mapsto \varphi_t(x)$ of the differential (16) for various values of the time t in the interval [0, 1]. The dashed arrows indicate the sequence of flows for increasing time.

The claim is shown by observing that any selection of ε_i with the property that $-\varepsilon_1(1-\varepsilon_2)^2 + \varepsilon_2 < 0$ would violate the inequality (15), reformulated according to (12), for any $0 < t < \bar{s}$, hence contradicting convexity of $\varphi_t(x)$. □

Example 1: Consider the scalar nonlinear system

$$\dot{x} = x^2 \qquad (16)$$

with $x(t) \in \mathbb{R}$. Since $d^2 f(x)/dx^2 = 2 > 0$, the underlying vector field f is a strongly convex function of the state in $\mathbb{R}$. Therefore, by Corollary 1 the *flow* $\varphi_t(x)$ of the differential (16), which is described by the function

$$\varphi_t(x) = \frac{x}{1-tx} \qquad (17)$$

is a strongly convex function of the initial condition $x \in \mathbb{R}$. Indeed, the second-order derivative of the flow with respect to the initial condition yields

$$\frac{d^2\varphi_t}{dx^2}(x) = \frac{2t}{(1-tx)^3}$$

which is positive definite for all $t \in (0,1/x)$ if $x > 0$ and for all $t > 0$ if $x < 0$. The Fig. 1 shows the graph of the function $x \mapsto \varphi_t(x)$ as in (17) for several values of time uniformly distributed in the interval [0, 1] and for $x \in [-0.5, 0.5]$, hence such that $1 - tx > 0$. △

Remark 1: As entailed by the statement Theorem 1, whenever a scalar vector field f is strongly convex in a neighborhood Ω_1 of a certain point $x_1^\circ \in \mathbb{R}$ and strongly concave in a neighborhood Ω_2 of x_2°, then the convexity and concavity properties of the flow φ_t hold in Ω_1 and Ω_2, respectively. This aspect is illustrated via the following example. ▲

Example 2: Consider the scalar nonlinear system

$$\dot{x} = -x^3 \qquad (18)$$

with $x(t) \in \mathbb{R}$. It is straightforward to observe that the vector field f is a locally strongly convex function in a neighborhood of any $x < 0$ and a locally strongly concave function around any $x > 0$. As implied by Corollary 1 similar properties are inherited

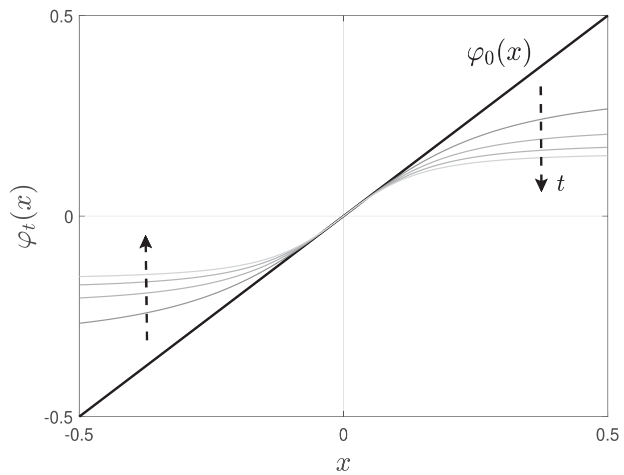

Fig. 2. Graph of the flow $x \mapsto \varphi_t(x)$ of the differential (18) for various values of the time t in the interval [0, 20]. The dashed arrows indicate the sequence of flows for increasing time.

by the corresponding flow φ_t, the graph of which is depicted in Fig. 2 for various values of time uniformly distributed in the interval [0, 20]. △

Finally, the convexity and concavity properties of the function $t \mapsto \varphi_t(x)$—namely convexity (concavity) of the flow with respect to time for a given initial condition—are characterized in the following statement.

Theorem 2: Consider the system (1) and fix $x^\circ \in \mathbb{R}^n$. Suppose that, for some integer $i \in \{1, \ldots, n\}$,

$$\left\langle \frac{\partial f_i(x^\circ)}{\partial x}, f(x^\circ) \right\rangle > 0 \quad (< 0, \text{resp.}). \tag{19}$$

Then the ith component of the flow φ_t^i is a strongly convex (concave, resp.) function of $t \in \mathbb{R}$, locally around x°. ◦

Proof: The claim is proved by computing the second-order derivative of the scalar function $\varphi_t^i(x)$ for fixed $x \in \mathbb{R}^n$. In fact, one has that the first-order derivative with respect to time is, by definition of *flow*, such that

$$\frac{\partial \varphi_t^i}{\partial t}(x) = f_i(\varphi_t(x)). \tag{20}$$

Therefore, the second-order derivative is obtained as

$$\begin{aligned}\frac{\partial^2 \varphi_t^i}{\partial t^2}(x) &= \frac{\partial f_i}{\partial x}(\varphi_t(x))^\top \frac{\partial \varphi_t}{\partial t}(x) \\ &= \frac{\partial f_i}{\partial x}(\varphi_t(x))^\top f(\varphi_t(x)).\end{aligned} \tag{21}$$

The proof is then concluded by observing that the condition (19) ensures that the second-order derivative (21) is strictly positive (negative, resp.) in a neighborhood of $t = 0$, by continuity with respect to time and by observing that $\varphi_0(x) = x$ by definition of *flow*. □

The section is concluded by discussing two technical lemmas that are instrumental for relating the properties of convexity (concavity) of a vector field with those of stability of an underlying equilibrium point, which is the main objective of Section IV.

Lemma 2: Consider the system (1). Suppose that there exists a set $\mathcal{U} \subseteq \mathbb{R}^n$, containing the origin, such that $f_i : \mathbb{R}^n \to \mathbb{R}$ is a strongly convex function in $\mathcal{U}$, for some i. Then for any $x_0 \in \mathcal{U} \setminus \{0\}$ and any $t_\ell > 0$ there exist $\alpha > 0$ and $t_u > 0$, depending on x_0, with the property that

$$\varphi_t^i(x_0) - \frac{\partial \varphi_t^i}{\partial x}(0) x_0 \geqslant \frac{\alpha}{2} \|x_0\|^2 > 0 \tag{22}$$

for all $t \in [t_\ell, t_u]$ such that $\varphi_t(x_0) \in \mathcal{U}$. ◦

Proof: By Theorem 1, strong convexity of f_i implies strong convexity of $x \mapsto \varphi_t(x)$ for any sufficiently small t strictly greater than any (even arbitrarily small) t_ℓ and for x in the set $\mathcal{U} \subset \mathbb{R}^n$. Therefore, by definition of strong convexity of φ_t and by adapting the definition in (5) and (15), it follows that

$$\varphi_t(x') - \varphi_t(x) - \frac{\partial \varphi_t}{\partial x}(x)(x' - x) \geqslant \frac{\alpha}{2} \|x' - x\|^2 \tag{23}$$

for a certain strictly positive α. Moreover, the inequality (23) immediately yields (22) by letting $x' = x_0$, $x = 0$ and by recalling that $\varphi_t^i(x) = \varphi_t^i(0) = 0$ for all $t \geqslant 0$.

Lemma 3: Consider the system (1). Suppose that there exists a set $\mathcal{U} \subseteq \mathbb{R}^n$, containing the origin, such that $f_i : \mathbb{R}^n \to \mathbb{R}$ is a strongly concave function in $\mathcal{U}$, for some i. Then for any $x_0 \in \mathcal{U} \setminus \{0\}$ and any $t_\ell > 0$ there exist $\alpha > 0$ and $t_u > 0$, depending on x_0, with the property that

$$\varphi_t^i(x_0) - \frac{\partial \varphi_t^i}{\partial x}(0) x_0 \leqslant -\frac{\alpha}{2} \|x_0\|^2 < 0 \tag{24}$$

for all $t \in [t_\ell, t_u]$ such that $\varphi_t(x_0) \in \mathcal{U}$. ◦

The proof of Lemma 3 can be obtained by following arguments identical to those of Lemma 2, hence it is omitted.

IV. Stability Properties of Convex or Concave Vector Fields

The connection between the properties of convexity or concavity of the underlying vector field and of stability of an equilibrium point is discussed in this section by building on the results introduced in the previous section.

A. Nonlinear Systems Without Linear Terms

The first statements deal with the case in which the vector field of the nonlinear system (1) does not possess linear terms locally around the equilibrium point at $x = 0$, that is the linearization of the system (1) around the origin is

$$\dot{\delta x} = \frac{\partial f}{\partial x}(x)\Big|_{x=0} \delta x := A \delta x \tag{25}$$

with $A = 0$. In particular, the following statement shows that strong convexity (concavity) of a component of the vector field prevents asymptotic stability of the underlying equilibrium point (see Fig. 3 for a graphical illustration).

Theorem 3: Consider the nonlinear system (1) and suppose that $A = 0$ in (25). Suppose that there exist an integer $i \in \{1, \ldots, n\}$ and a nonempty open set $\mathcal{U} \subset \mathbb{R}^n$, containing the origin, with the property that $f_i : \mathbb{R}^n \to \mathbb{R}$ is either a strongly convex or a strongly concave function for all $x \in \mathcal{U}$. Then, the origin cannot be an asymptotically stable equilibrium point of system (1). ◦

Proof: Consider the case of strongly convex functions. To begin with suppose that there exists a component f_i of the vector field f in (1) that is locally strongly convex with respect to x. Then, by the results of Theorem 1, the same property is inherited by the flow φ_t^i with respect to the initial condition $x_0 \in \mathcal{U}$. Moreover, by Lemma 2, the inequality (22) holds for

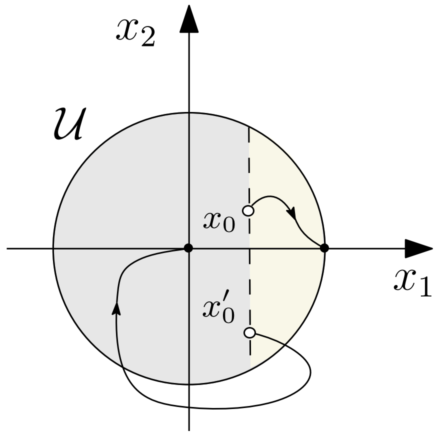

Fig. 3. Graphical description of the statement of Theorem 3. The gray region defines the set $\mathcal{U}$. The trajectories ensuing from the initial conditions x_0 and x_0' show that the equilibrium may be stable but not attractive or *vice-versa*, respectively.

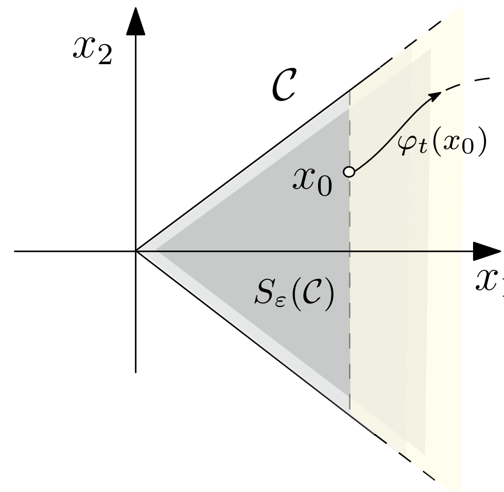

Fig. 4. Graphical interpretation of the statement of Corollary 2 with $i = 1$. The light gray region defines a positively invariant set $\mathcal{C} \subset \mathcal{K}_1^+$, while $S_\varepsilon(\mathcal{C})$ is described by the dark gray region. The trajectory ensuing from x_0 cannot leave the light (yellow) region by invariance of $\mathcal{C}$ and by strong convexity of the component $f_1(x)$.

any $t \geqslant t_\ell$ in some nonempty interval $[t_\ell, t_u] \subset \mathcal{I}_{x_0}$, where t_ℓ can be arbitrarily small, and for any $x \in \mathcal{U}$. Consider then the time evolution of the *sensitivity* matrix

$$S_{x_0}(t) := \frac{\partial \varphi_t}{\partial x_0}(x_0) \tag{26}$$

$S_{x_0} : \mathbb{R} \to \mathbb{R}^{n\times n}$, describing the partial derivative of the flow with respect to the initial condition. It can be shown (see, e.g., [35, Sec. 3.3]) that the sensitivity matrix S_{x_0} satisfies

$$\begin{aligned} \dot{x}(t) &= f(x(t)), & x(0) &= x_0 \\ \dot{S}_{x_0}(t) &= \tfrac{\partial f}{\partial x}(x(t))S_{x_0}(t), & S_{x_0}(0) &= I. \end{aligned} \tag{27}$$

Therefore, the sensitivity matrix $S_0(t)$, i.e., the time evolution of the derivative of the flow from the zero initial condition satisfies $\dot{S}_0(t) = (\partial f(0)/\partial x)S_0(t) = AS_0(t)$, since $\varphi_t(0) = 0$ for all $t \geqslant 0$. Thus, since by assumption $A = 0$, it follows that $S_0 \equiv I$. Let $\bar{x}_0 = [\,\star \;\; \dots \;\; \star \;\; \mu \;\; \star \;\; \dots \;\; \star\,]^\top \in \mathcal{U}$, where $\star$ denotes a generic real number and $\mu > 0$, in the ith position of $\bar{x}_0$, denotes an arbitrary positive constant. Then, since $(\partial \varphi_t^i/\partial x)(0)$ is equal to the ith row of the sensitivity matrix $S_0(t)$, namely $(\partial \varphi_t^i/\partial x)(0) = e_i^\top S_0(t) = e_i^\top$, where e_i denotes the ith element of the canonical basis of $\mathbb{R}^n$, it follows from (22) that

$$\varphi_t^i(\bar{x}_0) \geqslant e_i^\top \bar{x}_0 + \frac{\alpha}{2}\|\bar{x}_0\|^2 \geqslant e_i^\top \bar{x}_0 + \frac{\alpha}{2}\mu^2 > \bar{x}_0^i \tag{28}$$

for all $t \in [t_\ell, t_u]$. Thus, for all $x_0 \in \mathcal{U}$ such that $x_0^i > 0$, one has that $\varphi_t^i(x_0) > x_0^i$, locally with respect to time. Therefore, stability of the equilibrium point with respect to a value of ε, in the (ε, δ)-argument for Lyapunov stability, selected with the property that $\mathcal{U} \subseteq B_\varepsilon(0)$ would require that $\varphi_t(x_0) \in \mathcal{U}$ for all $t \geqslant 0$. However, the latter property in turn implies that the origin cannot be attractive, since φ_t^i is increasing, for any time, in the set $\mathcal{U}$. Conversely, attractivity of the equilibrium point implies that $\varphi_t(x_0)$ must necessarily leave $\mathcal{U}$ at a certain time, hence contradicting stability. Therefore, the proof is concluded by noting that the two properties of stability and attractivity of the origin cannot hold simultaneously. □

Remark 2: The arguments employed in the proof of Theorem 3 suggest the intuition and the rationale behind the corresponding formal statement: the property of convexity (concavity, resp.) of f_i *steers away* from the origin the ith component of the state trajectory x_i for positive (negative, resp.) initial conditions $x_i(0)$. This behavior motivates the following refinement of the sufficient conditions for instability stated in Theorem 3, the proof of which follows immediately from that of the latter. ▲

To provide a concise statement of the result, let $\mathcal{K}_i^+ \subset \mathbb{R}^n$ be the *half-space* defined as $\mathcal{K}_i^+ := \{x \in \mathbb{R}^n : x_i \geqslant 0\}$. Similarly, define $\mathcal{K}_i^- \subset \mathbb{R}^n$ as $\mathcal{K}_i^- := \{x \in \mathbb{R}^n : x_i \leqslant 0\}$. Moreover, given a closed set $\mathcal{C}$, containing the origin, consider the *set operator* S defined as the set $S_\varepsilon(\mathcal{C}) := \{x \in \mathbb{R}^n : B_\varepsilon(x) \subset \mathcal{C}\}$. Intuitively, $S_\varepsilon(\mathcal{C})$ yields a set *reduced* by a factor ε with respect to the set $\mathcal{C}$ (see Fig. 4 for a graphical description of the sets $\mathcal{C}$ (light gray region) and $S_\varepsilon(\mathcal{C})$ (dark gray region)).

Corollary 2: Consider the nonlinear system (1) and suppose that $A = 0$ in (25). Suppose that there exist a closed cone $\mathcal{C}$, containing the origin and with nonempty interior, positively invariant[2] for the system (1), and an integer $i \in \{1, \dots, n\}$ such that either of the following conditions holds.

i) $\mathcal{C} \subset \mathcal{K}_i^+$ and for any $\varepsilon > 0$ the function $f_i : \mathbb{R}^n \to \mathbb{R}$ is strongly convex in the set $S_\varepsilon(\mathcal{C})$;

ii) $\mathcal{C} \subset \mathcal{K}_i^-$ and for any $\varepsilon > 0$ the function $f_i : \mathbb{R}^n \to \mathbb{R}$ is strongly concave in the set $S_\varepsilon(\mathcal{C})$.

Then, the origin is an unstable equilibrium point of (1). ○

Proof: The claim is explicitly shown in the case of convexity, namely assuming that item *(i)* holds, while case *(ii)* can be immediately derived by considering identical arguments for concave functions. Therefore, select $x_0 \in \mathcal{C}$ and ε with the properties that $x_{0,i}$ is positive and x_0 belongs to the interior of $S_\varepsilon(\mathcal{C})$. Note that the corresponding trajectory of the system (1) cannot leave the set $S_\varepsilon(\mathcal{C}) \cap \{x \in \mathcal{C} : x_i \geqslant x_{0,i}\}$ from the boundary

[2]The set $\mathcal{C}$ is positively invariant for (1) if $x(0) \in \mathcal{C}$ implies that $x(t) \in \mathcal{C}$ for all $t \geqslant 0$.

defined by $x_i = x_{0,i}$, by strong convexity of the function f_i in $S_\varepsilon(\mathcal{C})$, while it cannot leave $\mathcal{C}$ from any other boundary, by invariance of $\mathcal{C}$ with respect to (1). Thus, the trajectories ensuing from initial conditions in $\mathcal{C}$ cannot converge to the origin since $\varphi_t^i > x_{0,i}$ for all $t > 0$. Thus, instability of the equilibrium at the origin is concluded by iterating the reasoning in (28) with respect to a sequence of times $\{t_j\}_{j=1}^\infty$, such that $t_j \in [t_\ell, t_u]$, and by defining at each iteration the initial condition $\tilde{x}_0^j$ recursively as $\varphi_{t_{j-1}}(\tilde{x}_0^{j-1})$, with $\tilde{x}_0^0 = \bar{x}_0$. The latter selection ensures that

$$\begin{aligned}\varphi^i_{\sum_{j=1}^N t_j}(\bar{x}_0) &= \varphi^i_{t_N}(\tilde{x}_0^N) \geqslant e_i^\top \tilde{x}_0^N + \frac{\alpha}{2}\|\tilde{x}_0^N\|^2 \\ &= \varphi^i_{t_{N-1}}(\tilde{x}_0^{N-1}) + \frac{\alpha}{2}\|\tilde{x}_0^N\|^2 \\ &\geqslant e_i^\top \tilde{x}_0^{N-1} + \frac{\alpha}{2}\|\tilde{x}_0^{N-1}\|^2 + \frac{\alpha}{2}\|\tilde{x}_0^N\|^2 \\ &\;\;\vdots \\ &\geqslant e_i^\top \bar{x}_0 + \frac{\alpha}{2}\sum_{j=1}^N \|\tilde{x}_0^j\|^2 > \frac{\alpha}{2}N\mu^2 \end{aligned} \tag{29}$$

hence showing the claim. □

Example 3: Consider a nonlinear system described by the equations

$$\begin{aligned}\dot{x}_1 &= -x_1x_2^2 + x_1^3 + x_2^3 \\ \dot{x}_2 &= \alpha x_1 x_2 - x_2^2\end{aligned} \tag{30}$$

with $\alpha \in \mathbb{R}$. Consider the function $f_1(x) = -x_1x_2^2 + x_1^3 + x_2^3$ and note that the corresponding Hessian matrix is

$$\frac{\partial^2 f_1}{\partial x^2}(x) = \begin{bmatrix} 6x_1 & -2x_2 \\ -2x_2 & 6x_2 \end{bmatrix}$$

the determinant of which is given by the polynomial $4x_2(9x_1 - x_2)$. Therefore, the function f_1 is strongly convex in $S_\varepsilon(\mathcal{C})$, for any $\varepsilon > 0$ and $\mathcal{C} = \{x \in \mathcal{K}_1^+ : x_2 \geqslant 0, 9x_1 - x_2 \geqslant 0\}$. Moreover, by inspecting the dynamics in (30) it can be observed that the set $\mathcal{C}_\alpha := \{x \in \mathcal{K}_1^+ : x_2 \geqslant 0, \alpha x_1 - x_2 \geqslant 0\}$ is invariant for the system (30), for any $\alpha \in (0, 9)$. In fact, on the boundary described by $x_2 = \alpha x_1$ one has that $\dot{x}_2 = 0$ and $\dot{x}_1|_{x_2=\alpha x_1} = (1 - \alpha + \alpha^3)x_1^3 \geqslant 0$ for any $\alpha \in (0, 9)$ and for all $x \in \mathcal{K}_1^+$. Hence, Corollary 2 implies that the origin is an unstable equilibrium point of (30) for any $\alpha \in (0, 9)$. Fig. 5 depicts the vector field of the system (30) together with a trajectory characterized by $x_0 \in \mathcal{C}$, light (yellow) region. △

Remark 3: The conditions stated in Corollary 2 resemble those of the celebrated Chetaev instability theorem (see, e.g., [35]), without however requiring the need for the explicit computation of a function $V : \mathbb{R}^n \to \mathbb{R}$ with certain properties, namely positive on a set that contains the origin on its boundary and, in a neighborhood of the origin, strictly increasing over time along the trajectories of the underlying system. It is worth observing that, differently from Chetaev's theorem, the property of invariance of the set is not implied by the properties of the function V, hence it is explicitly assumed in Corollary 2. Nonetheless, the proof of Corollary 2 suggests that the trajectories cannot leave the set $\mathcal{C}$ from the boundary described by $x_i = 0$, in fact $x_i = x_{0,i}$, (due to convexity or concavity properties of the function f_i in $\mathcal{C}$). Therefore, if $x_i = 0$ is a boundary of the set $\mathcal{C}$, then one may simply verify that the trajectories cannot leave from the remaining boundaries of the set $\mathcal{C}$, as illustrated below. ▲

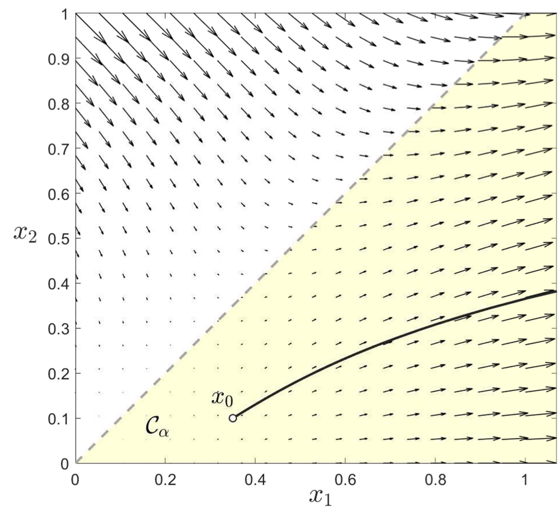

Fig. 5. Phase plot of the system (30) with $\alpha = 1$. The light (yellow) region describes the set $\mathcal{C}$ defined in Example 3.

Example 4: Consider a nonlinear system described by the equations

$$\begin{aligned}\dot{x}_1 &= -x_1x_2^2 - x_1^5 \\ \dot{x}_2 &= x_1^3 + x_2^3 - x_3^3 \\ \dot{x}_3 &= -x_3^3 x_1^2.\end{aligned} \tag{31}$$

The Hessian matrix of the function $f_2(x)$ is given by $(\partial^2 f_2(x)/\partial x^2) = 6\ \mathrm{diag}(x_1, x_2, -x_3)$, hence the function is convex in the set $\mathcal{C} = \{x \in \mathbb{R}^3 : x_1 \geqslant 0, x_2 \geqslant 0, x_3 \leqslant 0\}$ and strongly convex in $S_\varepsilon(\mathcal{C})$ for any $\varepsilon > 0$. Note that the set $\mathcal{C}$ is invariant for (31), since the trajectories cannot leave $\mathcal{C}$ from the boundary described by $x_2 = 0$, on which $\dot{x}_2 = x_1^3 - x_3^3 \geqslant 0$ in $\mathcal{C}$ (as expected from the comment at the end of Remark 3), as well as from the boundaries defined by $x_1 = 0$ and $x_3 = 0$, on which $\dot{x}_1 = 0$ and $\dot{x}_3 = 0$, respectively. Therefore, Corollary 2 allows concluding that the origin is an unstable equilibrium point of the system (31). △

The discussion in Remark 3 hints at the connections between the instability results based on convexity properties of the vector field and the well-known instability sufficient conditions based on the construction of a Chetaev's function. The following statement suggests a possible systematic construction of the latter function whenever a certain component of the underlying vector field is strongly convex in a set that contains the origin.

Proposition 1: Consider the nonlinear system (1) and suppose that $A = 0$ in (25). Suppose that there exist an integer $i \in \{1, \ldots, n\}$ and a non-empty open set $\mathcal{U} \subset \mathbb{R}^n$, containing the origin, with the property that $f_i : \mathbb{R}^n \to \mathbb{R}$ is strongly convex[3] for all $x \in \mathcal{U}$. Then there exists $r > 0$ such that $V : \mathbb{R}^n \to \mathbb{R}$ defined as

$$V(x) := x_i + \|x\|^2 \tag{32}$$

[3]The case of strongly concave function can be immediately obtained by changing the sign of the first term of the function (32).

is a Chetaev function for (1) in $B_r(0) \cap \mathcal{K}_i^+$. ∘

Proof: To begin with, note that the function V is such that $V(0) = 0$ and $V(x) > 0$ for any $x \in \mathcal{K}_i^+$, in which $x_i \geqslant 0$ by definition and which is invariant for (1). Then, since $f_i(x)$ is a strongly convex function of $x \in \mathcal{U}$ and the latter contains the origin, by suitably adapting the inequality (5) letting x_2 therein be replaced by a generic x and x_1 by $0 \in \mathcal{U}$, it follows that $f_i(x) \geqslant (\alpha/2)\|x\|^2$ for some $\alpha > 0$. The inequality is obtained by recalling that, in addition, $(\partial f_i/\partial x)(0) = 0$. The time-derivative of the function V along the trajectories of the system (1) is then given by

$$\dot{V} = f_i(x) + 2x^\top f(x) \geqslant \frac{\alpha}{2}\|x\|^2 + 2x^\top f(x).$$

Therefore, by recalling that f does not possess linear terms in x, there exists $r > 0$ such that $\dot{V} > 0$ for all $x \in B_r(0) \in \mathcal{K}_i^+$, which proves the claim. □

B. Nonlinear Systems With Linear Terms and Critical Cases

Consider now the general case in which A is different from the zero matrix. As it is well known, in such a scenario, the spectrum of the matrix A plays a crucial role in determining stability or instability properties of the equilibrium point for the original nonlinear system. In fact, it can be shown that the equilibrium point is locally exponentially stable whenever $\sigma(A) \subset \mathbb{C}^-$. Conversely, the presence of an eigenvalue $\lambda \in \sigma(A)$ with the property that $\mathrm{Re}(\lambda) > 0$ implies that the equilibrium point of the nonlinear system is unstable. However, it is well known that the stability properties of the equilibrium point are not entirely captured by the behavior of the linearized model. In the so-called *critical cases*—namely whenever $\sigma(A) \subset \mathbb{C}^- \cup \mathbb{C}^0$ and $\sigma(A) \cap \mathbb{C}^0 \neq \emptyset$—the stability analysis must necessarily include the study of the nonlinear behavior of the system.

Theorem 4: Consider the nonlinear system (1) and suppose that $\sigma(A) \subset \mathbb{C}^- \cup \mathbb{C}^0$ in (25). Suppose that there exist an integer $i \in \{1, \ldots, n\}$ and a nonempty open set $\mathcal{U} \subset \mathbb{R}^n$, containing the origin, with the properties that

i) $f_i : \mathbb{R}^n \to \mathbb{R}$ is either a strongly convex or a strongly concave function for all $x \in \mathcal{U}$.

ii) $e_i^\top \exp(At) = [\,0 \;\; \ldots \;\; m_{ii}(t) \;\; \ldots \;\; 0\,]$, with $m_{ii} : \mathbb{R} \to \mathbb{R}$ such that $m_{ii}(t) = 1$ for all $t \geqslant 0$.

Then, the origin cannot be an asymptotically stable equilibrium point of system (1). ∘

Proof: Consider the case of convex functions. For all $x_0 \in \mathcal{U}$ such that $x_0^i > 0$, one has that $\varphi_t^i(x_0) > e_i^\top \exp(At)x_0 = [0 \ldots m_{ii}(t) \ldots 0]x_0 = x_0^i$, locally with respect to time, hence showing that the ith component of the state cannot converge to the equilibrium point, by relying on arguments identical to those in the proof of Theorem 3. □

Example 5: Consider a nonlinear system described by

$$\begin{aligned}\dot{x}_1 &= -\gamma x_1 + x_2 - x_1^3 \\ \dot{x}_2 &= -x_1 x_2 - x_1^2 - x_2^2 \end{aligned} \tag{33}$$

with $\gamma > 0$. It can be observed that $f_2(x_1, x_2) = -x_1x_2 - x_1^2 - x_2^2$ is a strongly concave function on $\mathbb{R}^2$, while item *(ii)* of the statement of Theorem 4 holds since

$$A = \begin{bmatrix} -\gamma & 1 \\ 0 & 0 \end{bmatrix}$$

hence $e_2^\top \exp(At) = [0 \;\; 1]$, which satisfies the structure of the item *(ii)* of the statement with $m_{22} \equiv 1$. Therefore, by Theorem 4, it can be concluded that the origin is an unstable equilibrium point of (33) for any $\gamma > 0$ and any initial condition of the form (x_0^1, x_0^2), with negative x_0^2 and *arbitrary* $x_0^1 \in \mathbb{R}$ induces a diverging evolution for $\varphi_t^2(x_0)$. Figs. 6 and 7 depict the phase plot of the system (33) for two different values of the parameter γ, namely $\gamma = 1$ and $\gamma = 0.1$, respectively. As expected, the trajectories ensuing from initial conditions with the property that $x_{0,2} < 0$ diverge, whereas for positive values convergence to the origin depends on the value of γ, hence on the rate of convergence of the state x_1, as shown by the comparison of Figs. 6 and 7. △

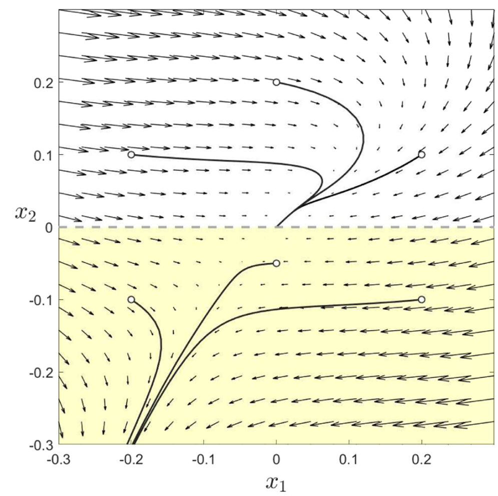

Fig. 6. Phase plot of the system (33) with $\gamma = 1$ for several initial conditions.

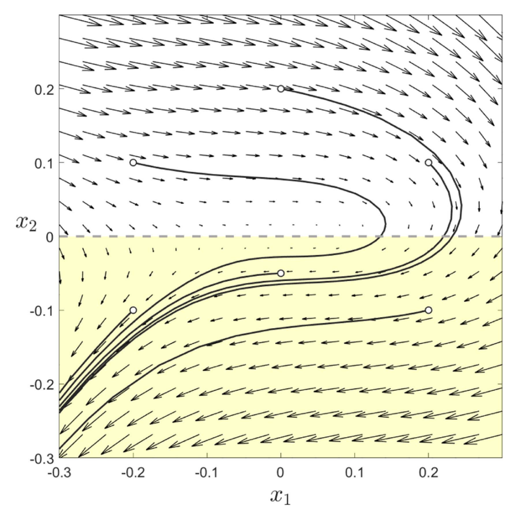

Fig. 7. Phase plot of the system (33) with $\gamma = 0.1$ for several initial conditions.

Remark 4: Whenever the matrix A possesses real, simple, eigenvalues in $\mathbb{C}^- \cup \mathbb{C}^0$, the inequality in item *(ii)* holds *for all* time *if and only if* $A_i = 0$, namely the ith row of the matrix A, is equal to zero. In fact, let $\mathcal{V} \in \mathbb{R}^{n\times n}$ be such that $A = \mathcal{V}^{-1}\text{diag}(\lambda_1,\ldots,\lambda_n)\mathcal{V}$—hence $\exp(At) = \mathcal{V}^{-1}\text{diag}(\exp(\lambda_1 t),\ldots,\exp(\lambda_n t))\mathcal{V} := \mathcal{V}^{-1}\Lambda\mathcal{V}$—and denote the columns of $\mathcal{V}$ and the rows of $\mathcal{V}^{-1}$ with v_i and w_i, respectively, namely $\mathcal{V} = [v_1 \; v_2 \; \ldots \; v_n]$ and $\mathcal{V}^{-1} = [w_1^\top \; w_2^\top \; \ldots \; w_n^\top]^\top$. Therefore,

$$S_0^{ii}(t) = e_i^\top \mathcal{V}^{-1}\Lambda\mathcal{V}e_i = w_i\Lambda v_i = \sum_{j=1}^{n} w_{ji}v_{ji}\exp(\lambda_j t) \quad (34)$$

hence $S_0^{ii}(0) = w_i v_i = 1$, by definition of matrix inverse. Moreover by inspecting (34) it appears that $S_0^{ii}(t) < 1$ for $t > 0$ unless $\lambda_i = 0$ and $v_i = w_i^\top = e_i$. Thus, since v_i is an eigenvector associated with the eigenvalue λ_i it follows that $Av_i = \lambda_i v_i$, hence $Av_i = Ae_i = A_i = 0$. ▲

V. Connections With Brockett's Theorem on Continuous Stabilizability

The main objective of this section consists in extending the previous conditions based on convexity and concavity of the underlying vector field to the setting of controlled system. This is achieved in particular by establishing connections with the conditions stated in Brockett's theorem [32]. Toward this end, consider a nonlinear control system described by the equations

$$\dot{x} = f(x,u), \quad x(0) = x_0 \quad (35)$$

with f continuously differentiable around the origin and such that $f(0,0) = 0$—hence assume for simplicity that the uncontrolled system possesses an equilibrium at the origin—with $x(t) \in \mathbb{R}^n$ denoting the state and $u(t) \in \mathbb{R}^m$ describing the controlled input. In particular, Brockett [32] provided necessary conditions for the existence of a smooth time-invariant feedback control input $u : \mathbb{R}^n \to \mathbb{R}^m$ such that the origin is a locally asymptotically stable (LAS) equilibrium for the closed-loop system $\dot{x} = f(x,u(x))$. The following statements attempt at connecting items *(ii)* and *(iii)* of Brockett's theorem (see [32, Th. 1] for more details) with properties of convexity and concavity of the components of $f(x,u)$. Such a connection possesses then an interesting consequence in permitting a constructive verification of such items, which may be particularly useful especially in the case of item *(ii)* of such a theorem. To this end, consider the linearization of system (35) around the origin, namely

$$\dot{\delta x} = \left.\frac{\partial f}{\partial x}\right|_{(0,0)} \delta x + \left.\frac{\partial f}{\partial u}\right|_{(0,0)} \delta u := A\delta x + B\delta u. \quad (36)$$

Theorem 5: Consider the nonlinear control system (35) and suppose that $A = 0$ in (36). Suppose that there exists an integer $i \in \{1,\ldots,n\}$ with the properties that $f_i : \mathbb{R}^n \to \mathbb{R}$ is either a strongly convex or a strongly concave function of x and that $(\partial f_i/\partial u) \equiv 0$. Then, the origin cannot be locally stabilized via a continuously differentiable, time-invariant feedback control law. ○

Proof: The condition on the gradient of the function f_i with respect to u ensures that the former does not in fact depend on the latter, hence $f_i(x,u) \equiv f_i(x)$. Moreover, since $x = 0$ is an equilibrium point of $\dot{x} = f(x,0)$, it follows that necessarily $f_i(0) = 0$. Therefore, strong convexity (concavity, resp.) implies that the image of the function f_i belongs to the set of positive (negative, resp.) real numbers. In fact, by considering the exact second-order expansion of the function $f_i : \mathbb{R}^n \to \mathbb{R}$ one has that

$$f_i(x) = f_i(0) + \frac{\partial f_i}{\partial x}(0)x + \frac{1}{2}x^\top \frac{\partial^2 f_i}{\partial x^2}(z)x = \frac{1}{2}x^\top \frac{\partial^2 f_i}{\partial x^2}(z)x > 0 \quad (< 0, \text{resp.}) \quad (37)$$

for all $x \in \mathbb{R}^n$, where z belongs to the line segment between the origin and x, with the first equality obtained by recalling that the origin is an equilibrium point and that $A = (\partial f/\partial x)(0,0) = 0$. Thus, since the component f_i can only take positive (negative, resp.) values, the mapping $(x,u) \mapsto f(x,u)$ cannot be surjective to any open set containing the origin. □

Remark 5: The statement and the proof of Theorem 5 entail that the property of convexity, or equivalently of concavity, of the vector field prevents item *(iii)* of [32, Th. 1] from being verified. Nonetheless, it is worth observing that the same condition is also sufficient to imply that item *(ii)* cannot be satisfied, by relying on the results and discussions of Section IV-A. Moreover, the conditions stated in Theorem 5 can be further relaxed to a *local* version, namely by requiring that the function f_i be strongly convex (or concave) in a certain neighborhood $\mathcal{U}$ of the origin, together with the property that it does not depend on u. In fact, by relying on the results of Theorem 3, the state trajectory x must first leave $\mathcal{U}$ to potentially converge to the origin, hence the origin cannot be an asymptotically stable equilibrium point for any selection of the control input u (see also the following example). ▲

Example 6: Consider the nonlinear system described by

$$\dot{x}_1 = u$$
$$\dot{x}_2 = -\alpha x_2^3 + \beta\left(x_1^2 + x_2^2\right) \quad (38)$$

with $\alpha > 0$ and $\beta \in \mathbb{R}$, $\beta \neq 0$, arbitrary coefficients. Note that $f_2(x_1,x_2) = -\alpha x_1^3 + \beta x_1^2 + x_2^2$ is a strongly convex (concave, resp.) function on a neighborhood of the origin in $\mathbb{R}^2$ provided β is positive (negative, resp.). Therefore, the hypotheses of Theorem 5 are satisfied with $i = 2$, and hence the origin cannot be stabilized by a continuously differentiable feedback control law. △

The following statement provides a direct extension of Corollary 2 to the case of *controlled* nonlinear systems.

Corollary 3: Consider the nonlinear control system (35) and suppose that $A = 0$ in (36). Suppose that there exist a closed cone $\mathcal{C} \subset \mathbb{R}^n$, containing the origin and with nonempty interior, invariant for (35), and an integer $i \in \{1,\ldots,n\}$ such that $(\partial f_i/\partial u) \equiv 0$ and either of the following conditions holds.

i) $\mathcal{C} \subset \mathcal{K}_i^+$ and for any $\varepsilon > 0$ the function $f_i : \mathbb{R}^n \to \mathbb{R}$ is strongly convex in the set $S_\varepsilon(\mathcal{C})$;

ii) $\mathcal{C} \subset \mathcal{K}_i^-$ and for any $\varepsilon > 0$ the function $f_i : \mathbb{R}^n \to \mathbb{R}$ is strongly concave in the set $S_\varepsilon(\mathcal{C})$.

Then, the origin of the system (35) cannot be locally asymptotically stabilized by a continuously differentiable feedback control law. ○

Proof: The claim is shown by relying on arguments identical to those employed in the proof of Theorem 5, combined with invariance of the set $\mathcal{C}$ for any $u \in \mathbb{R}^m$. In fact, considering for simplicity the case of a strongly convex function f_i, the conditions of the statement imply that the flow of the ith component of the state φ_t^i is lower-bounded for any time by the corresponding initial condition $e_i^\top x_0$ for any $x_0 \in \mathcal{C}$. Thus, there cannot be a selection of the control input that satisfies item *(ii)* of [32, Th. 1]. □

The claims of Corollary 3 are illustrated by means of the following example.

Example 7: Consider a nonlinear control system described by

$$\dot{x}_1 = x_1^3 - x_2^3$$
$$\dot{x}_2 = x_2 u. \tag{39}$$

Note that, since $(\partial^2 f_1(x)/\partial x^2) = 6\,\mathrm{diag}(x_1, -x_2)$, the function f_1 is strongly convex in $S_\varepsilon(\mathcal{C}_1)$ with $\mathcal{C}_1 := \{x \in \mathcal{K}_1^+ : x_2 \leqslant 0\}$, i.e., $\mathcal{C}_1$ defines the fourth quadrant. Moreover, f_1 is strongly concave in $S_\varepsilon(\mathcal{C}_2)$ with $\mathcal{C}_2 := \{x \in \mathcal{K}_1^- : x_2 \geqslant 0\}$, namely the second quadrant. It is interesting to observe that $\mathcal{C}_1$ and $\mathcal{C}_2$ are invariant and, by relying on the comment at the end of Remark 3, it is sufficient to verify that the trajectories cannot cross the boundary described by $x_2 = 0$, since therein $\dot{x}_2 = 0$ for any continuous u. Therefore, by Corollary 3 the zero equilibrium point of (39) cannot be stabilized by any continuously differentiable feedback control. △

Finally, consider the case of input-affine nonlinear systems described by the equation

$$\dot{x} = f(x) + g(x)u, \quad x(0) = x_0. \tag{40}$$

The following statement relates in a more constructive fashion the property of convexity (or, equivalently, concavity) of a combination of the components of the vector field with the existence of a continuously differentiable stabilizing feedback.

Theorem 6: Consider the nonlinear input-affine control system (40) and suppose that $\partial f(0)/\partial x = 0$. Suppose that there exists a continuously differentiable function $\lambda : \mathbb{R}^n \to \mathbb{R}$ such that

i) $\frac{\partial \lambda}{\partial x}(x) f(x)$ is a strongly convex (concave) function of x;
ii) $\frac{\partial \lambda}{\partial x}(x) g(x) = 0$, for all $x \in \mathbb{R}^n$.
iii) for any $r > 0$ there exists $x_r \in B_r(0)$ such that $\lambda(x_r) > 0$ $(\lambda(x_r) < 0)$.

Then, the origin cannot be locally asymptotically stabilized by any continuously differentiable feedback control law. ◦

Proof: By defining the *virtual* output $s := \lambda(x)$, $s(t) \in \mathbb{R}$, one has that the time evolution of s is described by the scalar differential equation

$$\dot{s} = \frac{\partial \lambda}{\partial x}\dot{x} = \frac{\partial \lambda}{\partial x}(x) f(x)$$

which follows from items *(i)* and *(ii)* of the statement. Since the latter vector field is strongly convex (or concave) by assumption, the *flow* $\varphi_t^s(x_0)$ is in turn strongly convex (or concave) by Theorem 1. Therefore, unboundedness of the output trajectories with respect to $s = \lambda(x)$ can be concluded by relying on the results of Section IV-A and by the existence of an initial condition, arbitrarily close to the origin, such that $s(0)$ is positive (or negative, resp.). □

Remark 6: The above interpretation establishes connections between the intuition of Theorem 6 and the notion of *Control Chetaev Function*, discussed in [30]. Furthermore, the choice of a linear combination of the components as the virtual output $s := \lambda^\top x$, namely provided *(i')* $\lambda^\top f(x)$ is strongly convex (concave) and *(ii')* $\lambda^\top g(x) = 0$ for all $x \in \mathbb{R}^n$, may be further related to item *(iii)* of [32, Th. 1]. In fact, a necessary condition for the existence of a continuously differentiable stabilizing feedback is that the system of (algebraic) equations $f(x) + g(x)u = \xi$ admit a solution for some (x, u) and for any sufficiently small $\xi \in \mathbb{R}^n$. Moreover, if the latter condition holds, then necessarily for any $\lambda \in \mathbb{R}^n$ also the (scalar algebraic) equation $\lambda^\top(f(x) + g(x)u) = \mu$ must have a solution for any $\mu \in \mathbb{R}$, sufficiently small. In fact, any μ can be obtained as $\lambda^\top \xi = \mu$, via a suitable selection of the (arbitrary) $\xi \in \mathbb{R}^n$. Note now that if there exist a vector $\lambda \in \mathbb{R}^n$ with the properties in items *(i')* and *(ii')* above, then the (scalar) equation $\lambda^\top f(x) = \mu$ cannot be satisfied for any $\mu > 0$ in the case of concavity (since the function $\lambda^\top f(x)$ is zero at the origin and with the properties that $\lambda^\top(\partial f(0)/\partial x) = 0$ and strongly concave) and any $\mu < 0$ in the case of convexity, via a symmetric argument. This aspect is illustrated by the following example. ▲

Example 8: Consider a nonlinear system described by

$$\dot{x}_1 = -x_1^3 + x_2^2 + u$$
$$\dot{x}_2 = x_1^2 + x_1^3 - u. \tag{41}$$

Since $\lambda^\top x$ with $\lambda = [1 \ \ 1]^\top$ satisfies the conditions of item *(i)* (since $\lambda^\top f(x) = x_1^2 + x_2^2$ is strongly convex in $\mathbb{R}^2$) and item *(ii)* (since $\lambda^\top g(x) = u - u = 0$), it follows by Theorem 6 that the zero equilibrium of the system (41) cannot be stabilized by any smooth feedback control law. △

VI. Conclusion

In this article, the connection between the properties of stability and instability of equilibrium points in nonlinear systems and the property of convexity/concavity (of the components) of the underlying vector field has been explored. In particular, it has been shown that the properties of convexity (concavity) of the underlying vector field and of the corresponding flow, as a function of the initial condition and for fixed (sufficiently small) time, are equivalent. Such a result, which is interesting *per se*, has been then instrumental for stating and proving several instability theorems. Finally, an interpretation of the celebrated Brockett's theorem in terms of convexity and concavity of a component of the vector field has been established. The theoretical results have been illustrated by means of several numerical examples.

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