Linear Output Regulation for Unknown Stable Systems With Uncertain Minimum-Phase Actuators

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***Abstract*—Consider a linear stable system—described by the transfer function $P(s)$ of unknown parameters, unknown order, and unknown relative degree—under a linear exosystem generating biased multisinusoidal references and/or disturbances with at most q different known frequencies ω_i, $i = 1,\ldots,q$. It has been recently established that a linear regulator with minimal order $(2q+1)$ exists under the knowledge of the positive or negative signs of i) $P(0)$ and ii) either $\Re(P(j\omega_i))$ or $\Im(P(j\omega_i))$, for any $i = 1,\ldots,q$ [j is the imaginary unit]. This technical note explores the case in which the measurable input u to the aforementioned system is provided by an unknown linear actuator. It is actually shown that the regulator design can be naturally extended to such a scenario, provided that the actuator process is minimum-phase, of known relative degree $\rho \geq 1$, and with known sign of the high-frequency gain.**

***Index Terms*—Linear actuator, linear output regulation, minimum phase, stable system, uncertain system.**

I. Introduction

When both output reference signals and disturbances are generated by linear exogenous systems named exosystems, a popular approach to achieve output tracking in linear systems relies on the formulation of the problem as a linear output regulation problem [1], [6]. A fundamental result, in this framework, is the *internal model principle*: given a stabilizable and detectable linear system, an output feedback regulating control exists if and only if the spectrum of the exosystem has no intersection with the set of the system zeroes. Furthermore, the resulting compensator has to contain an internal model of the reference/disturbance. In the case in which the process is uncertain and the exosystem is known, a relevant question regards which types of uncertainties [17] and which types of single-input, single-output plants (see [9], [10] for multiple-input, multiple-output systems) allow for the design of a feedback regulator that is able to cancel biased multisinusoidal disturbances. It is definitely known that, as long as the plant is stable, a suitably tuned integral control solves the problem for constant disturbances, provided that the transfer function $P(s)$ of the system does not vanish at the origin ($s = 0$) of the complex plane [16] (see [7] for a general design approach including nonlinear systems and [12] for the design of a novel nonlinear PID control that guarantees desired state limitations for a wide class of nonlinear systems): the sign of the integral action strictly depends on the sign of $P(0)$ and the integral gain has to be sufficiently small. On the other hand, if the exosystem has order two, i.e., it generates sinusoidal references/disturbances with a single frequency ω, then a small-gain regulator exists, provided that the system is stable and the phase of $P(j\omega)$ has an uncertainty less than 180° [3], [4].[1] The above results are then generalized in [13] to biased multisinusoidal signals: if the plant is stable and the linear exosystem generates references and/or disturbances containing at most q different known (positive) frequencies ω_i, $i = 1,\ldots,q$, then a linear regulator with minimal order $(2q+1)$ exists under the knowledge of the positive or negative signs of i) $P(0)$ and ii) either $\Re(P(j\omega_i))$ or $\Im(P(j\omega_i))$, for any $i = 1,\ldots,q$. Further developments can be found in [20] and [14].

The goal of this technical note is to present a natural extension of the design of [13] to the case in which the measurable input u to the system in [13] is provided by a minimum-phase linear actuator process with unknown order, unknown parameters—except for the sign of the high-frequency gain—and known relative degree $\rho \geq 1$. Covering this case is challenging and certainly deserves attention, since there is no guarantee that the same sign-structure of the controller for the actuator-free case can be still used to solve the problem in the presence of the actuator, as illustrated by the following motivating example.

Motivating example. Denote by $\mathcal{L}[f(t)](s)$ the Laplace transform of the Laplace transformable signal $f(t): \mathbb{R}_{\geq 0} \to \mathbb{R}$ in terms of the complex variable s. Consider the second-order dynamic stable plant Σ in [13] described by the transfer function

$$P(s) = 1/(s+1)^2$$

subject to the action of the u-matched disturbance

$$d(t) = 1 + \sin(t/2 + 0.3) - \sin(2t)$$

as represented by the top scheme of Fig. 1, with the output reference being set to zero in the output tracking error expression: $e(t) = y(t) - y_r(t)$. Since $P(0) > 0$, $\Re(P(j/2)) = 0.48 > 0$, $\Re(P(2j)) = -0.12 < 0$, the dynamic output feedback controller (8) of [13] [version (A)]:

$$\mathcal{L}[u(t)](s) = -\varepsilon \left[\frac{1}{s} + \frac{s}{(s^2+1/4)} - \frac{s}{(s^2+4)}\right] \mathcal{L}[e(t)](s)$$

successfully applies, for sufficiently small (positive) ε, to guarantee exponential convergence to zero of the output regulation error $e(t) = y(t)$ in spite of the presence of $d(t)$. Now assume that the input $u(t)$ to the system Σ is instead provided by the uncertain (unmodeled) two-relative-degree, stable, third-order dynamic, minimum phase actuator process (see Fig. 1, scheme at the bottom, in the absence of the disturbance d_a) whose transfer function is given by ($\zeta \in \mathbb{R}_{>0}$)

$$F(s) = (s+\zeta)/(s+1)^3.$$

The dynamic relationship between the output $y(t)$ and the input $v(t)$ reads (subscript e stands for *extended*):

$$P_e(s) = F(s)P(s) \ = \ (s+\zeta)/(s+1)^5$$

[1] Sufficient conditions are also presented in [5] to guarantee, through a fractional-order derivative controller, the stability of the closed-loop system, and consequently, the disturbance rejection even if the uncertainty on the phase is greater than 180°; see also [2] for the case of periodic disturbances.

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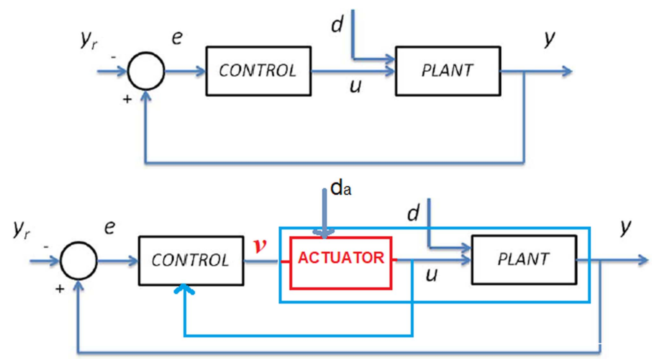

Fig. 1. Block diagram of the regulator problem in the original version of [13] (at the top) and in the extended version of this note (at the bottom), i.e., in the presence of an unmodeled actuator ($d = Dw$ and $d_a = Ew$ are the system state disturbance and the actuator state disturbance, respectively).

which still complies with the scenario in [13], since it describes an uncertain linear stable system under the action of an exosystem generating biased multisinusoidal references and/or disturbances with at most q different known frequencies. Now, no information about $\Re(P_e(j/2))$, $\Re(P_e(2j))$ is available. Indeed, $\Re(P_e(j/2)) < 0$ for any $\zeta > 41/76$, while $\Re(P_e(2j)) > 0$ for any $\zeta > 76/41$, so that the previously designed controller (exhibiting a positive sign for the second term and a negative one for the third term within the square brackets) does not have a sign-structure that follows the design rules of [13] when $\zeta > 41/76$. In particular, root locus analysis tools show how, for $\zeta = 2$, $\mathcal{L}[u(t)](s)$ with $\varepsilon \in (0, 0.749)$ solves the problem in the absence of the actuator dynamics, whereas no controller of the above sign-structure is able to solve the problem in presence of the actuator.

II. Problem Statement

As in [13], consider the class of linear time-invariant systems with one input and one output [A, b, c, D are matrices/vectors of suitable dimensions]

$$\dot{x} = Ax + bu + Dw$$
$$y = cx \tag{1}$$

with $x \in \mathbb{R}^m$ and $w \in \mathbb{R}^{2q+1}$ ($u \in \mathbb{R}$, $y \in \mathbb{R}$). Further assume that the output y has to track the reference $y_r = -rw$ (r is a row vector with $2q+1$ components), leading to the definition of the output regulation error $e = y + rw$. Such an error is measured and has to be regulated to zero. Both the system state disturbance $d = Dw$ and the output reference $y_r = -rw$ rely on the vector w that is assumed to be generated by the linear exosystem

$$\dot{w} = Rw \tag{2}$$

characterized by the R-matrix with spectrum $\sigma(R) = \{0, \pm j\omega_1, \ldots, \pm j\omega_q\}$, where $\omega_i > 0$ and $\omega_i \neq \omega_j$, for any $i \neq j$, $i, j = 1, \ldots, q$. In this regard, recall from [13] (see also [18]) that necessary and sufficient conditions for the solution of the regulator problem are as follows:

1) the pair (A, b) is stabilizable;
2) the pair (A, c) is detectable;
3) $P(s)$ is such that $P(\lambda) \neq 0$ for any $\lambda \in \sigma(R)$;

with this last condition being equivalent to the existence of a unique pair of matrices (Γ_r, γ_r) solution to the regulator equations: $\Gamma_r R = A\Gamma_r + b\gamma_r + D$, $c\Gamma_r + r = 0$ and which the state and input references, $x_r = \Gamma_r w$, $u_r = \gamma_r w$, correspond to. Here, matrix A in (1) is additionally supposed to be Hurwitz, with system (1) being assumed to be reachable and observable, though no other information is available for system (1). In particular, neither the matrices/vectors A, b, c nor the order m and the relative degree ρ_s of (1) are assumed to be known. Let $P(s) = c(s\mathbb{I} - A)^{-1}b$ finally denote the proper transfer function of system (1). The considered scenario and the corresponding output regulation problem—including its notation—is thus coinciding with the one depicted in [13], once the order $2q+1$ and the eigenvalues of the exosystem (2) are considered to be known and the following assumptions are introduced: i) $P(0)$ is different from zero, with known sign; ii) for any ω_i, $i = 1, \ldots, q$, either (A) $\Re(P(j\omega_i))$ or (B) $\Im(P(j\omega_i))$ is different from zero, with known sign. The related controller is the linear regulator with minimal order $(2q+1)$ that reads[2]:

$$\mathcal{R}(s) = -\varepsilon \frac{d_1 s^{2q} + \cdots + d_{2q+1}}{s\prod_{i=1}^{q}(s^2 + \omega_i^2)}$$
$$p_1\left(\omega_i^2\right) = (\omega_i^2)^{-1}[(-1)^i \mathrm{sgn}\{\Re(P(j\omega_i))\} \tag{3}$$
$$- \mathrm{sgn}\{P(0)\}] \tag{A}$$
$$p_1\left(\omega_i^2\right) = (\omega_i^2)^{-1}[-\mathrm{sgn}\{P(0)\}] \tag{B}$$
$$p_2\left(\omega_i^2\right) = 0 \tag{A}$$
$$p_2\left(\omega_i^2\right) = (\omega_i)^{-1}(-1)^{i+1}\mathrm{sgn}\{\Im(P(j\omega_i))\} \tag{B}$$
$$p_1(\sigma) = (-1)^q d_1 \sigma^{q-1} + (-1)^{q-1} d_3 \sigma^{q-2} + \cdots - d_{2q-1}$$
$$p_2(\sigma) = (-1)^{q-1} d_2 \sigma^{q-1} + (-1)^{q-2} d_4 \sigma^{q-2} + \cdots + d_{2q}$$

where $d_1, \ldots, d_{2q+1}$—solutions to the equations above– - are the constants defined in [13, Th. 3.3]. Now, differently from [13] (addressing the scenario reported in Fig. 1, block diagram at the top), consider the case (Fig. 1, block diagram at the bottom) in which the measurable input signal u is generated by the (reachable and observable[3]) linear actuator process—possibly perturbed by the actuator state disturbance $d_a = Ew$, with E being a matrix of suitable dimension—described by the equations

$$\dot{z} = Fz + \gamma g v + Ew$$
$$u = hz \tag{4}$$

with $z \in \mathbb{R}^n$, $v \in \mathbb{R}$. Without loss of generality, system (4) is assumed to be in the observer canonical form, namely, exhibiting

$$F = \begin{bmatrix} -f_1 & 1 & 0 & \cdots & 0 \\ -f_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -f_{n-1} & 0 & 0 & \cdots & 1 \\ -f_n & 0 & 0 & \cdots & 0 \end{bmatrix}$$
$$g = [0, \ldots, 1, g_{\rho+1}, \ldots, g_n]^{\mathrm{T}}, \quad h = [1, 0, \ldots, 0].$$

For such a system, the order n and the matrices/vectors F, g, h are unknown, along with the high-frequency gain γ. Only the sign of γ—positive without loss of generality—and the relative degree $\rho \geq 1$ of the actuator process (4) are assumed to be known. Furthermore, the zeroes of the transfer function $H(s) = \gamma h(s\mathbb{I} - F)^{-1}g$, namely, the roots of the polynomial $\pi_f(s) = s^{n-\rho} + g_{\rho+1}s^{n-\rho-1} + \cdots + g_n$, are

[2]Such a controller is not guaranteed to have the same modular structure of the dynamic output feedback controller [13, (8)].

[3]Stabilizable and detectable plants/actuators can be dealt with as well.

assumed to belong to open left-side of the complex plane (*minimum phase* assumption). Introduce the following definition.

Definition 2.1: Given the previously described linear system (1), (4) under the action of the linear exosystem (2), the related linear output regulation problem is solvable if there exists a linear proper dynamic controller

$$\mathcal{L}[v(t)](s) = G_e(s)\mathcal{L}[e(t)](s) - G_u(s)\mathcal{L}[u(t)](s) \tag{5}$$

such that i) the closed-loop error system is exponentially stable when $w(0) = 0$; and ii) the regulation error $e(t)$ exponentially converges to zero for any $(x(0), z(0), w(0))$, as t tends to infinity.

III. Main Result

The following theorem states the contribution of this note.

Theorem 3.1: Let k be a positive control gain. Let a_i, $i = 1, \ldots, \rho - 1$ (when $\rho > 1$) and a_φ be positive reals, with the polynomial $q(s) = s^{\rho-1} + a_1 s^{\rho-2} + \ldots + a_{\rho-1}$ being Hurwitz (when $\rho > 1$). Let $d_1, \ldots, d_{2q+1}$ be the constants in (3). If a filter $\mathcal{F}(s)$ is set as

$$\mathcal{F}(s) = \frac{a_{\rho-1}k^{\rho-1}(s^{\rho-1} + a_1 s^{\rho-2} + \cdots + a_{\rho-1})}{s^{\rho-1} + a_1 k s^{\rho-2} + \cdots + a_{\rho-1}k^{\rho-1}} \tag{6}$$

for $\rho > 1$ and $\mathcal{F}(s) = 1$ for $\rho = 1$, then there exists a positive real k^* such that, for any $k \geq k^*$, the output regulation problem of Definition 2.1 for (1) and (4) under (2) is solvable by (5) with

$$G_e(s) = \mathcal{F}(s)\left[-\frac{a_\varphi}{k} \cdot \frac{d_1 s^{2q} + \cdots + d_{2q+1}}{s\prod_{i=1}^{q}(s^2 + \omega_i^2)}\right]$$
$$G_u(s) = \sqrt[4]{k}\mathcal{F}(s). \tag{7}$$

Proof: The proof is divided into eight separate parts. The first one concerns the μ-filtered transformation ζ for (4) in the case in which $\rho > 1$: the aim is to get, on the basis of the Hurwitz nature of the polynomial $s^{\rho-1} + a_1 s^{\rho-2} + \cdots + a_{\rho-1}$, a new ζ-subsystem [namely, (17)] with relative degree equal to one with respect to the new input μ_1. Such a step has to be skipped when $\rho = 1$. The second step regards a suitable scaling of the μ-dynamics with the aim of making a power-increasing k-dependent scaling factor appear in the lower system level [refer to (20)]. The third step concerns the explicit definition of intermediate reference signals for system (1) and the ζ-subsystem, with the former being taken from [13] and the latter relying on the minimum-phase properties of the actuator system. The fourth step regards a suitable change of coordinates highlighting the zero-dynamics for the ζ-system [namely, (17)] and the related definition of additional control signals [namely, (27)]. The fifth step regards the definition of the tracking errors and the derivation of the related error system. The sixth step concerns the presentation of the key idea (inspired by [19]), that is the derivation of an equivalent disturbance that just acts on the upper error system [namely, (21)]; such an equivalent disturbance has to be counteracted by the multisinusoidal generator of the controller [namely, the block of estimators (20)]. The seventh step regards the derivation of the state space representation for the intermediate signal $z_{1,\mathrm{ref}}$, which is instrumental to derive the entire error system in the state space. The last step finally uses a composite Lyapunov function for the entire error system [namely, (41)], which is obtained as the sum of the sub-Lyapunov functions for each linear error subsystem.

A. Filtered Transformation for (4) When $\rho > 1$

For $\rho > 1$, introduce the linear stable filter of order $\rho - 1$ ($\mu = [\mu_1, \ldots, \mu_{\rho-1}]^{\mathrm{T}}$):

$$\dot{\mu} = M\mu - b_c v \tag{8}$$

where the first column coefficients of the matrix M within the expressions:

$$M = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{\rho-2} & 0 & 0 & \cdots & 1 \\ -a_{\rho-1} & 0 & 0 & \cdots & 0 \end{bmatrix}, \; b_c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{9}$$

define the Hurwitz polynomial:

$$q(s) = s^{\rho-1} + a_1 s^{\rho-2} + \cdots + a_{\rho-1}. \tag{10}$$

Then, introduce the filtered transformation

$$\zeta = z - \gamma T\mu \tag{11}$$

with T being an $n \times (\rho - 1)$ constant matrix to be chosen hereafter, with the aim of making the new ζ-subsystem own relative degree equal to one with respect to the new input μ_1 (see [15] for an analogous transformation). To this purpose, from (4) and (8) compute

$$\dot{\zeta} = Fz + \gamma(g - Tb_c)v - \gamma TM\mu + Ew \tag{12}$$

and, according to (4), decompose F as

$$F = F_u + F_c \tag{13}$$

where

$$F_u = \begin{bmatrix} -f_1 & 0 & \cdots & 0 \\ -f_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -f_{n-1} & 0 & \cdots & 0 \\ -f_n & 0 & \cdots & 0 \end{bmatrix}$$

$$F_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

so that (12) becomes

$$\dot{\zeta} = F_1 z_1 + F_c\zeta + \gamma(F_c T - TM)\mu + \gamma(g - Tb_c)v + Ew \tag{14}$$

with F_1 denoting the first column of F_u. Choose T, with its entire first row being made of zero elements, in order to simultaneously satisfy

$$(F_c T - TM) = \bar{g}[1, 0, \ldots, 0], \quad Tb_c = g \tag{15}$$

where the coefficients of $\bar{g} = [1, \bar{g}_2, \ldots, \bar{g}_n]^{\mathrm{T}}$ are extracted from the Hurwitz polynomial [recall (16)]

$$\pi(s) = q(s) \times \pi_f(s) \doteq s^{n-1} + \bar{g}_2 s^{n-2} + \cdots + \bar{g}_n. \tag{16}$$

Such a choice, which leads to $T_{\rho-1} = g$, $F_c T_j = T_{j-1}$ for $j = 2, \ldots, \rho - 1$ [T_l are the columns of T, $l = 1, \ldots, \rho - 1$], is actually feasible, owing to the structure of F_c and M, g in (9) and (4), respectively. In accordance with (13)–(15) and with the definition of F_1, we get

$$\dot{\zeta} = F\zeta + \gamma\bar{g}\mu_1 + Ew$$
$$u = h\zeta. \tag{17}$$

B. Scaling for (8) and (9) When $\rho > 1$

Define the change of coordinates[4] $(i = 3, \dots, \rho - 1)$:

$$\begin{aligned}\bar{\mu}_1 &= \mu_1 \\ \bar{\mu}_2 &= \mu_2 + (k-1)a_1\mu_1 \\ \bar{\mu}_i &= \mu_i + (k^{i-1}-1)a_{i-1}\mu_1 + \sum_{j=1}^{i-2} a_j(k^j-1)\mu_1^{(i-j-1)} \end{aligned} \tag{18}$$

and design the control input

$$v = \bar{v} - (k^{\rho-1}-1)a_{\rho-1}\mu_1 - \sum_{j=1}^{\rho-2} a_j(k^j-1)\mu_1^{(\rho-j-1)} \tag{19}$$

in which $\bar{v}$ is yet to be defined. On the basis of (8) and (18)–(19), we get $[\bar{\mu} = [\bar{\mu}_1, \dots, \bar{\mu}_{\rho-1}]^{\mathrm{T}}]$

$$\begin{aligned}\dot{\bar{\mu}} &= \begin{bmatrix} -a_1 k & 1 & 0 & \cdots & 0 \\ -a_2 k^2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{\rho-2}k^{\rho-2} & 0 & 0 & \cdots & 1 \\ -a_{\rho-1}k^{\rho-1} & 0 & 0 & \cdots & 0 \end{bmatrix} \bar{\mu} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \bar{v} \\ &\doteq \bar{M}\bar{\mu} + b_c \bar{v}. \end{aligned} \tag{20}$$

Notice that no filter is required when $\rho = 1$. In that case, $q(s) = 1$, $\pi(s) = \pi_f(s)$, $\bar{g} = g$, and ζ, μ_1 are replaced by z, v, respectively.

C. Intermediate Reference Signals

Consider (1) and define the tracking error $\tilde{x} = x - x_r$, where the reference state x_r, by definition, satisfies $\dot{x}_r = Ax_r + bu_r$, $y_r = cx_r$, in terms of the reference input u_r. The $\tilde{x}$-error system thus reads

$$\begin{aligned}\dot{\tilde{x}} &= A\tilde{x} + b(z_1 - u_r) \\ e &= c\tilde{x}. \end{aligned} \tag{21}$$

Denote by $z_{1,\mathrm{ref}}$ and $\mu_{1,\mathrm{ref}}$ the two reference signals that play the role of intermediate controls in (21) and (17), respectively. In particular, define $z_{1,\mathrm{ref}}$ for (21) as the control input that solves the problem of [13] for subsystem (1) with input $u = z_1$ and satisfies

$$\begin{aligned}\mathcal{L}[z_{1,\mathrm{ref}}(t)](s) &= -\bar{\varepsilon}\frac{d_1 s^{2q} + \cdots + d_{2q+1}}{s\prod_{i=1}^{q}(s^2+\omega_i^2)}\mathcal{L}[e(t)](s) \\ &\doteq G_{\mathrm{ref}}(s)\mathcal{L}[e(t)](s) \end{aligned} \tag{22}$$

with $\bar{\varepsilon}$ being any sufficiently small positive number such that all the poles of $G_{\mathrm{ref}}(s)P(s)/(1 + G_{\mathrm{ref}}(s)P(s))$ have negative real part. On the other hand, design $\mu_{1,\mathrm{ref}}$ for (17) on the basis of the u-feedback as

$$\mu_{1,\mathrm{ref}} = -k_1 u + \nu \tag{23}$$

in which ν is yet to be designed.

D. Change of Coordinates (Zero-Dynamics for the ζ-System)

In accordance with [21] (see [11, Definition 6.4]; see also [8]), there exists a sufficiently large positive real k_1^* such that, for any $k_1 \geq k_1^*$ in (23), the triple $(F - k_1\gamma\bar{g}h, \gamma\bar{g}, h)$ is strictly positive real. To highlight it, introduce the change of coordinates in [15] $\eta_i = \zeta_{i+1} - \bar{g}_{i+1}z_1$, $i = 1, \dots, n-1$, $y_\eta = z_1$ and let η denote the column vector whose components are $\eta_1, \dots, \eta_{n-1}$. The ζ-system in (17), when expressed in such new coordinates, becomes

$$\begin{aligned}\dot{\eta} &= \Gamma_G \eta + \beta_G y_\eta + Z_w w \\ \dot{y}_\eta &= \eta_1 - (f_1 + k_1\gamma - \bar{g}_2)y_\eta + \gamma\nu \\ &\quad + \gamma(\mu_1 - \mu_{1,\mathrm{ref}}) + z_w w \end{aligned} \tag{24}$$

where Z_w and z_w (coming from the (η, y_η)-change of coordinates) are a suitable matrix and a suitable row vector, respectively; Γ_G is the Hurwitz $(n-1)\times(n-1)$ matrix in companion form (4) with characteristic polynomial $\pi(s)$; β_G is the column vector with the $n-1$ components $\bar{g}_3 - \bar{g}_2^2 - f_2 + f_1\bar{g}_2, \dots, \bar{g}_n - \bar{g}_{n-1}\bar{g}_2 - f_{n-1} + f_1\bar{g}_{n-1}, -\bar{g}_n\bar{g}_2 - f_n + f_1\bar{g}_n$. Now, on the basis of (24), the input ν_{ref}, which is able to guarantee $y_\eta \equiv z_{1,\mathrm{ref}}$ when $w = 0$, $\mu_1 = \mu_{1,\mathrm{ref}}$ and initial conditions are compatible, takes the explicit expression

$$\nu_{\mathrm{ref}} = \left[\dot{z}_{1,\mathrm{ref}} - \eta_{\mathrm{ref},1} + (f_1 + k_1\gamma - \bar{g}_2)z_{1,\mathrm{ref}}\right]/\gamma \tag{25}$$

with $\eta_{\mathrm{ref},1}$ being the first component of the vector η_{ref} obeying the ordinary differential equation

$$\dot{\eta}_{\mathrm{ref}} = \Gamma_G \eta_{\mathrm{ref}} + \beta_G z_{1,\mathrm{ref}}. \tag{26}$$

In accordance with the last row of (20) and with (25), impose in (19) and in (23) the signals

$$\begin{aligned}\bar{v} &= a_{\rho-1}k^{\rho-1}\mu_{1,\mathrm{ref}} \\ \nu &= (f_1 + k_1\gamma - \bar{g}_2)z_{1,\mathrm{ref}}/\gamma. \end{aligned} \tag{27}$$

E. Tracking Errors and Related Error System

Introduce the tracking errors:

$$\begin{aligned}\tilde{\mu}_1 &= \mu_1 - \mu_{1,\mathrm{ref}} \\ \tilde{\mu}_j &= \bar{\mu}_j - \sum_{i=1}^{j-1} a_i k^i \mu_{1,\mathrm{ref}}^{(j-1-i)} - \mu_{1,\mathrm{ref}}^{(j-1)}, \ j = 2, \dots, \rho-1 \end{aligned} \tag{28}$$

and their k^{j-1}-scaled version

$$\tilde{\mu}_j^{[\mathrm{s}]} = \tilde{\mu}_j k^{1-j}, \ j = 1, \dots, \rho-1 \tag{29}$$

so that, from (20), the ordinary differential equation satisfied by the column vector $\tilde{\mu}^{[\mathrm{s}]}$ with components $\tilde{\mu}_1^{[\mathrm{s}]}, \dots, \tilde{\mu}_{\rho-1}^{[\mathrm{s}]}$ reads

$$\dot{\tilde{\mu}}^{[\mathrm{s}]} = kM\tilde{\mu}^{[\mathrm{s}]} - b_c \sum_{j=1}^{\rho-1} a_{\rho-1-j}k^{1-j}\mu_{1,\mathrm{ref}}^{(j)} \tag{30}$$

with k—in front of the stable matrix M of (9)—allowing us to resort to the classical high-gain design techniques. On the other hand, define the tracking errors:

$$\tilde{y}_\eta = y_\eta - z_{1,\mathrm{ref}}, \quad \tilde{\eta} \ = \ \eta - \eta_{\mathrm{ref}}. \tag{31}$$

From (24) to (26) and the second expression in (27), they satisfy the error system ($\tilde{\eta}_1$ is the first component of $\tilde{\eta}$):

$$\begin{aligned}\dot{\tilde{\eta}} &= \Gamma_G\tilde{\eta} + \beta_G\tilde{y}_\eta + Z_w w \\ \dot{\tilde{y}}_\eta &= \tilde{\eta}_1 - (f_1 + k_1\gamma - \bar{g}_2)\tilde{y}_\eta - [\dot{z}_{1,\mathrm{ref}} - \eta_{\mathrm{ref},1}] \\ &\quad + \gamma(\mu_1 - \mu_{1,\mathrm{ref}}) + z_w w. \end{aligned} \tag{32}$$

F. Key Idea (Equivalent Disturbance on the Upper Error System)

Let us proceed to allow the multisinusoidal generator of the controller—namely, the block of estimators (22)—to successfully counteract an equivalent disturbance being forced to appear just within the

[4] $f^{(l)}$ denotes the l^{th} derivative of the general time function f.

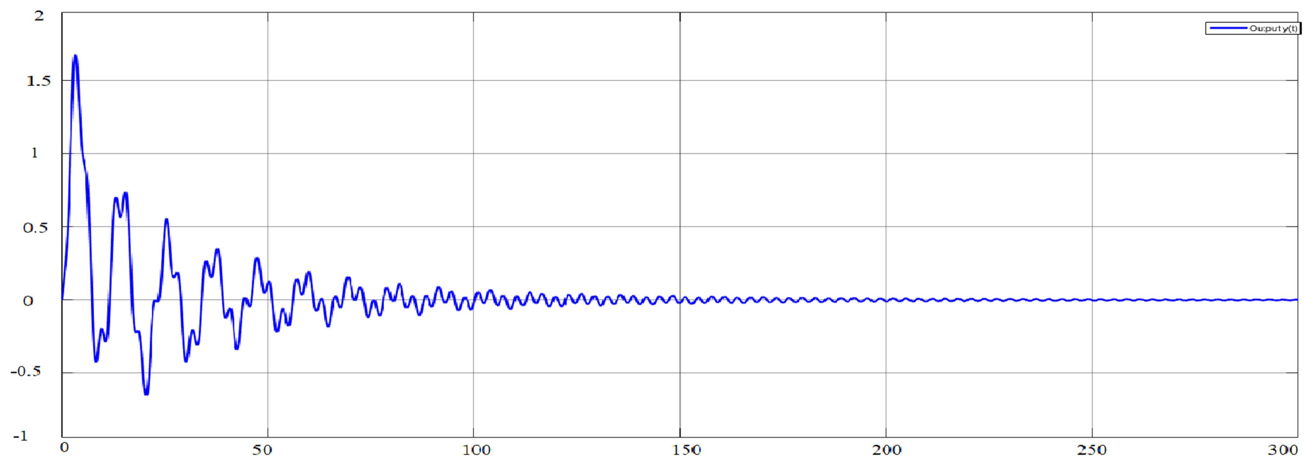

Fig. 2. Time profile for the output $y(t)$.

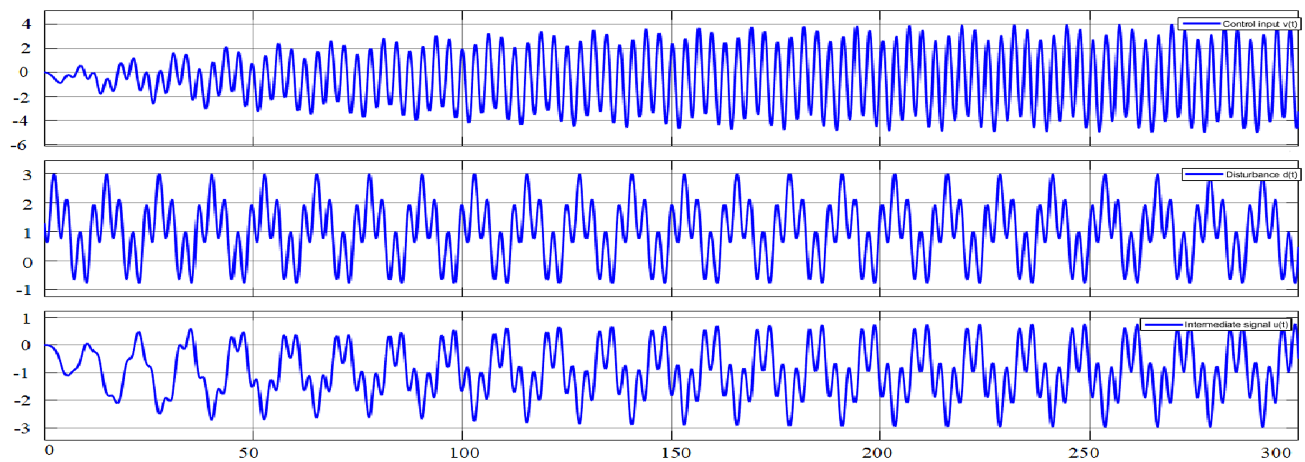

Fig. 3. Time profiles for the input $v(t)$, disturbance $d(t)$, and intermediate signal $u(t)$.

$\tilde{x}$-system (21). To this purpose, by recalling the role of the intermediate reference signal $z_{1,\mathrm{ref}}$, system (21) is rewritten as

$$\dot{\tilde{x}} = A\tilde{x} + b(z_{1,\mathrm{ref}} + \overbrace{\zeta_{\mathrm{ss},1} - u_r}^{\text{equivalent disturbance}}) + b(\overbrace{z_1 - z_{1,\mathrm{ref}} - \zeta_{\mathrm{ss},1}}^{\text{new error } \tilde{y}_\eta - \zeta_{\mathrm{ss},1}}) \tag{33}$$

where $\zeta_{\mathrm{ss}} = [\zeta_{\mathrm{ss},1}, \zeta_{\mathrm{ss},\eta}^{\mathrm{T}}]^{\mathrm{T}} = [\zeta_{\mathrm{ss},1}, \zeta_{\mathrm{ss},\eta,1}, \ldots, \zeta_{\mathrm{ss},\eta,n-1}]^{\mathrm{T}}$ is the uncompensated steady-state solution to (32), (26), and (30), i.e., the vector satisfying

$$\begin{aligned} \dot{\zeta}_{\mathrm{ss},\eta} &= \Gamma_G \zeta_{\mathrm{ss},\eta} + \beta_G \zeta_{\mathrm{ss},1} + Z_w w \\ \zeta_{\mathrm{ss},1} &= (f_1 + k_1\gamma - \bar{g}_2)^{-1} [\zeta_{\mathrm{ss},\eta,1} - \dot{u}_r + \eta_{r,1} \\ &\quad + \gamma \tilde{\mu}_{r,1}^{[\mathrm{s}]} + z_w w] \end{aligned} \tag{34}$$

with $\eta_{r,1}$ being the first component of the η_r-generator

$$\dot{\eta}_r = \Gamma_G \eta_r + \beta_G (u_r - \zeta_{\mathrm{ss},1}) \tag{35}$$

and $\tilde{\mu}_{r,1}^{[\mathrm{s}]}$ being the first component of the $\tilde{\mu}_r^{[\mathrm{s}]}$-generator

$$\dot{\tilde{\mu}}_r^{[\mathrm{s}]} = kM\tilde{\mu}_r^{[\mathrm{s}]} - b_c \sum_{j=1}^{\rho-1} a_{\rho-1-j} k^{1-j} \mu_{1,r}^{(j)}. \tag{36}$$

Here, $\mu_{1,r}$, according to (23) and the definition of the equivalent disturbance in (33), is the multisinusoidal component of $\mu_{1,\mathrm{ref}}$—component generated by the exosystem (2)—given by

$$\mu_{1,r} = -k_1 u_r + (f_1 + k_1\gamma - \bar{g}_2)(u_r - \zeta_{\mathrm{ss},1})/\gamma. \tag{37}$$

The structure of system (34)–(37) guarantees that, for sufficiently large k_1 and k, the solution to (34)–(37) is a multisinusoidal signal generated[5] by (2).

G. State Space Representation for $z_{1,\mathrm{ref}}$

Now, write $z_{1,\mathrm{ref}}$ as generated by

$$\begin{aligned} z_{1,\mathrm{ref}} &= -\bar{\varepsilon} h_\Pi \Pi \\ \dot{\Pi} &= R_c \Pi + de \end{aligned} \tag{38}$$

[5]If $w = 0$, then $\zeta_{\mathrm{ss}} \equiv 0$.

with $\Pi \in \mathbb{R}^{2q+1}$, (R_c, d, h_Π) in form

$$R_c = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -\theta_1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ -\theta_2 & 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\theta_q & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$d = [d_1, \ldots, d_{2q+1}]^{\mathrm{T}}, \quad h_\Pi = [1, 0, \ldots, 0]$$

and θ_i, $i = 1, \ldots, q$, being the coefficients of the polynomial $\prod_{i=1}^{q}(s^2 + \omega_i^2) = s^{2q} + \theta_1 s^{2q-2} + \cdots + \theta_q$. On the other hand, the multisinusoidal component $z_{1,r} = -\zeta_{\mathrm{ss},1} + u_r$ of $z_{1,\mathrm{ref}}$ generated by (2) similarly satisfies

$$z_{1,r} = -\bar{\varepsilon} h_\Pi G$$
$$\dot{G} = R_c G \tag{39}$$

from suitable initial conditions.

H. Lyapunov Analysis (Entire Error System)

In accordance with the previous step, introduce the tracking errors:

$$\tilde{y}_e = \tilde{y}_\eta - \zeta_{\mathrm{ss},1}, \quad \tilde{\eta}_e = \tilde{\eta} - \zeta_{\mathrm{ss},\eta}, \quad \tilde{\Pi} = \Pi - G$$
$$\Delta_e = \eta_{\mathrm{ref}} - \eta_r, \quad \tilde{\mu}_e^{[\mathrm{s}]} = \tilde{\mu}^{[\mathrm{s}]} - \tilde{\mu}_r^{[\mathrm{s}]} \tag{40}$$

so that, from (26), (30), (32)–(39), the error system in the new error coordinates (40) takes the form ($e = c\tilde{x}, h_\eta = [1, 0, \ldots, 0]^{\mathrm{T}}$)

$$\dot{\tilde{x}} = A\tilde{x} - b\bar{\varepsilon} h_\Pi \tilde{\Pi} + b\tilde{y}_e$$
$$\dot{\tilde{\Pi}} = R_c\tilde{\Pi} + de$$
$$\dot{\tilde{\eta}}_e = \Gamma_G \tilde{\eta}_e + \beta_G \tilde{y}_e$$
$$\dot{\tilde{y}}_e = h_\eta \tilde{\eta}_e - (f_1 + k_1\gamma - \bar{g}_2)\tilde{y}_e + \gamma \tilde{\mu}_{e,1}^{[\mathrm{s}]} + \bar{\varepsilon} h_\Pi de + \bar{\varepsilon} h_\Pi R_c \tilde{\Pi} - h_\eta \Delta_e$$
$$\dot{\Delta}_e = \Gamma_G \Delta_e - \beta_G \bar{\varepsilon} h_\Pi \tilde{\Pi}$$
$$\dot{\tilde{\mu}}_e^{[\mathrm{s}]} = kM\tilde{\mu}_e^{[\mathrm{s}]} - b_c \sum_{j=1}^{\rho-1} a_{\rho-1-j} k^{1-j} \mu_{1,e}^{(j)} \tag{41}$$

where $\mu_{1,e} = \mu_{1,\mathrm{ref}} - \mu_{1,r}$ owns jth time derivative

$$\mu_{1,e}^{(j)} = -k_1 \tilde{y}_e^{(j)} + \left(\frac{f_1 - \bar{g}_2}{\gamma}\right)\left(\bar{\varepsilon} h_\Pi R_c^j \tilde{\Pi} - \sum_{i=0}^{j-1} \bar{\varepsilon} h_\Pi R_c^i d e^{(j-1-i)}\right). \tag{42}$$

Let S denote the block-matrix $[A, -b\bar{\varepsilon}h_\Pi; dc, R_c]$ characterizing the exponential stability, in accordance with [13], of the upper linear subsystem with variables $(\tilde{x}, \tilde{\Pi})$ and perturbed by $\tilde{y}_e$. Now use the composite quadratic Lyapunov function that is obtained as the sum of the sub-Lyapunov functions for each linear error subsystem, namely, i) the quadratic Lyapunov function of [15] for the $(\tilde{\eta}_e, \tilde{y}_e)$-subsystem and ii) the quadratic Lyapunov functions including the solutions to the Lyapunov equations $P_j A_j + A_j^{\mathrm{T}} P_j = -\mathbb{I}$ for $j = x, G, \mu$ and $A_x = S$, $A_G = \Gamma_G$, and $A_\mu = M$. The time derivative of such Lyapunov function, not reported here for the sake of brevity, is made negative definite by sufficiently large values of k_1 [stabilizing the $(\tilde{\eta}_e, \tilde{y}_e)$-subsystem and forcing the solution to (34)–(37) to be a multisinusoidal signal] and by sufficiently large values of k, as a function of k_1 at least of order $\mathcal{O}(k_1^{>3})$ for $k_1 \to +\infty$, as well as by sufficiently small values of $\bar{\varepsilon}$ [stabilizing the upper linear subsystem with variables $(\tilde{x}, \tilde{\Pi})$ and concurrently allowing k to dominate the related mixed terms]. The expression (5)–(7) is finally reporting the Laplace transform of the state-space controller (8)–(9), (19), (23), (27), (38), once k_1 is set as $\sqrt[4]{k}$ and a_φ/k replaces the sufficiently small $\bar{\varepsilon}(f_1 + k_1\gamma - \bar{g}_2)/\gamma$.

Remark 3.1: The controller presented in Theorem 3.1 is a two-tier controller. The first tier is constituted by a high-gain linear controller that makes the actuator output follow the nominal control signal being generated by the second tier. Such a second tier is a small-gain controller that is designed at the first stage without accounting for the actuator dynamics. More specifically, controller (5)–(7) is a linear combination of $G_e(s)$ and $G_u(s)$, with $G_e(s)$ acting on the regulation error e with poles at $\{0, \pm j\omega_1, \ldots, \pm j\omega_q\}$ and $G_u(s)$ relying on the variable u. While the former has a gain that reduces as k increases, a high-gain action characterizes the latter as k increases.

Remark 3.2: The choice regarding the functions $\sqrt[4]{k}$ and k/a_φ (a_φ is an additional positive gain) is not mandatory. In fact, the last steps of the proof keeps on holding true even when the expressions of $\sqrt[4]{k}$ and k/a_φ in the statement of Theorem 3.1 are replaced by the most general: i) $\varphi_1(k) : (0, +\infty) \to (0, +\infty)$, restriction[6] to $(0, +\infty)$ of a $\mathcal{K}_\infty$—class function of k that satisfies $\lim_{k \to +\infty} \varphi_1(k)^3/k = 0$; ii) $\varphi_2(k) : (0, +\infty) \to (0, +\infty)$, restriction to $(0, +\infty)$ of a $\mathcal{K}_\infty$—class function of k.

Remark 3.3: Controller (5)–(7) involves $2 + \rho + 2q$ gains, namely, k; a_i, $i = 1, \ldots, \rho - 1$; a_φ; $d_1, \ldots, d_{2q+1}$, whose explicit role is determined by the previously reported proof. First, a_i, $i = 1, \ldots, \rho - 1$, within $\mathcal{F}(s)$ [such that the polynomial $q(s) = s^{\rho-1} + a_1 s^{\rho-2} + \cdots + a_{\rho-1}$ is Hurwitz] are the positive reals characterizing the stable filter (9) of order $\rho - 1$; a_φ characterizes the law which the gain a_φ/k reduces with k through; $d_1, \ldots, d_{2q+1}$ are the coefficients in (3) that characterize controller (3) solving the output regulator problem in the actuator-free case; k is the (possibly high) gain in the scaling transformation (18) that also comprehensively collects the actions of k_1 in (23) and ε in (22). Second, the aforementioned $(G_e(s), G_u(s))$-structure of the controller makes the proposed approach directly extend [13] (namely, the actuator-free case). Indeed, once the action of the filter $\mathcal{F}(s)$ is neglected, (7) has the same structure—with a_φ/k playing the role of ε—of the controller in [13] solving the regulation problem for (1) under the input u, whereas the filter in (8) and (9) [that lead to the definition of $\mathcal{F}(s)$ and is different from the one in [15]], complies with the scaling action imposed, as of (20), onto the filter dynamics, which allows us to apply the classical high-gain analysis tools to the $\tilde{\mu}_e^{[\mathrm{s}]}$-subsystem.

IV. Motivating Example (Continued)

Consider the motivating example of Section I for $\zeta = 2$ under the same disturbance $d(t)$. Theorem 3.1 now applies. The performance of the resulting controller (5)–(7) with

$$\mathcal{F}(s) = \frac{a_1 k(s + a_1)}{s + a_1 k}$$
$$G_e(s) = -\frac{a_\varphi}{k}\mathcal{F}(s)\left[\frac{1}{s} + \frac{s}{(s^2 + 1/4)} - \frac{s}{(s^2 + 4)}\right]$$
$$G_u(s) = \sqrt[4]{k}\mathcal{F}(s)$$

[6]The restriction of a function is a new function, obtained by choosing a smaller domain for the original function.

and $a_1 = 2, k = 16, a_\varphi = 5$ (all the initial conditions of the process and the controller are set to zero) is illustrated by Figs. 2 and 3, which report the time profiles for $y(t)$, $v(t)$, $d(t)$, $u(t)$, respectively. Exponential convergence to zero of $y(t)$ is achieved, with $u(t)$ being able to nullify asymptotically the effect of the disturbance $d(t)$. Anyway, the rate of convergence relies on the stability properties of the upper subsystem that are dominated by the design of [13], as well as by the conservative choice of a_φ/k playing the role of the sufficiently small ε in (3).

V. Conclusion

The regulator design approach of [13] has been extended to cover the case in which the measurable input to the process (1) is provided by the linear minimum-phase actuator (4) with known sign of the high-frequency gain and known relative degree $\rho \geq 1$. The resulting control (5), reported in Theorem 3.1, is a linear combination of $G_e(s)$ and $G_u(s)$, with $G_e(s)$ acting on the regulation error e with poles at $\{0, \pm j\omega_1, \ldots, \pm j\omega_q\}$ and $G_u(s)$ relying on the variable u. The presented scenario covers the case of an uncertain (unmodeled) actuator process, in which no complete knowledge of the information on the transfer function of the overall system is given, while just partial knowledge of such information—i.e., restricted to the transfer function of the original plant—is provided.

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