# Finite-Time Control of Markov Jump Lur'e Systems With Singular Perturbations

Jun Cheng, Ju H. Park, *Senior Member, IEEE*, and Zheng-Guang Wu

***Abstract*—This article investigates the asynchronous controller design issue for Markov jump Lur'e systems with singular perturbations within a fixed time interval. For broader practical applications, the variation of system modes is regulated by a switched nonhomogeneous Markov process that corresponded to both lower level nonhomogeneous stochastic jumping and higher level deterministic switching. By resorting to the hidden nonhomogeneous-Markov model, an asynchronous control law that relies on the detected mode information is proposed. In light of the mode-dependent average dwell time strategy and mode-dependent stochastic system theory, the singularly perturbed parameter-independent conditions are attained to ensure the stochastic finite-time boundedness of the resulting systems. With respect to a linear matrix inequality optimization problem, a solution to the maximum singular perturbation parameter is obtained. Finally, the feasibility and applicability of the devised theoretical results are verified by simulation examples.**

***Index Terms*—Average dwell time, finite-time bounded, nonhomogeneous Markov chain, singularly perturbed Lur'e system.**

## I. Introduction

As a special class of dynamic systems, singularly perturbed systems (SPSs) have shown their powerful capability in describing systems with the cooccurrence of the slow states and the fast states. More specially, the discrepancy of these states is revealed by a singularly perturbed parameter (SPP). SPSs can well model the practical dynamics with multitime-scale phenomena, which have been experimentally exploited in [1]. Until now, most of the existing results on SPSs are focused on linearity, and little effort has been devoted to nonlinearity that is described by Takagi–Sugeno fuzzy model [2]. Different from Takagi–Sugeno fuzzy systems, Lur'e systems are described by both the linear terms and the nonlinearity ones [3], which restricts the dynamic behaviors in the Hurwitz angle domain. Thus, it is an interesting topic to study singularly perturbed Lur'e systems (SPLSs) in the control systems community, and a great many results have been delivered [4]. However, in the most reported results of SPLSs, the nonlinearity is presupposed to rely only on the slow states, which brings some conservatism in the continuous-time domain. It is a typical case that nonlinearity may rely on both slow states and fast states. So far, there are very few results that have been launched in the investigation of discrete-time SPLSs.

In practice, random occurred faults are commonly suffered in the structure and parameters of the industrial systems, which is the source of instability. Note that the Markov process can well depict the jumping among a finite number of system modes, and Markov jumping systems have been forwarded in many fields, such as power systems and robotic systems [5]. It is noteworthy that transition probabilities (TPs) are essential to reflect the mode jumping sequences. Most aforementioned results assume that the Markov jumping process is homogeneous, which indicates TPs are time unchanged [6]. However, this assumption is unrealistic in view of the engineering point. For example, the airspeed of the helicopter varies along with different weather. Accordingly, the nonhomogeneous Markov process has been proposed in [7], in which the TPs are time-varying. Very recently, a piecewise homogeneous Markov process has been studied, in which TPs are varying but unchanged within a certain time interval [5], [8]. In [9], a more general piecewise nonhomogeneous Markov process has been addressed, in which the time-varying TPs are subjected to a higher level Markov process. To this end, in light of the deterministic switching law owes a priority regulator in regulating the feedback information [5], one may be curious about how to describe a switched nonhomogeneous Markov process, in which the higher level signal is modeled by a dwell-time constraint instead of the Markov process. Unfortunately, the switched nonhomogeneous Markov process remains unsettled, which motivates us in this study.

From another perspective, most of the abovementioned results are concerned with the Lyapunov asymptotic stability, which neglects the transient dynamics. From the practical viewpoint, the researchers are only interested in transient dynamics over a fixed time interval, such as electronic circuits and flight systems. To overcome this limitation, the finite-time stability has attracted the increasing research interest [10], [11]. Note that in reality, by resorting to finite-time stability, the system states are limited to stay in a prescribed threshold within a certain time interval, which presents a better transient performance. Although many fruitful achievements have been made for transient dynamics [12], to our knowledge, no literature corresponds to the finite-time stability for SPLSs, not to mention to the resulting dynamics suffering the nonhomogeneous Markov process. Therefore, it is of theoretical and practical significance to explore the transient dynamics for Markov jumping SPLSs (MJSPLSs), which inspired the current work.

Inspired by the abovementioned observation, the goal of this work is to forward the finite-time control for MJSPLSs by proposing a switched nonhomogeneous Markov process strategy. To carry out the developed issue, the main challenges are highlighted as follows:

Manuscript received 28 February 2022; revised 28 August 2022; accepted 16 January 2023. Date of publication 19 January 2023; date of current version 26 October 2023. The work of J. H. Park was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (Ministry of Science and ICT) under Grant 2019R1A5A8080290. This work was supported in part by the National Natural Science Foundation of Guangxi Province under Grant 2020GXNSFFA297003, and in part by the National Natural Science Foundation of China under Grant 62173100. Recommended by Associate Editor U. Topcu. *(Corresponding authors: Jun Cheng; Ju H. Park.)*

Jun Cheng is with the School of Mathematics and Statistics, Guangxi Normal University, Guilin 541006, China (e-mail: jcheng@gxnu.edu.cn).

Ju H. Park is with the Department of Electrical Engineering, Yeungnam University, Kyongsan 38541, Republic of Korea (e-mail: jessie@ynu.ac.kr).

Zheng-Guang Wu is with the College of Electrical and Information Engineering, Hunan University, Changsha 410082, China (e-mail: nashwzhg@126.com).

Color versions of one or more figures in this article are available at https://doi.org/10.1109/TAC.2023.3238296.

Digital Object Identifier 10.1109/TAC.2023.3238296

1) How to better model the complicated behavior of the Markov process?
2) How to analyze the stochastic finite-time stability for MJSPLSs and solve the finite-time control problem?
3) How to deal with the asynchronous phenomenon between the plant and the controller?

To facilitate these challenges, we endeavor to develop a unified asynchronous finite-time control methodology for MJSPLSs subject to the switched nonhomogeneous Markov process. The main contributions are summarized as fourfolds.

1) As a first attempt, a switched nonhomogeneous Markov model is developed to regulate the system modes switching, in which the time-varying TPs are subjected to a higher level deterministic switching signal.
2) Different from the reported Lyapunov asymptotic stability of continuous-time SPLSs, the finite-time control issue is first exploited for discrete-time SPLSs, where the state trajectories remain bounded within a certain time interval.
3) Benefit from the hidden nonhomogeneous-Markov model (HNMM), the asynchronous control law that relies on the detected modes can be developed.
4) By establishing a novel Lyapunov functional, SPP-independent sufficient conditions are attained such that the closed-loop dynamic is stochastic finite-time bounded (SFTB).

The rest of this article is organized as follows. Section II formulates the preliminaries. Section III presents the SFTB for discrete-time SPLSs. Simulation examples are applied in Section IV. Finally, Section V concludes this article.

*Notation:* $\mathscr{E}\{\mathscr{Q}\}$ refers to the expectation of $\mathscr{Q}$. $\lambda_{\max}(\mathscr{Q})/\lambda_{\min}(\mathscr{Q})$ signifies the maximum/minimum eigenvalues of $\mathscr{Q}$. $\mathrm{He}(\mathscr{Q})$ means $\mathscr{Q}+\mathscr{Q}^\top$. $\mathrm{diag}\{\cdots\}$ implies a block diagonal matrix. $\bar{\Psi}$ refers to the sample space, $\mathcal{F}$ stands for the $\sigma$-algebra of subsets of $\bar{\Psi}$, $\mathbb{P}$ symbolizes the probability measure on $\mathcal{F}$. $\|\cdot\|$ implies the Euclidean norm.

## II. Problem Formulation

In a fixed probability space $(\bar{\Psi},\mathcal{F},\mathbb{P})$, $\{\delta_k, k\geq 0\}$ is identified as a general right-continuous nonhomogeneous Markov process on the probability space subject to a finite-state space $\mathcal{N}=\{1,2,\ldots,\mathscr{N}\}$ with generator $\Psi^{(\sigma_k)}(k)=[\psi_{mn}^{(\sigma_k)}(k)]_{\mathscr{N}\times\mathscr{N}}$ expressed as

$$\psi_{mn}^{(\sigma_k)}(k)=\Pr\{\delta_{k+1}=n\mid\delta_k=m\}\ \forall m,n\in\mathcal{N} \tag{1}$$

where $\psi_{mn}^{(\sigma_k)}(k)\geq 0$ and $\sum_{n\in\mathcal{N}}\psi_{mn}^{(\sigma_k)}(k)=1$. The switched nonhomogeneous transition probability (SNTP) matrices $\Psi^{\sigma_k}(k)$ are time-varying matrices, which can be characterized by the following polytope structure:

$$\begin{aligned}\Psi^{(\sigma_k)}(k)&=\Psi^{(\sigma_k)}(\hbar(k))=\sum_{\iota=1}^{J_1}\hbar_l(k)\Psi^{(\sigma_k,\iota)}\\ \Psi^{(\sigma_k,\iota)}&=\left[\psi_{mn}^{(\sigma_k,\iota)}\right]_{\mathscr{N}\times\mathscr{N}}\end{aligned} \tag{2}$$

where $\hbar_\iota(k)\geq 0$ and $\sum_{\iota=1}^{J_1}\hbar_\iota(k)=1$, $\iota=\{1,2,\ldots,J_1\}$. Specifically, $\Psi^{(\sigma_k,\iota)}$ are vertex matrices, and $J_1$ implies the number of vertices of $\Psi^{(\sigma_k)}$. It is clear that $\psi_{mn}^{(\sigma_k)}(k)=\sum_{\iota=1}^{J_1}\hbar_\iota(k)\psi_{mn}^{(\sigma_k,\iota)}$. Specifically, $\{\sigma_k, k\geq 0\}$ is recognized as a high-level deterministic switching function, which meets the mode-dependent average dwell time switching and taking values over a set $\mathcal{M}=\{1,2,\ldots,\mathscr{M}\}$. In this study, the model (2) is called SNTPs, and switching signal $\sigma_k$ is employed to describe the deterministic switching of TPMs in $\Psi^{(\sigma_k)}(k)$.

*Remark 1:* The piecewise homogeneous transition probabilities have been exploited in [8], which are time-dependent but invariant within a certain time interval. Motivated by the aforementioned work, it is remarkable that the MJSPLS (1) with SNTPs (2) is characterized by both nonhomogeneous stochastic jumping process $\delta_k$ and deterministic switching signal $\sigma_k$ simultaneously. $\sigma_k$ is identified as a higher level switching signal, which releases a sequence of signals to govern suitable feedback switching signals among the lower level nonhomogeneous Markov process $\delta_k$. In light of the mode-dependent average dwell time approach having the better capability in compensating the effect of arbitrary switching, the merits of the switched nonhomogeneous Markov process can be easily recognized. When the switching signal $\sigma_k$ is transferred into a homogeneous Markov chain and $J_1=1$, model (1) corresponds to the piecewise homogeneous case [9]. When the switching signal $\sigma_k$ is fixed as a constant signal, model (1) degrades to the nonhomogeneous case [7]. Thus, the investigation of NSTPs is more general and complex.

Consider a class of discrete-time MJSPLSs described by

$$\begin{cases}x_1(k+1)=A_{11}(\delta_k)x_1(k)+\varepsilon A_{12}(\delta_k)x_2(k)+B_1(\delta_k)u(k)\\ \qquad\qquad +C_1(\delta_k)\varphi(\delta_k,\zeta(k))+F_1(\delta_k)\omega(k)\\ x_2(k+1)=A_{21}(\delta_k)x_1(k)+\varepsilon A_{22}(\delta_k)x_2(k)+B_2(\delta_k)u(k)\\ \qquad\qquad +C_2(\delta_k)\varphi(\delta_k,\zeta(k))+F_2(\delta_k)\omega(k)\\ \qquad \zeta(k)=H_1(\delta_k)x_1(k)+\varepsilon H_2(\delta_k)x_2(k)\\ \qquad z(k)=D_1(\delta_k)x_1(k)+\varepsilon D_2(\delta_k)x_2(k)\end{cases} \tag{3}$$

where $x_1(k)\in\mathbb{R}^{n_s}$, $x_2(k)\in\mathbb{R}^{n_f}$ are the slow and fast state vectors, respectively. $u(k)\in\mathbb{R}^{n_u}$, and $z(k)\in\mathbb{R}^{n_z}$, respectively, denote the control input and the controlled output. $\varphi(\cdot)$ stands for a memoryless nonlinear function. $\omega(k)\in\mathbb{R}^w$ is the external disturbance. Specifically, the Lur'e nonlinearity $\varphi(\delta_k,\zeta(k))$ is a parameter-dependent memoryless nonlinearity. $\varepsilon\in(0,\bar{\varepsilon}]$ is an SPP, and $\bar{\varepsilon}<1$ symbolizes the upper bound of SPP. $A_{ij}(\delta_k)$, $B_i(\delta_k)$, $C_i(\delta_k)$, $H_i(\delta_k)$, $D_i(\delta_k)$, and $F_i(\delta_k)$ $(i,j=1,2)$ are known matrices with suitable dimensions.

*Assumption 1:* For given scalar $d>0$, the external disturbance $\omega(k)$ is norm-bounded over interval $[\mathcal{T}_1,\mathcal{T}_2]$ and meets $\mathbb{W}_{[\mathcal{T}_1,\mathcal{T}_2],d}\triangleq\{\omega(k):\sum_{k=\mathcal{T}_1}^{\mathcal{T}_2}\omega^\top(k)\omega(k)\leq d\}$.

*Assumption 2:* ([13]) The nonlinear function $\varphi(\delta_k,\zeta(k))$ satisfies a cone-bounded sector criteria and is decentralized being associated with each scalar $\delta_k=m\in\mathcal{N}$, $\varphi_m(\zeta(k))=\varphi(\delta_k,\zeta(k))$, which meets the following two conditions: 1) $\varphi_m(0)=0$; and 2) there exist diagonal matrices $\Omega_m>0$

$$\varphi_{m,l}(\zeta(k))[\varphi_m(\zeta(k))-\Omega_m\zeta(k)]_l\leq 0. \tag{4}$$

By resorting to Assumption 2, for any diagonal matrices $\Lambda_m>0\quad\forall m\in\mathcal{N}$, it is clear that $\mathbf{SC}(m,\zeta(k),\Lambda_m)\triangleq\varphi_m^\top(\zeta(k))\Lambda_m[\varphi_m(\zeta(k))-\Omega_m\zeta(k)]\leq 0$. Obviously, from (4), it yields $[\Omega_m\zeta(k)]_l[\varphi_m(\zeta(k))-\Omega_m\zeta(k)]_l\leq 0$, one further derives the condition $0\leq\varphi_m^\top(\zeta(k))\Lambda_m\zeta_m(\zeta(k))\leq\varphi_m^\top(\zeta(k))\Lambda_m\Omega_m\zeta(k)\leq\zeta^\top(k)\Omega_m\Lambda_m\Omega_m\zeta(k)$ holds.

*Remark 2:* In [4], Lur'e nonlinearities are parameter-$\varepsilon$-independent and some simple storage functions are proposed in Lyapunov functionals. To our knowledge, the investigation of SPLSs has not been extended to the discrete-time domain. To reduce the aforementioned gap, the dynamic behavior of discrete-time SPLSs subject to a stochastic switching process is studied within a certain interval. In contrast with the existing results, Lur'e nonlinearity $\varphi(\delta_k,\zeta(k))$ is both mode-dependent and SPP $\varepsilon$-dependent, which signifies that the investigated dynamic (3) is more general.

Let $x(k) = [x_1^\top(k)\ x_2^\top(k)]^\top$, the MJSPLS (3) can be reformulated as

$$\begin{cases} x(k+1) = & A(\delta_k)E_\varepsilon x(k) + B(\delta_k)u(k) \\ & +C(\delta_k)\varphi(\delta_k,\zeta(k)) + F(\delta_k)\omega(k) \\ z(k) = & D(\delta_k)E_\varepsilon x(k), \zeta(k) = H(\delta_k)E_\varepsilon x(k) \end{cases} \tag{5}$$

where $A(\delta_k) = \begin{bmatrix} A_{11}(\delta_k) & A_{12}(\delta_k) \\ A_{21}(\delta_k) & A_{22}(\delta_k)\end{bmatrix}, B(\delta_k) = \begin{bmatrix} B_1(\delta_k) \\ B_2(\delta_k)\end{bmatrix}, C(\delta_k) = \begin{bmatrix} C_1(\delta_k) \\ C_2(\delta_k)\end{bmatrix}, F(\delta_k) = \begin{bmatrix} F_1(\delta_k) \\ F_2(\delta_k)\end{bmatrix}, D(\delta_k) = [D_1(\delta_k)\ D_2(\delta_k)], H(\delta_k) = [H_1(\delta_k)\ H_2(\delta_k)]$, $E_\varepsilon = \mathrm{diag}\{I_{n_s}, \varepsilon I_{n_f}\}$.

*Assumption 3:* Notice that for all $\pi_k \in \mathcal{S} = \{1, 2, \ldots, \mathscr{S}\}$, we consider the pair $(\delta_k, \pi_k)$ as HNMM, from which the mode detection probabilities $\phi_{ms}(k)$ are devised as

$$\phi_{ms}(k) = \Pr\{\pi_k = s \mid \delta_k = m\} \tag{6}$$

where $\phi_{ms}(k) \in [0,1]$ and $\sum_{s\in\mathcal{S}} \phi_{ms}(k) = 1$. The time-varying mode detection probability matrix (DPM) $\Phi(k) = [\phi_{ms}(k)]_{m\in\mathcal{N}, s\in\mathcal{S}}$ can be described by the following polytope:

$$\Phi(k) = \Phi(\nu(k)) = \sum_{\ell=1}^{J_2} \nu_\ell(k)\Phi^{(\ell)},\ \Phi^{(\ell)} = \left[\phi_{ms}^{(\ell)}\right]_{\mathscr{S}\times\mathscr{S}}$$

where $\nu_\ell(k) \geq 0$ and $\sum_{\ell=1}^{J_2} \nu_\ell(k) = 1$, $\ell = \{1, 2, \ldots, J_2\}$. $\Phi^{(\ell)}$ indicate the vertex matrices, and $J_2$ signifies the number of vertices of $\Phi(k)$. It is clear that $\phi_{ms}(k) = \sum_{\ell=1}^{J_2} \nu_\ell(k)\phi_{ms}^{(\ell)}$.

*Remark 3:* To reveal the asynchronous framework between original system mode and controller mode, the hidden homogeneous-Markov model has been widely studied in literature [3], [14], where mode detection probabilities of the controller are presupposed to be time-invariant. It is noteworthy that such assumption is not verified in real applications. Inspired by the work of [15], the mode detection probabilities are time-varying, which rely on the current moment. Thus, the HNMM is addressed in this work, where the detection probabilities of controller mode $\pi_k$ are time-varying and belong to a polytope set.

As the above discussion implies, the plant mode $\delta_k$ can be only observed by the HNMM (6) with the detector $\pi_k$. To carry out the aforementioned special requirement, an asynchronous controller for MJSPLS (5) is designed as

$$\begin{cases} u(k) = K(\pi_k)E_\varepsilon x(k) + L(\pi_k)\varphi(\pi_k, \varsigma(k)) \\ \varsigma(k) = H(\pi_k)E_\varepsilon x(k) \end{cases} \tag{7}$$

where $K(\pi_k)$ and $L(\pi_k)$ are the controller gains to be solved.

Combining (5) and (7), for $\delta_k = m$ and $\pi_k = s$, one has

$$\begin{cases} x(k+1) = \mathscr{A}_{ms}E_\varepsilon x(k) + C_m\varphi_m(\zeta(k)) \\ \qquad\qquad +B_m L_s \varphi_s(\varsigma(k)) + F_m\omega(k) \\ z(k) = D_m E_\varepsilon x(k) \end{cases} \tag{8}$$

where $\mathscr{A}_{ms} = A_m + B_m K_s$.

In what follows, some helpful definitions are recalled.

*Definition 1:* ([12]) For a switching signal $\sigma_k$ and any $k \in [0, \mathcal{T}]$, define $N_{\sigma p}(k, \mathcal{T})$ and $\mathcal{G}_p(k, \mathcal{T})$ as the switching number and the whole activating time of the $p$th subsystem during an interval $[k, \mathcal{T}]$. One derives $\sigma(k)$ subject to mode-dependent average dwell time $\tau_{ap}$, where $N_{0p} > 0$ is mode-dependent chatter bounds, and $\tau_{ap}$ meets

$$N_{\sigma p}(k, \mathcal{T}) \leq N_{0p} + \frac{\mathcal{G}_p(k, \mathcal{T})}{\tau_{ap}}. \tag{9}$$

*Definition 2:* ([10]) The MJSPLS (8) is SFTB with respect to (w.r.t.) $(c_1, c_2, \mathcal{R}, [\mathcal{T}_1, \mathcal{T}_2], \mathbb{W}_{[\mathcal{T}_1,\mathcal{T}_2],d})$ for all $\omega(t) \in \mathbb{W}_{[\mathcal{T}_1,\mathcal{T}_2],d}$, where matrix $\mathcal{R} > 0$, and parameters $0 < c_1 < c_2$, such that for all $k \in [\mathcal{T}_1, \mathcal{T}_2]$

$$x^\top(\mathcal{T}_1)\mathcal{R}x(\mathcal{T}_1) \leq c_1 \Rightarrow x^\top(k)\mathcal{R}x(k) \leq c_2\ \forall k \in [\mathcal{T}_1, \mathcal{T}_2].$$

Our objective is to design an asynchronous controller (7), such that the resulting system is SFTB with assigned $\mathcal{H}_\infty$ performance $\gamma(\gamma > 0)$, similar to [16], $z(k)$ satisfies $\sum_{k=0}^{\mathcal{T}} \mathscr{E}\{\|z(k)\|^2\} < \gamma^2 \sum_{k=0}^{\mathcal{T}} \|\omega(k)\|^2$.

## III. Main Result

In what follows, for brevity, we define

$$V_1(x(k), \delta_k, \sigma_k) = x^\top(k) \sum_{\iota=1}^{J_1} \hbar_\iota(k)(P_m^{(p,\iota)})^{-1}x(k)$$

$$V_2(x(k), \delta_k, \sigma_k) = \varphi_m^\top(\zeta(k)) \sum_{\iota=1}^{J_1} \hbar_\iota(k)(\Delta_m^{(p,\iota)})^{-1}\Omega_m H_m \times E_\varepsilon x(k), \mathscr{G}_m^{(p,\iota)} = \{P_m^{(p,\iota)}, \Delta_m^{(p,\iota)}\}$$

$$\mathscr{T}_p = \mathcal{T}\ln(\mu_p)/(\ln(\lambda_1 c_2) - \ln((\lambda_2 + \lambda_3)c_1 + \gamma^2 d) - \mathcal{T}_p)$$

$$\mathcal{T}_p = \mathcal{T}\ln(1 + \alpha_p) + \sum_{p=1}^{N} N_{0p}\ln(\mu_p)$$

$$\Gamma(k) = \gamma^2\omega^\top(k)\omega(k) - z^\top(k)z(k), E_0 = \mathrm{diag}\{I_{n_s}, 0\}$$

$$\lambda_1 = \min_{m\in\mathcal{N}, p\in\mathcal{S}, \iota=1,\ldots,J_1} (\lambda_{\min}(\mathcal{R}^{-\frac{1}{2}}(P_m^{(p,\iota)})^{-1}\mathcal{R}^{-\frac{1}{2}}))$$

$$\lambda_2 = \max_{m\in\mathcal{N}, p\in\mathcal{S}, \iota=1,\ldots,J_1} \left(\lambda_{\max}(\mathcal{R}^{-\frac{1}{2}}(P_m^{(p,\iota)})^{-1}\mathcal{R}^{-\frac{1}{2}}\right)$$

$$\lambda_3 = \max_{m\in\mathcal{N}, p\in\mathcal{S}, \iota=1,\ldots,J_1} \lambda_{\max}\Big(\mathcal{R}^{-\frac{1}{2}}(2E_\varepsilon H_m^\top \times \Omega_m\Delta_m^{(p,\iota)}\Omega_m H_m E_\varepsilon)^{-1}\mathcal{R}^{-\frac{1}{2}}\Big)\Big)$$

$$\overline{\gamma} = \gamma \max_{p\in\mathscr{S}}(\exp\{2\mathcal{T}\ln(1 + \alpha_p) + \sum_{p=1}^{N} N_{0p}\ln(\mu_p)\})^{1/2}$$

$$\Sigma_{\iota\jmath\ell msp}^{1,1} = \mathrm{diag}\{\upsilon_1^2 Z_{\iota msp} - \mathbf{He}(\upsilon_1\mathcal{W}_1), -2\Delta_m^{(p,\iota)}, -\gamma^2 I\}$$

$$\Sigma_{\iota\jmath\ell msp}^{1,2} = \mathrm{diag}\{\upsilon_1^2 Z_{\iota msp} - \mathbf{He}(\upsilon_1\mathcal{U}_{\overline{\varepsilon}}), -2\Delta_m^{(p,\iota)}, -\gamma^2 I\}$$

$$\Sigma_{\iota\jmath\ell msp}^{1,3} = \mathrm{diag}\{\upsilon_1^2 Z_{\iota msp} - \mathbf{He}(\upsilon_1\mathcal{U}_{\overline{\varepsilon}}), -2\Delta_m^{(p,\iota)}, -\gamma^2 I\}$$

$$\Sigma_{\iota\jmath\ell msp}^{2,\wp} = \left[\sqrt{2}\mathscr{F}\mathcal{X}_{ms}^{\wp\top}\ 2\mathcal{X}_{ms}^{\wp\top}\ \Sigma_{\iota\jmath\ell msp}^{22,\wp}\ \Sigma_{\iota\jmath\ell msp}^{22,\wp}\ \Sigma_{\iota\jmath\ell msp}^{23,\wp}\right]$$

$$\Sigma_{\iota\jmath\ell msp}^{3} = \mathrm{diag}\{-P_1^{(p,\jmath)}, \ldots, -P_{\mathcal{N}}^{(p,\jmath)}, -\tau_{\iota\jmath msp}I, -Q_{\iota\jmath msp} - S_{\iota\jmath msp}, -I\}, E_{\overline{\varepsilon}} = \mathrm{diag}\{I_{n_s}, \overline{\varepsilon}I_{n_f}\}$$

$$\Sigma_{\iota\jmath\ell msp}^{22,j} = [\Omega_s H_s \Theta_j\ 0\ 0]^\top (j = 1,2,3), \Sigma_{\iota\jmath\ell msp}^{23,1} = [D_m\Theta_1\ 0\ 0]$$

$$\Sigma_{\iota\jmath\ell msp}^{23,2} = [D_m\Theta_2\ 0\ 0], \Sigma_{\iota\jmath\ell msp}^{23,3} = [D_m\Theta_3\ 0\ 0]$$

$$\mathcal{X}_{ms}^1 = [(A_m\Theta_1 + B_m\overline{K}_s E_0)\ C_m\ F_m]$$

$$\mathcal{X}_{ms}^i = [(A_m\Theta_i + B_m\overline{K}_s E_{\overline{\varepsilon}})\ C_m\ F_m], (i = 2,3)$$

$$\mathcal{U}_{\overline{\varepsilon}} = \mathcal{W}_1 + \overline{\varepsilon}\mathcal{W}_2, \Theta_f = \sum_{f=1}^{3} \overline{\varepsilon}^{f-1}G_f, G_1 = \mathrm{diag}\{\mathcal{U}_1, 0\}$$

$$G_2 = \begin{bmatrix} \mathcal{U}_3 & \mathcal{U}_5^\top \\ \mathcal{U}_5 & \mathcal{U}_2 \end{bmatrix} G_3 = \mathrm{diag}\{0, \mathcal{U}_4\}\ \mathcal{W}_1 = \begin{bmatrix} \mathcal{U}_1 & 0 \\ \mathcal{U}_5 & \mathcal{U}_2 \end{bmatrix}$$

$$\mathcal{W}_2 = \begin{bmatrix} \mathcal{U}_3 & \mathcal{U}_5^\top \\ 0 & \mathcal{U}_4 \end{bmatrix} \mathscr{F} = \begin{bmatrix} \sqrt{\psi_{m1}^{p,\iota}} & \cdots & \sqrt{\psi_{m\mathcal{N}}^{p,\iota}} \end{bmatrix}.$$

In this section, the SFTB of closed-loop dynamic (8) will be presented, and the upper bound of SPP will be calculated.

*Theorem 1:* Given constants $c_2 > c_1 > 0$, $d > 0$, $\alpha_p > 0$, $\gamma, \mu_p \geq 1$, $p \in \mathcal{S}$, matrices $P_m^{(p,\iota)} > 0$, $\Delta_m^{(p,\iota)} > 0$, $m \in \mathcal{N}, p \in \mathcal{S}, \iota = 1, \ldots, J_1$, and three $\mathcal{K}_\infty$ functions $\lambda_1$, $\lambda_2$, and $\lambda_3$, if there exists Lyapunov functionals

$$V(x(k), \delta_k, \sigma_k) = \sum_{l=1}^{2} V_l(x(k), \delta_k, \sigma_k) \tag{10}$$

with matrices

$$(\mathscr{G}_m^{(p,\iota)})^{-1} > 0,\ (\mathscr{G}_m^{(p,\iota)})^{-1} \leq \mu_p (\mathscr{G}_m^{(q,\iota)})^{-1} (p \neq q) \tag{11}$$

the closed-loop dynamic (8) is SFTB w.r.t. $(c_1, c_2, \mathcal{R}, [0, \mathcal{T}], \mathbb{W}_{[0,\mathcal{T}],d})$ and meets an $\mathcal{H}_\infty$ performance index $\overline{\gamma}$, such that $\forall \delta_k = m, \sigma_k = p$

$$\begin{aligned} \lambda_1 x^\top(k) \mathcal{R} x(k) &\leq V(x(k), \delta_k, \sigma_k) \\ &\leq (\lambda_2 + \lambda_3) x^\top(k) \mathcal{R} x(k) \end{aligned} \tag{12}$$

$$\begin{aligned} V(x(k+1), \delta_{k+1}, \sigma_k) &\leq (1 + \alpha_p) V(x(k), \delta_k, \sigma_k) \\ &\quad + \Gamma(k) \end{aligned} \tag{13}$$

$$(\lambda_2 + \lambda_3) c_1 + \gamma^2 d < \exp(-\mathcal{T}_p)\, \lambda_1 c_2. \tag{14}$$

Thus, the switching signal $\sigma(k)$ satisfying

$$\tau_{ap} > \tau_{ap}^* = \max[\mathscr{T}_p, \ln(\mu_p)/\ln(1+\alpha_p)] \tag{15}$$

thus, we have $\sum_{k=0}^{\mathcal{T}} \mathscr{E}\{z^\top(k) z(k)\} < \overline{\gamma}^2 \sum_{k=0}^{\mathcal{T}} \omega^\top(k)\omega(k)$.

*Proof:* Please see the Appendix A. ■

*Remark 4:* It is noteworthy that a novel Lyapunov functional $V(x(k), \delta_k, \sigma_k)$ has been proposed, in which the matrices $P_m^{(p,\iota)}$ and $\Delta_m^{(p,\iota)}$ simultaneously rely on parameter $\hbar_\iota(k)$, mode $\delta_k = m$, and switching signal $\sigma_k = p$. Apparently, by absorbing some useful information (i.e., $\hbar_\iota(k)$, $\delta_k$, $\sigma_k$) in Lyapunov functional $V(x(k), \delta_k, \sigma_k)$, one can achieve more freedom in searching for feasible solutions, and acquire better system performance. In addition, the switching signal $\sigma_k$ is subjected to the mode-dependent average dwell time giving more flexibility in searching for a feasible solution, which can be degenerated to the case of average dwell time by letting $\alpha_p = \alpha$, $\tau_{ap} = \tau_a$, $N_{\sigma p} = N_\sigma$.

*Remark 5:* Most existing results on SPLSs are concerned with the Lyapunov asymptotic stability over an infinite time horizon. In practical application, there is a lack of investigation on SFTB for SPLSs, not to mention to the nonhomogeneous stochastic switching process. To fill this gap, the SFTB analysis has been made for MJSPLSs in Theorem 1.

Notice that Theorem 1 gives the SFTB with assigned $\mathcal{H}_\infty$ performance for system (8), based on Theorem 1, the desired controller gains will be derived in Theorem 2.

*Theorem 2:* Given constants $\overline{\varepsilon} > 0$, $c_2 > c_1 > 0$, $\alpha_p > 0$, $\gamma$, $\mu_p \geq 1$, $\tau_{\iota\jmath mp} > 0$, matrices $P_m^{(p,\iota)} > 0$, $\Delta_m^{(p,\iota)} > 0$, $Z_{\iota msp} > 0$, $Q_{\iota\jmath\,\mathrm{msp}}$, $S_{\iota\jmath\mathrm{msp}} > 0$, $\overline{K}_s$, $\overline{L}_s$, the closed-loop dynamic (8) is SFTB w.r.t. $(c_1, c_2, \mathcal{R}, [0, \mathcal{T}], \mathbb{W}_{[0,\mathcal{T}],d})$ and meets an $\mathcal{H}_\infty$ performance index $\overline{\gamma}$, such that for any $m \in \mathcal{N}$, $p \in \mathcal{M}$, $\iota, \jmath \in \{1, 2, \ldots, J_1\}$, $\ell \in \{1, 2, \ldots, J_2\}$, the switching signal $\sigma_k$ satisfies (15), condition (14) holds and

$$\mathcal{U}_1 > 0 \tag{16}$$

$$\begin{bmatrix} \mathcal{U}_1 + \overline{\varepsilon}\mathcal{U}_3 & \overline{\varepsilon}\mathcal{U}_5^\top \\ \overline{\varepsilon}\mathcal{U}_5 & \overline{\varepsilon}\mathcal{U}_2 \end{bmatrix} > 0 \tag{17}$$

$$\begin{bmatrix} \mathcal{U}_1 + \overline{\varepsilon}\mathcal{U}_3 & \overline{\varepsilon}\mathcal{U}_5^\top \\ \overline{\varepsilon}\mathcal{U}_5 & \overline{\varepsilon}\mathcal{U}_2 + \overline{\varepsilon}^2\mathcal{U}_4 \end{bmatrix} > 0 \tag{18}$$

$$\begin{bmatrix} -\lambda_2\mathcal{R} & I \\ * & -P_m^{(p,\iota)} \end{bmatrix} < 0, \begin{bmatrix} -P_m^{(p,\iota)} & \sqrt{\lambda_1} P_m^{(p,\iota)}\mathcal{R} \\ * & -\mathcal{R} \end{bmatrix} < 0 \tag{19}$$

$$\begin{bmatrix} -\lambda_3\mathcal{R} & E_0 H_m^\top \Omega_m \\ * & -\Delta_m^{(p,\iota)} \end{bmatrix} < 0, \begin{bmatrix} -\lambda_3\mathcal{R} & E_{\overline{\varepsilon}} H_m^\top \Omega_m \\ * & -\Delta_m^{(p,\iota)} \end{bmatrix} < 0 \tag{20}$$

$$\begin{bmatrix} -\mu_p \mathscr{G}_m^{(q,\iota)} & \mathscr{G}_m^{(q,\iota)} \\ * & -\mathscr{G}_m^{(p,\iota)} \end{bmatrix} < 0 \tag{21}$$

$$\begin{bmatrix} -(1+\alpha_p)P_m^{(p,\iota)} & \begin{bmatrix} \sqrt{\phi_{m1}^{(\ell)}} P_m^{(p,\iota)} & \cdots \sqrt{\phi_{m\mathscr{S}}^{(\ell)}} P_m^{(p,\iota)} \end{bmatrix} \\ * & -\mathrm{diag}\{Z_{\iota m1p}, \ldots, Z_{\iota m\mathscr{S}p}\} \end{bmatrix} < 0 \tag{22}$$

$$\begin{bmatrix} \upsilon_2^2 Q_{\iota\jmath msp} - \mathbf{He}(\upsilon_2 Y_s) & \sqrt{2}\mathscr{F}\overline{L}_s^\top B_m^\top \\ * & -\mathrm{diag}\{P_1^{(p,\jmath)}, \ldots, P_{\mathcal{N}}^{(p,\jmath)}\} \end{bmatrix} < 0 \tag{23}$$

$$\begin{bmatrix} -\tau_{\iota\jmath mp} & \tau_{\iota\jmath mp} E_0 [\sqrt{\psi_{m1}^{p,\iota}} H_m^\top \Omega_m & \cdots & \sqrt{\psi_{m\mathcal{N}}^{p,\iota}}] \\ * & -\mathrm{diag}\{\Delta_1^{(p,\jmath)}, \ldots, \Delta_{\mathcal{N}}^{(p,\jmath)}\} \end{bmatrix} < 0 \tag{24}$$

$$\begin{bmatrix} -\tau_{\iota\jmath mp} & \tau_{\iota\jmath mp} E_{\overline{\varepsilon}} [\sqrt{\psi_{m1}^{p,\iota}} H_m^\top \Omega_m & \cdots & \sqrt{\psi_{m\mathcal{N}}^{p,\iota}}] \\ * & -\mathrm{diag}\{\Delta_1^{(p,\jmath)}, \ldots, \Delta_{\mathcal{N}}^{(p,\jmath)}\} \end{bmatrix} < 0 \tag{25}$$

$$\begin{bmatrix} \upsilon_3^2 S_{\iota\jmath msp} - \mathbf{He}(\upsilon_3 Y_s) & 2\overline{L}_s^\top B_m^\top \\ * & -\tau_{\iota\jmath mp} I \end{bmatrix} < 0 \tag{26}$$

$$\begin{bmatrix} \Sigma_{\iota\jmath\ell msp}^{1,\wp} & \Sigma_{\iota\jmath\ell msp}^{2,\wp} \\ * & \Sigma_{\iota\jmath\ell msp}^{3} \end{bmatrix} < 0, (\wp = 1, 2, 3) \tag{27}$$

Besides, if the conditions (14) and (15) and the linear matrix inequalities (LMIs) (16)–(27) are feasible, the asynchronous controller gains are

$$K_s = \overline{K}_s (\mathcal{W}_1 + \varepsilon\mathcal{W}_2)^{-\top},\ L_s = \overline{L}_s Y_s^{-1}. \tag{28}$$

*Proof:* Please see Appendix B. ■

Notice that $\varepsilon \in [0, \overline{\varepsilon}]$ is an SPP in Theorem 2, and the upper-bound of SPP can be maximized as the following optimization problem:

$$\max\ \overline{\varepsilon}$$

$$\alpha_p > 0, \gamma, \mu_p \geq 1, \tau_{\iota\jmath mp} > 0, c_2 > c_1 > 0$$

$$P_m^{(p,\iota)} > 0, \Delta_m^{(p,\iota)} > 0, Z_{\iota\mathrm{msp}} > 0$$

$$Q_{\iota\jmath\mathrm{msp}}, S_{\iota\jmath\mathrm{msp}} > 0, \overline{K}_s, \overline{L}_s$$

$$\forall m \in \mathcal{N}, p \in \mathcal{M} \forall \iota, \jmath \in \{1, \ldots, J_1\} \forall \ell \in \{1, \ldots, J_2\}$$

$$\text{subject to} \begin{cases} LMIs\ (16)\text{–}(27)\ \text{and}\ (14) \\ P_m^{(p,\iota)} > 0, \Delta_m^{(p,\iota)} > 0 \\ Z_{\iota\mathrm{msp}} > 0, S_{\iota\jmath\mathrm{msp}} > 0 \\ \forall m \in \mathcal{N}, p \in \mathcal{M} \\ \forall \iota, \jmath \in \{1, \ldots, J_1\} \\ \forall \ell \in \{1, \ldots, J_2\}. \end{cases}$$

TABLE I
UPPER BOUND OF SPP FOR DIFFERENT CASES

| | $\bar{\varepsilon}_{\max}$ | $[K_1 \mid K_2]$ |
|---|---|---|
| Case 1 | 0.1772 | [-0.1622 -8.4863 -3.2738 -2.6179 \|-0.4206 -5.5878 -2.5396 0.8852] |
| Case 2 | 0.0826 | [-0.1026 -7.0592 -2.4017 -0.9521 \|-0.0448 -7.6408 -2.0799 -1.2166] |
| Case 3 | 0.0505 | [-0.1703 -7.1016 -1.8600 -1.7067 \|-0.1703 -7.1013 -1.8605 -1.7055] |

## IV. ILLUSTRATIVE EXAMPLE

*Example 1:* (Numerical Example) The parameters of the MJSPLS with two modes are described as follows:

$$A_1 = \begin{bmatrix} 0.4 & 0.3 & -1.6 & 0.3 \\ 0 & 2.4 & 0 & 0.9 \\ 0.3 & 0 & 0.9 & 1.8 \\ 1.7 & 1.3 & 0.6 & 1.2 \end{bmatrix}, B_1 = \begin{bmatrix} 0.1 \\ 0.3 \\ 0 \\ 0.2 \end{bmatrix}, C_1 = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \end{bmatrix}$$

$$F_1 = H_1 = [0.1\ 0.1\ 0.1\ 0.1]^\top, D_1 = [1\ 1\ 1\ 1]$$

$$A_2 = \begin{bmatrix} 0.2 & 0.3 & -1.9 & 0.24 \\ 0 & 1.3 & 0 & -0.6 \\ 0.3 & 0 & 0.9 & 2.1 \\ 2.4 & 1.5 & 0.3 & 1.8 \end{bmatrix}, B_2 = \begin{bmatrix} 0.1 \\ 0.2 \\ 0 \\ 1 \end{bmatrix}, C_2 = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \end{bmatrix}$$

$$F_2 = H_2 = [0.1\ 0.1\ 0.1\ 0.1]^\top, D_2 = [1\ 1\ 1\ 1].$$

For the MJSPLSs, the singularly perturbed matrix is selected as $E_\varepsilon = \mathrm{diag}\{1, 1, \varepsilon, \varepsilon\}$. Moreover, the nonhomogeneous stochastic switching processes $\delta_k$ governed by the switching signal $\sigma_k$ is unveiled as $\Psi^{(1,1)} = \begin{bmatrix} 0.1 & 0.9 \\ 0.5 & 0.5 \end{bmatrix}, \Psi^{(2,1)} = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}, \Psi^{(1,2)} = \begin{bmatrix} 0.5 & 0.5 \\ 0.35 & 0.65 \end{bmatrix}$, $\Psi^{(2,1)} = \begin{bmatrix} 0.05 & 0.95 \\ 0.95 & 0.05 \end{bmatrix}$.

Meanwhile, other parameters are chosen as $\gamma = 1.5$, $(c_1, c_2, \mathcal{R}, [\mathcal{T}_1, \mathcal{T}_2], \mathbb{W}_{[\mathcal{T}_1,\mathcal{T}_2],d}) = (0.2, 15, \mathrm{diag}\{1,1,1,1\}, [0,10], 0.01)$, $\Omega_1 = 1.2$, $\Omega_2 = 1.8$. $\lambda_1 = 2.5$, $\alpha_p = 0.001$, $\mu_p = 1.01$ $(p = 1, 2)$. $\mu_p = 1.01$ $(p = 1, 2)$, and $N_{0p} = 0$.

The purpose of this work is to design an asynchronous control strategy with uncertain detected mode $\pi_k$ such that the inverted pendulum model is SFTB. Next, we divide the DPM into three cases as follows.

*Case 1:* Ideal mode-dependent detector, i.e., $\delta(k) = \pi(k)$.

*Case 2:* The time-invariant DPM $\Phi^1 = \Phi^2 = \begin{bmatrix} 0.3 & 0.7 \\ 0.95 & 0.05 \end{bmatrix}$.

*Case 3:* The time-varying DPM that is characterized in a polytope set $\Phi^1 = \begin{bmatrix} 0.3 & 0.7 \\ 0.95 & 0.05 \end{bmatrix}, \Phi^2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.2 & 0.8 \end{bmatrix}$.

Table I demonstrates the upper bound of SPP for three cases. From Table I we get that case 1 gained larger values of upper bound of SPP $\bar{\varepsilon}$. Furthermore, it can be easily observed from cases 2 and 3 that when a more reliable information is obtained, the lower value of SPP $\bar{\varepsilon}$ can be acquired.

In what follows, to evaluate the manner of closed-loop MJSPLS (8) for the case 3, the initial values are given as $\vartheta(0) = [-0.1\ 0.2\ 0.1\ 0.2]^\top$, and $\omega(k) = 0.1\sin(4\pi k)$, the simulation results can be seen in Figs. 1–4. The evolutions of deterministic switching signal $\sigma_k$, system mode $\delta_k$, and controller mode $\pi_k$ are shown in Fig. 1. The state responses of closed-loop systems (100 realizations) are plotted in Fig. 2. The 100 realizations of the evolution of $z(k)$, $u(k)$ and $x^\top(k)\mathcal{R}x(k)$ are depicted in Fig. 3. Indicated by these figures, one manifests the validity of the control scheme.

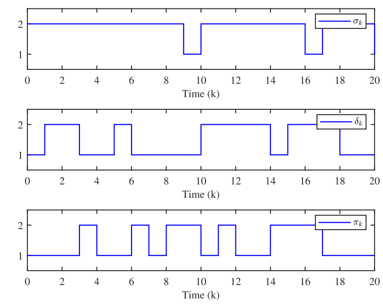


Fig. 1. Evolutions of $\sigma_k$, $\delta_k$, and $\pi_k$.

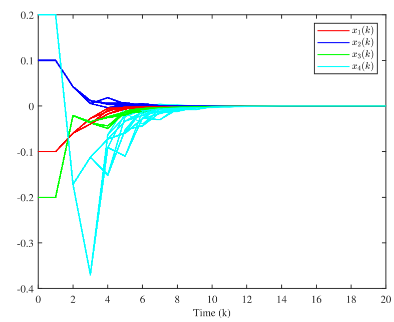


Fig. 2. Responses of the state $x(k)$ in Example 1 (100 realizations).

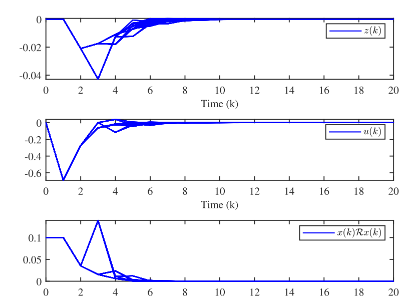


Fig. 3. Evolution of $z(k)$, $u(k)$, and $x^\top(k)\mathcal{R}x(k)$ in Example 1 (100 realizations).

*Example 2:* (Practical Example) Consider an inverted pendulum controlled by a dc motor with two modes, the dynamic equation of motions is yielded as

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = \frac{g}{l}\sin x_1(t) + \frac{\mathscr{N}\mathscr{K}_m}{ml^2}x_3(t) \\ \mathscr{L}_a\dot{x}_3(t) = -\mathscr{K}_b\mathscr{N}x_2(t) - \mathscr{R}_m x_3(t) + u(t) + \omega(k) \end{cases} \tag{29}$$

where the physical meanings of $\mathscr{K}_m$, $\mathscr{K}_b$, $\mathscr{N}$, $\mathscr{L}_a$, and $\mathscr{R}_m (m = 1, 2)$ in (29) are presented in Table II. The parameters are given as $\mathscr{K}_m = 0.1\,\mathrm{Nm/A}$, $\mathscr{K}_b = 0.1\,\mathrm{Vs/rad}$, $\mathscr{N} = 10$, $\mathscr{L}_a = 5\,\mathrm{mH}$, $g = 9.8\,\mathrm{m/s^2}$, $l = 1\,\mathrm{m}$, and $\mathscr{R}_1 = 1$, $\mathscr{R}_2 = 0.5$, set $\varepsilon = \mathscr{L}_a$ is the SPP, $\varphi_m(\zeta(k)) = 0.5\Omega_m \sin(x_1(k))(\sin(x_1(k)) + 0.1)$ and $\varphi_s(\varsigma(k)) = 0.5\Omega_s \sin(x_1(k))(\sin(x_1(k)) + 0.1)$ with $H_m = H_s = [0.1\ 0.1\ 0.1]$ and $\Omega_1 = 1.2$, $\Omega_2 = 1.8$. Letting $x(k) = [x_1^\top(k)\ x_2^\top(k)\ x_3^\top(k)]^\top$, and discretizing the continuous-time system (29) with sampling period

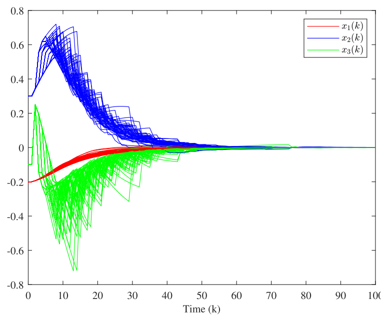


Fig. 4. State trajectories of $x(k)$ in Example 2 (50 realizations).

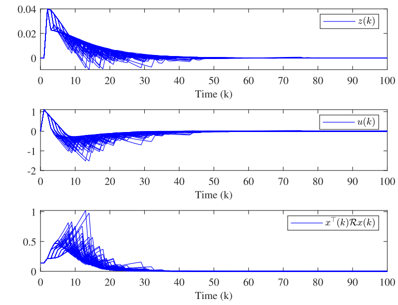


Fig. 5. Evolution of $z(k)$, $u(k)$, and $x^\top(k)\mathcal{R}x(k)$ in Example 2 (50 realizations).

TABLE II
PARAMETERS MEANING

| Parameter | Physical Meaning |
|---|---|
| $\mathscr{K}_m$ | motor torque constant |
| $\mathscr{K}_b$ | back emf constant |
| $\mathscr{N}$ | gear ratio |
| $\mathscr{L}_a$ | inductance |
| $\mathscr{R}_q(q=1,2)$ | resistance |

$T_s = 0.02\,s$, the discrete-time MJSPLS can be inferred as

$$\begin{aligned} x(k) = A_m E_\varepsilon x(k) + B_m u(k) + C_m \varphi_m(\zeta(k)) \\ + F_m \omega(k),\ (m=1,2) \end{aligned} \tag{30}$$

where

$$A_1 = \begin{bmatrix} 0.9878 & 0.0495 & 0.0183 \\ -0.4848 & 0.9695 & 0.6237 \\ 0.1792 & -0.6237 & 7.0952 \end{bmatrix},\quad C_1 = \begin{bmatrix} 0.0122 \\ 0.4848 \\ -0.1792 \end{bmatrix}$$

$$B_1 = F_1 = [0.0003\ 0.0183\ 0.6270]^\top$$

$$A_2 = \begin{bmatrix} 0.9878 & 0.0494 & 0.0212 \\ -0.4844 & 0.9666 & 0.7768 \\ 0.2075 & -0.7768 & 11.7722 \end{bmatrix},\quad C_2 = \begin{bmatrix} 0.0122 \\ 0.4844 \\ -0.2075 \end{bmatrix}$$

$$B_2 = F_2 = [0.0004\ 0.0212\ 0.7804]^\top,\ E_\varepsilon = \mathrm{diag}\{1, 1, \varepsilon\}.$$

In what follows, different from the existing piecewise-homogeneous Markov process [2], a generalized nonhomogeneous Markov process $\delta_k$ governed by the switching signal $\sigma_k$ is unveiled as $\Psi^{(1,1)} = \begin{bmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{bmatrix}$, $\Psi^{(2,1)} = \begin{bmatrix} 0.5 & 0.5 \\ 0.9 & 0.1 \end{bmatrix}$, $\Psi^{(1,2)} = \begin{bmatrix} 0.05 & 0.95 \\ 0.45 & 0.55 \end{bmatrix}$, $\Psi^{(2,1)} = \begin{bmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{bmatrix}$, and $\Phi^1 = \begin{bmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{bmatrix}$, $\Phi^2 = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$. $D_\ell = H_\ell = [0.1\quad 0.1\quad 0.1], (\ell = 1, 2)$. The finite-time control issue will be addressed for the inverted pendulum model by selecting the parameter of $(c_1, c_2, \mathcal{R}, \mathcal{T}, d)$ are $c_1 = 1.1$, $c_2 = 15$, $\mathcal{R} = \mathrm{diag}\{1, 1, 1\}$, $\mathcal{T} = 20$, $d = 1$. Other parameters are chosen as $\alpha_p = 0.05$, $\mu_p = 1.15$ $(p = 1, 2)$, and $N_{0p} = 0$. By resorting to Theorem 1, it is clear that for any switching signal meeting $\tau_{ap} = 2.8646 > \max[(\mathcal{T}\ln(\mu_p))/(\ln(c_2) - \ln(\lambda_1 c_1 + \gamma^2 d)) - (\mathcal{T}\ln(1+\alpha_p) + \sum_{p=1}^N N_{0p}\ln(\mu_p)), \ln(\mu_p)/\ln(1+\alpha_p)]$, the closed-loop dynamic (8) is SFTB.

In the sequel, letting $\varepsilon = 0.005$, $\gamma = 1.5$, and $\lambda_1 = 1.01$, by solving the LMIs of Theorem 2, the asynchronous controller gains can be attained. To evaluate the manner of closed-loop MJSPLS (8), the initial values are given as $\vartheta(0) = [-0.5\ 0.7\ -0.5]^\top$, and $\omega(k) = \sin(4\pi k)$. Based on the derived controller gains, the simulation results are shown in Figs. 4 and 5, in which Fig. 4 shows the 50 realizations of state trajectories of closed-loop dynamic (8), Fig. 5 displays the 50 realizations of the trajectory of $z(k)$, $u(k)$, and $x^\top(k)\mathcal{R}x(k)$. From which one can observe visually that the evolution of $x^\top(k)\mathcal{R}x(k)$ in closed-loop dynamic (8) meets $x^\top(k)\mathcal{R}x(k) \le c_2 = 15$. Eventually, one concludes that the closed-loop dynamic (8) is SFTB w.r.t. $(c_1, c_2, \mathcal{R}, [0, \mathcal{T}], \mathbb{W}_{[0,\mathcal{T}],d})$.

## V. CONCLUSION

The asynchronous controller design issue has been addressed for MJSPLSs in this work. The resulting dynamic parameters have been modeled by a nonhomogeneous Markov chain, where time-varying TP matrices are governed by a deterministic switching law. Based on a novel Lyapunov functional, the SPP-independent sufficient conditions have been exhibited within a fixed interval. In the end, two computational examples have been adopted to show the feasibility and applicability of the gained theoretical results.

## APPENDIX A

In light of condition (13), it yields

$$\begin{aligned} \mathscr{E}\{\Delta V(k)\} &= \mathscr{E}\{V(x(k+1), \delta_{k+1}, \sigma_k) \mid \delta_k = m \\ &\quad \sigma_k = p\} - \mathscr{E}\{V(x(k), \delta_k, \sigma_k)\} \\ &\le \alpha_p \mathscr{E}\{V(x(k), \delta_k, \sigma_k)\} + \Gamma(k). \end{aligned} \tag{31}$$

Summing up the inequality (31) from $k_t$ to $k$, one gains

$$\begin{aligned} &\mathscr{E}\{V(x(k), \delta_k, \sigma_k) \mid \delta_{k_t}, \sigma_{k_t}\} \\ &\le (1+\alpha_p)^{(k-k_t)} \mathscr{E}\{V(x(k_t), \delta_{k_t}, \sigma_{k_t})\} \\ &\quad + \sum_{l=k_t}^{k-1} (1+\alpha_p)^{(k-l-1)} \mathscr{E}\{\Gamma(l)\}. \end{aligned} \tag{32}$$

Generally, one assumes $\sigma_{k_t} = p$ and $\sigma_{k_t - 1} = q$ at switching instant $k_t$. For $\mu_p \ge 1$, it yields from (11) that

$$\mathscr{E}\{V(x(k_t), \delta_{k_t}, \sigma_{k_t})\} \le \mu_p \mathscr{E}\{V(x(k_t), \delta_{k_t}, \sigma_{k_t - 1})\}. \tag{33}$$

Recalling the inequalities (32) and (33), one derives

$$\begin{aligned} &\mathscr{E}\{V(x(k), \delta_k, \sigma_k) \mid \delta_{k_t}, \sigma_{k_t}\} \\ &< (1+\alpha_p)^{(k-k_t)} \mu_p \mathscr{E}\{V(x(k_t), \delta_{k_t}, \sigma_{k_t - 1})\} \\ &\quad + \sum_{l=k_t}^{k-1} (1+\alpha_p)^{(k-l-1)} \mathscr{E}\{\Gamma(l)\}. \end{aligned} \tag{34}$$

For $\forall k \in \{1, 2, \ldots, \mathcal{T}\}$, it follows from Definition 1 and (34) that

$$
\begin{aligned}
&\mathscr{E}\{V(x(k), \delta_k, \sigma_k)\} \\
&< (1+\alpha_p)^{(k-k_t)} \mu_{\sigma_{k_t}} \mathscr{E}\{V(x(k_t), \delta_{k_t}, \sigma_{k_{t-1}})\} \\
&\quad + \sum_{l=k_t}^{k-1} (1+\alpha_{\sigma_{k_t}})^{(k-l-1)} \mathscr{E}\{\Gamma(l)\} < \cdots \\
&< \prod_{p=1}^{N} \exp\{N_{\sigma p}(k_0, k) \ln(\mu_p)\} \exp\left\{ \sum_{p=1}^{N} \mathcal{G}_p(k_0, k) \right. \\
&\quad \left. \times \ln(1+\alpha_p) \right\} \left[ V(x(k_0), \delta_{k_0}, \sigma_{k_0}) \right. \\
&\quad \left. + \gamma^2 \sum_{l=k_0}^{k-1} \omega^\top(l)\omega(l) \right]. \qquad (35)
\end{aligned}
$$

Since $N_{\sigma p}(k_0, k) \leq N_{0p} + \mathcal{G}_p(k_0, k)/\tau_{ap}$, $V(x_{k_0}, \delta_0, \sigma_0) \leq \max_{m\in\mathcal{N}, p\in\mathcal{S}, \iota=1,\ldots,J_1} (\lambda_{\max}(\mathcal{R}^{-\frac{1}{2}} (P_m^{(p,\iota)})^{-1} \mathcal{R}^{-\frac{1}{2}} + \lambda_{\max}(\mathcal{R}^{-\frac{1}{2}} (P_m^{(p,\iota)})^{-1} \mathcal{R}^{-\frac{1}{2}}) + \lambda_{\max}(\mathcal{R}^{-\frac{1}{2}} (E_\varepsilon H_m^\top \Omega_m (\Delta_m^{(p,\iota)})^{-1} \Omega_m H_m E_\varepsilon)^{-1} \mathcal{R}^{-\frac{1}{2}})))x^\top(k)\mathcal{R}x(k) = (\lambda_2 + \lambda_3)c_1$, and recalling $\sum_{l=k_0}^{\mathcal{T}} \omega^\top(l)\omega(l) \leq d$, which gives rise to

$$
\begin{aligned}
&\mathscr{E}\{V(x(k), \delta_k, \sigma_k)\} \\
&\leq \exp\left\{ \sum_{p=1}^{N} N_{0p} \ln(\mu_p) \right\} \exp\left\{ \mathcal{T} \max\left[ \frac{\ln(\mu_p)}{\tau_{ap}} \right.\right. \\
&\quad \left.\left. + \ln(1+\alpha_p) \right] \right\} \left( (\lambda_2+\lambda_3)c_1 + \gamma^2 d \right). \qquad (36)
\end{aligned}
$$

Based on condition (14), one gets $\ln(\lambda_1 c_2) - \ln((\lambda_2 + \lambda_3)c_1 + \gamma^2 d) - \mathcal{T}_p > 0$. Meanwhile, for the second term of (36), one has $\mathcal{T} \max[\ln(\mu_p)/\tau_{ap} + \ln(1+\alpha_p)] < \ln(\lambda_1 c_2) - \ln((\lambda_2+\lambda_3)c_1 + \gamma^2 d) - \sum_{p=1}^{N} N_{0p} \ln(\mu_p)$. Recalling (10), it is easy to achieve that $V(x(k), \delta_k, \sigma_k) \geq \min_{m\in\mathcal{N}, p\in\mathcal{S}, \iota=1,\ldots,J_1} (\lambda_{\min}(\mathcal{R}^{-\frac{1}{2}} (P_m^{(p,\iota)})^{-1} \mathcal{R}^{-\frac{1}{2}}))x^\top(k)\mathcal{R} \times x(k) = \lambda_1 x^\top(k)\mathcal{R}x(k)$. Combining (36), which implies $x^\top(k)\mathcal{R}x(k) < c_2$. According to Definition 2, one concludes that closed-loop dynamic (8) is SFTB w.r.t. $(c_1, c_2, \mathcal{R}, [\mathcal{T}_1, \mathcal{T}_2], \mathbb{W}_{[\mathcal{T}_1, \mathcal{T}_2], d})$.

In what follows, the $\mathcal{H}_\infty$ performance for closed-loop dynamic (8) with $\omega(k) \neq 0$ will be presented. By the similar derivation, one achieves

$$
\begin{aligned}
&\mathscr{E}\{V(x(k), \delta_k, \sigma_k)\} < \prod_{p=1}^{N} \exp\{N_{\sigma p}(k_0, k) \ln(\mu_p)\} \\
&\quad \times \exp\left\{ \sum_{p=1}^{N} \mathcal{G}_p(k_0, k) \ln(1+\alpha_p) \right\} V(x(k_0), \delta_{k_0}, \sigma_{k_0}) \\
&\quad + \sum_{l=0}^{k-1} \left[ \prod_{p=1}^{N} \exp\{(N_p(l,k)) \ln(\mu_p)\} \right. \\
&\quad \left. \times \exp\left\{ \sum_{p=1}^{N} \mathcal{G}_p(l,k) \ln(1+\alpha_p) \right\} \Gamma(l) \right]. \qquad (37)
\end{aligned}
$$

In light of $\tau_{ap} > \ln(\mu_p)/\ln(1+\alpha_p)$, which suffices to guarantee $N_{\sigma p}(l,k)\ln(\mu_p) \leq N_{0p}\ln(\mu_p) + \mathcal{G}_p(l,k)\ln(1+\alpha_p)$. Under the zero-initial condition, recalling the relationship (37), which indicates

$$
\begin{aligned}
&\sum_{l=0}^{k-1} \left[ \prod_{p=1}^{N_\sigma} \exp\{(N_p(l,k)) \ln(\mu_p)\} \right. \\
&\quad \left. \times \exp\left\{ \sum_{p=1}^{N_\sigma} \mathcal{G}_p(l,k) \ln(1+\alpha_p) \right\} z^\top(l) z(l) \right] \\
&< \gamma^2 \sum_{l=0}^{k-1} \exp\left\{ \sum_{p=1}^{N} N_{0p} \ln(\mu_p) \right. \\
&\quad \left. + 2\mathcal{G}_p(l,k)\ln(1+\alpha_p) \right\} \omega^\top(l)\omega(l). \qquad (38)
\end{aligned}
$$

Recalling $\mu_p \geq 1$ and $\alpha_p > 0$, letting $k - 1 = \mathcal{T}$, which renders $\sum_{k=0}^{\mathcal{T}} \mathscr{E}\{z^\top(k)z(k)\} < \overline{\gamma}^2 \sum_{k=0}^{\mathcal{T}} \omega^\top(k)\omega(k)$, this completes the proof. ■

## APPENDIX B

According to LMIs (16)–(18) and [4, Lemmas 1.2-1.3], which gives rise to $E_\varepsilon \mathcal{U}_\varepsilon = \mathcal{U}_\varepsilon^\top E_\varepsilon > 0$ for any $\varepsilon \in (0, \overline{\varepsilon}]$. Meanwhile, with respect to the well-known inequality $(\upsilon\mathcal{U}_\varepsilon^\top - Z_{\iota\mathrm{msp}}) Z_{\iota\mathrm{msp}}^{-1} (\upsilon\mathcal{U}_\varepsilon - Z_{\iota\mathrm{msp}}) \geq 0$, we can get $\forall \upsilon > 0$

$$
\begin{bmatrix} \widehat{\Sigma}^1_{\iota\jmath\ell\mathrm{msp}} & \widehat{\Sigma}^2_{\iota\jmath\ell\mathrm{msp}} \\ * & \Sigma^3_{\iota\jmath\ell\mathrm{msp}} \end{bmatrix} < 0, (\wp = 1, 2, 3) \qquad (39)
$$

where $\widehat{\Sigma}^1_{\iota\jmath\ell\,\mathrm{msp}} = \mathrm{diag}\{-\mathcal{U}_\varepsilon^\top Z_{\iota\mathrm{msp}}^{-1} \mathcal{U}_\varepsilon, -2\Delta_m^{(p,\iota)}, -\gamma^2 I\}$, $\widehat{\Sigma}^2_{\iota\jmath\ell\,\mathrm{msp}} = [\mathscr{F}\widehat{\mathcal{X}}_{ms}^\top \;\; \widehat{\mathcal{X}}_{ms}^\top \;\; \widehat{\Sigma}^{22}_{\iota\jmath\ell\mathrm{msp}} \;\; \widehat{\Sigma}^{22}_{\iota\jmath\ell\mathrm{msp}} \;\; \widehat{\Sigma}^{23}_{\iota\jmath\ell\mathrm{msp}}]$, $\widehat{\Sigma}^{22}_{\iota\jmath\ell\mathrm{msp}} = [\Omega_s H_s E_\varepsilon \mathcal{U}_\varepsilon \;\; 0 \;\; 0]$, $\widehat{\Sigma}^{23}_{\iota\jmath\ell\mathrm{msp}} = [D_m E_\varepsilon \mathcal{U}_\varepsilon \;\; 0 \;\; 0]$, $\widehat{\mathcal{X}}_{ms} = [\sqrt{2}\mathscr{A}_{ms} E_\varepsilon \mathcal{U}_\varepsilon \;\; \sqrt{2}\Delta_m^{(p,\iota)} C_m \;\; \sqrt{2}F_m]$.

Next, define $\mathscr{J}(k) = V(x(k+1), \delta_{k+1}, \sigma_k \mid \delta_k = m, \sigma_k = p) - (1+\alpha_p)V(x(k), \delta_k, \sigma_k) - \Gamma(k)$.

In light of (10), which yields

$$
\begin{aligned}
&\mathscr{E}\{V_1(x(k+1), \delta_{k+1}, \sigma_k \mid \delta_k = m, \sigma_k = p)\} \\
&= \mathscr{E}\left\{ x^\top(k+1) \sum_{n\in\mathcal{N}} \psi_{mn}^{(p,\iota)}(k) \sum_{\iota=1}^{J_1} \hbar_\iota(k+1) \right. \\
&\quad \left. \times (P_n^{(p,\iota)})^{-1} x(k+1) \right\}. \qquad (40)
\end{aligned}
$$

Set $\hbar_\iota(k+1) = \varpi_\jmath(k)$ with $\varpi_\jmath(k) \geq 0$ and $\sum_{\jmath=1}^{J_1} \varpi_\jmath(k) = 1$, one has $\sum_{\jmath=1}^{J_1} \varpi_\jmath(k)(P_n^{(p,\jmath)})^{-1} = \sum_{\iota=1}^{J_1} \hbar_\iota(k+1)(P_n^{(p,\iota)})^{-1}$. Equation (40) can be rewritten as

$$
\begin{aligned}
&\mathscr{E}\{V_1(x(k+1), \delta_{k+1}, \sigma_k \mid \delta_k = m, \sigma_k = p)\} \\
&\leq \mathscr{E}\{2[\mathscr{A}_{ms} E_\varepsilon x(k) + C_m \varphi_m(\zeta(k)) \\
&\quad + F_m \omega(k)]^\top \sum_{\ell=1}^{J_2} \nu_\ell(k) \sum_{s\in\mathcal{S}} \phi_{ms}^{(\ell)} \sum_{\iota=1}^{J_1} \sum_{\jmath=1}^{J_1} \hbar_\iota(k) \varpi_\jmath(k) \\
&\quad \times \mathcal{P}_{\iota\jmath m}^{(p)} [\mathscr{A}_{ms} E_\varepsilon x(k) + C_m \varphi_m(\zeta(k)) \\
&\quad + F_m \omega(k)] + 2\varphi_s^\top(\varsigma(k)) L_s^\top B_m^\top \sum_{\ell=1}^{J_2} \nu_\ell(k) \sum_{s\in\mathcal{S}} \phi_{ms}^{(\ell)} \\
&\quad \times \sum_{\iota=1}^{J_1} \sum_{\jmath=1}^{J_1} \hbar_\iota(k) \varpi_\jmath(k) \mathcal{P}_{\iota\jmath m}^{(p)} B_m L_s \varphi_s(\varsigma(k))\}. \qquad (41)
\end{aligned}
$$

where $\mathcal{P}_{\iota\jmath m}^{(p)} = \sum_{n\in\mathcal{N}} \psi_{mn}^{(p,\iota)} (P_n^{(p,\jmath)})^{-1}$.

Besides, by exploiting the condition (23), we can get

$$\begin{aligned}&2\varphi_s^\top(\varsigma(k))L_s^\top B_m^\top \mathcal{P}_{\imath\jmath m}^{(p)} B_m L_s \varphi_s(\varsigma(k))\\&\leq x^\top(k)E_\varepsilon H_s^\top \Omega_s^\top Q_{\imath\jmath\mathrm{msp}}^{-1}\Omega_s H_s E_\varepsilon x(k).\end{aligned} \tag{42}$$

Substituting (42) into (41), which leads to

$$\begin{aligned}&\mathscr{E}\{V_1(x(k+1),\delta_{k+1},\sigma_k \mid \delta_k=m,\sigma_k=p)\}\\&\leq \sum_{\imath=1}^{J_1}\sum_{\jmath=1}^{J_1}\sum_{\ell=1}^{J_2}\hbar_\imath(k)\varpi_\jmath(k)\nu_\ell(k)\mathscr{E}\Bigg\{2\sum_{s\in\mathcal{S}}\phi_{ms}^{(\ell)}\\&\quad\times[\mathscr{A}_{ms}E_\varepsilon x(k)+C_m\varphi_m(\zeta(k))+F_m\omega(k)]^\top\mathcal{P}_{\imath\jmath m}^{(p)}\\&\quad\times[\mathscr{A}_{ms}E_\varepsilon x(k)+C_m\varphi_m(\zeta(k))+F_m\omega(k)]\\&\quad+\sum_{s\in\mathcal{S}}\phi_{ms}^{(\ell)}x^\top(k)E_\varepsilon H_s^\top\Omega_s Q_{\imath\jmath\mathrm{msp}}^{-1}\Omega_s H_s E_\varepsilon x(k)\Bigg\}.\end{aligned} \tag{43}$$

On the other hand, by analysis of (24)–(26), for any $\varepsilon\in(0,\bar{\varepsilon}]$, which indicates

$$\sum_{n\in\mathcal{N}}\psi_{mn}^{(p,\iota)}E_\varepsilon H_n^\top\Omega_n(\Delta_n^{(p,\jmath)})^{-1}\Omega_n H_n E_\varepsilon\leq\tau_{\imath\jmath mp}^{-1}I. \tag{44}$$

$$4\tau_{\imath\jmath mp}^{-1}L_s^\top B_m^\top B_m L_s < S_{\imath\jmath\mathrm{msp}}^{-1}. \tag{45}$$

By adopting the similar operations to $V_2(x(k+1),\delta_{k+1},\sigma_k)$, the following inequality can be gained:

$$\begin{aligned}&\mathscr{E}\{V_2(x(k+1),\delta_{k+1},\sigma_k \mid \delta_k=m,\sigma_k=p)\}\\&\leq \sum_{\imath=1}^{J_1}\sum_{\jmath=1}^{J_1}\sum_{\ell=1}^{J_2}\hbar_\imath(k)\varpi_\jmath(k)\nu_\ell(k)\mathscr{E}\Bigg\{4\sum_{s\in\mathcal{S}}\phi_{ms}^{(\ell)}\\&\quad\times[\mathscr{A}_{ms}E_\varepsilon x(k)+C_m\varphi_m(\zeta(k))+F_m\omega(k)]^\top\\&\quad\times\tau_{\imath\jmath\mathrm{msp}}^{-1}[\mathscr{A}_{ms}E_\varepsilon x(k)+C_m\varphi_m(\zeta(k))+F_m\omega(k)]\\&\quad+\sum_{s\in\mathcal{S}}\phi_{ms}^{(\ell)}x^\top(k)E_\varepsilon H_s^\top\Omega_s S_{\imath\jmath\mathrm{msp}}^{-1}\Omega_s H_s E_\varepsilon x(k).\end{aligned} \tag{46}$$

It follows from (21) that $-(1+\alpha_p)(P_m^{(p,\iota)})^{-1}+\sum_{s\in\mathcal{S}}\phi_{ms}^{(\ell)} Z_{\iota\mathrm{msp}}^{-1}<0$. Taking (41)–(46) into account, one has

$$\begin{aligned}\mathscr{E}\{\mathscr{J}(k)\}\leq\eta^\top(k)\sum_{\imath=1}^{J_1}\sum_{\jmath=1}^{J_1}\sum_{\ell=1}^{J_2}\sum_{s\in\mathcal{S}}\hbar_\imath(k)\\\times\varpi_\jmath(k)\nu_\ell(k)\phi_{ms}^{(\ell)}\overline{\Sigma}_{\imath\jmath\ell\mathrm{msp}}\eta(k)s\end{aligned} \tag{47}$$

where $\eta(k)=[x^\top(k)\quad\varphi^\top(\zeta(k))\quad\omega^\top(k)]^\top$, $\overline{\Sigma}_{\imath\jmath\ell\mathrm{msp}}=\overline{\Sigma}_{\imath\jmath\ell\mathrm{msp}}^1+\overline{\Sigma}_{\imath\jmath\ell\mathrm{msp}}^{2\top}(2\mathcal{P}_{\imath\jmath m}^{(p)}+4\tau_{\imath\jmath\mathrm{msp}}^{-1}I)\overline{\Sigma}_{\imath\jmath\ell\mathrm{msp}}^2+\overline{\Sigma}_{\imath\jmath\ell\mathrm{msp}}^{3\top}(Q_{\imath\jmath\mathrm{msp}}^{-1}+S_{\imath\jmath\mathrm{msp}}^{-1})\overline{\Sigma}_{\imath\jmath\ell\mathrm{msp}}^3+\overline{\Sigma}_{\imath\jmath\ell\mathrm{msp}}^{4\top}\overline{\Sigma}_{\imath\jmath\ell\mathrm{msp}}^4$, $\overline{\Sigma}_{\imath\jmath\ell\mathrm{msp}}^1=\mathrm{diag}\{-Z_{\iota\mathrm{msp}}^{-1},\ -2(\Delta_m^{(p,\iota)})^{-1},\ -\gamma^2 I\}$, $\overline{\Sigma}_{\imath\jmath\ell\mathrm{msp}}^2=[\mathscr{A}_{ms}E_\varepsilon\ \ C_m\ \ F_m]$, $\overline{\Sigma}_{\imath\jmath\ell\mathrm{msp}}^3=[\Omega_s H_s E_\varepsilon\ \ 0\ \ 0]$, $\overline{\Sigma}_{\imath\jmath\ell\mathrm{msp}}^4=[D_m E_\varepsilon\ 0\ 0]$.

According to (28), it holds that $\overline{K}_s=K_s\mathcal{U}_\varepsilon$ and $\overline{L}_s=L_sY_s$, where $\mathcal{U}_\varepsilon=\mathcal{W}_1+\varepsilon\mathcal{W}_2$. By means of the Schur complement to (47), premultiplying and postmultiplying $\overline{\Sigma}_{\imath\jmath\ell\mathrm{msp}}$ with $\mathrm{diag}\{\mathcal{U}_\varepsilon^\top,(\Delta_m^{(p,\iota)})^{-1},I,\ldots,I\}$ and its transpose, (39) can be ensured. Besides, based on the Schur complement, conditions (19) and (20) equal to $\lambda_1\mathcal{R}<(P_m^{(p,\iota)})^{-1}<\lambda_2\mathcal{R}$, then condition (12) can be guaranteed. By following a similar derivation of Theorem 1, one derives $x^\top(k)\mathcal{R}x(k)\leq c_2$. This completes the proof. ■

## References


[1] H. Shen, F. Li, S. Xu, and V. Sreeram, "Slow state variables feedback stabilization for semi-Markov jump systems with singular perturbations," *IEEE Trans. Autom. Control*, 2008, vol. 63, no. 8, pp. 2709–2714, 2008.

[2] Y. Wang, C. K. Ahn, H. Yan, and S. Xie, "Fuzzy control and filtering for nonlinear singularly perturbed Markov jump systems," *IEEE Trans. Cybern.*, vol. 51, no. 1, pp. 297–308, 2021.

[3] J. Song, Y. Niu, and J. Xu, "An event-triggered approach to sliding mode control of Markovian jump Lur'e systems under hidden mode detections," *IEEE Trans. Syst., Man, Cybern. Syst.*, vol. 50, no. 4, pp. 1514–1525, 2020.

[4] C. Yang, Q. Zhang, J. Sun, and T. Chai, "Lur'e Lyapunov function and absolute stability criterion for Lur'e singularly perturbed systems," *IEEE Trans. Autom. Control*, vol. 56, no. 11, pp. 2666–2671, 2011.

[5] P. Bolzern, P. Colaneri, and G. D. Nicolao, "Markov jump linear systems with switching transition rates: Mean square stability with dwell-time," *Automatica*, vol. 46, no. 6, pp. 1081–1088, 2010.

[6] Z. Shu, J. Lam, and J. Xiong, "Static output-feedback stabilization of discrete-time Markovian jump linear systems: A system augmentation approach," *Automatica*, vol. 46, no. 4, pp. 687–694, 2010.

[7] S. Aberkane, "Stochastic stabilization of a class of nonhomogeneous Markovian jump linear systems," *Syst. Control Lett.*, vol. 60,pp. 156–160, 2011.

[8] L. Zhang, "$H_\infty$ estimation for discrete-time piecewise homogeneous Markov jump linear systems," *Automatica*, vol. 45, no. 11, pp. 2570–2576, 2009.

[9] Y. Wang, H. Pu, P. Shi, C. K. Ahn, and J. Luo, "Sliding mode control for singularly perturbed Markov jump descriptor systems with nonlinear perturbation," *Automatica*, vol. 127, 2021, Art. no. 109515.

[10] F. Amato, M. Ariola, and C. Cosentino, "Finite-time stability of linear time-varying systems: Analysis and controller design," *IEEE Trans. Autom. Control*, vol. 55, no. 4, pp. 1003–1008, 2010.

[11] F. Li, C. Du, C. Yang, L. Wu, and W. Gui, "Finite-time asynchronous sliding mode control for Markovian jump systems," *Automatica*, vol. 109, 2019, Art.no. 108503.

[12] C. Guan, Z. Fei, H. R. Karimi, and P. Shi, "Finite-time synchronization for switched neural networks via quantized feedback control," *IEEE Trans. Syst., Man, Cybern. Syst.*, vol. 51, no. 5, pp. 2873–2884, 2019.

[13] Y. Zhu, L. Zhang, and W. X. Zheng, "Distributed $\mathcal{H}_\infty$ filtering for a class of discrete-time Markov jump Lur'e systems with redundant channels," *IEEE Trans. Ind. Electron.*, vol. 63, no. 3, pp. 1876–1885, 2016.

[14] O. V. d.V. Costa, M. D. Fragoso, and M. G. Todorov, "A detector-based approach for the $h_2$ control of Markov jump linear systems with partial information," *IEEE Trans. Autom. Control*, vol. 60, no. 5, pp. 1219–1234, 2014.

[15] Y. Tian, H. Yan, H. Zhang, J. Cheng, and H. Shen, "Asynchronous output feedback control of hidden semi-Markov jump systems with random mode-dependent delays," *IEEE Trans. Autom. Control*, vol. 67, no. 8, pp. 4107–4114, 2022.

[16] Y. Liu, X. Chen, J. Lu, and W. Gui, "Non-weighted $l_2/L_2$-gain of asynchronously switched systems," *Nonlinear Analysis, Hybrid Syst.*, vol. 43, Art. no. 101105.