A Small-Gain Theory for Abstract Systems on Topological Spaces

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***Abstract*—We develop a small-gain theory for systems described by set-valued maps between topological spaces. We introduce an abstract notion of stability unifying the continuity properties underlying different existing concepts, such as Lyapunov stability of equilibria, sets, or motions, (incremental) input–output stability, asymptotic gain properties, and continuity with respect to fast-switching inputs. Then, we prove that a feedback interconnection enjoying a given abstract small-gain property is stable. While, in general, the proposed small-gain property cannot be decomposed as the union of stability of the subsystems and a contractiveness condition, we show that it is implied by standard assumptions in the context of input-to-state stable systems. Finally, we provide application examples illustrating how the developed theory can be used for the analysis of interconnected systems and design of control systems.**

***Index Terms*—Abstract systems, small-gain theorem, stability theory.**

I. Introduction

IN ALL their different facets and variants, small-gain theorems constitute one of the most powerful classes of tools for the analysis of interconnected systems and the design of control schemes. The development of small-gain results can be traced back at least to the 60's in the context of input–output operators between (extended) normed spaces of signals. See, for instance, [1, Ch. III], [2], [3], and the subsequent extensions to nonlinear [4], [5], [6], stochastic [7], and monotone [8] systems. The seminal work [9] developed a small-gain theory for *input-to-state stable* (ISS) systems described by nonlinear differential equations [10], [11]. The results in [9] were followed by extensive research efforts aimed at extending the small-gain theory to different domains and problems. In particular, [12] considered ISS systems with saturations, [13] provided characterizations in terms of Lyapunov functions, [14] extended the results of [9] to general "ISS operators," [15] to *integral* ISS systems, and [16], [17], [18] to systems not necessarily ISS. Extensions to time-varying and possibly nonuniformly ISS systems appeared in [19] and [20], whereas [21], [22], [23], and [24] considered discrete-time systems, and [25] abstract systems satisfying a "weak semigroup property" (see also [26]). More recently, small-gain results have been developed also for switching and hybrid systems [18], [27], [28], [29], for infinite-dimensional systems described by partial differential equations [30], [31], for finite [29], [31], [32], [33], [34] and infinite [24], [35] networks, and for stochastic systems [36]. See also [37] for a review.

The aforementioned small-gain results typically differ in terms of the class of systems considered and the *stability* requirements for the systems involved in the interconnection, but they all share a common paradigm from which a "small-gain principle" can be drawn: the interconnection of stable systems satisfying a certain "small-gain condition" is itself stable. In qualitative terms

$$\begin{array}{c}\textit{stability of the subsystems}\\ + \\ \textit{small-gain property}\end{array} \;\Rightarrow\; \begin{array}{c}\textit{stability of the}\\ \textit{interconnection}\end{array} \qquad (1)$$

In this article, we develop a small-gain theory extending the small-gain principle (1) to set-valued maps between topological spaces. This leads to the following three main contributions.

1) It unifies different existing theorems developed for metric spaces of trajectories and gives new insights on the topological nature of the small-gain principle.
2) It enables the study of interconnections out of reach of existing small-gain theorems (e.g., formed by systems that do not satisfy ISS-like conditions).
3) It extends the small-gain principle to general maps between topological spaces not necessarily representing trajectories of a dynamical system (See Example 2).

In this respect, we point out that the main methodological corpus of this work is composed of Items 1) and 3), which establish a common framework for the small-gain principle. Instead, item 2) is illustrated through examples (see Section VI). Indeed, the application of the presented results to specific cases requires

Manuscript received 31 May 2022; revised 2 December 2022; accepted 5 March 2023. Date of publication 14 March 2023; date of current version 28 July 2023. This work was supported in part by the European Union's Horizon 2020 Research and Innovation Programme under Grant 739551 (KIOS CoE) and in part by the Italian Ministry for Research in the framework of the 2017 Program for Research Projects of National Interest (PRIN) under Grant 2017YKXYXJ. Recommended by Associate Editor S. N. Dashkovskiy. *(Corresponding author: Michelangelo Bin.)*

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Color versions of one or more figures in this article are available at https://doi.org/10.1109/TAC.2023.3256760.

Digital Object Identifier 10.1109/TAC.2023.3256760

the definition of suitable topological spaces and a preliminary analysis, both of which are problem-specific and, hence, not treated here systematically.

Going into the specifics of the framework and the main result of the article, we describe systems in terms of set-valued maps between arbitrary sets. These sets can be endowed with different topologies turning them into topological spaces. For each choice of such topologies, we define "stability" as a property similar to upper semicontinuity generalizing the continuity properties implied by the usual notions of Lyapunov stability of equilibria, sets, or motions, global or local (incremental) stability, and asymptotic gain. In particular, the continuity conditions underlying any of these properties can be obtained in terms of the proposed notion of stability for a specific choice of the involved topological spaces.

Given a feedback interconnection of two systems of this kind, we introduce an abstract small-gain property, and we prove a small-gain theorem stating that such property implies stability of the interconnection. The proposed small-gain property is an abstraction of the joint condition *"stability of the subsystems + small-gain property"* of (1) that, however, does not admit a similar decomposition but is a unique requirement. Nevertheless, we show that, in an ordinary ISS context, *"stability of the subsystems + small-gain condition"* implies the proposed small-gain property.

Finally, in this connection, we emphasize that the presented results only concern the continuity conditions implied by the global stability and asymptotic gain properties [11] and not directly ISS. While for finite-dimensional systems these properties imply (local) ISS, this is not generally true, for instance, for hybrid (even of finite dimension) [38, Remark 3.3] and infinite-dimensional [39], [40] systems. Therefore, in an ISS context, the conclusions that can be drawn on the feedback interconnection from the proposed theory are in general weaker than ISS. Nevertheless, we remark that this is not necessarily a shortcoming of the proposed theory, as it deals with spaces where ordinary notions of uniform convergence or boundedness may not make sense. Moreover, stronger properties, such as "uniform asymptotic gain" [11], may be obtained by suitably redefining or extending the input and output spaces and their topologies, as discussed in Section V-E.

The article is organized as follows. In Section II, we introduce the basic notions of systems and interconnections. In Section III, we define the notion of stability and connect it to the usual global stability and asymptotic gain properties in metric spaces. In Section IV, we define the small-gain property, we establish the main result, and we show that ISS implies the proposed small-gain property. In Section V, we discuss further connections between stability and other existing notions. Finally, in Section VI, we present three examples illustrating how the proposed theory can be used to handle interconnections of systems falling outside the scope of existing small-gain theorems.

Notations and Preliminaries. We denote by $\mathbb{R}$ and $\mathbb{N}$ the set of real and natural numbers, respectively ($0 \in \mathbb{N}$). If $\sim$ is a relation on a set S and $s \in S$, we let $S_{\sim s} := \{z \in S : z \sim s\}$. By $F : X \rightrightarrows Y$ we denote a set-valued map from X to Y. If $S \subseteq X$, we let $F(S) := \cup_{s\in S} F(s)$. Accordingly, $F(\varnothing) = \varnothing$. If $X = A \times B$, $S \subseteq A$, and $Z \subseteq B$, then $F(S,Z) = F(S \times Z)$. We denote by $\operatorname{dom} F$ the set of $x \in X$ for which $F(x)$ is nonempty, and by $\operatorname{ran} F := F(\operatorname{dom} F)$ the range of F. For $V \subseteq Y$, we denote by $F^{\mathrm{L}}(V) := \{x \in X : F(x) \cap V \neq \varnothing\}$ and $F^{\mathrm{U}}(V) := \{x \in X : F(x) \subseteq V\}$ the *lower* and *upper* inverse, respectively. If $F(x)$ is a singleton for all $x \in \operatorname{dom} F$, we identify F with the function $f : \operatorname{dom} F \to Y$ satisfying $f(x) \in F(x)$ for all $x \in \operatorname{dom} F$. The graph of a map F is defined as $\operatorname{graph} F := \{(x,y) \in X \times Y : y \in F(x)\}$.

A *topological space* is a pair $(\mathcal{X}, \tau)$ where $\mathcal{X}$ is a set and τ is a collection of subsets of $\mathcal{X}$ which contains $\varnothing$ and $\mathcal{X}$ itself and is closed under finite intersections and arbitrary unions. The elements of τ are called *open sets*. A *neighborhood of a point* $x \in \mathcal{X}$ is a subset of $\mathcal{X}$ containing an open set containing x. A *neighborhood of a set* $X \subseteq \mathcal{X}$ is a subset of $\mathcal{X}$ containing a neighborhood of every point of X. The set of all the neighborhoods of $X \subseteq \mathcal{X}$ is denoted by $\mathcal{N}_\tau(X)$ (or $\mathcal{N}_\tau(x)$ if $X = \{x\}$). When τ is clear from the context we omit it and, for instance, we write $\mathcal{X}$ for $(\mathcal{X}, \tau)$ and $\mathcal{N}(\cdot)$ for $\mathcal{N}_\tau(\cdot)$. If not otherwise specified, we shall assume every $X \subseteq \mathcal{X}$ to be endowed with the *subset topology* $\tau_X := \{O \cap X : O \in \tau\}$ and, if $(\mathcal{X}_1, \tau_{\mathcal{X}_1}), \dots, (\mathcal{X}_n, \tau_{\mathcal{X}_n})$ are topological spaces, their product $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ will be assumed to be endowed with the *product topology* denoted by $\tau_{\mathcal{X}_1} \otimes \cdots \otimes \tau_{\mathcal{X}_n}$. A *net* on a set X is a map $x : I \to X$ from a *directed set* I to X. We denote nets also by $(x_j)_{j\in I}$.

For $t > 0$, we denote by $\mathcal{C}_t(X)$ the set of continuous functions $[0,t) \to X$, and we let $\mathcal{C}_{(0,\infty]}(X) := \cup_{t\in(0,\infty]} \mathcal{C}_t(X)$. A continuous function $k : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-K if it is strictly increasing and $k(0) = 0$. We denote by $k^{-1} : \operatorname{ran} k \to \mathbb{R}_{\geq 0}$ the inverse of k. Notice that, if k is of class-K, there always exists $\epsilon > 0$ so that $[0,\epsilon) \subseteq \operatorname{ran} k = \operatorname{dom} k^{-1}$. Hence, $k^{-1}(s)$ exists for all sufficiently small $s > 0$.

II. Systems

In this section, we introduce the basic notions we use to model systems and their interconnections.

A. Systems as Mappings

Throughout the article, systems are represented by set-valued maps between sets.

Definition 1 (Systems): A *system* is a triple $(\mathcal{D}, \mathcal{Y}, \Psi)$ in which $\mathcal{D}$ and $\mathcal{Y}$ are sets and $\Psi : \mathcal{D} \rightrightarrows \mathcal{Y}$ is a set-valued map.

The set $\mathcal{D}$ is called the *input space*, and its elements the *inputs* of the system. The set $\mathcal{Y}$ is called the *output space*, and its elements the *outputs* of the system. A system with $\operatorname{dom} \Psi = \varnothing$ is called *trivial*. Since $d \notin \operatorname{dom} \Psi$ implies $\Psi(d) = \varnothing$, then $\Psi(D) = \Psi(D \cap \operatorname{dom} \Psi)$, for all $D \subseteq \mathcal{D}$.

The notion of systems provided by Definition 1 resembles that of [2] and [6], with the difference that here $\mathcal{D}$ and $\mathcal{Y}$ are generic sets, and not necessarily normed spaces of signals. Moreover, Definition 1 also fits the *behavioral* framework of [41], as the set $\operatorname{graph} \Psi$ is a behavior on $\mathcal{D} \times \mathcal{Y}$ in the sense of [41, Def. 1.2.1]. In this connection, we observe that seeing Ψ as a map $\mathcal{D} \rightrightarrows \mathcal{Y}$, instead of a map $\Psi^{\mathrm{L}} : \mathcal{Y} \rightrightarrows \mathcal{D}$, is a matter of convention as $\operatorname{graph} \Psi$ and $\operatorname{graph} \Psi^{\mathrm{L}}$ are isomorphic.

Definition 1 is sufficiently general to include most of the usual definitions of interest in control theory, such as transfer

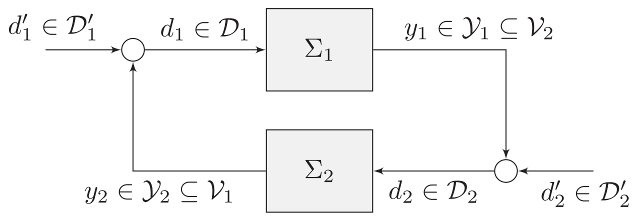

Fig. 1. Feedback interconnection of Σ_1 and Σ_2.

functions, ordinary/partial differential equations or inclusions, and hybrid systems, as shown in the following example.

Example 1: Consider the hybrid inclusions [42]

$$\begin{cases} \dot{x} & \in & F(x,u) & \quad (x,u) \in C \\ x^+ & \in & G(x,u) & \quad (x,u) \in D \end{cases} \tag{2}$$

with $C, D \subseteq \mathbb{R}^n \times \mathbb{R}^m$, $n, m \in \mathbb{N}$, and $F, G : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$. With $X_0 \subseteq \{x \in \mathbb{R}^n : \exists u \in \mathbb{R}^m,\ (x,u) \in C \cup D\}$ and $\mathcal{U}$ the set of *hybrid inputs* [38] on $\mathbb{R}^m$, let $\mathcal{D} := X_0 \times \mathcal{U}$. Moreover, let $\mathcal{Y}$ be the set of *hybrid arcs* [38], [42] on $\mathbb{R}^n$. For each $(x_0, u) \in \mathcal{D}$, let $\Psi(x_0, u) \subseteq \mathcal{Y}$ be the set of all $x \in \mathcal{Y}$ such that (x, u) is a solution pair to (2) with x originating at x_0. Then, $(\mathcal{D}, \mathcal{Y}, \Psi)$ is a system in the sense of Definition 1.

In addition, Definition 1 extends beyond dynamical systems. It can be used to model algebraic maps, solution mapping of optimization problems, or other relations capturing only some specific aspects of dynamics. For instance, Example 2 hereafter deals with limit sets, used in control to characterize the steady-state trajectories of a system [43], [44].

Example 2: In the setting of Example 1, fix $u = 0$ and let $X_0 \subset \mathbb{R}^n$ be compact. Let $\mathcal{S}(X_0)$ be the set of all complete solutions x of (2) originating in X_0 and corresponding to $u = 0$. Suppose that $\mathcal{S}(X_0) \neq \varnothing$ and, for each $\tau \geq 0$, define the *reachable tail* $\mathcal{R}^\tau(X_0) := \{x(t,j) \in \mathbb{R}^n : x \in \mathcal{S}(X_0),\ (t,j) \in \operatorname{dom} x,\ t + j \geq \tau\}$. Then, $\{\mathcal{R}^\tau(X_0) : \tau \geq 0\}$ is a filter base [45, Sec. 1.6] whose (possibly empty) set of cluster points $\Omega(X_0) := \cap_{\tau \geq 0} \overline{\mathcal{R}^\tau(X_0)}$ is the Ω*-limit set* of (2) from X_0 (and with $u = 0$). Let $\mathcal{D}$ be the set of all compact subsets X_0 of $\mathbb{R}^n$, $\mathcal{Y} = \mathbb{R}^n$, and $\Psi : \mathcal{D} \rightrightarrows \mathcal{Y}$ the set-valued map $\Psi(X_0) := \Omega(X_0)$. Then, $(\mathcal{D}, \mathcal{Y}, \Psi)$ is a system in the sense of Definition 1 representing the mapping between sets of initial conditions and the corresponding attractor.

B. Interconnections

Let $\Sigma_1 = (\mathcal{D}_1, \mathcal{Y}_1, \Psi_1)$ and $\Sigma_2 = (\mathcal{D}_2, \mathcal{Y}_2, \Psi_2)$ be systems. As a first case, assume that $\mathcal{D}_1 = \mathcal{D}_2$ and let $\mathcal{D} := \mathcal{D}_1 = \mathcal{D}_2$. The *parallel interconnection* of Σ_1 and Σ_2 is defined as the system $(\mathcal{D}, \mathcal{Y}, \Psi)$ with $\mathcal{Y} := \mathcal{Y}_1 \times \mathcal{Y}_2$, $\Psi : \mathcal{D} \rightrightarrows \mathcal{Y}$, $d \mapsto \Psi(d) := \Psi_1(d) \times \Psi_2(d)$, and $\operatorname{dom} \Psi = \operatorname{dom} \Psi_1 \cap \operatorname{dom} \Psi_2$.

As a second case, assume that $\mathcal{Y}_1 \subseteq \mathcal{D}_2$. The *series interconnection* of Σ_1 and Σ_2 is defined as the system $(\mathcal{D}, \mathcal{Y}, \Psi)$ with $\mathcal{D} := \mathcal{D}_1, \mathcal{Y} := \mathcal{Y}_2, \Psi : \mathcal{D} \rightrightarrows \mathcal{Y}, d \mapsto \Psi(d) := \Psi_2(\Psi_1(d))$, and $\operatorname{dom} \Psi = \{d \in \mathcal{D}_1 : \Psi_1(d) \cap \operatorname{dom} \Psi_2 \neq \varnothing\}$.

Finally, assume that, for some sets $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{D}_1'$, and $\mathcal{D}_2'$ such that $\mathcal{Y}_1 \subseteq \mathcal{V}_2$ and $\mathcal{Y}_2 \subseteq \mathcal{V}_1$, we have $\mathcal{D}_1 = \mathcal{V}_1 \times \mathcal{D}_1'$ and $\mathcal{D}_2 = \mathcal{V}_2 \times \mathcal{D}_2'$ (see Fig. 1). Then, a *feedback interconnection* of Σ_1 and Σ_2 is a system $\Sigma = (\mathcal{D}, \mathcal{Y}, \Psi)$ with $\mathcal{D} := \mathcal{D}_1' \times \mathcal{D}_2'$, $\mathcal{Y} := \mathcal{Y}_1 \times \mathcal{Y}_2$, $\operatorname{dom} \Psi \subseteq \mathcal{D}$ and, for all $d = (d_1, d_2) \in \operatorname{dom} \Psi$,

$$\Psi(d) := \{(y_1, y_2) \in \mathcal{Y} : y_1 \in \Psi_1(y_2, d_1),\ y_2 \in \Psi_2(y_1, d_2)\}. \tag{3}$$

Let $\mathcal{F}(\Sigma_1, \Sigma_2)$ denote the set of all feedback interconnections of Σ_1 and Σ_2. Let $\preceq$ be the partial order on $\mathcal{F}(\Sigma_1, \Sigma_2)$ such that, for every two $(\mathcal{D}, \mathcal{Y}, \Psi), (\mathcal{D}, \mathcal{Y}, \Psi') \in \mathcal{F}(\Sigma_1, \Sigma_2)$, $(\mathcal{D}, \mathcal{Y}, \Psi) \preceq (\mathcal{D}, \mathcal{Y}, \Psi')$ if and only if $\operatorname{graph} \Psi \subseteq \operatorname{graph} \Psi'$. Then, every totally ordered subset T of $\mathcal{F}(\Sigma_1, \Sigma_2)$ has an upper bound in $\mathcal{F}(\Sigma_1, \Sigma_2)$ given by $(\mathcal{D}, \mathcal{Y}, \overline{\Psi})$ with

$$\operatorname{graph} \overline{\Psi} = \bigcup_{(\mathcal{D}, \mathcal{Y}, \Psi) \in T} \operatorname{graph} \Psi.$$

Since $(\mathcal{F}(\Sigma_1, \Sigma_2), \preceq)$ is also upward directed, as for every $(\mathcal{D}, \mathcal{Y}, \Psi), (\mathcal{D}, \mathcal{Y}, \Psi') \in \mathcal{F}(\Sigma_1, \Sigma_2)$, the system $(\mathcal{D}, \mathcal{Y}, \overline{\Psi}) \in \mathcal{F}(\Sigma_1, \Sigma_2)$ obtained with $\operatorname{graph} \overline{\Psi} = \operatorname{graph} \Psi \cup \operatorname{graph} \Psi'$ is an upper bound for both $(\mathcal{D}, \mathcal{Y}, \Psi), (\mathcal{D}, \mathcal{Y}, \Psi')$, then Zorn's Lemma guarantees the following.

Lemma 1: $\mathcal{F}(\Sigma_1, \Sigma_2)$ has a unique maximal element.

We call such a unique maximal element *the* feedback interconnection of Σ_1 and Σ_2.

The parallel interconnection of any two nontrivial systems is nontrivial when $\operatorname{dom} \Psi_1 \cap \operatorname{dom} \Psi_2 \neq \varnothing$. The series interconnection is nontrivial if $\operatorname{dom} \Psi_2 \cap \operatorname{ran} \Psi_1 \neq \varnothing$. Nontriviality of feedback interconnections is instead not straightforward, as illustrated by the following example.

Example 3: Consider the systems $\Sigma_1 := (\mathcal{U} \times X_{0,1}, \mathcal{X}, \Psi_1)$, $\Sigma_2 := (\mathcal{U} \times X_{0,2}, \mathcal{X}, \Psi_2)$ in which $X_{0,1}, X_{0,2} \subseteq \mathbb{R}$, $\mathcal{U} = \mathcal{X} = \mathcal{C}_{(0,\infty]}(\mathbb{R})$, and Ψ_1 and Ψ_2 are defined as the solution maps of the differential equations

$$\dot{x}_1 = \begin{cases} -1 & \text{if } x_1 u_1 > 0 \\ 1 & \text{otherwise,} \end{cases} \qquad x_1(0) \in X_{0,1}, \tag{4a}$$

$$\dot{x}_2 = u_2 x_2, \qquad x_2(0) \in X_{0,2}. \tag{4b}$$

Consider the feedback interconnection of Σ_1 and Σ_2 given by the system $\Sigma = (X_0, \mathcal{X}^2, \Psi)$ with $X_0 := X_{0,1} \times X_{0,2}$, and Ψ the solution map of

$$\dot{x}_1 = \begin{cases} -1 & \text{if } x_1 x_2 > 0 \\ 1 & \text{otherwise} \end{cases} \qquad \dot{x}_2 = x_1 x_2. \tag{5}$$

When Σ_1 is considered alone, the set $\{(u_1, x_1) \in \operatorname{dom} \Psi_1 : x_1 = 0\}$ may be nonempty, and hence, $x_1(0) = 0$ may be a possible initial condition. In fact, $x_1(t) = t$ for all $t \geq 0$ is a solution of (4a) corresponding to $x_1(0) = 0$ and $u_1(t) = -1$ for all $t \geq 0$. Likewise, any $x_2(0) > 0$ is a feasible initial condition for (4b) for all continuous u_2. However, for every feedback interconnection of Σ_1 and Σ_2, necessarily $\{(x_1, x_2) \in X_0 : x_1 = 0 \text{ and } x_2 > 0\} \not\subseteq \operatorname{dom} \Psi$, so that we cannot have simultaneously $x_1(0) = 0$ and $x_2(0) > 0$.

With reference to (3), define the projections

$$\Upsilon_1(d) := \{y_1 \in \mathcal{Y}_1 : \exists y_2 \in \mathcal{Y}_2,\ (y_1, y_2) \in \Psi(d)\},$$
$$\Upsilon_2(d) := \{y_2 \in \mathcal{Y}_2 : \exists y_1 \in \mathcal{Y}_1,\ (y_1, y_2) \in \Psi(d)\}. \tag{6}$$

Clearly,

$$\Psi(d) \subseteq \Upsilon_1(d) \times \Upsilon_2(d), \quad \forall d \in \mathcal{D}. \tag{7}$$

The following lemma, proved in Appendix A, provides an alternative characterization of Υ_1 and Υ_2 that plays an important role in the main result given later in Section IV.

Lemma 2: For all $d \in \mathcal{D}$, $\Upsilon_1(d) \subseteq \Psi_1(\Upsilon_2(d), d_1)$, $\Upsilon_2(d) \subseteq \Psi_2(\Upsilon_1(d), d_2)$, and

$$\Upsilon_1(d) = \{y_1 \in \mathcal{Y}_1 \,:\, y_1 \in \Psi_1(\Psi_2(y_1, d_2), d_1)\},$$
$$\Upsilon_2(d) = \{y_2 \in \mathcal{Y}_2 \,:\, y_2 \in \Psi_2(\Psi_1(y_2, d_1), d_2)\}. \quad (8)$$

Remark 1: In view of (7) and (8), the elements of $\Psi(d)$ are fixed points of the maps $\Psi_1(\Psi_2(\cdot, d_2), d_1)$ and $\Psi_2(\Psi_1(\cdot, d_1), d_2)$, respectively. This, however, is only a necessary condition since, in general, $\Upsilon_1(d) \times \Upsilon_2(d) \not\subseteq \Psi(d)$. Namely, pairs of fixed points of the aforementioned maps need not be elements of $\Psi(d)$, although they always belong to $\Upsilon_1(d) \times \Upsilon_2(d)$.

III. Stability

In this section, we introduce a notion of stability for systems satisfying Definition 1. We then discuss its relationship with the properties of global stability and asymptotic gain. This enables us to make a direct connection with ISS systems. Instead, connections with other notions, such as Lyapunov stability and incremental stability, are discussed later in Section V.

A. Stability as a Topological Notion

As for continuity, stability is defined with reference to a topology $\tau_{\mathcal{D}}$ defined on the input space $\mathcal{D}$ and a topology $\tau_{\mathcal{Y}}$ defined on the output space $\mathcal{Y}$. Different choices of these topologies lead to different notions of stability.

Definition 2 (Stability): Let $(\mathcal{D}, \tau_{\mathcal{D}})$ and $(\mathcal{Y}, \tau_{\mathcal{Y}})$ be topological spaces. A system $(\mathcal{D}, \mathcal{Y}, \Psi)$ is said to be *stable at* $D \subseteq \mathcal{D}$ *with respect to* $(\tau_{\mathcal{D}}, \tau_{\mathcal{Y}})$ (or, briefly, $(\tau_{\mathcal{D}}, \tau_{\mathcal{Y}})$*-stable at* D) if, for every $Y \in \mathcal{N}(\Psi(D))$, there exists $U \in \mathcal{N}(D)$, such that $\Psi(U) \subseteq Y$.

When $\tau_{\mathcal{D}}$ and $\tau_{\mathcal{Y}}$ are clear from the context they are omitted, and we say that the system is stable at D. If $D = \{d\}$ is a singleton, we say that the system is stable at d instead of at $\{d\}$. Stability at $D \subseteq \mathcal{D}$ is implied by upper semicontinuity of Ψ at each point of D. Indeed, every $Y \in \mathcal{N}(\Psi(D))$ contains a set of the form $\cup_{d\in D} Y_d$ with $Y_d \in \mathcal{N}(\Psi(d))$. If, for each $d \in D$, Ψ is upper semicontinuous at d, we can find $O_d \in \mathcal{N}(d)$ such that $\Psi(O_d) \subseteq Y_d$. Then, $O := \cup_{d\in D} O_d \in \mathcal{N}(D)$ and $\Psi(O) \subseteq \cup_{d\in D} Y_d \subseteq Y$. Nevertheless, the converse does not hold. Namely, stability at D does not imply upper semicontinuity of Ψ at each point of D. Indeed, as a trivial counterexample, notice that every system is stable at $\mathcal{D}$ (since $\mathcal{D}$ is a neighborhood of itself) without any relation to upper semicontinuity of Ψ at any point of $\mathcal{D}$.

Remark 2: We underline that the notion of stability given in Definition 2 is aimed at generalizing the continuity properties implied by the notion of Lyapunov stability of autonomous differential/difference equations and global stability à la [11] for systems with input, both of which are properties of the map Ψ, and not convergence or attractiveness, which are instead properties of the specific inputs and outputs of Ψ.

Remarkably, parallel interconnections are stable at a point if so are the interconnected systems, whereas series interconnections of stable systems are stable at both points and sets. Specifically, let $\Sigma_1 = (\mathcal{D}_1, \mathcal{Y}_1, \Psi_1)$ and $\Sigma_2 = (\mathcal{D}_2, \mathcal{Y}_2, \Psi_2)$ be systems, and let $\Sigma_p = (\mathcal{D}_p, \mathcal{Y}_p, \Psi_p)$ and $\Sigma_s = (\mathcal{D}_s, \mathcal{Y}_s, \Psi_s)$ denote their parallel and series interconnection (see Section II-B) whenever they make sense. Let $\mathcal{D}_1, \mathcal{D}_2, \mathcal{Y}_1$, and $\mathcal{Y}_2$ be endowed with some topologies, and $\mathcal{Y}_p = \mathcal{Y}_1 \times \mathcal{Y}_2$ with the product topology (in the parallel case, $\mathcal{D}_p = \mathcal{D}_1 = \mathcal{D}_2$; hence $\mathcal{D}_1$ and $\mathcal{D}_2$ are given the same topology). Then, the following holds.[1]

Proposition 1: If Σ_1 and Σ_2 are stable at $d \in \mathcal{D}_p$, then Σ_p is stable at d. If Σ_1 is stable at $D \subseteq \mathcal{D}_s$, and Σ_2 is stable at $\Psi_1(D)$, then Ψ_s is stable at D.

Proposition 1 is proved in Appendix B. Stability of feedback interconnections between Σ_1 and Σ_2 is instead more delicate, and it is indeed the main object of this article. We conclude this section with the following technical lemma (proved in Appendix C), which is used by several forthcoming results.

Lemma 3: Let $\mathcal{A}$ and $\mathcal{B}$ be topological spaces, and let $A \subseteq \mathcal{A}$ and $B \subseteq \mathcal{B}$. Every neighborhood of $A \times B$ in the product space $\mathcal{A} \times \mathcal{B}$ contains a neighborhood of $A \times B$ of the form $U \times V$, where $U \in \mathcal{N}(A)$ and $V \in \mathcal{N}(B)$. In particular, a system $(\mathcal{D}, \mathcal{Y}, \Psi)$ with $\mathcal{D} = \mathcal{A} \times \mathcal{B}$ is stable at $A \times B \subseteq \mathcal{A} \times \mathcal{B}$ if and only if for every $Y \in \mathcal{N}(\Psi(A \times B))$, there exist $U \in \mathcal{N}(A)$ and $V \in \mathcal{N}(B)$ such that $\Psi(U \times V) \subseteq Y$.

B. Connections With Global Stability

Let $(X, |\cdot|)$, $(U, |\cdot|)$, and $(Y, |\cdot|)$ be seminormed linear spaces (for ease of notation, we denote all seminorms by $|\cdot|$). Let $(\mathbb{T}_{\mathcal{U}}, \geq)$ and $(\mathbb{T}_{\mathcal{Y}}, \geq)$ be (possibly different) directed sets, and let $(\mathcal{U}, |\cdot|)$ and $(\mathcal{Y}, |\cdot|)$ be seminormed linear spaces (under the pointwise operations) of functions $\mathbb{T}_{\mathcal{Y}} \to Y$ and $\mathbb{T}_{\mathcal{U}} \to U$, respectively. The elements of X may represent initial/boundary conditions or parameters. The elements of $\mathcal{U}$ represent exogenous input signals, and those of $\mathcal{Y}$ the system's outputs. Let $\mathcal{D} := X \times \mathcal{U}$, and suppose a system $\Sigma = (\mathcal{D}, \mathcal{Y}, \Psi)$ is defined such that, for some class-K functions α and κ,

$$\forall (x, u, y) \in \operatorname{graph} \Psi, \quad |y| \leq \max\{\alpha(|x|), \kappa(|u|)\}. \quad (9)$$

When $|y| := \sup_{t\in\mathbb{T}_{\mathcal{Y}}} |y(t)|$ and $|u| := \sup_{t\in\mathbb{T}_{\mathcal{U}}} |u(t)|$, Condition (9) is a global stability property implied by ISS. If, instead, $\mathbb{T}_{\mathcal{U}} = \mathbb{R}_{\geq 0}$ and $|u| := (\int_0^\infty |u(t)|^2 dt)^{1/2}$, we obtain an "integral" variant of global stability, implied by integral ISS [46]. In general, different notions of stability can be obtained for different choices of the seminorms [46]. In any case, (9) implies stability, in the sense of Definition 2, of Σ at $D^\star := \{(x, u) \in X \times \mathcal{U} \,:\, |x| = 0, |u| = 0\}$ with respect to $(\tau_{\mathcal{D}}, \tau_{\mathcal{Y}})$, where $\tau_{\mathcal{Y}}$ is the topology induced on $\mathcal{Y}$ by its seminorm, and $\tau_{\mathcal{D}}$ is the product topology on $\mathcal{D}$ induced by the seminorms on X and $\mathcal{U}$. Indeed, (9) implies $\Psi(D^\star) \subseteq \{y \in \mathcal{Y} \,:\, |y| = 0\}$, and every neighborhood Y of $\Psi(D^\star)$ contains a set of the form $B_\epsilon := \{y \in \mathcal{Y} \,:\, |y| < \epsilon\}$ for some $\epsilon > 0$ small enough so as $\epsilon \in \operatorname{ran} \alpha \cap \operatorname{ran} \kappa$. As $\tau_{\mathcal{D}}$ is induced by the seminorm $|(x, u)| := \max\{|x|, |u|\}$, then the set $U := \{(x, u) \in \mathcal{D} \,:\, |(x, u)| < \min\{\alpha^{-1}(\epsilon), \kappa^{-1}(\epsilon)\}\}$ is a neighborhood of $D^\star$, and $\Psi(U) \subseteq B_\epsilon \subseteq Y$.

[1] Proposition 1 concerns parallel and series interconnections of "input–output" systems in the sense that we do not consider initial conditions as extra inputs as in Example 1. Thus, in case of dynamical systems, Proposition 1 has to be intended for fixed initial conditions. Nevertheless, the extension to the case where initial conditions are taken into account is straightforward.

C. Connections With Asymptotic Gain

In this section, we show that the asymptotic gain property implies stability according to Definition 2 for a specific choice of the topology of the input and output spaces. In particular, let $(Z, |\cdot|)$ be a seminormed linear space, $(\mathbb{T}_{\mathcal{Z}}, \geq)$ be a directed set, and $\mathcal{Z}$ be a linear space (under the pointwise operations) of functions $\mathbb{T}_{\mathcal{Z}} \to Z$. For all $\zeta \in \mathcal{Z}$, define

$$\limsup |\zeta| := \inf_{t \in \mathbb{T}_{\mathcal{Z}}} \sup_{s \geq t} |\zeta(s)|$$

(we set $\limsup |\zeta| := \infty$ if $\limsup |\zeta|$ does not exist in $\mathbb{R}$). For every $\varepsilon \geq 0$, define the set

$$O_\varepsilon := \{\zeta \in \mathcal{Z} \,:\, \limsup |\zeta| < \varepsilon\}$$

which is nonempty whenever $\varepsilon > 0$ since it contains the zero function. The collection $\beta := \{\mathcal{Z}\} \cup \{O_\varepsilon \,:\, \varepsilon > 0\}$ generates a topology $\tau_{\mathcal{Z}}$ on $\mathcal{Z}$ that we call the *limsup topology*. Moreover, β is a base for $(\mathcal{Z}, \tau_{\mathcal{Z}})$ [45, Prop. 1.2.1].

In the same setting of Section III-B, suppose that all $u \in \mathcal{U}$ and $y \in \mathcal{Y}$ are bounded and there exists a class-K function ρ such that

$$\forall (x, u, y) \in \operatorname{graph} \Psi, \quad \limsup |y| \leq \rho\left(\limsup |u|\right). \quad (10)$$

Condition (10) is an asymptotic gain property (implied by ISS). Let $\tau_{\mathcal{U}}$ and $\tau_{\mathcal{Y}}$ denote the limsup topologies on $\mathcal{U}$ and $\mathcal{Y}$, respectively, and let τ_X be any topology on X. Then, for every $S \subseteq X$, (10) implies that Ψ is stable at $D^\star := S \times \{u \in \mathcal{U} \,:\, \limsup |u| = 0\}$ with respect to $(\tau_X \otimes \tau_{\mathcal{U}}, \tau_{\mathcal{Y}})$. Indeed, (10) implies that $\Psi(D^\star) \subseteq \{y \in \mathcal{Y} \,:\, \limsup |y| = 0\}$. Hence, every neighborhood V of $\Psi(D^\star)$ contains a set of the form $Y_\varepsilon := \{y \in \mathcal{Y} \,:\, \limsup |y| < \varepsilon\}$ for a sufficiently small $\varepsilon > 0$ satisfying $\varepsilon \in \operatorname{ran} \rho$. Therefore, with $\delta := \rho^{-1}(\varepsilon)$, the set $U_\delta := \{u \in \mathcal{U} \,:\, \limsup |u| < \delta\}$ is such that $X \times U_\delta \in \mathcal{N}(D^\star)$ and, in view of (10), $y \in Y_\varepsilon \subseteq V$ for all $y \in \Psi(X \times U_\delta)$.

IV. Abstract Small-Gain Theory

A. The Small-Gain Property

Consider two systems $\Sigma_1 = (\mathcal{D}_1, \mathcal{Y}_1, \Psi_1)$ and $\Sigma_2 = (\mathcal{D}_2, \mathcal{Y}_2, \Psi_2)$ such that $\mathcal{D}_1 = \mathcal{V}_1 \times \mathcal{D}'_1$ and $\mathcal{D}_2 = \mathcal{V}_2 \times \mathcal{D}'_2$, with $\mathcal{Y}_2 \subseteq \mathcal{V}_1$ and $\mathcal{Y}_1 \subseteq \mathcal{V}_2$. Moreover, consider a feedback interconnection $\Sigma = (\mathcal{D}, \mathcal{Y}, \Psi)$ of Σ_1 and Σ_2 (not necessarily the maximal one) obtained as specified in Section II-B with $\mathcal{D} := \mathcal{D}'_1 \times \mathcal{D}'_2$, $\mathcal{Y} := \mathcal{Y}_1 \times \mathcal{Y}_2$, and Ψ defined as in (3) (see Fig. 1). We associate with Σ the maps $\Gamma_{12} : \mathcal{Y}_1 \times \mathcal{D} \rightrightarrows \mathcal{Y}_1$ and $\Gamma_{21} : \mathcal{Y}_2 \times \mathcal{D} \rightrightarrows \mathcal{Y}_2$ defined as

$$\Gamma_{12}(y_1, d) := \Psi_1(\Psi_2(y_1, d_2), d_1)$$
$$\Gamma_{21}(y_2, d) := \Psi_2(\Psi_1(y_2, d_1), d_2).$$

Moreover, for $(i, j) = (1, 2), (2, 1)$ and $n \in \mathbb{N}_{\geq 1}$, we define the maps Γ_{12}^n and Γ_{21}^n according to the following recursion:

$$\Gamma_{ij}^1(y_i, d) := \Gamma_{ij}(y_i, d),$$
$$\Gamma_{ij}^{n+1}(y_i, d) := \Gamma_{ij}(\Gamma_{ij}^n(y_i, d), d), \quad \forall n \in \mathbb{N}_{\geq 1},$$

for all $y_i \in \mathcal{Y}_i$ and all $d \in \mathcal{D}$. The maps Γ_{12} and Γ_{21} satisfy the following property.

Lemma 4: For every $(i, j) \in \{(1, 2), (2, 1)\}$, $D \subseteq \mathcal{D}$, $y_i \in \Upsilon_i(D)$, and $n \in \mathbb{N}_{\geq 1}$, it holds that $y_i \in \Gamma_{ij}^n(y_i, D)$.

Proof: Pick arbitrarily $D \subseteq \mathcal{D}$ and $y_i \in \Upsilon_i(D)$. Then, there exists $d \in D$ such that $y_i \in \Upsilon_i(d)$. In view of (8),

$$y_i \in \Gamma_{ij}(y_i, d). \quad (11)$$

Suppose that, for some $n \in \mathbb{N}_{\geq 1}$,

$$y_i \in \Gamma_{ij}^n(y_i, d). \quad (12)$$

Then, (12) implies that $\Gamma_{ij}(y_i, d) \subseteq \Gamma_{ij}(\Gamma_{ij}^n(y_i, d), d) = \Gamma_{ij}^{n+1}(y_i, d)$. Thus, (11) implies that $y_i \in \Gamma_{ij}^{n+1}(y_i, d)$. As (11) is (12) with $n = 1$, we conclude by induction that (12), and hence $y_i \in \Gamma_{ij}^n(y_i, D)$, hold for every $n \in \mathbb{N}_{\geq 1}$. Since D and $y_i \in \Upsilon_i(D)$ were arbitrary, the claim follows. ∎

We endow $\mathcal{D}$ with a topology $\tau_{\mathcal{D}}$ and $\mathcal{Y}$ with a topology $\tau_{\mathcal{Y}}$. Then, the following definition formalizes the *small-gain condition* used in this article to establish stability, with respect to $(\tau_{\mathcal{D}}, \tau_{\mathcal{Y}})$, of the feedback interconnection Σ.

Definition 3 (Small-gain property): The feedback interconnection Σ is said to satisfy the *small-gain property at $D^\star \subseteq \mathcal{D}$ with respect to $(\tau_{\mathcal{D}}, \tau_{\mathcal{Y}})$* if, for every $Y \in \mathcal{N}(\Psi(D^\star))$, there exists $D \in \mathcal{N}(D^\star)$, such that:

$$\forall y = (y_1, y_2) \in \mathcal{Y} \setminus Y, \;\; \exists n_1, n_2 \in \mathbb{N}_{\geq 1},$$
$$\text{s.t. } \Gamma_{12}^{n_1}(y_1, D) \times \Gamma_{21}^{n_2}(y_2, D) \subseteq Y. \quad (13)$$

Remark 3: Condition (13) in Definition 3 is a "contraction" requirement for the maps Γ_{12} and Γ_{21} that plays in our setting the role of *"stability of the subsystems + small-gain property"* of typical ISS contexts [cf., (1)]. However, we stress that, unlike (1), Definition 3 is a single condition that, up to the authors' knowledge, cannot be expressed in terms of the composition of stability of Σ_1 and Σ_2 and a contraction property. Indeed, in general, the set $D^\star$ may not even be a product of the form $D_1^\star \times D_2^\star$. Nevertheless, later in Section IV-C, we show that ISS implies Definition 3.

Remark 4: Computing Γ_{12} and Γ_{21} may be difficult, if not impossible, for nontrivial interconnections. Fortunately, their computation is generally not needed for checking whether the condition (13) holds. Indeed, (13) can be usually checked by using some known property of the systems involved. In the following, we provide several instances where this is the case, i.e., Propositions 2 and 3, and the examples in Section VI.

B. Main Result

In this section, we prove the main result of this article, establishing that a feedback interconnection of two systems satisfying the small-gain property of Definition 3 is stable. As in the previous section, we consider two systems $\Sigma_1 = (\mathcal{D}_1, \mathcal{Y}_1, \Psi_1)$ and $\Sigma_2 = (\mathcal{D}_2, \mathcal{Y}_2, \Psi_2)$, where $\mathcal{D}_1 = \mathcal{V}_1 \times \mathcal{D}'_1$ and $\mathcal{D}_2 = \mathcal{V}_2 \times \mathcal{D}'_2$, with $\mathcal{Y}_2 \subseteq \mathcal{V}_1$ and $\mathcal{Y}_1 \subseteq \mathcal{V}_2$. Then, we consider a feedback interconnection $\Sigma = (\mathcal{D}, \mathcal{Y}, \Psi)$ of Σ_1 and Σ_2 as specified in

Section II-B, with $\mathcal{D} := \mathcal{D}_1' \times \mathcal{D}_2'$, $\mathcal{Y} := \mathcal{Y}_1 \times \mathcal{Y}_2$, and Ψ defined as in (3). Finally, we endow $\mathcal{D}$ with a topology $\tau_{\mathcal{D}}$ and $\mathcal{Y}$ with a topology $\tau_{\mathcal{Y}}$.

Theorem 1 (Small-gain Theorem): If Σ satisfies the small-gain property at $D^\star \subseteq \mathcal{D}$ with respect to $(\tau_{\mathcal{D}}, \tau_{\mathcal{Y}})$, then it is $(\tau_{\mathcal{D}}, \tau_{\mathcal{Y}})$-stable at $D^\star$.

Proof: Pick $Y \in \mathcal{N}(\Psi(D^\star))$ arbitrarily, and let $D \in \mathcal{N}(D^\star)$ be such that (13) holds. Equation (7) implies that

$$\begin{aligned}\Psi(D) &\subseteq \bigcup_{d\in D} (\Upsilon_1(d) \times \Upsilon_2(d)) \subseteq \Upsilon_1(D) \times \Upsilon_2(D)\\ &= ((\Upsilon_1(D)\times\Upsilon_2(D)) \setminus Y) \cup ((\Upsilon_1(D) \times \Upsilon_2(D)) \cap Y)\\ &\subseteq ((\Upsilon_1(D)\times\Upsilon_2(D)) \setminus Y) \cup Y. \qquad (14)\end{aligned}$$

For every $(y_1, y_2) \in (\Upsilon_1(D) \times \Upsilon_2(D)) \setminus Y$, Lemma 4 implies that

$$y_1 \in \Gamma_{12}^{n_1}(y_1, D), \quad y_2 \in \Gamma_{21}^{n_2}(y_2, D), \quad \forall n_1, n_2 \in \mathbb{N}_{\geq 1}. \quad (15)$$

For each $y = (y_1, y_2) \in (\Upsilon_1(D) \times \Upsilon_2(D)) \setminus Y$, let n_1 and n_2 be such that (13) holds. Then, (13) and (15) imply that

$$\begin{aligned}(\Upsilon_1(D) \times \Upsilon_2(D)) \setminus Y &= \bigcup_{y\in(\Upsilon_1(D)\times\Upsilon_2(D))\setminus Y} \{y_1\} \times \{y_2\}\\ &\subseteq \bigcup_{y\in(\Upsilon_1(D)\times\Upsilon_2(D))\setminus Y} \Gamma_{12}^{n_1}(y_1, D) \times \Gamma_{21}^{n_2}(y_2, D)\\ &\subseteq Y.\end{aligned}$$

Hence, we deduce from (14) that $\Psi(D) \subseteq Y$.

In this way, we have shown that, for every $Y \in \mathcal{N}(\Psi(D^\star))$, there exists $D \in \mathcal{N}(D^\star)$, such that $\Psi(D) \subseteq Y$. Namely, Σ is stable at $D^\star$. ■

C. Connections With Other Small-Gain Theorems

In this section, we establish a relationship between the small-gain property given in Definition 3 and some existing small-gain theorems applying to ISS systems. Specifically, we show that *"stability of the subsystems + small-gain property"* in (1) (with stability meaning ISS) implies the small-gain condition of Definition 3. For simplicity, we focus on stability of the origin. Nevertheless, the same arguments can be extended to stability of sets or motions, by resorting to the corresponding notions described later in Section V.

With $(i,j) \in \{(1,2),(2,1)\}$, let $(X_i, |\cdot|)$, $(U_i, |\cdot|)$, and $(Y_i, |\cdot|)$ be seminormed linear spaces (as before, we denote all seminorms by $|\cdot|$). Let $(\mathbb{T}_{\mathcal{U}_i}, \geq)$ and $(\mathbb{T}_{\mathcal{Y}_i}, \geq)$ be directed sets, and $\mathcal{U}_i$ and $\mathcal{Y}_i$ be linear spaces (under the pointwise operations) of bounded functions $\mathbb{T}_{\mathcal{Y}_i} \to Y_i$ and $\mathbb{T}_{\mathcal{U}_i} \to U_i$, respectively. Finally, let $\Psi_i : \mathcal{Y}_j \times X_i \times \mathcal{U}_i \rightrightarrows \mathcal{Y}_i$. With $\mathcal{D}_i' := X_i \times \mathcal{U}_i$ and $\mathcal{D}_i := \mathcal{Y}_j \times \mathcal{D}_i'$, the triple $\Sigma_i = (\mathcal{D}_i, \mathcal{Y}_i, \Psi_i)$ is a system in the sense of Definition 1. The set X_i may represent initial/boundary conditions in an initial value problem, while $\mathcal{U}_i$ contains exogenous inputs.

We define $O_{X_i} := \{x_i \in X_i : |x_i| = 0\}$, $O_{\mathcal{U}_i} := \{u_i \in \mathcal{U}_i : \sup_{t\in\mathbb{T}_{\mathcal{U}_i}} |u_i(t)| = 0\}$, and $O_{\mathcal{Y}_i} := \{y_i \in \mathcal{Y}_i : \sup_{t\in\mathbb{T}_{\mathcal{Y}_i}} |y_i(t)| = 0\}$. Moreover, we assume that there exist class-K functions $\alpha_i, \varrho_i, \kappa_i$ such that

$$\sup_{t\in\mathbb{T}_{\mathcal{Y}_i}} |y_i(t)| \leq \max\left\{ \alpha_i(|x_i|),\ \varrho_i\left(\sup_{t\in\mathbb{T}_{\mathcal{Y}_j}} |y_j(t)|\right) \kappa_i\left(\sup_{t\in\mathbb{T}_{\mathcal{U}_i}} |u_i(t)|\right)\right\}, \quad (16a)$$

$$\limsup |y_i| \leq \max\left\{ \varrho_i(\limsup |y_j|), \kappa_i(\limsup |u_i|)\right\} \quad (16b)$$

hold for all $(y_j, x_i, u_i, y_i) \in \operatorname{graph} \Psi_i$. Condition (16a) relates to global stability and (16b) to asymptotic gain. Both are implied by ISS. Finally, we suppose that the following small-gain condition holds:

$$\varrho_{ij}(s) := \varrho_i(\varrho_j(s)) < s, \quad \forall s \in \mathbb{R}_{>0}. \quad (17)$$

First, we show that Conditions (16a) and (17) imply the small-gain property of Definition 3 with respect to the uniform seminorm topology. To this end, for $i = 1, 2$, we define on $\mathcal{U}_i$ and $\mathcal{Y}_i$ the seminorms $|u_i| := \sup_{t\in\mathbb{T}_{\mathcal{U}_i}} |u_i(t)|$ and $|y_i| := \sup_{t\in\mathbb{T}_{\mathcal{Y}_i}} |y_i(t)|$, respectively. The product topology $\tau_{\mathcal{D}}$ on $\mathcal{D} := \mathcal{D}_1' \times \mathcal{D}_2'$ is generated by the seminorm $|(x_1, u_1, x_2, u_2)| := \max_{i=1,2}\{|x_i|, |u_i|\}$. Likewise, the product topology $\tau_{\mathcal{Y}}$ on $\mathcal{Y} := \mathcal{Y}_1 \times \mathcal{Y}_2$ is generated by $|(y_1, y_2)| := \max\{|y_1|, |y_2|\}$. Consider a feedback interconnection $\Sigma = (\mathcal{D}, \mathcal{Y}, \Psi)$ of Σ_1 and Σ_2, defined according to Section II-B, and let $O_{\mathcal{D}} := O_{X_1} \times O_{\mathcal{U}_1} \times O_{X_2} \times O_{\mathcal{U}_2}$. Then, the following holds (the proof is given in Appendix D).

Proposition 2: Suppose that, for all $(i,j) \in \{(1,2),(2,1)\}$, (17) holds and every $(y_j, x_i, u_i, y_i) \in \operatorname{graph} \Psi_i$ satisfies (16a). Then, Σ satisfies the small-gain property of Definition 3 at $O_{\mathcal{D}}$ with respect to $(\tau_{\mathcal{D}}, \tau_{\mathcal{Y}})$.

In view of Proposition 2, we can apply Theorem 1 to conclude that the interconnection Σ is stable at $O_{\mathcal{D}}$. This implies the following continuity property:

$$\begin{aligned}&\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall (x_1, u_1, x_2, u_2, y_1, y_2) \in \operatorname{graph} \Psi,\\ &\max\{|x_1|, |x_2|, |u_1|, |u_2|\} < \delta \Rightarrow \max\{|y_1|, |y_2|\} < \varepsilon. \quad (18)\end{aligned}$$

Next, we show that, for the previously defined feedback interconnection Σ of Σ_1 and Σ_2, the bounds (16b) and (17) also imply the small-gain condition of Definition 3 with respect to the limsup topology on $\mathcal{U}_i$ and $\mathcal{Y}_i$ (and any topology on X_i). To this end, we endow X_i with an arbitrary topology τ_{X_i}, we give $\mathcal{U}_i$ and $\mathcal{Y}_i$ the respective limsup topologies, as described in Section III-C, and we let $\tau_{\mathcal{D}}'$ and $\tau_{\mathcal{Y}}'$ be the product topologies on $\mathcal{D}$ and $\mathcal{Y}$, respectively. For $i = 1, 2$, define $L_{\mathcal{U}_i} := \{u_i \in \mathcal{U}_i : \limsup |u_i| = 0\}$, $L_{\mathcal{Y}_i} := \{y_i \in \mathcal{Y}_i : \limsup |y_i| = 0\}$, and let $L_{\mathcal{D}} := X_1 \times L_{\mathcal{U}_1} \times X_2 \times L_{\mathcal{U}_2}$. Then, the following result holds (the proof is in Appendix E).

Proposition 3: Suppose that, for all $(i,j) \in \{(1,2),(2,1)\}$, (17) holds and every $(y_j, x_i, u_i, y_i) \in \operatorname{graph} \Psi_i$ satisfies (16b). Then, Σ satisfies the small-gain property of Definition 3 at $L_{\mathcal{D}}$ with respect to $(\tau_{\mathcal{D}}', \tau_{\mathcal{Y}}')$.

In view of Proposition 3, we can use Theorem 1 to deduce from (16b) and (17) that the interconnection Σ is also $(\tau_{\mathcal{D}}', \tau_{\mathcal{Y}}')$-stable at $L_{\mathcal{D}}$. This implies that [cf., (18)]

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall (x_1, u_1, x_2, u_2, y_1, y_2) \in \operatorname{graph} \Psi,$$
$$\max_{i=1,2} \limsup |u_i| < \delta \Rightarrow \max_{i=1,2} \limsup |y_i| < \varepsilon. \tag{19}$$

To summarize, Theorem 1 allows us to conclude from the global stability and asymptotic gain properties (16) (hence, from ISS), and the small-gain condition (17), that the feedback interconnection Σ of the two subsystems Σ_1 and Σ_2 satisfies the continuity conditions (18) and (19), which are the same continuity conditions implied by ISS of Σ.

V. Connections With Other Stability Notions

In this section, we present some relevant cases, obtained for a specific choice of $(\mathcal{D}, \tau_{\mathcal{D}})$, $(\mathcal{Y}, \tau_{\mathcal{Y}})$, and Ψ, that connect the stability notion given by Definition 2 with more common notions of stability used in control and systems theory.

A. Lyapunov Stability of Motions

Consider a differential equation of the form

$$\dot{x} = f(x), \tag{20}$$

with $x \in \mathbb{R}^n$, $n \in \mathbb{N}$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $X_0 \subseteq \mathbb{R}^n$, and $\mathcal{X} := \mathcal{C}_{[0,\infty]}(\mathbb{R}^n)$. Let

$$\rho_\infty(x, y) := \begin{cases} \sup_{t \in \operatorname{dom} x} |x(t) - y(t)|, & \text{if } \operatorname{dom} x = \operatorname{dom} y, \\ \infty, & \text{otherwise,} \end{cases}$$

and let $\rho(x, y) := \min\{\rho_\infty(x, y),\ c\}$ for an arbitrary $c > 0$.

Lemma 5: ρ is a metric on $\mathcal{X}$.

Proof: Clearly, ρ is symmetric and $\rho(x, y) = 0 \iff x = y$. It remains to show that $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for arbitrary $x, y, z \in \mathcal{X}$. If $\operatorname{dom} x \neq \operatorname{dom} y$, then no $z \in \mathcal{X}$ satisfies $\operatorname{dom} z = \operatorname{dom} x$ and $\operatorname{dom} z = \operatorname{dom} y$. Hence, $\rho(x, z) + \rho(z, y) \geq c = \rho(x, y)$. If, instead, $\operatorname{dom} x = \operatorname{dom} y$, then either $\operatorname{dom} z = \operatorname{dom} x = \operatorname{dom} y$, in which case the inequality follows by the definition of ρ_∞, or $\rho(x, z) + \rho(z, y) = 2c \geq \rho(x, y)$. ■

Let X_0 be given the Euclidean topology, and $\mathcal{X}$ be given the topology induced by ρ. Moreover, let Ψ be the solution map of (20), mapping initial conditions $x_0 \in X_0$ to maximal solutions $x \in \mathcal{X}$ satisfying $x(0) = x_0$. Let $x_0^* \in X_0$ be such that $\Psi(x_0^*)$ is single-valued and $x^* \in \Psi(x_0^*)$ is complete (i.e., $\operatorname{dom} x^* = [0, \infty)$). Then, system $(X_0, \mathcal{X}, \Psi)$ is stable at x_0^* in the sense of Definition 2 if and only if

$$\forall \epsilon > 0,\ \exists \delta(\epsilon) > 0,$$
$$|x_0 - x_0^*| < \delta(\epsilon) \Rightarrow \forall x \in \Psi(x_0),\ \rho(x^*, x) < \epsilon. \tag{21}$$

Notice that, by definition of ρ, (21) implies that, for all initial conditions inside a sufficiently small neighborhood of x_0^*, the corresponding solutions are complete. We observe that, when Ψ is single-valued and every solution of (20) originating in X_0 is complete, condition (21) (which, we recall, is Definition 2 in this context) equals Lyapunov stability of the motion $x^* = \Psi(x_0^*)$.

B. Lyapunov Stability of Sets

In the same setting of Section V-A, let $\mathcal{A} \subseteq X_0$ be nonempty, closed, and forward-invariant (i.e., $\operatorname{ran} x \subseteq \mathcal{A}$ for all $x \in \Psi(\mathcal{A})$). With $c > 0$ arbitrary, define

$$\rho_{\mathcal{A}}(x) := \min\left\{\sup_{t \in \operatorname{dom} x} |x(t)|_{\mathcal{A}},\ c\right\},$$

where $|x|_{\mathcal{A}} := \inf_{a \in \mathcal{A}} |x - a|$ denotes the distance of x to $\mathcal{A}$. Then, $\rho_{\mathcal{A}}$ induces a topology on $\mathcal{X}$ as specified by the following lemma.

Lemma 6: For every $\epsilon \geq 0$, let $O_\epsilon := \{x \in \mathcal{X} : \rho_{\mathcal{A}}(x) < \epsilon\}$. Then, $\tau_{\mathcal{A}} := \{O_\epsilon : \epsilon \geq 0\}$ is a topology on $\mathcal{X}$.

Proof: Both $\varnothing = O_0$ and $\mathcal{X} = O_{c+1}$ belong to $\tau_{\mathcal{A}}$. Moreover, since $\mathcal{X} = O_\epsilon$ for all $\epsilon > c$, then, for every $\epsilon_1, \epsilon_2 \geq 0$, $O_{\epsilon_1} \cap O_{\epsilon_2} = O_{\min\{\epsilon_1, \epsilon_2\}} \in \tau_{\mathcal{A}}$. Finally, pick $E \subseteq \mathbb{R}_{\geq 0}$ arbitrary. Since $\epsilon \mapsto O_\epsilon$ is increasing (in the sense of inclusion), if $E \subseteq [0, c]$, then $\cup_{\epsilon \in E} O_\epsilon = O_{\sup E}$; otherwise, $\cup_{\epsilon \in E} O_\epsilon = \mathcal{X}$. In both cases, $\cup_{\epsilon \in E} O_\epsilon \in \tau_{\mathcal{A}}$. ■

A sequence $(x_n)_{n \in \mathbb{N}}$ converges in $\tau_{\mathcal{A}}$ to a limit x if, for every $\epsilon > 0$, there exists $n^*(\epsilon) \in \mathbb{N}$, such that $\rho_{\mathcal{A}}(x_n) < \rho_{\mathcal{A}}(x) + \epsilon$ for all $n \geq n^*(\epsilon)$. Clearly, if $(x_n)_{n \in \mathbb{N}}$ converges to x, it converges to every other z satisfying $\rho_{\mathcal{A}}(z) = \rho_{\mathcal{A}}(x)$. In particular, if x_n converges to an x satisfying $\rho_{\mathcal{A}}(x) = 0$ (namely, $\rho_{\mathcal{A}}(x_n) \to 0$), we say that $(x_n)_{n \in \mathbb{N}}$ converges to $\mathcal{A}$ and write $x_n \to_{\tau_{\mathcal{A}}} \mathcal{A}$.

The topology $\tau_{\mathcal{A}}$ has the following property, establishing the equivalence between convergence of x to $\mathcal{A}$ in the usual sense and convergence of the "tails" of x to $\mathcal{A}$ in $\tau_{\mathcal{A}}$.

Proposition 4: Let $x \in \mathcal{X}$ be such that $\operatorname{dom} x = [0, \infty)$. Then, $\lim_{t \to \infty} |x(t)|_{\mathcal{A}} = 0$ if and only if there exists a sequence $(t_n)_{n \in \mathbb{N}}$ in $\mathbb{R}_{\geq 0}$ such that the sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathcal{X}$ with terms $x_n(\cdot) := x(t_n + \cdot)$ converges to $\mathcal{A}$ in $\tau_{\mathcal{A}}$.

Proof: (*Only If*) If $|x(t)|_{\mathcal{A}} \to 0$, for every sequence $(\epsilon_n)_{n \in \mathbb{N}}$ with $\epsilon_n \in (0, c)$ satisfying $\epsilon_n \to 0$, there exists a sequence $(t_n)_{n \in \mathbb{N}}$ in $\mathbb{R}_{\geq 0}$ such that $|x(s)|_{\mathcal{A}} < \epsilon_n$ for all $s \geq t_n$. This implies $\rho_{\mathcal{A}}(x_n) = \sup_{t \geq 0} |x_n(t)|_{\mathcal{A}} = \sup_{t \geq t_n} |x(t)|_{\mathcal{A}} \to_n 0$. Namely, $x_n \to_{\tau_{\mathcal{A}}} \mathcal{A}$.

(*If*) Let $(t_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}_{\geq 0}$ such that $x_n \to_{\tau_{\mathcal{A}}} \mathcal{A}$. Then, for every $\epsilon \in (0, c)$, there exists $n^*(\epsilon) \in \mathbb{N}$ such that $\rho_{\mathcal{A}}(x_n) < \epsilon$ for all $n \geq n^*(\epsilon)$. This implies that $|x(s)|_{\mathcal{A}} < \epsilon$ for all $s \geq t_n$ and all $n \geq n^*(\epsilon)$. Thus, $\lim_{t \to \infty} |x(t)|_{\mathcal{A}} = 0$. ■

Let $\tau'_{\mathcal{A}}$ be the topology on X_0 generated by the family $\{X_0\} \cup \{Q_\delta : \delta > 0\}$, in which $Q_\delta := \{x_0 \in X_0 : |x_0|_{\mathcal{A}} < \delta\}$. Then, $(X_0, \mathcal{X}, \Psi)$ is $(\tau'_{\mathcal{A}}, \tau_{\mathcal{A}})$-stable at $\mathcal{A}$ in the sense of Definition 2 if and only if for each $\epsilon > 0$, there exists $\delta > 0$, such that $|x_0|_{\mathcal{A}} < \delta$ implies $\sup_{t \in \operatorname{dom} x} |x(t)|_{\mathcal{A}} < \epsilon$ for all $x \in \Psi(x_0)$. In turn, this coincides with the usual notion of Lyapunov stability of a closed forward invariant set $\mathcal{A}$ [47].

C. Incremental Input–Output Stability

Given a pseudometric space (S, ρ), for every $\epsilon > 0$ define the set $O_\epsilon := \{(a, b) \in S \times S : \rho(a, b) < \epsilon\}$. Then, the family $\beta_{\delta S} := \{O_\epsilon : \epsilon > 0\} \cup \{S \times S\}$ generates a topology $\tau_{\delta S}$ on $S \times S$ that we call the *incremental topology*. Moreover, $\beta_{\delta S}$ is a base for $(S \times S, \tau_{\delta S})$ [45, Prop. 1.2.1].

In the same setting of previous Section III-B, suppose now that instead of (9) we have

$$\forall (x, u, y), (x', u', y') \in \operatorname{graph} \Psi,$$
$$|y - y'| \leq \max\{\alpha(|x - x'|),\ \kappa(|u - u'|)\}. \tag{22}$$

Condition (22) is the equivalent of (9) for *incremental ISS* systems [48]. Define the system $\delta\Sigma := (\delta\mathcal{D}, \delta\mathcal{Y}, \delta\Psi)$, where

$\delta\mathcal{D} := \mathcal{D}\times\mathcal{D}$, $\delta\mathcal{Y} := \mathcal{Y}\times\mathcal{Y}$, and $\delta\Psi : \delta\mathcal{D} \rightrightarrows \delta\mathcal{Y}$, $(d_1, d_2) \mapsto \Psi(d_1)\times\Psi(d_2)$. Endow $\delta\mathcal{D}$ and $\delta\mathcal{Y}$ with the respective incremental topologies. Then, by the same arguments used in Section III-B, one can show that (22) implies that the system $\delta\Sigma$ is stable at the diagonal set $\delta D^\star := \{((x,u),(x',u')) \in \delta\mathcal{D} : |x-x'| = 0,\ |u-u'| = 0\}$.

Remark 5: $\{O_\epsilon : \epsilon > 0\}$ is a special case of a *diagonal uniformity* [49, Sec. 3.15], generalizing in topology the notion of *uniform continuity*. Indeed, in this sense, input–output stability (9) relates to continuity in the same way as its incremental version (22) relates to uniform continuity.

D. Weak Stability and Switching Controls

Many applications of control, such as PWM-based regulation of electric machines [50] or sliding-mode control [51], employ control signals that switch between quantized values (see also the example in Section VI-C). As the switching frequency increases, the controlled system's trajectories get closer to those one would obtain with a control law given by the average of the switched signal. This is a fundamental phenomenon that, in the same setting of Section III-B, can be characterized in terms of stability according to Definition 2, in which the input space is endowed with the *weak topology*.

Specifically, consider the same setting of Section III-B with $\mathbb{T}_\mathcal{Y} = \mathbb{T}_\mathcal{U} = [0,t]$ for some $t>0$, $X = Y = \mathbb{R}^n$, $n\in\mathbb{N}_{>0}$, $U = \mathbb{R}$,[2] $\mathcal{U} = \mathcal{L}^\infty[0,t]$ the space of essentially bounded functions $[0,t]\to U$ with the seminorm $|u| := \inf\{c\geq 0 : |u(s)| \leq c$ for almost all $s\in[0,t]\}$, and $\mathcal{Y}$ the space of absolutely continuous functions $[0,t]\to Y$ with the uniform norm. Let Ψ be the solution map of

$$\dot{y} = f(y) + g(y)u$$

where f and g are continuously differentiable. For simplicity, suppose that $\Psi(x,u)$ is single valued at each $(x,u)\in\operatorname{dom}\Psi$. Let $((x_i,u_i))_{i\in\mathcal{I}}$ be an equibounded net in $\operatorname{dom}\Psi$. If, for some $(x^*,u^*)\in\operatorname{dom}\Psi$, $x_i\to x^*$ and $u_i\to u^*$ weakly, then $\Psi(x_i,u_i)\to\Psi(x^*,u^*)$ uniformly (this can be deduced by the same arguments used in [52, Th. 1]). Hence, with $\tau_\mathcal{Y}$ the topology of uniform convergence on $\mathcal{Y}$, τ_X the Euclidean topology on X, $\tau_\mathcal{U}$ the weak topology on $\mathcal{U}$, and $\tau_\mathcal{D}$ the product topology on $\mathcal{D} := X\times\mathcal{U}$, the system $(\mathcal{D},\mathcal{Y},\Psi)$ is $(\tau_\mathcal{D},\tau_\mathcal{Y})$-stable at (x^*,u^*).

In particular, let $Q\subseteq U$ be a compact (possibly finite) set where "implementable" control inputs can take values (for instance, Q contains the upper and lower values of a PWM signal), and let $\operatorname{co}Q$ denote its convex hull. Let $u^*\in\operatorname{co}Q$ be a nominal, but possibly "not implementable" (i.e., $u^*\notin Q$), steady value for the control input on $[0,t]$ and, with $(T_i)_{i\in\mathbb{N}}$ a sequence of periods $T_i\in(0,t)$ satisfying $T_i\to 0$, let $(u_i)_{i\in\mathbb{N}}$ be a sequence of periodic piecewise-continuous control inputs of period T_i, with values in Q, and satisfying

$$\frac{1}{T_i}\int_0^{T_i} u_i(\tau)d\tau \to_i u^*. \tag{23}$$

[2]Generalizations to the case in which $U=\mathbb{R}^m$ for some $m>1$ are straightforward but not considered for simplicity.

Then, by denoting by u^* also the constant function equal to u^* at every $s\in[0,t]$, the following result holds.

Lemma 7: $u_i\to u^*$ weakly.

Lemma 7, proved in Appendix F, allows to conclude that we can substitute a given arbitrary reference control input u^*, constant on subsequent intervals of the form $[t_k,t_{k+1})$, with a second control input u with quantized values possibly very different from u pointwise (provided that u is periodic on each $[t_k,t_{k+1})$ with mean value equal to $u^*(t_k)$). The resulting effect is a deviation of the corresponding solution y from the one produced by u^* that can be made arbitrarily small on each compact interval by taking the period of u sufficiently small.

E. Uniform Asymptotic Gain

The asymptotic gain condition (10) does not say anything about the convergence rate of y. Typically, stronger conditions asking for *uniformity of convergence* may be useful, for instance, to establish uniform ISS. In the same setting of Section III-B and III-C, let (X,τ_X) be compact, and let $U^\star := \{u\in\mathcal{U} : |u| = 0\}$. By following [11], we say that the system $(X\times\mathcal{U},\mathcal{Y},\Psi)$ has the *uniform asymptotic gain property* if there exists a class-K function κ such that

$$\begin{aligned}&\forall\varepsilon>0,\ \exists T(\varepsilon)\in\mathbb{T}_\mathcal{Y},\\ &\quad\forall(x,u,y)\in\operatorname{graph}\Psi,\ \sup_{s\geq T(\varepsilon)}|y(s)|\leq\varepsilon+\kappa(|u|).\end{aligned} \tag{24}$$

Now, we construct an extended system Σ', obtained by "including time" as input in Σ, for which Condition (24) implies stability in the sense of Definition 2. Let $\mathcal{T} := \mathbb{T}_\mathcal{Y}\cup\{\infty\}$, in which ∞ satisfies $t<\infty$ for all $t\in\mathbb{T}_\mathcal{Y}$. Then, we define the family of intervals $\mathcal{I} := \{I_s\subseteq\mathcal{T} : s\in\mathcal{T}\setminus\{\infty\}\}$, in which $I_s := \{t\in\mathcal{T} : t\geq s\}$, and we let $\tau_\mathcal{T}$ be the topology generated by $\mathcal{I}$. In this topology, I_s is a neighborhood of ∞ for every $s\in\mathcal{T}\setminus\{\infty\}$. Next, for every $y\in\mathcal{Y}$, define $|y(\infty)| := \limsup|y|$. Then, we define the system $\Sigma' := (\mathcal{D}',\mathcal{Y}',\Psi')$, by letting $\mathcal{D}' := \mathcal{T}\times X\times\mathcal{U}$, $\mathcal{Y}' := \mathbb{R}_{\geq 0}$, $\operatorname{dom}\Psi' := \mathcal{T}\times\operatorname{dom}\Psi$, and $\Psi'(t,x,u) := \{|y(t)| : y\in\Psi(x,u)\}$. With $\tau_\mathcal{D} := \tau_\mathcal{T}\otimes\tau_X\otimes\tau_\mathcal{U}$ and $\tau_{\mathcal{Y}'}$ the standard Euclidean topology on $\mathbb{R}_{\geq 0}$, the following result holds.

Proposition 5: If Σ has the uniform asymptotic gain property (24), then Σ' is $(\tau_\mathcal{D},\tau_{\mathcal{Y}'})$-stable at $\{\infty\}\times X\times U^\star$.

Proof: If $(X\times U^\star)\cap\operatorname{dom}\Psi = \varnothing$, the result is true; otherwise, first notice that $\Psi'(\{\infty\}\times X\times U^\star) = \{0\}$. Indeed, every $s\in\Psi'(\{\infty\}\times X\times U^\star)$ satisfies $s = |y(\infty)| = \limsup|y|$ for some $y\in\Psi(X,U^\star)$, and (24) implies $\limsup|y| = 0$. Pick a $\tau_{\mathcal{Y}'}$-neighborhood W of 0. Then, for sufficiently small $\varepsilon>0$, $[0,2\varepsilon]\subseteq W$ and $\varepsilon\in\operatorname{ran}\kappa$. Let $V_\varepsilon := I_{T(\varepsilon)}\times X\times U_\varepsilon$, with T the same as in (24), and $U_\varepsilon := \{u\in\mathcal{U} : |u|<\kappa^{-1}(\varepsilon)\}$. Then, V_ε is a neighborhood of $\{\infty\}\times X\times U^\star$ and, in view of (24), $\Psi'(V_\varepsilon) = \{|y(t)| : t\geq T(\varepsilon),\ y\in\Psi(X,U_\varepsilon),\}\subseteq[0,2\varepsilon]\subseteq W$. ■

F. Practical Stability

In this section, we consider the case in which, as in [9], a further *bias* term appears in properties of the kind of global stability (9) and asymptotic gain (10). We specifically develop the asymptotic gain case as a simple example to introduce a methodology that can be extended to other similar contexts. In particular, in the same setting of Section III-C, suppose that, instead of (10), the following property holds:

$$\forall(x,u,y)\in \operatorname{graph}\Psi,\ \limsup|y|\le \rho(\limsup|u|)+b, \quad (25)$$

in which $b>0$ is a bias term. While the presence of b ruins stability when $\mathcal{Y}$ has the limsup topology, stability still applies if $\mathcal{Y}$ is given the topology $\tau^b_{\mathcal{Y}}$ generated by the family $\{\mathcal{Y}\}\cup\{O_\varepsilon : \varepsilon > b\}$ where, as in Section III-C, $O_\varepsilon := \{y\in\mathcal{Y} : \limsup|y|<\varepsilon\}$. Indeed, in view of (25), for every $S\subseteq X$, $D^\star := S\times\{u\in\mathcal{U} : \limsup|u|=0\}$ maps to $\Psi(D^\star)\subseteq\{y\in\mathcal{Y} : \limsup|y|\le b\}$, and every $\tau^b_{\mathcal{Y}}$-neighborhood Y of $\Psi(D^\star)$ contains a set of the form $O_{b+\epsilon}$ for all sufficiently small $\epsilon\in\operatorname{ran}\rho\cap\mathbb{R}_{>0}$. Then, $D := X\times\{u\in\mathcal{U} : \limsup|u|<\rho^{-1}(\epsilon)\}$ is a neighborhood of $D^\star$ and $\Psi(D)\subseteq Y$.

VI. Application Examples

A. RMS Rejection of Unknown Disturbances

Consider the system

$$\dot{x} = f(x) + w + u,$$

in which $x(t)\in\mathbb{R}$ is the state variable, $w:\mathbb{R}_{\ge 0}\to\mathbb{R}$ is a bounded unmeasured disturbance, $u(t)\in\mathbb{R}$ is a control input, and f is an uncertain Lipschitz function whose nominal value $f^\star$ has Lipschitz constant $\ell>0$.

Define the *asymptotic root mean square* of x as

$$|x|_{\mathrm{aRMS}} := \limsup_{t\to\infty}\sqrt{\frac{1}{t}\int_0^t x(s)^2 ds}.$$

Given an arbitrary but fixed $\varepsilon>0$, we consider the problem of designing a controller ensuring that $|x|_{\mathrm{aRMS}}\le\varepsilon$ at front of *every* bounded disturbance w unknown a priori and for all sufficiently small deviations of f from $f^\star$.

We assume to measure the state x of the plant subject to a small, bounded and smooth additive disturbance ν. In particular, we assume to measure $y := x+\nu$ with $\nu\in\mathcal{V}$, in which $\mathcal{V}$ denotes the set of bounded continuously differentiable functions $\mathbb{R}_{\ge 0}\to\mathbb{R}$ with bounded derivative.

We propose the following controller:

$$\begin{aligned}\dot{\eta} &= \alpha\max\{0, y^2-\varepsilon^2\}\eta, \qquad \eta(0)\neq 0,\\ u &= -ky-\alpha y\eta^2,\end{aligned}$$

in which $\alpha>0$ and $k\ge\ell+3$ are arbitrary (larger values of α lead to faster convergence). The closed-loop system can be written in terms of y as

$$\dot{y} = f(y-\nu) - ky - \alpha y\eta^2 + w + \dot{\nu} \quad (26a)$$

$$\dot{\eta} = \alpha\max\{0, y^2-\varepsilon^2\}\eta, \qquad \eta(0)\neq 0. \quad (26b)$$

Denote by $\mathrm{Lip}(\ell+1)$ the set of Lipschitz functions $\mathbb{R}\to\mathbb{R}$ with Lipschitz constant strictly less than $\ell+1$. Consider (26a) and notice that, since $k\ge\ell+3$, we have

$$\frac{d}{dt}\left(\frac{1}{2}y^2\right) \le -y^2 - \alpha\eta^2y^2 + \bar{w}^2/4 \quad (27a)$$

$$\le -y^2 + \bar{w}^2/4 \quad (27b)$$

for all $f\in\mathrm{Lip}(\ell+1)$, in which $\bar{w} := |w| + (\ell+1)|\nu| + |\dot{\nu}|$ is bounded. Hence, for all $f\in\mathrm{Lip}(\ell+1)$, η continuous, w continuous and bounded, and $\nu\in\mathcal{V}$, we have that y (hence, x) is bounded. Therefore, the corresponding solution of (26a) is defined for all $t\ge 0$ and bounded.

Take $\mathcal{F} := \mathrm{Lip}(\ell+1)$, $X_0 := \mathbb{R}$, $\mathcal{H}$ the set of absolutely continuous functions $\mathbb{R}_{\ge 0}\to\mathbb{R}$, $\mathcal{W}$ the set of bounded continuous functions $\mathbb{R}_{\ge 0}\to\mathbb{R}$, $\mathcal{D}_1 = \mathcal{H}\times X_0\times\mathcal{F}\times\mathcal{W}\times\mathcal{V}$, and $\mathcal{Y}_1$ the set of bounded continuous functions $\mathbb{R}_{\ge 0}\to\mathbb{R}$. We model (26a) as a system $\Sigma_1 = (\mathcal{D}_1,\mathcal{Y}_1,\Psi_1)$ in which, for each $(\eta,x_0,f,w,\nu)\in\mathcal{D}_1$, $\Psi_1(\eta,x_0,f,w,\nu)$ equals the maximal solution of (26a) satisfying $y(0)=x_0+\nu(0)$ (implying $x(0)=x_0$) and subject to the inputs w, ν, and η.

Similarly, with $H_0 := \mathbb{R}\setminus\{0\}$, we model (26b) as a system $\Sigma_2 = (\mathcal{D}_2,\mathcal{Y}_2,\Psi_2)$ in which $\mathcal{D}_2 = \mathcal{Y}_1\times H_0$, $\mathcal{Y}_2=\mathcal{H}$, and, for each $(y,\eta_0)\in\mathcal{D}_2$, $\Psi_2(y,\eta_0)$ is the solution of (26b) originating at η_0 and subject to y, i.e.,

$$t\mapsto \exp\left(\int_0^t \alpha\max\left\{0, y(s)^2-\varepsilon^2\right\}ds\right)\eta(0), \quad (28)$$

which is defined for all $t\ge 0$.

We consider the feedback interconnection of Σ_1 and Σ_2, resulting in the system $\Sigma = (\mathcal{D},\mathcal{Y},\Psi)$ defined as specified in Section II-B, with $\mathcal{D} = X_0\times\mathcal{F}\times\mathcal{W}\times\mathcal{V}\times H_0$ and $\mathcal{Y} = \mathcal{Y}_1\times\mathcal{Y}_2$. We give X_0, $\mathcal{F}$, H_0, $\mathcal{W}$, $\mathcal{V}$, and $\mathcal{Y}_2$ the trivial topology and $\mathcal{Y}_1$ the topology $\tau_{\mathcal{Y}_1}$ generated by the collection $\{\mathcal{Q}_\mu : \mu>\varepsilon\}$ with $\mathcal{Q}_\mu := \{y\in\mathcal{Y}_1 : |y|_{\mathrm{aRMS}}<\mu\}$. We then give $\mathcal{D}$ and $\mathcal{Y}$ the respective product topologies $\tau_{\mathcal{D}}$ and $\tau_{\mathcal{Y}}$.

Next, we prove that system Σ satisfies the small-gain property at $D^\star := X_0\times\{f^\star\}\times\{0\}\times\{0\}\times H_0\subseteq\mathcal{D}$ with respect to $(\tau_{\mathcal{D}},\tau_{\mathcal{Y}})$, where, we recall, $f^\star\in\mathcal{F}$ is the nominal value for f, and where we denoted by 0 the zero function $t\mapsto 0$. Notice that (27b) implies $\Psi(D^\star)\subseteq\overline{\mathcal{Q}}_0\times\mathcal{Y}_2$, in which $\overline{\mathcal{Q}}_0 := \{y\in\mathcal{Y}_1 : |y|_{\mathrm{aRMS}}=0\}$. Then, since $\mathcal{Y}_2$ has the trivial topology, it suffices to show that for every $\mu>\varepsilon$, there exists $D\in\mathcal{N}(D^\star)$, and for every $\bar{y}\in\mathcal{Y}_1\setminus\mathcal{Q}_\mu$, there exists $n_{\bar{y}}$, such that $\Gamma_{12}^{n_{\bar{y}}}(\bar{y},D)\subseteq\mathcal{Q}_\mu$. Pick $\mu>\varepsilon$ and let $D := \mathcal{D} = X_0\times\mathcal{F}\times\mathcal{W}\times\mathcal{V}\times H_0$, which is in $\mathcal{N}(D^\star)$. Pick $\bar{y}\in\mathcal{Y}_1\setminus\mathcal{Q}_\mu$ arbitrarily. Then, $|\bar{y}|_{\mathrm{aRMS}}\ge\mu>\varepsilon$, which implies that, given any $\bar{\mu}\in(\varepsilon,\mu)$, for all $T>0$, there exists $t\ge T$ such that $\int_0^t\bar{y}(s)^2ds\ge\bar{\mu}^2t$. In turn, this implies that for all $T>0$, there exists $t\ge T$, such that $\int_0^t\max\{0,\bar{y}(s)^2-\varepsilon^2\}ds\ge\int_0^t(\bar{y}(s)^2-\varepsilon^2)ds\ge(\bar{\mu}^2-\varepsilon^2)t$. In view of (28), this implies that every $\eta\in\Psi_2(\bar{y},H_0)$ satisfies $|\eta(t)|\to\infty$. Therefore, in view of (27a), and since for each $(w,\nu)\in\mathcal{W}\times\mathcal{V}$, $\bar{w}$ is bounded, we obtain $y(t)\to 0$ and $|y|_{\mathrm{aRMS}}=0<\mu$ for all $y\in\Psi_1(\Psi_2(\bar{y},H_0),X_0,\mathcal{F},\mathcal{W},\mathcal{V}) = \Gamma_{12}(\bar{y},D)$. This proves that $\Gamma_{12}^{n_{\bar{y}}}(\bar{y},D)\subseteq\mathcal{Q}_\mu$ for $n_{\bar{y}}=1$.

As D does not depend on μ, the previous analysis and Theorem 1 imply that, for every $(x(0),\eta(0))\in X_0\times H_0$, the solutions of (26) obtained with $f\in\mathrm{Lip}(\ell+1)$ and subject to $(w,\nu)\in\mathcal{W}\times\mathcal{V}$ satisfy $|y|_{\mathrm{aRMS}}\le\varepsilon$. Finally, since

$$|y-\nu|_{\mathrm{aRMS}} \le \left(1+\frac{\epsilon}{2\varepsilon}\right)|y|_{\mathrm{aRMS}} + \left(1+\frac{2\varepsilon}{\epsilon}\right)|\nu|_{\mathrm{aRMS}}$$

for all $\epsilon>0$, we conclude that, for every $\epsilon>0$, there exists $\delta(\epsilon) := \frac{\epsilon^2}{2\epsilon+4\varepsilon}>0$, such that for every pair $(x(0),\eta(0))\in X_0\times H_0$ of initial conditions, the solutions of (26) obtained with $f\in\mathrm{Lip}(\ell+1)$ and subject to $(w,\nu)\in\mathcal{W}\times\mathcal{V}$ with $|\nu|_{\mathrm{aRMS}}\le\delta(\epsilon)$

satisfy $|x|_{\text{aRMS}} = |y - \nu|_{\text{aRMS}} \le \varepsilon + \epsilon$. This implies that the property $|x|_{\text{aRMS}} \le \varepsilon$ is obtained robustly with respect to the measurement uncertainty ν. Namely, if $\nu = 0$, then $|x|_{\text{aRMS}} \le \varepsilon$; otherwise, the above-claimed continuity property between $|\nu|_{\text{aRMS}}$ and $|x|_{\text{aRMS}}$ holds.

B. Robust Global Attractiveness Without ISS

Consider the forced *Susceptible-Infected* system

$$\dot{S} = -\beta SI, \quad \dot{I} = \beta SI - \gamma I + v, \tag{29}$$

with $\beta > \gamma > 0$, $S(0), I(0) \ge 0$, and $v \in \mathcal{V}$, where $\mathcal{V}$ is the set of bounded continuous functions $\mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$. It is well known that, when $v = 0$, the set $\mathcal{A} := \{(S, I) \in \mathbb{R}^2_{\ge 0} : I = 0\}$ is globally attractive. Yet $\mathcal{A}$ is not Lyapunov stable. Moreover, the I subsystem is not ISS with respect to the input (S, v), nor the S subsystem is ISS with respect to I. Hence, attractiveness of $\mathcal{A}$ cannot be concluded by means of canonical small-gain arguments for ISS systems. Instead, as detailed in the rest of this section, it can be proved by using Theorem 1. In particular, we prove following stronger "robust attractiveness" property:

$$\begin{aligned} &\forall \varepsilon > 0,\ \exists \bar{\nu} > 0,\ \forall (S(0), I(0), v) \in \mathbb{R}^2_{\ge 0} \times \mathcal{V}, \\ &\qquad \limsup |v| < \bar{\nu} \Rightarrow \limsup |(S, I)|_{\mathcal{A}} < \varepsilon \end{aligned} \tag{30}$$

where $|(S, I)|_{\mathcal{A}}$ denotes the distance of (S, I) to $\mathcal{A}$.

Let $\mathcal{Y}_1$ be the set of bounded nonincreasing continuous functions $\mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$, and $\mathcal{Y}_2$ the set of continuous functions $I : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ that either are Lipschitz or satisfy $I(t) \to \infty$. Define the system $\Sigma_1 := (\mathcal{D}_1, \mathcal{Y}_1, \Psi_1)$, in which $\mathcal{D}_1 = \mathcal{Y}_2 \times \mathbb{R}_{\ge 0}$ and Ψ_1 is the solution map of $\dot{S} = -\beta SI$ mapping pairs $(I, S(0)) \in \mathcal{D}_1$ to complete solutions $S \in \mathcal{Y}_1$. Define the system $\Sigma_2 := (\mathcal{D}_2, \mathcal{Y}_2, \Psi_2)$, in which $\mathcal{D}_2 = \mathcal{Y}_1 \times \mathbb{R}_{\ge 0} \times \mathcal{V}$ and Ψ_2 the solution map of $\dot{I} = \beta SI - \gamma I + v$ mapping triples $(S, I(0), v) \in \mathcal{D}_2$ to complete solutions $I \in \mathcal{Y}_2$. Then, system (29) can be seen as a feedback interconnection $\Sigma = (\mathcal{D}, \mathcal{Y}, \Psi)$ of the previous two systems (see Section II-B), with $\mathcal{D} = \mathbb{R}^2_{\ge 0} \times \mathcal{V}$, $\mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2$, and Ψ mapping triples $(S(0), I(0), v)$ to complete solutions (S, I) of (29). We endow $\mathbb{R}^2_{\ge 0}$ and $\mathcal{Y}_1$ with the respective trivial topologies, and we give $\mathcal{Y}_2$ and $\mathcal{V}$ the respective limsup topologies (see Section III-C).

Let $D^\star := \{(S_0, I_0, v) \in \mathcal{D} : I_0 = 0, v = 0\}$. Then, every $(S, I) \in \Psi(D^\star)$ satisfies $(S(t), I(t)) \in \mathcal{A}$ for all $t \in \mathbb{R}_{\ge 0}$. Moreover, Σ satisfies the small-gain property of Definition 3 at $D^\star$. Since $\mathcal{Y}_1$ and $\mathbb{R}^2_{\ge 0}$ have the trivial topology, to show this it suffices to show that for every $\varepsilon > 0$, there exists $\bar{\nu} = \bar{\nu}(\varepsilon) > 0$, and for every $\bar{I} \in \mathcal{Y}_2 \setminus O_\varepsilon$ (as in Section III-C, $O_\varepsilon := \{I \in \mathcal{Y}_2 : \limsup |I| < \varepsilon\}$), there exists n_I, such that

$$I = \Gamma_{21}^{n_I}(\bar{I}, (S_0, I_0, v)) \Rightarrow \limsup |I| < \varepsilon \tag{31}$$

for all $S_0, I_0 \in \mathbb{R}_{\ge 0}$ and $v \in \mathcal{V}$ satisfying $\limsup |v| < \bar{\nu}$. Pick $\varepsilon > 0$, let $\bar{\nu} := \varepsilon\gamma$, and pick $\bar{I} \in \mathcal{Y}_2 \setminus O_\varepsilon$. Since $\bar{I} \notin O_\varepsilon$, then $\limsup |\bar{I}| \ge \varepsilon > 0$. Since $\bar{I}$ is continuous, $\bar{I}(t) \ge 0$ for all t, and either $\bar{I}(t) \to \infty$ or it is Lipschitz, we conclude that $\bar{S}(t) \to 0$ for all $\bar{S} \in \Psi_1(\bar{I}, \mathbb{R}_{\ge 0})$. From (29), we then obtain that, for all such $\bar{S}$, we have $\limsup |I| \le \limsup |v|/\gamma < \varepsilon$ for every $I \in \Psi_2(\bar{S}, \mathbb{R}_{\ge 0}, v)$ with $v \in \mathcal{V}$ satisfying $\limsup |v| < \bar{\nu}$. Hence, (31) holds with $n_I = 1$.

Therefore, by using Theorem 1, we finally conclude that Σ is stable at $D^\star$. As the set $\mathbb{R}^2_{\ge 0}$ of initial conditions has the trivial topology, and $|(S, I)|_{\mathcal{A}} = |I|$, this implies (30).

C. Automatic Frequency Regulation in PWM Control

Consider an electrical motor described by the linear system

$$\dot{x} = Ax + Bu, \qquad\qquad y = Cx, \tag{32}$$

with $x(t) \in \mathbb{R}^n$, $n \in \mathbb{N}$, $u(t), y(t) \in \mathbb{R}$, and A Hurwitz. The output y represents the rotor's angular velocity, and u is the input voltage. We consider a control system that can only generate quantized switching voltages taking the value V or $-V$ at a given time instant, with $V > 0$. This is typical of controllers implemented by power converters [50]. Given an ideal input profile $u^\star : \mathbb{R}_{\ge 0} \to [-V, V]$ (which can take any value in-between $-V$ and V), the controller approximates $u^\star$ by means of a function $\hat{u}_T : \mathbb{R}_{\ge 0} \to \{-V, V\}$ satisfying

$$\frac{1}{T_k} \int_{t_k}^{t_{k+1}} \hat{u}_T(s) ds = u^\star(t_k), \quad \forall k \in \mathbb{N}, \tag{33}$$

where $T_k := t_{k+1} - t_k \in [T_{\min}, \infty)$, with $T_{\min} > 0$. The sequence $T = (T_k)_{k \in \mathbb{N}}$ represents a time-varying switching period, and it is a degree of freedom. Moreover, due to the presence of unavoidable uncertainty in the controller implementation, we suppose that the actual generated control is $u_T := \hat{u}_T + \nu$, with $\nu : \mathbb{R}_{\ge 0} \to \mathbb{R}$ a bounded additive perturbation. The controller measures the error $e_T := y^\star - y_T$ between the ideal output $y^\star$ that would be produced by (32) with $u^\star$ and $x(0) = 0$, and the actual output y_T produced by u_T for some $x(0) \in \mathbb{R}^n$. The control goal is to tune the sequence T_k online to eventually reduce the error below a prespecified threshold $\varepsilon^\star > 0$.[3]

The error e_T is bounded, and by means of arguments similar to those used in proving Lemma 7, it can be shown that, when $\nu = 0$, for every $\varepsilon > 0$, there exists $\bar{\sigma} = \bar{\sigma}(\varepsilon) > 0$ such that, for every y_T obtained with a sequence T satisfying $T_k \le \bar{\sigma}(\varepsilon)$ for all but at most finite k, $\limsup |e_T| \le \varepsilon$. We assume that $T_{\min} < \bar{\sigma}(\varepsilon^\star)$, so as the set of sequences T for which $\limsup |e_T| \le \varepsilon^\star$ when $\nu = 0$ is nonempty.

We focus on decision strategies (continuous- or discrete-time) adapting T_k iteratively in such a way that

P1. $T_k \le T_h$, for $k \ge h$;

P2. there exist $\alpha \in \mathbb{N}$ and $\omega : [T_{\min}, \infty) \to [T_{\min}, \infty)$ satisfying $\lim_{m \to \infty} \omega^m(s) = T_{\min}$ for all $s \in [T_{\min}, \infty)$ such that, if $|e_T(t)| > \varepsilon^\star$ for some $t \in [t_{k-1}, t_{k+\alpha}]$, then $T_{k+\alpha} = t_{k+\alpha+1} - t_{k+\alpha} \le \omega(T_{k-1})$.

Let $\mathcal{T}$ denote the set of bounded sequences $(T_k)_{k \in \mathbb{N}}$ in $[T_{\min}, \infty)$, $\mathcal{E}$ denote the space of bounded continuous functions $e : \mathbb{R}_{\ge 0} \to \mathbb{R}$, and $\mathcal{V} := \mathcal{E}$. We can model the closed-loop system as in Fig. 2 in terms of the feedback interconnection $\Sigma = (\mathcal{D}, \mathcal{Y}, \Psi)$ between $\Sigma_1 = (\mathcal{D}_1, \mathcal{Y}_1, \Psi_1)$ ($\mathcal{D}_1 = \mathcal{E} \times [T_{\min}, \infty)$ and $\mathcal{Y}_1 = \mathcal{T}$), mapping error signals $e_T \in \mathcal{E}$ and

[3]Clearly, $T_k = T_{\min}$ for each k would be the best choice in terms of asymptotic bound. Nevertheless, higher switching frequencies are associated with higher power consumption and possibly with more significant drawbacks due to switching, such as unwanted vibrations and flickering phenomena. Hence, it makes sense to seek larger switching periods guaranteeing the desired bound on the error.

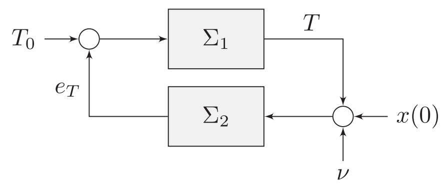

Fig. 2. Closed-loop system in the example of Section VI-C.

initial conditions $T_0 \in [T_{\min}, \infty)$ to sequences $T \in \mathcal{T}$, and $\Sigma_2 = (\mathcal{D}_2, \mathcal{Y}_2, \Psi_2)$ ($\mathcal{D}_2 = \mathcal{T} \times \mathbb{R}^n \times \mathcal{V}$ and $\mathcal{Y}_2 = \mathcal{E}$), mapping sequences $T \in \mathcal{T}$, initial conditions $x(0) \in \mathbb{R}^n$, and perturbations $\nu \in \mathcal{V}$ to error signals $e_T \in \mathcal{E}$. According to Section II-B, $\mathcal{D} = [T_{\min}, \infty) \times \mathbb{R}^n \times \mathcal{V}$, $\mathcal{Y} = \mathcal{T} \times \mathcal{E}$, and Ψ is given by (3).

We give $[T_{\min}, \infty)$, $\mathbb{R}^n$, and $\mathcal{T}$ the trivial topology, $\mathcal{E}$ the topology $\tau_{\mathcal{E}}$ generated by the collection $\{E_\mu : \mu > \varepsilon^\star\}$, where $E_\mu := \{e \in \mathcal{E} : \limsup |e| < \mu\}$ (cf., Section V-F), and $\mathcal{V}$ the uniform-norm topology. We let $D^\star := \{(T_0, x(0), \nu) \in \mathcal{D} : T_0 = T_{\min},\ x(0) = 0,\ \nu = 0\}$. As $T_{\min} < \bar{\sigma}(\varepsilon^\star)$, as assumed before, in view of **P1** we have $\limsup |e| \le \varepsilon^\star$ for all $(T, e) \in \Psi(D^\star)$. Moreover, since $\mathcal{T}$, $[T_{\min}, \infty)$, and $\mathbb{R}^n$ have the trivial topology, to show that the interconnection satisfies the small-gain property of Definition 3 at $D^\star$, it suffices to show that for every $\mu > \varepsilon^\star$ there exists $\bar{\nu} = \bar{\nu}(\varepsilon) > 0$, and for every $\bar{e} \in \mathcal{E} \setminus E_\mu$, there exists $m_e \in \mathbb{N}$, such that, for every $(T_0, x(0), \nu) \in \mathcal{D}$ with $|\nu|_\infty < \bar{\nu}$,

$$e = \Gamma_{21}^{m_e}(\bar{e}, (T_0, x(0), \nu)) \Rightarrow \limsup |e| < \mu. \tag{34}$$

Pick arbitrarily $\mu > 0$ and $\bar{e} \in \mathcal{E} \setminus E_\mu$ and notice that, by linearity, for all $(T_0, x(0), \nu) \in \mathcal{D}$, $e := \Gamma_{21}(\bar{e}, (T_0, x(0), \nu)) = \Psi_2(\Psi_1(\bar{e}, T_0), x(0), \nu)$ can be written as $e = e_0 + e_\nu$ with $e_0 = \Psi_2(\Psi_1(\bar{e}, T_0), 0, 0)$ and $\limsup |e_\nu| \le c|\nu|_\infty$ for some $c > 0$ depending only on A, B, and C. Take $\bar{\nu} < (\mu - \varepsilon^\star)/c$, and pick $(T_0, x(0), \nu) \in \mathcal{D}$ such that $|\nu|_\infty < \bar{\nu}$. This implies $\limsup |e_\nu| < \mu - \varepsilon^\star$. As $\bar{e} \notin E_\mu$, $\limsup |\bar{e}| \ge \mu > \varepsilon^\star$. Hence, for each $h, \alpha \in \mathbb{N}$ there exists $\ell > h$ such that $\sup |\bar{e}|_{[t_{\ell-1}, t_{\ell+\alpha}]} > \varepsilon^\star$. Therefore, it follows from **P1** and **P2** that $T = \Psi_1(\bar{e}, T_0)$ satisfies $T_k < \bar{\sigma}(\varepsilon^\star)$ for all large enough k. As a consequence, we conclude that necessarily $\limsup |e_0| \le \varepsilon^\star$, which implies $\limsup |e| < \mu$. Hence, the small-gain property (34) holds with $m_e = 1$.

Therefore, in view of Theorem 1, we claim that Σ is stable at $D^\star$. As $[T_{\min}, \infty)$ and $\mathbb{R}^n$ have the trivial topology, this means that, regardless of the actual value of the initial conditions $x(0)$ and T_0, the controller produces an error whose limsup is larger than $\varepsilon^\star$ by an amount that continuously increases with the size of the disturbance ν (in particular, if $\nu = 0$, $\limsup |e| \le \varepsilon^\star$).

We underline that we made no strong stability assumption, such as ISS, to achieve the above-mentioned result. Moreover, how to characterize Σ_1 in terms of ISS is also unclear. Indeed, a map Ψ_1 satisfying **P1** and **P2** may not be continuous in general. For example, take Ψ_1 as the function mapping every $(e, T_0) \in \mathcal{D}_1$ to a sequence T with initial condition T_0 and satisfying

$$T_k = \begin{cases} T_{k-1}/2 & \text{if } \sup_{t \in [t_{k-1}, t_k]} |e(t)| > \varepsilon^\star \\ T_{k-1} & \text{otherwise.} \end{cases} \tag{35}$$

Clearly, the sequence generated in this way fulfills **P1** and **P2**. Consider the metric $|T - T'| := \sup_{k \in \mathbb{N}} |T_k - T'_k|$ on $\mathcal{T}$, and the one induced by the uniform norm on $\mathcal{E}$. Pick $T_0 := T_{\min} + 2$ and let $e \in \mathcal{E}$ be the constant function $e(t) := \varepsilon^\star$. Then, in view of (35), $T = \Psi_1(T_0, e)$ satisfies $T_k = T_0 = T_{\min} + 2$ for all $k \in \mathbb{N}$. Now, pick $e_\epsilon(t) := \varepsilon^\star + \epsilon$ for some $\epsilon > 0$. Then, $e_\epsilon \to e$ uniformly as $\epsilon \to 0$. However, in view of (35), $T^\epsilon := \Psi_1(T_0, e_\epsilon)$ satisfies $T_k^\epsilon \le T_{\min} + 1$ for sufficiently large k. Hence, $|\Psi_1(T_0, e) - \Psi_1(T_0, e_\epsilon)| \ge 1$ for all $\epsilon > 0$, which implies $\Psi_1(T_0, e_\epsilon) \not\to \Psi_1(T_0, e)$ as $\epsilon \to 0$. Thus, Ψ is not continuous.

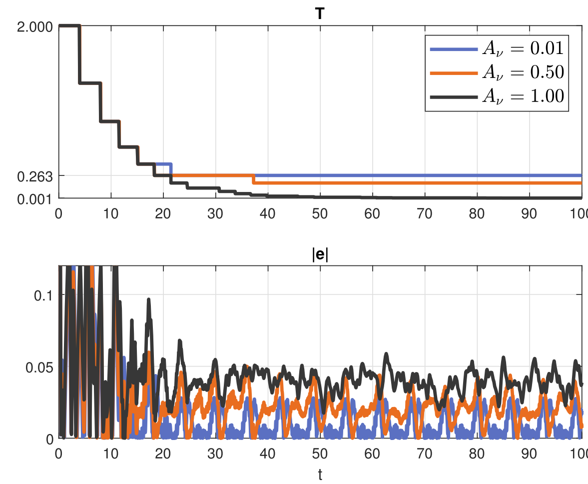

Fig. 3. Simulation of Policy **I1**–**I2**.

Fig. 3 shows a simulation obtained with the following decision policy for T (m, ρ, and σ are auxiliary variables):

I1 start with $m(t_0) = p(t_0) = 0$ and $\sigma(t_0) = T_0$;

I2 for every $k \in \mathbb{N}$:

- Define $t_{k+1} := t_k + \sigma(t_k)$.
- Integrate the following equations over $[t_k, t_{k+1}]$:

$$\dot{m} = 0, \quad \dot{\sigma} = 0, \quad \dot{\rho} = \max\left\{0,\ e_T^2 - \varepsilon^{\star 2}\right\}.$$

- If $t_{k+1} - m(t_{k+1}) \ge 3$, update the variables as

$$\sigma(t_{k+1}) \leftarrow \begin{cases} \max\left\{T_{\min}, \frac{2}{3}\sigma(t_{k+1})\right\} & \text{if } \rho(t_{k+1}) > 0 \\ \sigma(t_{k+1}) & \text{otherwise} \end{cases}$$

$$\rho(t_{k+1}) \leftarrow 0, \quad m(t_{k+1}) \leftarrow t_{k+1}.$$

The kth term T_k of the sequence T generated in this way is given by $T_k = t_{k+1} - t_k = \sigma(t_k)$. The error signal e_T is obtained as $e_T = C\hat{x} - y$ with $\dot{\hat{x}} = A\hat{x} + Bu^\star$ and $\hat{x}(0) = 0$.

Specifically, Fig. 3 shows three simulations obtained with $\hat{u}_T$ defined as a PWM signal with variable period (according to T) and, within each period, a duty cycle chosen in such a way that (33) holds, $T_{\min} = 0.001$, $\varepsilon^\star = 0.05$, $V = 2$, and ν given by the interpolation of a pseudorandom signal uniformly sampled from $[0, A_\nu]$, where A_ν equals, respectively, 0.01, 0.5, and 1 in the three simulations. As shown in the figure, the asymptotic amplitude of the error gradually increases with $|\nu|_\infty$. This is consistent with our results that claim continuity of such increase. The sequence T, instead suffers from an evident discontinuity (in the previously defined uniform norm) when the amplitude of ν provokes spikes of $e(t)$ above $\varepsilon^\star$. In such case, indeed, T_k decreases to $T_{\min}$ despite the actual value of ν, whereas in the other two cases with $A_\nu = 0.01$ and $A_\nu = 0.5$, only a small variation of T_k is observed. This behavior is caused by the fact that, as for (35), also in this case Ψ_1 is discontinuous. Indeed, while discontinuity of Ψ_1 does not invalidate our results, it nevertheless implies that we have no guarantee that "small" changes of ν reflect on "small" changes of T (with respect to the previously defined metric).

VII. Conclusions

In this article, we proposed a small-gain theory for interconnections of abstract systems described by set-valued maps between topological spaces. For systems of this kind, stability is defined as a continuity property generalizing and unifying the continuity conditions underlying commonly used stability notions, including Lyapunov stability of motions or sets, (incremental) input–output stability, and asymptotic gain properties. Given a feedback interconnection of two subsystems, the main result of the paper (Theorem 1) establishes the following implication:

$$\textit{small-gain property} \quad \Rightarrow \quad \begin{array}{l}\textit{stability of the}\\ \textit{feedback interconnection}\end{array}$$

where the *"small-gain property"* is formally defined in Definition 3 and represents an abstraction, in the context of topological spaces, of the joint condition *"stability of the subsystems + small-gain condition"* of ordinary small-gain theories for input–output operators or ISS systems. While the proposed small-gain property does not admit, in general, a similar decomposition, we proved in Section III that it is always implied by ISS.

The main contribution of the article is methodological, as the presented results provide a common framework for small-gain theories and extend the "small-gain principle" beyond interconnections of systems defined between metric spaces of trajectories. Yet, the application of Theorem 1 to practical problems might not be straightforward. The examples of Section VI do suggest that the developed small-gain theory can provide a useful tool to study complex interconnections uncovered by other existing paradigms. However, its application does require the definition of suitable topological spaces and a preliminary analysis that are problem-specific and may not be easy. In this respect, further research is required.

Moreover, the developed theory focuses on continuity at a set or point, which is a local property. How to cover properties, such as uniform convergence or Lagrange stability, within the same setting of this article is still an open problem deserving further research.

Appendix A

Proof of Lemma 2: Let $(i,j) \in \{(1,2),(2,1)\}$. If $d \notin \operatorname{dom}\Psi$, then $\Psi(d) = \varnothing$ and $\Upsilon_i(d) = \varnothing$. Hence, the claim of the lemma is vacuously true. Pick $d = (d_1, d_2) \in \operatorname{dom}\Psi$ and $y_i \in \Upsilon_i(d)$. Then, there exists $y_j \in \mathcal{Y}_j$ such that $y_i \in \Psi_i(y_j, d_i)$ and $y_j \in \Psi_j(y_i, d_j)$. This, in turn, implies that $y_j \in \Upsilon_j(d)$. Hence, $y_i \in \Psi_i(\Upsilon_j(d), d_i)$. Since y_i was arbitrary, we conclude that $\Upsilon_i(d) \subseteq \Psi_i(\Upsilon_j(d), d_i)$.

Let $S_i(d) := \{y_i \in \mathcal{Y}_i \,:\, y_i \in \Psi_i(\Psi_j(y_i, d_j), d_i)\}$. To prove (8), we have to show that $\Upsilon_i(d) = S_i(d)$. Pick $y_i \in \Upsilon_i(d)$. Then, there exists $y_j \in \mathcal{Y}_j$ such that $y_i \in \Psi_i(y_j, d_i)$ and $y_j \in \Psi_j(y_i, d_j)$. This, in turn, implies that $y_i \in \Psi_i(\Psi_j(y_i, d_j), d_i)$. Thus, $y_i \in S_i(d)$. Hence, we conclude that $\Upsilon_i(d) \subseteq S_i(d)$. Conversely, pick $y_i \in S_i(d)$. Then, there exists $y_j \in \Psi_j(y_i, d_j)$ such that $y_i \in \Psi_i(y_j, d_i)$. By the definition (3), this implies that $(y_1, y_2) \in \Psi(d)$. Hence, $y_i \in \Upsilon_i(d)$, which proves $S_i(d) \subseteq \Upsilon_i(d)$. ■

Appendix B

Proof of Proposition 1: Pick $Y \in \mathcal{N}(\Psi_p(d))$. Since $\Psi_p(d) = \Psi_1(d) \times \Psi_2(d)$, by Lemma 3 we can find $O_1 \in \mathcal{N}(\Psi_1(d))$ and $O_2 \in \mathcal{N}(\Psi_2(d))$ such that $O_1 \times O_2 \subseteq Y$. If Σ_1 and Σ_2 are stable at d, we can find open sets $U_1, U_2 \in \mathcal{N}(d)$ such that $\Psi_1(U_1) \subseteq O_1$ and $\Psi_2(U_2) \subseteq O_2$. Then, $U := U_1 \cap U_2 \in \mathcal{N}(d)$ and $\Psi_p(U) = \cup_{d \in U}(\Psi_1(d) \times \Psi_2(d)) \subseteq \Psi_1(U_1) \times \Psi_2(U_2) \subseteq O_1 \times O_2 \subseteq Y$.

For the second claim, pick $Y \in \mathcal{N}(\Psi_s(D))$. As $\Psi_s(D) = \Psi_2(\Psi_1(D))$, and Σ_2 is stable at $\Psi_1(D)$, there exists $U_2 \in \mathcal{N}(\Psi_1(D))$ such that $\Psi_2(U_2) \subseteq Y$. As Σ_1 is stable at D, there exists $U_1 \in \mathcal{N}(D)$ such that $\Psi_1(U_1) \subseteq U_2$. Thus, $\Psi_s(U_1) \subseteq Y$. The claim then follows from the arbitrariness of Y. ■

Appendix C

Proof of Lemma 3: Pick $A \subseteq \mathcal{A}$ and $B \subseteq \mathcal{B}$. If $A = \varnothing$ or $B = \varnothing$, the first claim is vacuously true, since $A \times B = \varnothing$. Thus, we assume $A \neq \varnothing$ and $B \neq \varnothing$. As sets of the form $U \times V$, where U and V are open in $\mathcal{A}$ and $\mathcal{B}$, respectively, form a base for the product topology of $\mathcal{A} \times \mathcal{B}$, for every point $(a,b) \in \mathcal{A} \times \mathcal{B}$, every $Y \in \mathcal{N}((a,b))$ contains a set of the form $U_a \times V_b$, with $U_a \in \mathcal{N}(a)$ and $V_b \in \mathcal{N}(b)$. Then, every $Y \in \mathcal{N}(A \times B)$ contains a set of the form

$$O := \bigcup_{(a,b) \in A \times B} (U_a \times V_b)\,, \quad U_a \in \mathcal{N}(a),\ V_b \in \mathcal{N}(b),$$

which belongs itself to $\mathcal{N}(A \times B)$. Hence, the first claim follows by noticing that $O = O_A \times O_B$, where $O_A := \bigcup_{a \in A} U_a$ and $O_B = \bigcup_{b \in B} V_b$. Indeed, $(x,y) \in O$ only if there exists $(a,b) \in A \times B$ such that $(x,y) \in U_a \times V_b$, which implies $x \in \cup_{a \in A} U_a$ and $y \in \cup_{b \in B} V_b$. Hence, $(x,y) \in O_A \times O_B$. The converse is shown by a similar argument.

The "if" part of the second claim is obvious, since $U \in \mathcal{N}(A)$ and $V \in \mathcal{N}(B)$ imply $U \times V \in \mathcal{N}(A \times B)$. The "only if" part, instead, directly follows from the first claim of the lemma proved above. Indeed, if $Y \in \mathcal{N}(\Psi(A \times B))$ and $O \in \mathcal{N}(A \times B)$ is such that $\Psi(O) \subseteq Y$, we can find $O_A \in \mathcal{N}(A)$ and $O_B \in \mathcal{N}(B)$ such that $O_A \times O_B \in \mathcal{N}(A \times B)$ and $O_A \times O_B \subseteq O$, so that $\Psi(O_A \times O_B) \subseteq \Psi(O) \subseteq Y$. ■

Appendix D

Proof of Proposition 2: Conditions (16a) and (17) imply $\Psi(O_{\mathcal{D}}) \subseteq O_{\mathcal{Y}_1} \times O_{\mathcal{Y}_2}$. Hence, for every $Y \in \mathcal{N}(\Psi(O_{\mathcal{D}}))$, there exists $\varepsilon > 0$, sufficiently small so that $\varepsilon \in \operatorname{ran}\alpha_i \cap \operatorname{ran}\kappa_i \cap \operatorname{ran}\varrho_i \circ \alpha_j \cap \operatorname{ran}\varrho_i \circ \kappa_j$ for all $(i,j) = (1,2),(2,1)$, such that $|(y_1,y_2)| < \varepsilon$ implies $(y_1,y_2) \in Y$. With $\delta(\cdot) := \min_{(i,j)=(1,2),(2,1)}\{\alpha_i^{-1}(\cdot),\ \kappa_i^{-1}(\cdot),\ (\varrho_i \circ \alpha_j)^{-1}(\cdot),\ (\varrho_i \circ \kappa_j)^{-1}(\cdot)\}$, let

$$D = \{(x_1,u_1,x_2,u_2) \in \mathcal{D} \,:\, |(x_1,u_1,x_2,u_2)| < \delta(\varepsilon/2)\}\,.$$

Then $D \in \mathcal{N}(O_{\mathcal{D}})$. Next, pick $(y_1, y_2) \in \mathcal{Y}$ arbitrarily, and notice that (16a) implies that every $y_i' \in \Gamma_{ij}(y_i, D)$ satisfies $|y_i'| \le \max\{\varrho_{ij}(|y_i|), \varepsilon/2\}$, with ϱ_{ij} defined in (17). Since (17) implies $\varrho_{ij}(\varepsilon/2) < \varepsilon/2$, by induction one obtains

$$\forall n \ge 1, \quad \forall y_i' \in \Gamma_{ij}^n(y_1, D), \quad |y_i'| \le \max\{\varrho_{ij}^n(|y_i|), \varepsilon/2\}.$$

In view of (17), there exists n_y such that $\varrho_{ij}^{n_y}(|y_i|) < \varepsilon/2$ for both $i = 1, 2$. Hence, we conclude that, for all $y_1' \in \Gamma_{12}^{n_y}(y_1, D)$ and all $y_2' \in \Gamma_{21}^{n_y}(y_2, D)$, $|(y_1', y_2')| \le \varepsilon/2 < \varepsilon$, i.e., $(y_1', y_2') \in Y$. By arbitrariness of Y, we then conclude that the small-gain condition of Definition 3 holds at $O_{\mathcal{D}}$ with respect to $(\tau_{\mathcal{D}}, \tau_{\mathcal{Y}})$. ■

Appendix E

Proof of Proposition 3: The proof follows the same arguments used to prove Proposition 2. In particular, (16b) and (17) imply that $\Psi(L_{\mathcal{D}}) \subseteq L_{\mathcal{Y}_1} \times L_{\mathcal{Y}_2}$. Hence, by definition of the limsup topology, every neighborhood $Y \in \mathcal{N}(\Psi(L_{\mathcal{D}}))$ contains a set of the form $\{(y_1, y_2) \in \mathcal{Y} : \limsup |y_i| < \varepsilon, \forall i = 1, 2\}$ for $\varepsilon > 0$ sufficiently small so that $\varepsilon \in \operatorname{ran} \kappa_i \cap \operatorname{ran} \varrho_i \circ \kappa_j$ for all $(i, j) = (1, 2), (2, 1)$. Then, with $\delta(\cdot) := \min_{(i,j)=(1,2),(2,1)}\{\kappa_i^{-1}(\cdot), (\varrho_i \circ \kappa_j)^{-1}(\cdot)\}$, the set

$$D := \Big\{(x_1, u_1, x_2, u_2) \in \mathcal{D} : \limsup |u_i| < \delta(\varepsilon/2),\ i = 1, 2\Big\}$$

is in $\mathcal{N}(L_{\mathcal{D}})$, and, in view of (16b) and (17), for every $i = 1, 2$ and every $y_i' \in \Gamma_{ij}(y_i, D)$, with $y_i \in \mathcal{Y}_i$ arbitrary, we have

$$\limsup |y_i'| < \max\{\varrho_{ij}(|y_i|), \varepsilon/2\}.$$

Thus, in view of (17), proceeding as in the proof of Proposition 2, we can find $n_y \in \mathbb{N}_{\ge 1}$ such that $\Gamma_{12}^{n_y}(y_1, D) \times \Gamma_{21}^{n_y}(y_2, D) \subseteq Y$, which implies (13). ■

Appendix F

Proof of Lemma 7: Recall that $\mathcal{U} = \mathcal{L}^\infty[0, t]$. Then, $u_i \to u^\star$ weakly if (see [52, Def. 10.1.1])

$$\int_0^t \phi(s) u_i(s) ds \to \int_0^t \phi(s) u^\star ds = u^\star \int_0^t \phi(s) ds \tag{36}$$

for every integrable $\phi : [0, t] \to \mathbb{R}$. Let ϕ be the indicator function on a given interval $[a, b] \subseteq [0, t]$. Then, with $\omega_i(\tau) := \max\{m \in \mathbb{N} : mT_i \le \tau\}$, we can write

$$\int_0^t \phi(s) u_i(s) ds = \sum_{k=0}^{\omega_i(b-a)-1} \int_{a+kT_i}^{a+(k+1)T_i} u_i(s) ds + \int_{a+\omega_i(b-a)T_i}^{b} u_i(s) ds. \tag{37}$$

As $T_i \to 0$, we have $\omega_i(b-a)T_i \to b - a$. Hence, the second term of the right-hand side of (37) vanishes as $i \to \infty$. Since u_i is T_i-periodic, then $\int_{a+kT_i}^{a+(k+1)T_i} u_i(s) ds = \int_a^{a+T_i} u_i(s) ds$. Hence, in view of (23), the first term of (37) satisfies

$$\sum_{k=0}^{\omega_i(b-a)-1} \int_{a+kT_i}^{a+(k+1)T_i} u_i(s) ds = \frac{\omega_i(b-a)T_i}{T_i} \int_a^{a+T_i} u_i(s) ds \to (b-a)u^\star = u^\star \int_0^t \phi(s) ds.$$

Thus, (36) holds for the indicator function ϕ. In turn, this implies that (36) holds for all finite linear combinations of indicator functions (i.e., all simple functions).

Now, let ϕ be a generic integrable function. Then, there exists a sequence φ_k of simple functions such that $\int_0^t |\phi(s) - \varphi_k(s)| ds \to_k 0$. With $q := \sup_{u \in \mathrm{co} Q} |u|$, we have

$$\left|\int_0^t \phi(s) u_i(s) ds - u^\star \int_0^t \phi(s) ds\right| = \left|\int_0^t \phi(s)(u_i(s) - u^\star) ds\right| \le 2q \int_0^t |\phi(s) - \varphi_k(s)| ds + \left|\int_0^t \varphi_k(s)(u_i(s) - u^\star) ds\right|$$

for all $k \in \mathbb{N}$, where we have used the fact that $u^\star \in \mathrm{co} Q$ and $\operatorname{ran} u_i \subseteq Q$ for all i. Then, given any $\epsilon > 0$, we can find $\bar{k}$ such that $\int_0^t |\phi(s) - \varphi_{\bar{k}}(s)| ds < \epsilon/(4q)$ and then $\iota(\epsilon, \bar{k})$ such that $|\int_0^t \varphi_{\bar{k}}(s)(u_i(s) - u^\star) ds| < \epsilon/2$ for all $i \ge \iota(\epsilon, \bar{k})$ (which is possible since $\varphi_{\bar{k}}$ is a simple function). In turn, this implies that $|\int_0^t \phi(s) u_i(s) ds - u^\star \int_0^t \phi(s) ds| < \epsilon$ for all $i \ge \iota(\epsilon, \bar{k})$. For arbitrariness of $\epsilon > 0$, (36) follows. ■

Acknowledgment

The authors would like to thank Prof. David Angeli for his precious comments and suggestions.

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