Characterization of Input–Output Negative Imaginary Systems in a Dissipative Framework

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***Abstract*—In this article, we define the notion of stable input–output negative imaginary (IONI) systems. This new class captures and unifies all the existing stable subclasses of negative imaginary (NI) systems and is capable of distinguishing between the strict subclasses (e.g., strongly strictly negative imaginary, output strictly negative imaginary (OSNI), input strictly negative imaginary, etc.) in the literature. In addition to a frequency-domain definition, the proposed IONI class has been characterized in a time-domain dissipative framework in terms of a new quadratic supply rate $w(u, \bar{u}, \dot{\bar{y}})$. This supply rate consists of the system's input (u), an auxiliary input ($\bar{u}$) that is a filtered version of the system's input, and the time-derivative of an auxiliary output of the system ($\dot{\bar{y}}$). This supply rate corrects earlier supply rate attempts in the literature, which were only expressed in terms of the input (u) and the time-derivative of the system's output ($\dot{y}$). In this article, IONI systems are proved to be a class of dissipative systems with respect to the proposed supply rate $w(u, \bar{u}, \dot{\bar{y}})$. Subsequently, an equivalent frequency-dependent $(Q(\omega), S(\omega), R(\omega))$ dissipative supply rate is also proposed for IONI systems. These findings reveal the connections between the NI property and classical dissipativity in both the time domain and frequency domain. We also provide linear matrix inequality (LMI) tests on the state-space matrices to check whether a system belongs to the IONI class or any of its important subclasses. Finally, the derived results are specialized for OSNI systems since such systems exhibit interesting closed-loop stability properties when connected, in a positive feedback loop, to NI systems without poles at the origin. Several illustrative numerical examples are provided to make the results intuitive and useful.**

***Index Terms*—Dissipativity, input–output negative imaginary (IONI) systems, input–output passive systems, output strictly negative imaginary (OSNI) systems, quadratic supply rate, storage function.**

Manuscript received 26 November 2021; accepted 25 January 2022. Date of publication 9 February 2022; date of current version 30 January 2023. This work was supported by the Engineering and Physical Sciences Research Council under Grant EP/R008876/1. All research data supporting this publication are directly available within this publication. Recommended by Associate Editor S. Galeani. *(Corresponding author: Parijat Bhowmick.)*

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Color versions of one or more figures in this article are available at https://doi.org/10.1109/TAC.2022.3149938.

Digital Object Identifier 10.1109/TAC.2022.3149938

I. Introduction

NEGATIVE imaginary (NI) systems theory was introduced in [1] and was primarily inspired by the "positive position feedback control" of highly resonant mechanical systems with colocated position sensors and force actuators [2]. NI theory has formalized and unified some well-known vibration control techniques (e.g., graphical techniques and integral resonant control schemes) developed for lightly damped flexible structures using positive position feedback [1], [3]. NI theory offers a stand-alone robust control analysis and synthesis framework, similar to passivity and small-gain methodologies [4]. The NI system property is closely related to counterclockwise input–output dynamics in a nonlinear setting [5] and input–output Hamiltonian systems in both a linear and a nonlinear setting [6], [7]. NI control theory can be considered to be an energy-based control methodology [8], and consequently, it has a strong connection with dissipative theory [9]. These connections will be investigated in detail in this article. NI systems theory has gained popularity owing to its simple robust stability condition that depends only on the dc loop gain. Hence, the theory can be easily applied to practical systems without having an exact mathematical model [10]–[13]. NI theory finds potential applications in vibration control of lightly damped flexible structures [1], cantilever beams [14], large space structures [15], and robotic manipulators [15], in control of nanopositioning systems [16], in control of large vehicle platoons [17], etc.

In this article, the notion of input–output negative imaginary (IONI) systems is defined via a new frequency-domain definition that eliminates the difficulties identified in [18]. This new definition differs substantially from adjacent concepts in [18]–[21]. The IONI class proposed here includes stable NI systems and the existing strict subclasses of the NI class, e.g., strictly negative imaginary (SNI) [1], strongly strictly negative imaginary (SSNI) [22] (denoted by $\mathrm{SSNI}_{(\alpha=1,\beta=1)}$ in this article), SSNI [23] (denoted by $\mathrm{SSNI}_{(\alpha=2,\beta=1)}$[1] in this article), and output strictly negative imaginary (OSNI) [8], [18]. It also creates a valid input strictly negative imaginary (ISNI) system class (with $\varepsilon > 0, \delta \geq 0, \alpha \in \mathbb{N}, \beta \in \mathbb{N}$). The meaning of the parameters δ, ε, α, and β will be explained in Definition 6. A set-theoretic relationship among the subclasses of IONI systems is illustrated in the Venn diagram shown in Fig. 2.

The connections between NI systems theory and classical dissipativity have not yet been thoroughly explored. In the case of passive systems, a complete characterization exists in the literature, which was built on Willems's dissipative framework [9] and Hill–Moylan's (Q, S, R)-dissipative framework [24]–[26].

[1] For real, rational and proper transfer functions, $\mathrm{SSNI}_{(\alpha=2,\beta=1)}$ is the bigger set that contains $\mathrm{SSNI}_{(\alpha=1,\beta=1)}$.

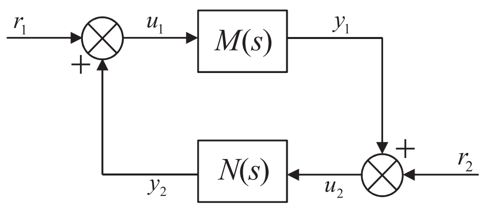

Fig. 1. Interconnection of NI systems with positive feedback.

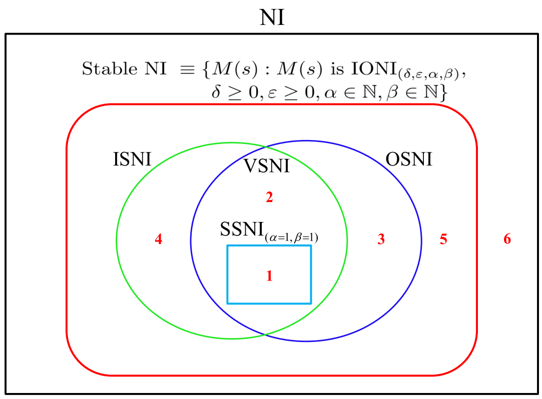

Fig. 2. Venn diagram shows the set-theoretic relationship amongst subclasses of the IONI and NI classes.

All strict and nonstrict passive systems can be shown to be dissipative in respect of a supply rate $w(u, y)$ that depends on the input (u) and the output (y) of the system. Different variants of the passivity theorem are available in the literature, which are proven using the (Q, S, R)-dissipative framework [25], [26]. In [27], Griggs *et al.* introduced a class of systems with "mixed" input–output passive and finite-gain properties. Griggs *et al.* [27], also proved finite-gain input–output stability of the closed-loop system having "mixed" properties using a frequency-domain dissipative approach. Inspired by the work presented in [27], Patra and Lanzon introduced in [19] the notion of "mixed" IONI and finite-gain properties along with a stand-alone frequency-domain definition for IONI systems on a finite frequency interval. Patra and Lanzon [19] also provided a frequency-domain $(Q(\omega), S(\omega), R(\omega))$-dissipative supply rate to characterize such systems. Later, Das *et al.* [20], [21] pursued a similar approach alike [19] to establish internal stability conditions for interconnected systems with "mixed" NI, passive and finite-gain properties.

Unlike [19]–[21], in this article, it is shown that the IONI systems are dissipative with respect to a new time-domain supply rate $w(u, \bar{u}, \dot{\bar{y}}) = 2\dot{\bar{y}}^\top u - \delta\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\bar{u}^\top\bar{u}$ by proving the existence of a positive semidefinite storage function $V(x)$. An auxiliary output $\bar{y} = y - Du$ is utilized to capture the full class of OSNI systems (i.e., including biproper cases), while the auxiliary input $\bar{u}$, which is a filtered version (as discussed later in Section V) of the actual input u, is used to capture an ISNI property. For a strictly proper OSNI system, this supply rate reduces to $2\dot{y}^\top u - \delta\,\dot{y}^\top\dot{y}$, which finds an interesting physical interpretation. For example, in the case of a spring–mass–damper system being OSNI, the term $\dot{y}^\top u$ gives the mechanical power input [velocity $(\dot{y}) \times$ force (u)], while the term $\dot{y}^\top\dot{y}$ represents the power dissipated in the damper $(d\,\dot{y}^2)$, and hence, the expression $\int_0^T (2\dot{y}^\top u - \delta\,\dot{y}^\top\dot{y})\mathrm{d}t$ gives the stored energy of the system, which is always nonnegative. However, for more general systems, the supply rate provides an abstraction of the net power inflow into the system, and often, it is not possible to find an exact physical interpretation.

Apart from the time-domain analysis, a frequency-domain $(Q(\omega), S(\omega), R(\omega))$-dissipative framework is also proposed in this article to characterize IONI systems. Thereafter, an equivalence is established between the time-domain and frequency-domain dissipative frameworks via applying Parseval's theorem. Furthermore, LMI-based state-space characterizations are derived for the IONI systems and each of its subclasses. We also specialize the above results to OSNI systems since such systems exhibit interesting closed-loop stability properties when connected (in a positive feedback loop) with NI systems that may contain complex conjugate poles on the imaginary axis excluding the origin.

II. Notation and Mathematical Preliminaries

The notation is standard throughout. The set of all natural numbers (excluding 0) is denoted by $\mathbb{N} = \{1, 2, 3, \ldots\}$. $\mathbb{R}_{\geq 0}$ denotes the set of all nonnegative real numbers. A^{-*} and $A^{-\top}$ represent shorthand for $(A^{-1})^*$ and $(A^{-1})^\top$ respectively. $\lambda_{\max}(A)$ denotes the maximum eigenvalue of a matrix A that has only real eigenvalues. Let $\mathscr{R}^{m\times n}$ be the set of all real, rational, and proper transfer function matrices of dimension $m \times n$, and let $\mathscr{RH}_\infty^{m\times n}$ denote the set of all asymptotically stable transfer function matrices in $\mathscr{R}^{m\times n}$. For $M(s) \in \mathscr{R}^{m\times m}$, the real-Hermitian and imaginary-Hermitian frequency response parts are given by $\frac{1}{2}[M(j\omega) + M(j\omega)^*]$ and $\frac{1}{2j}[M(j\omega) - M(j\omega)^*]$, respectively, where $M(j\omega)^* = M(-j\omega)^\top$. The $\mathscr{L}_2$-adjoint of a transfer function matrix $M(s)$, where $s \in \mathbb{C}$, is expressed as $M^\sim(s) = M(-s)^\top$. (A, B, C, D) denotes a state-space realization of a real, rational, and proper transfer function matrix $M(s) = D + C(sI - A)^{-1}B$. $\mathscr{L}_2^m(j\mathbb{R})$ denotes the frequency-domain Lebesgue space [19], [28] under the inner product $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(j\omega)^* g(j\omega)\,\mathrm{d}\omega < \infty$ when $f, g \in \mathscr{L}_2^m(j\mathbb{R})$. For a signal $f \in \mathscr{L}_2^m(j\mathbb{R})$, the norm is given by $\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} f(j\omega)^* f(j\omega)\,\mathrm{d}\omega} < \infty$. A dynamical system is said to be initially relaxed if it has zero initial condition, i.e., $x(0) = 0$. The term "stable system" refers to an asymptotically stable system, i.e., the associated transfer function matrix belongs to $\mathscr{RH}_\infty$. The space of all real-valued, absolutely square integrable, time-domain functions is defined by $\mathbb{L}_2^m = \{f : \mathbb{R} \to \mathbb{R}^m : f(t) = 0 \text{ when } t < 0, \int_0^\infty f(t)^\top f(t)\,\mathrm{d}t < \infty\}$, while the space of all real-valued, locally square integrable, time-domain functions is defined by $\mathbb{L}_{2e}^m = \{f : \mathbb{R} \to \mathbb{R}^m : f(t) = 0 \text{ when } t < 0, \int_0^\top f(t)^\top f(t)\,\mathrm{d}t < \infty\ \forall T \in [0, \infty)\}$. An energy supply rate function $w(u, y)$ is an abstraction of the rate of energy inflow into a physical system that is expressed by the mapping $w : \mathbb{U} \times \mathbb{Y} \to \mathbb{R}$, where the input space $\mathbb{U} \in \mathbb{L}_{2e}^m$ and the output space $\mathbb{Y} \in \mathbb{L}_{2e}^p$, and satisfies the property $\int_0^\top w(u, y)\,\mathrm{d}t < \infty$ for all admissible $(u, y) \in \mathbb{U} \times \mathbb{Y}$ and $\forall T \in [0, \infty)$. In particular, $\int_0^T w(u, y)\,\mathrm{d}t < \infty\ \forall T \in [0, \infty]$ when $(u, y) \in \mathbb{L}_2^m \times \mathbb{L}_2^p$. Note that an energy supply rate can also be defined in the frequency domain for a stable system, and it remains equivalent to the corresponding time-domain supply rate via Parseval's theorem [4]. The symbol $\prod$ denotes the product operator. For a transfer function $M(s) \in \mathscr{R}^{m\times m}$, $\mathscr{L}^{-1}[M(s)]$ represents the impulse response (also called the Kernel function), where $\mathscr{L}^{-1}$ denotes the inverse Laplace operator. The symbol $\star$ denotes the time-domain convolution operator and the expression $y(t) = \mathscr{L}^{-1}[M(s)] \star u(t)$ indicates that the output signal $y(t)$ is being generated by the time-domain convolution of the impulse response of the system and an input $u(t)$. The spectral factorization [29], [30] of a transfer function matrix $F(s)$ is given by $F(s) = F_s^{\sim}(s)F_s(s)$, where $F_s(s)$ denotes the stable, minimum phase spectral factor of $F(s)$ and $F_s^{\sim}(s) = F_s(-s)^\top$ indicates the antistable, antiminimum phase spectral factor. Let S_1 and S_2 be two subsets of $\mathbb{R}$, then $S_1 \backslash S_2 = S_1 \cap S_2^c$ where S_2^c denotes the complementary set of S_2 in $\mathbb{R}$. $A \otimes B$ represents the Kronecker product of the matrices A and B. Let $\text{sym}[A] = A + A^\top$ for $A \in \mathbb{R}^{m\times m}$ and $\text{sqr}[B] = B^\top B$ for $B \in \mathbb{R}^{p\times q}$.

III. Technical Preliminaries

In this section, essential technical preliminaries, definitions, and lemmas are presented, which underpin the proofs of the main results of this article.

The finite-dimensional, causal, LTI systems studied in this article are described by[2]

$$M : \begin{cases} \dot{x} = Ax + Bu, & x(0) = x_0, \\ y = Cx + Du. \end{cases} \tag{1}$$

The admissible inputs $u(t)$ are considered to be in the space $\mathbb{L}_2^m$ such that the unique solution of the state trajectory $x(t)$ exists forward in time $t \geq 0$ and $x \in \mathbb{L}_{2e}^n$. Therefore, the output $y(t)$ also exists forward in time $t \geq 0$ and $y \in \mathbb{L}_{2e}^p$. Let us introduce the state transition function Φ, associated with M, being a mapping from $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{L}_2^m$ to $\mathbb{R}^n$. Here, $\Phi(t_1, t_0, x(t_0), u(t))$ denotes the state $x(t_1)$ at time t_1 when the system M starts from an initial state $x(t_0) \in \mathbb{R}^n$ at time $t = t_0$, and an admissible input $u(t)$ is applied on M for the time interval $t \in [t_0, t_1]$.

A. Dissipative Systems Notations and Definitions

Let us recall the notion of dissipativity of finite-dimensional, causal, LTI systems introduced in [9]. It is important to mention here that in the following definitions related to time-domain dissipativity, we have chosen to restrict the input space to $\mathbb{L}_2^m$ since, in this article, we aim to establish the equivalence between the time-domain dissipativity and frequency-domain dissipativity of stable NI systems where the frequency-domain dissipativity is characterized by only finite energy input signals $U \in \mathscr{L}_2^m(j\mathbb{R})$.

Definition 1 (Dissipative systems) [9]: A dynamical system M, given in (1), is said to be dissipative with respect to an energy supply rate $w(u, y)$ if there exists a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, called the storage function, such that

$$V\left(x(0)\right) + \int_0^T w(u, y)\, \mathrm{d}t \geq V\left(x(T)\right) \tag{2}$$

for any $T \in [0, \infty)$, any initial condition $x(0) \in \mathbb{R}^n$ and any admissible input $u \in \mathbb{L}_2^m$ where $x(T) = \Phi(T, 0, x(0), u(t))$ and $w(u, y)$ has been evaluated along any trajectory of (1).

[2]For simplicity of presentation, the dependence of x, u, and y on time $t \in \mathbb{R}_{\geq 0}$ is omitted.

Inequality (2) is known as the "dissipation inequality" in the sense of Willems. Note that for asymptotically stable LTI systems and for all input $u \in \mathbb{L}_2^m$, $\lim_{t\to\infty} x(t)$ is finite and also $x \in \mathbb{L}_2^m$; hence, $y \in \mathbb{L}_2^p$ implying $\int_0^\infty w(u, y)\, \mathrm{d}t < \infty$. In such cases, Willems's dissipation inequality implies

$$V\left(x(0)\right) + \int_0^\infty w(u, y)\, \mathrm{d}t \geq V\left(x(\infty)\right). \tag{3}$$

Furthermore, if $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a differentiable storage function, then the dissipation inequality (2) can be expressed in the differential form as

$$w(u, y) \geq \dot{V}(x) \tag{4}$$

where the "dot" represents the time-derivative.

Note that for finite-dimensional LTI systems with minimal state-space realizations, the storage function $V(x)$ can be characterized with a quadratic form $x^\top P x$, without loss of generality, where $P = P^\top > 0$ [9], [31]. Moreover, in an LTI setting, the storage function $V(x)$ can always be assumed to be a differentiable function of x [24], [32].

For a dissipative system with a completely controllable state space, the "required supply" is defined as [33]

$$V_r(x_1) = \inf_{\substack{x^* \to x_1 \\ u(\cdot),\, T \leq 0}} \int_T^0 w(u, y)\, \mathrm{d}t \tag{5}$$

where $x^* \in \mathbb{R}^n$ represents the point of minimum storage. In general, the origin of a state space is the point of minimum storage, where $V(x^*) = V(0) = 0$. The "required supply" is the least amount of energy required to excite a system to a desired state from the state of minimum energy level [34]. $V_r(x)$ is a possible storage function for any dissipative system with a reachable (from the origin) state space.

Definition 2 ((Q, S, R)-dissipativity in Hill–Moylan's framework) [24]: A dynamical system M, given by (1) with $x_0 = 0$, is said to be (Q, S, R)-dissipative if there exist $Q = Q^T \in \mathbb{R}^{p\times p}$, $S \in \mathbb{R}^{p\times m}$ and $R = R^T \in \mathbb{R}^{m\times m}$ such that

$$\int_0^T \left(y^\top Q y + 2y^\top S u + u^\top R u\right) \mathrm{d}t \geq 0 \tag{6}$$

for any $T \in [0, \infty)$ and all $u \in \mathbb{L}_2^m$.

If the supply rate function in Willems's framework is considered to be $w(u, y) = y^\top Q y + 2y^\top S u + u^\top R u$ where $Q = Q^T \in \mathbb{R}^{p\times p}$, $S \in \mathbb{R}^{p\times m}$ and $R = R^T \in \mathbb{R}^{m\times m}$, then (4) takes the form

$$y^\top Q y + 2y^\top S u + u^\top R u \geq \dot{V}(x). \tag{7}$$

So far we have discussed only time-domain dissipativity. However, dissipative characterization can also be expressed in the frequency domain. The following definition articulates the notion of frequency-domain $(Q(\omega), S(\omega), R(\omega))$-dissipativity, which may be regarded as a frequency-domain counterpart of the Hill–Moylan's (Q, S, R)-dissipativity.

Definition 3 ((Q(ω), S(ω), R(ω))-dissipativity) [19], [27]: Let $M(s) \in \mathscr{RH}_\infty^{p\times m}$ be the transfer function matrix of a causal system M with the input–output relationship $Y(s) = M(s)U(s)$, where $U \in \mathscr{L}_2^m(j\mathbb{R})$. Then, M is said to be $(Q(\omega), S(\omega), R(\omega))$-dissipative with respect to the frequency-dependent triplet $(Q(\omega), S(\omega), R(\omega))$ where $Q(\omega) = Q(\omega)^\top \in$

$\mathbb{R}^{p\times p}$, $S(\omega)\in\mathbb{C}^{p\times m}$ and $R(\omega)=R(\omega)^\top\in\mathbb{R}^{m\times m}$ $\forall\omega\in\mathbb{R}$ if

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}[Y(j\omega)^*Q(\omega)Y(j\omega)+Y(j\omega)^*S(\omega)U(j\omega)$$
$$+U(j\omega)^*S(\omega)^*Y(j\omega)+U(j\omega)^*R(\omega)U(j\omega)]\,\mathrm{d}\omega\geq 0 \quad (8)$$

for all $U\in\mathscr{L}_2^m(j\mathbb{R})$.

The following lemma on the characterization of output strictly passive (OSP) systems is recalled here, so that it can be used later in this article to define the OSNI systems property.

Lemma 1 [8], [18]: A system $F(s)\in\mathscr{RH}_\infty^{m\times m}$ with $F(s)+F^\sim(s)$ having full normal rank is OSP if and only if there exists $\delta_p>0$ such that

$$F(j\omega)+F(j\omega)^*\geq\delta_p F(j\omega)^*F(j\omega)\qquad\forall\omega\in\mathbb{R}\cup\{\infty\}. \quad (9)$$

B. Definitions of Negative Imaginary Systems

In this section, we recall the definitions of NI and SNI systems.

Definition 4 (NI System) [15], [35]: Let $M(s)$ be the real, rational and proper transfer function matrix of a square and causal system without any poles in the open right-half plane. $M(s)$ is said to be NI if the following three conditions hold:

1) $j[M(j\omega)-M(j\omega)^*]\geq 0$ for all $\omega\in(0,\infty)$ except the values of ω where $s=j\omega$ is a pole of $M(s)$;
2) if $s=j\omega_0$ with $\omega_0\in(0,\infty)$ is a pole of $M(s)$, then it is at most a simple pole and the residue matrix $\lim_{s\to j\omega_0}(s-j\omega_0)jM(s)$ is Hermitian and positive semidefinite; and
3) if $s=0$ is a pole of $M(s)$, then $\lim_{s\to 0}s^kM(s)=0$ for all $k\geq 3$ and $\lim_{s\to 0}s^2M(s)$ is Hermitian and positive semidefinite.

In the literature, there are extensions of the NI definition to improper nonrational systems [23], [36], [37] and NI theory has also been recently extended to discrete-time LTI systems [38]. However, in this article, we restrict our attention to only continuous-time, real, rational, and proper NI systems as per Definition 4.

Definition 5 (SNI System) [1]: Let $M(s)$ be the real, rational, and proper transfer function matrix of a square and causal system. $M(s)$ is said to be SNI if $M(s)$ has no poles in $\{s\in\mathbb{C}:\Re[s]\geq 0\}$ and $j[M(j\omega)-M(j\omega)^*]>0$ for all $\omega\in(0,\infty)$.

We now present a necessary and sufficient condition for internal stability of an NI-SNI positive feedback interconnection, as shown in Fig. 1.

Theorem 1 [35]: Let $M(s)$ be an NI system without poles at the origin and $N(s)$ be an SNI system. Then, the positive feedback interconnection of $M(s)$ and $N(s)$, as shown in Fig. 1, is internally stable if and only if

$$\det[I-M(\infty)N(\infty)]\neq 0, \quad (10a)$$

$$\lambda_{\max}\left[(I-M(\infty)N(\infty))^{-1}(M(\infty)N(0)-I)\right]<0, \quad (10b)$$

$$\lambda_{\max}\left[(I-N(0)M(\infty))^{-1}(N(0)M(0)-I)\right]<0. \quad (10c)$$

C. Relationship Between the Transmission Zeros of a Transfer Function Matrix and the Rank Deficiency of Its Imaginary-Hermitian Part

In the following, we establish a relationship between the transmission zeros of an NI transfer function matrix on the imaginary axis and the rank deficiency of its imaginary-Hermitian part at that frequency. This result will be used later in Section VI to prove the closed-loop stability of a positive feedback interconnection of an NI system without poles at the origin and an OSNI system.

Lemma 2: Let $G(s)\in\mathscr{R}^{m\times m}$ be an NI system with full normal rank. Suppose $s=j\omega_z$ with $\omega_z\in(0,\infty)$ is a transmission zero of $G(s)$ but not a pole. Then, $\det[G(j\omega_z)-G(j\omega_z)^*]=0$.

Proof: As $s=j\omega_z$ with $\omega_z\in(0,\infty)$ is a transmission zero of $G(s)$, there exists a nonzero vector $x\in\mathbb{C}^m$ such that $G(j\omega_z)x=0$. This then implies $x^*G(j\omega_z)x=0\Rightarrow x^*\frac{1}{2j}[G(j\omega_z)-G(j\omega_z)^*]x=0\Leftrightarrow x^*j[G(j\omega_z)-G(j\omega_z)^*]x=0$. For convenience, let $Z=j[G(j\omega_z)-G(j\omega_z)^*]$. Now, $Z=Z^*\geq 0$ as $G(s)$ is NI and $s=j\omega_z$ is not a pole of $G(s)$. Since $Z\geq 0$, there exists a unique matrix square root $Z^{\frac{1}{2}}\geq 0$ (see [39, p. 406]). Therefore, we have $x^*Z^{\frac{1}{2}}Z^{\frac{1}{2}}x=0\Leftrightarrow y^*y=0$ denoting $y=Z^{\frac{1}{2}}x$, which in turn implies $y=0$. Hence, $\det[Z]=0$ as $x\neq 0$. ■

Lemma 3: Let $G(s)\in\mathscr{R}^{m\times m}$ be an NI system. Then, $j[G(j\omega_z)-G(j\omega_z)^*]>0$ with $\omega_z\in(0,\infty)$ implies $\det[G(j\omega_z)]\neq 0$.

Proof: Suppose via contradiction that $\det[G(j\omega_z)]=0$. Then, there exists a nonzero vector $x\in\mathbb{C}^m$ such that $G(j\omega_z)x=0$, which ultimately implies $x^*[j\{G(j\omega_z)-G(j\omega_z)^*\}]x=0$, as shown in the proof of Lemma 2. But, the result violates the supposition that $j[G(j\omega_z)-G(j\omega_z)^*]>0$ [39, Ch. 7]. Hence, there does not exist any nonzero $x\in\mathbb{C}^m$ such that $G(j\omega_z)x=0$, that is, $\det[G(j\omega_z)]\neq 0$. ■

IV. IONI Systems

In this section, we define a unifying class of stable negative imaginary systems, termed as IONI systems,[3] that encompasses the existing strict forms of NI systems, namely, 1) SSNI systems introduced in [22] (denoted by $\text{SSNI}_{(\alpha=1,\beta=1)}$ in this article), 2) a different class of SSNI systems defined in [23] (denoted by $\text{SSNI}_{(\alpha=2,\beta=1)}$ in this article), and 3) OSNI systems defined in [18] and modified later in [8]. Moreover, the proposed IONI class also gives birth to two new subclasses of SNI systems, termed as the ISNI systems and very strictly negative imaginary (VSNI) systems in this article. The set-theoretic relationship among the subclasses of IONI systems is illustrated in the Venn diagram of Fig. 2.

Definition 6 (IONI Systems): Let $M(s)\in\mathscr{RH}_\infty^{m\times m}$. Then, $M(s)$ is said to be IONI with a level of output strictness $\delta\geq 0$, a level of input strictness $\varepsilon\geq 0$, and having an arrival rate specified by $\alpha\in\mathbb{N}$ and a departure rate specified by $\beta\in\mathbb{N}$ ($\text{IONI}_{(\delta,\varepsilon,\alpha,\beta)}$) if

$$j\omega[M(j\omega)-M(j\omega)^*]-\delta\,\omega^2\bar{M}(j\omega)^*\bar{M}(j\omega)$$
$$-\varepsilon\left(\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}\right)I_m\geq 0\quad\forall\omega\in\mathbb{R}\cup\{\infty\} \quad (11)$$

where $\bar{M}(j\omega)=M(j\omega)-M(\infty)$.

Remark 1: $\alpha\in\mathbb{N}$ (resp. $\beta\in\mathbb{N}$) is referred to as the arrival (resp. departure) rate as it determines the behavior of $j[M(j\omega)-M(j\omega)^*]$ as $\omega\to\infty$ (resp. $\omega\to 0$).

The following lemma shows that if inequality (11) is fulfilled for some $\delta_0\geq 0$ and $\varepsilon_0\geq 0$, it is also automatically fulfilled for any $\delta\in[0,\delta_0]$ and any $\varepsilon\in[0,\varepsilon_0]$.

[3]The IONI property is defined for finite-dimensional, causal, square, and asymptotically stable systems.

Lemma 4: Let $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}$, $\delta_0 \geq 0$ and $\varepsilon_0 \geq 0$. Let $M(s)$ be IONI$_{(\delta_0,\varepsilon_0,\alpha,\beta)}$. Then, $M(s)$ is IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ for all $\delta \in [0,\delta_0]$ and all $\varepsilon \in [0,\varepsilon_0]$.

Proof: This trivially follows from $j\omega[M(j\omega) - M(j\omega)^*] \geq \delta_0\,\omega^2 \bar{M}(j\omega)^* \bar{M}(j\omega) + \varepsilon_0\left(\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}\right) I_m \geq \delta\,\omega^2 \bar{M}(j\omega)^* \bar{M}(j\omega) + \varepsilon\left(\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}\right) I_m \;\forall \omega \in \mathbb{R}\cup\{\infty\}$, $\delta \in [0,\delta_0]$ and $\varepsilon \in [0,\varepsilon_0]$. ■

We will now classify IONI systems on the basis of the values of the parameters δ, ε, α, and β.

Definition 7 (Classification of IONI Systems): Let $M(s) \in \mathscr{RH}_\infty^{m\times m}$. Then, $M(s)$ is said to be:

1) stable NI if it belongs to $\{M(s) : M(s)$ is IONI$_{(\delta,\varepsilon,\alpha,\beta)}$, $\delta \geq 0, \varepsilon \geq 0, \alpha \in \mathbb{N}, \beta \in \mathbb{N}\}$;
2) ISNI if it belongs to $\{M(s) : M(s)$ is IONI$_{(\delta,\varepsilon,\alpha,\beta)}$, $\delta \geq 0, \varepsilon > 0, \alpha \in \mathbb{N}, \beta \in \mathbb{N}\}$;
3) SSNI$_{(\alpha=1,\beta=1)}$ if it belongs to $\{M(s) : M(s)$ is IONI$_{(\delta,\varepsilon,1,1)}, \delta \geq 0, \varepsilon > 0\}$;
4) SSNI$_{(\alpha=2,\beta=1)}$ if it belongs to $\{M(s) : M(s)$ is IONI$_{(\delta,\varepsilon,2,1)}, \delta \geq 0, \varepsilon > 0\}$;
5) VSNI if it belongs to $\{M(s) : M(s)$ is IONI$_{(\delta,\varepsilon,\alpha,\beta)}$, $\delta > 0, \varepsilon > 0, \alpha \in \mathbb{N}, \beta \in \mathbb{N}\}$; and
6) OSNI if it belongs to $\{M(s) : M(s)$ is IONI$_{(\delta,\varepsilon,\alpha,\beta)}$, $\delta > 0, \varepsilon \geq 0, \alpha \in \mathbb{N}, \beta \in \mathbb{N}$, $[M(s) - M^\sim(s)]$ has full normal rank$\}$.

The following lemma states the connections between the above classifications.

Lemma 5: The following five statements hold.

1) If $M(s)$ is ISNI, then it is also stable NI.
2) If $M(s)$ is OSNI, then it is also stable NI.
3) If $M(s)$ is VSNI, then it is also ISNI and OSNI.
4) If $M(s)$ is SSNI$_{(\alpha=1,\beta=1)}$, then it is also ISNI.
5) If $M(s)$ is SSNI$_{(\alpha=2,\beta=1)}$, then it is also ISNI.

Proof: All five cases are trivial consequences of Definition 7. ■

The following lemma gives a simpler, yet equivalent, characterization for each of the classes in Definition 7.

Lemma 6: Let $M(s) \in \mathscr{RH}_\infty^{m\times m}$. Then, $M(s)$ is:

1) stable NI if and only if $M(s)$ is IONI$_{(0,0,\diamond,\diamond)}$;[4]
2) ISNI if and only if there exist $\varepsilon > 0$, $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$ such that $M(s)$ is IONI$_{(0,\varepsilon,\alpha,\beta)}$;
3) SSNI$_{(\alpha=1,\beta=1)}$ if and only if there exists $\varepsilon > 0$ such that $M(s)$ is IONI$_{(0,\varepsilon,1,1)}$;
4) SSNI$_{(\alpha=2,\beta=1)}$ if and only if there exists $\varepsilon > 0$ such that $M(s)$ is IONI$_{(0,\varepsilon,2,1)}$;
5) VSNI if and only if there exist $\delta > 0$, $\varepsilon > 0$, $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$ such that $M(s)$ is IONI$_{(\delta,\varepsilon,\alpha,\beta)}$; and
6) OSNI if and only if $[M(s) - M^\sim(s)]$ has full normal rank and there exists $\delta > 0$ such that $M(s)$ is IONI$_{(\delta,0,\diamond,\diamond)}$.

Proof: We proof each statement separately.

1) Since "$M(s)$ is stable NI" is equivalent to $\exists\,\delta_0 \geq 0$, $\varepsilon_0 \geq 0$, $\alpha_0 \in \mathbb{N}$ and $\beta_0 \in \mathbb{N}$ such that $M(s)$ is IONI$_{(\delta_0,\varepsilon_0,\alpha_0,\beta_0)}$ via Definition 7, sufficiency trivially follows on choosing $\delta_0 = 0$, $\varepsilon_0 = 0$ and any $\alpha_0 \in \mathbb{N}$ and $\beta_0 \in \mathbb{N}$ whereas necessity follows from Lemma 4 on choosing $\delta = 0$ and $\epsilon = 0$ in Lemma 4.
2) Since "$M(s)$ is ISNI" is equivalent to $\exists\,\delta_0 \geq 0$, $\varepsilon_0 > 0$, $\alpha_0 \in \mathbb{N}$ and $\beta_0 \in \mathbb{N}$ such that $M(s)$ is IONI$_{(\delta_0,\varepsilon_0,\alpha_0,\beta_0)}$ via Definition 7, sufficiency trivially follows on choosing $\delta_0 = 0$ whereas necessity follows from Lemma 4 on choosing $\delta = 0$ in Lemma 4.
3) Since "$M(s)$ is SSNI$_{(\alpha=1,\beta=1)}$" is equivalent to $\exists\,\delta_0 \geq 0$ and $\varepsilon_0 > 0$ such that $M(s)$ is IONI$_{(\delta_0,\varepsilon_0,1,1)}$ via Definition 7, sufficiency trivially follows on choosing $\delta_0 = 0$ whereas necessity follows from Lemma 4 on choosing $\delta = 0$, $\alpha = 1$ and $\beta = 1$ in Lemma 4.
4) Since "$M(s)$ is SSNI$_{(\alpha=2,\beta=1)}$" is equivalent to $\exists\,\delta_0 \geq 0$ and $\varepsilon_0 > 0$ such that $M(s)$ is IONI$_{(\delta_0,\varepsilon_0,2,1)}$ via Definition 7, sufficiency trivially follows on choosing $\delta_0 = 0$ whereas necessity follows from Lemma 4 on choosing $\delta = 0$, $\alpha = 2$ and $\beta = 1$ in Lemma 4.
5) "$M(s)$ is VSNI" is directly equivalent to $\exists\,\delta_0 > 0$, $\varepsilon_0 > 0$, $\alpha_0 \in \mathbb{N}$ and $\beta_0 \in \mathbb{N}$ such that $M(s)$ is IONI$_{(\delta_0,\varepsilon_0,\alpha_0,\beta_0)}$ via Definition 7.
6) Since "$M(s)$ is OSNI" is equivalent to $[M(s) - M^\sim(s)]$ has full normal rank and $\exists\,\delta_0 > 0$, $\varepsilon_0 \geq 0$, $\alpha_0 \in \mathbb{N}$ and $\beta_0 \in \mathbb{N}$ such that $M(s)$ is IONI$_{(\delta_0,\varepsilon_0,\alpha_0,\beta_0)}$ via Definition 7, sufficiency trivially follows on choosing $\varepsilon_0 = 0$ and any $\alpha_0 \in \mathbb{N}$ and $\beta_0 \in \mathbb{N}$ whereas necessity follows from Lemma 4 on choosing $\varepsilon = 0$ in Lemma 4. ■

Note that the pointwise frequency-domain condition (11) can equivalently be expressed on the open positive frequency interval, that is, for all $\omega \in (0,\infty)$, as shown in Lemma 7.

Lemma 7: Let $M(s) \in \mathscr{RH}_\infty^{m\times m}$, $\bar{M}(s) = M(s) - M(\infty)$, $\delta \geq 0$, $\varepsilon \geq 0$, $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$. Then, (11) is equivalent to

$$j[M(j\omega) - M(j\omega)^*] - \delta\,\omega \bar{M}(j\omega)^* \bar{M}(j\omega) - \varepsilon\left(\frac{1}{\omega^{2\alpha-1} + \frac{1}{\omega^{2\beta-1}}}\right) I_m \geq 0 \quad \forall \omega \in (0,\infty). \tag{12}$$

Proof (Sufficiency): The inequality (12) implies $j[M(j\omega) - M(j\omega)^*] \geq 0 \quad \forall \omega \in (0,\infty)$ which, in turn, implies $M(0) = M(0)^\top$ and $M(\infty) = M(\infty)^\top$ [1]. Also, (12) implies $j[M(j\hat{\omega}) - M(j\hat{\omega})^*] - \delta\hat{\omega}\bar{M}(j\hat{\omega})^*\bar{M}(j\hat{\omega}) - \varepsilon\left(\frac{1}{\hat{\omega}^{2\alpha-1} + \frac{1}{\hat{\omega}^{2\beta-1}}}\right) I_m \leq 0 \quad \forall \hat{\omega} \in (-\infty,0)$ on letting $\hat{\omega} = -\omega$ and taking the transpose throughout. On multiplying this last inequality by $\hat{\omega}$ and (12) by ω, we get $j\omega[M(j\omega) - M(j\omega)^*] - \delta\,\omega^2 \bar{M}(j\omega)^* \bar{M}(j\omega) - \varepsilon\left(\frac{\omega}{\omega^{2\alpha-1} + \frac{1}{\omega^{2\beta-1}}}\right) I_m \geq 0 \quad \forall \omega \in \mathbb{R}$ since this inequality is trivially satisfied at $\omega = 0$. Then (11) holds because $j\omega[M(j\omega) - M(j\omega)^*] - \delta\,\omega^2 \bar{M}(j\omega)^* \bar{M}(j\omega) - \varepsilon\left(\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}\right) I_m = (j\omega\bar{M}(j\omega)) + (j\omega\bar{M}(j\omega))^* -$

[4]The symbol $\diamond$ stands for a "do not care value" because the associated term disappears from (11) when $\varepsilon = 0$.

$\delta(j\omega\bar{M}(j\omega))^*(j\omega\bar{M}(j\omega)) - \varepsilon\left(\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}\right)I_m$, which is clearly finite and positive semidefinite in the limit $\omega \to \infty$.

(Necessity). Trivial restriction. ■

The following lemma shows that the SNI set contains the same elements as the ISNI set.

Lemma 8: $M(s)$ is ISNI if and only if $M(s)$ is SNI.

Proof (Necessity): $M(s)$ is ISNI implies that there exist $\varepsilon > 0$, $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$ such that (11) holds. This implies that $j\omega[M(j\omega) - M(j\omega)^*] \geq \varepsilon\left(\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}\right)I_m > 0\ \forall\omega \in (0,\infty)$, which in turn implies that $M(s)$ is SNI.

(Sufficiency): $M(s)$ is SNI implies that $j[M(j\omega) - M(j\omega)^*] > 0\ \forall\omega \in (0,\infty)$. This then implies that there exist a sufficiently small $\varepsilon > 0$ and sufficiently large $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$ such that $j[M(j\omega) - M(j\omega)^*] \geq \varepsilon\left(\frac{1}{\omega^{2\alpha-1} + \frac{1}{\omega^{2\beta-1}}}\right)I_m$ $\forall\omega \in (0,\infty)$. This hence implies that $M(s)$ is ISNI. ■

The following lemma shows that the set of $\text{SSNI}_{(\alpha=1,\beta=1)}$ systems is contained within the set of $\text{SSNI}_{(\alpha=2,\beta=1)}$ systems and also within the set of VSNI systems.

Lemma 9: Let $M(s)$ be $\text{SSNI}_{(\alpha=1,\beta=1)}$. Then, $M(s)$ is also $\text{SSNI}_{(\alpha=2,\beta=1)}$ and VSNI.

Proof: Since $M(s)$ is $\text{SSNI}_{(\alpha=1,\beta=1)}$, there exists $\varepsilon > 0$ such that $j\omega[M(j\omega) - M(j\omega)^*] \geq \varepsilon\frac{\omega^2}{1+\omega^2}I_m\ \forall\omega \in \mathbb{R}\cup\{\infty\}$. Now $\frac{\omega^2}{1+\omega^2} \geq \left(\frac{2}{\sqrt{2}+1}\right)\frac{\omega^2}{1+\omega^4}\ \forall\omega\in\mathbb{R}\cup\{\infty\}$ since $\frac{1+\omega^2}{1+\omega^4} \leq \frac{1}{2}(\sqrt{2}+1)\ \forall\omega \in \mathbb{R}\cup\{\infty\}$. Let $\varepsilon_1 = \left(\frac{2}{\sqrt{2}+1}\right)\varepsilon$. It easily follows that $j\omega[M(j\omega) - M(j\omega)^*] \geq \varepsilon_1\left(\frac{\omega^2}{1+\omega^4}\right)I_m\ \forall\omega \in \mathbb{R}\cup\{\infty\}$, which implies that $M(s)$ is $\text{SSNI}_{(\alpha=2,\beta=1)}$. ■

Furthermore, let $\bar{M}(s) = M(s) - M(\infty)$, $\varepsilon_2 = \frac{1}{2}\varepsilon$, and $\delta_2 = \frac{\varepsilon_2}{||(s+1)\bar{M}(s)||_\infty^2}$. Since $(1+\omega^2)\bar{M}(j\omega)^*\bar{M}(j\omega) = [(1+j\omega)\bar{M}(j\omega)]^*[(1+j\omega)\bar{M}(j\omega)] \leq ||(s+1)\bar{M}(s)||_\infty^2 I_m$ $\forall\omega \in \mathbb{R}\cup\{\infty\}$, it follows that $\delta_2\bar{M}(j\omega)^*\bar{M}(j\omega) \leq \varepsilon_2\left(\frac{1}{1+\omega^2}\right)I_m\ \forall\omega\in\mathbb{R}\cup\{\infty\}$. Thus, $j\omega[M(j\omega)-M(j\omega)^*] \geq \varepsilon_2\left(\frac{\omega^2}{1+\omega^2}\right)I_m + \varepsilon_2\left(\frac{\omega^2}{1+\omega^2}\right)I_m \quad \geq \varepsilon_2\left(\frac{\omega^2}{1+\omega^2}\right)I_m + \delta_2\omega^2\bar{M}(j\omega)^*\bar{M}(j\omega)\ \forall\omega \in \mathbb{R}\cup\{\infty\}$. ■

The following lemma states that $\text{SSNI}_{(\alpha=2,\beta=1)}$ systems that have a strictly proper $s\bar{M}(s)$ are also VSNI. The interpretation of a strictly proper $s\bar{M}(s)$ is easy in a scalar setting as it would correspond to systems $\bar{M}(s)$ with a relative degree of two.

Lemma 10: Let $M(s)$ be $\text{SSNI}_{(\alpha=2,\beta=1)}$ with $\lim_{s\to\infty}[s\bar{M}(s)] = 0$ where $\bar{M}(s) = M(s) - M(\infty)$. Then, $M(s)$ is VSNI.

Proof: Since $M(s)$ is $\text{SSNI}_{(\alpha=2,\beta=1)}$, there exists $\varepsilon > 0$ such that $j\omega[M(j\omega) - M(j\omega)^*] \geq \varepsilon\left(\frac{\omega^2}{1+\omega^4}\right)I_m\ \forall\omega \in \mathbb{R}\cup\{\infty\}$. Since $\lim_{s\to\infty}[s\bar{M}(s)] = 0$, $(s^2+\sqrt{2}s+1)\bar{M}(s)$ is proper. Let $\varepsilon_1 = \frac{1}{2}\varepsilon$ and $\delta_1 = \frac{\varepsilon_1}{||(s^2+\sqrt{2}s+1)\bar{M}(s)||_\infty^2}$. Since $(1+\omega^4)\bar{M}(j\omega)^*\bar{M}(j\omega) = [((j\omega)^2+\sqrt{2}j\omega+1)\bar{M}(j\omega)]^*[((j\omega)^2+\sqrt{2}j\omega+1)\bar{M}(j\omega)] \leq ||(s^2+\sqrt{2}s+1)\bar{M}(s)||_\infty^2 I_m$ $\forall\omega \in \mathbb{R}\cup\{\infty\}$, it follows that $\delta_1\bar{M}(j\omega)^*\bar{M}(j\omega) \leq \frac{\varepsilon_1}{1+\omega^4}I_m\ \forall\omega \in \mathbb{R}\cup\{\infty\}$. Thus, $j\omega[M(j\omega) - M(j\omega)^*] \geq \varepsilon_1\left(\frac{\omega^2}{1+\omega^4}\right)I_m + \varepsilon_1\left(\frac{\omega^2}{1+\omega^4}\right)I_m \quad \geq \varepsilon_1\left(\frac{\omega^2}{1+\omega^4}\right)I_m + \delta_1\omega^2\bar{M}(j\omega)^*\bar{M}(j\omega)\ \forall\omega \in \mathbb{R}\cup\{\infty\}$, which implies that $M(s)$ is VSNI.

The following lemma shows that in the scalar case, the set of $\text{SSNI}_{(\alpha=2,\beta=1)}$ systems is contained within the set of VSNI systems.

Lemma 11: Let $M(s)$ be scalar and $\text{SSNI}_{(\alpha=2,\beta=1)}$. Then, $M(s)$ is also VSNI.

Proof: Since Lemma 9 guarantees that all $\text{SSNI}_{(\alpha=1,\beta=1)}$ systems are VSNI, we only need to consider scalar $\text{SSNI}_{(\alpha=2,\beta=1)}$ systems that are not $\text{SSNI}_{(\alpha=1,\beta=1)}$.

Since $M(s)$ is $\text{SSNI}_{(\alpha=2,\beta=1)}$, $\exists\,\varepsilon > 0$ such that $j\omega[M(j\omega) - M(j\omega)^*] \geq \varepsilon\left(\frac{\omega^2}{1+\omega^4}\right)\forall\omega \in \mathbb{R}\cup\{\infty\}$. Since $M(s)$ is not $\text{SSNI}_{(\alpha=1,\beta=1)}$, $\nexists\,\hat{\varepsilon} > 0$ such that $j\omega[M(j\omega) - M(j\omega)^*] \geq \hat{\varepsilon}\left(\frac{\omega^2}{1+\omega^2}\right)\ \forall\omega \in \mathbb{R}\cup\{\infty\}$. These two conditions together give $\lim_{\omega\to\infty}[j\omega[M(j\omega) - M(j\omega)^*]] = 0$, which in turn implies that $\lim_{\omega\to\infty}[j\omega\bar{M}(j\omega)] + \lim_{\omega\to\infty}[j\omega\bar{M}(j\omega)]^* = 0$, where $\bar{M}(s) = M(s) - M(\infty)$. Since $M(s)$ is scalar and $\lim_{\omega\to\infty}[j\omega\bar{M}(j\omega)]$ is real, we get $\lim_{\omega\to\infty}[j\omega\bar{M}(j\omega)] = 0$, which is equivalent to $\lim_{s\to\infty}[s\bar{M}(s)] = 0$. The proof is then complete by invoking Lemma 10. ■

The Venn diagram (see Fig. 2) expresses the set-theoretic relationship amongst different subclasses of IONI systems. The strict subclasses are determined via appropriate restrictions on the parameters $\delta \geq 0$, $\varepsilon \geq 0$, $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$ used in (11). Note that only scalar $\text{SSNI}_{(\alpha=2,\beta=1)}$ systems and MIMO $\text{SSNI}_{(\alpha=2,\beta=1)}$ systems that satisfy $\lim_{s\to\infty}[s\bar{M}(s)] = 0$ are guaranteed to be also VSNI via Lemma 10 and Lemma 11; hence, these systems [$\text{SSNI}_{(\alpha=2,\beta=1)}$] have not been illustrated through the Venn Diagram.

In the sequel, we will present six numerical examples corresponding to each region of the Venn diagram in Fig. 2 to illustrate different examples of IONI systems. Note that it is possible to check the strict conditions separately, one at a time, as explained in the next lemma.

Lemma 12: Let $\alpha\in\mathbb{N}$, $\beta\in\mathbb{N}$, $\delta_0 > 0$ and $\varepsilon_0 > 0$. Let $M(s)$ be $\text{IONI}_{(\delta_0,0,\alpha,\beta)}$ and $M(s)$ be $\text{IONI}_{(0,\varepsilon_0,\alpha,\beta)}$. Then, $M(s)$ is $\text{IONI}_{(\delta,\varepsilon,\alpha,\beta)}$ for all $\delta \in [0, \frac{1}{2}\delta_0]$ and all $\varepsilon \in [0, \frac{1}{2}\varepsilon_0]$.

Proof: Since $j\omega[M(j\omega) - M(j\omega)^*] \geq \delta_0\omega^2\bar{M}(j\omega)^*\bar{M}(j\omega)$ and $j\omega[M(j\omega) - M(j\omega)^*] \geq \varepsilon_0\left(\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}\right)I_m$ for all $\omega\in\mathbb{R}\cup\{\infty\}$, it easily follows that $j\omega[M(j\omega) - M(j\omega)^*] = \frac{1}{2}j\omega[M(j\omega)-M(j\omega)^*] + \frac{1}{2}j\omega[M(j\omega)-M(j\omega)^*] \geq \frac{1}{2}\ \delta_0\ \omega^2\ \bar{M}\ (j\omega)^*\ \bar{M}\ (j\omega) + \frac{1}{2}\varepsilon_0\left(\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}\right)I_m \geq \delta\omega^2\bar{M}(j\omega)^*\bar{M}(j\omega) + \varepsilon\left(\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}\right)I_m\ \forall\omega \in \mathbb{R}\cup\{\infty\}$, $\delta \in [0, \frac{1}{2}\delta_0]$ and $\varepsilon \in [0, \frac{1}{2}\varepsilon_0]$. ■

Example 1: Let $M_1(s)=\frac{s+4}{s^2+8s+17}$. Then, $j\omega[M_1(j\omega)-M_1(j\omega)^*]=\frac{2\omega^2(15+\omega^2)}{\omega^4+30\omega^2+289}$ and $\bar{M}_1(j\omega)^*\bar{M}_1(j\omega)=\frac{16+\omega^2}{\omega^4+30\omega^2+289}$ $\forall\omega\in\mathbb{R}\cup\{\infty\}$. $M_1(s)$ belongs to area 1 of the Venn diagram (see Fig. 2) because $\frac{2\omega^2(15+\omega^2)}{\omega^4+30\omega^2+289}\geq\frac{\delta\omega^2(16+\omega^2)}{\omega^4+30\omega^2+289}$ $\forall\omega\in\mathbb{R}\cup\{\infty\}$ when $\delta\in(0,\frac{30}{16}]$; $\frac{2\omega^2(15+\omega^2)}{\omega^4+30\omega^2+289}\geq\varepsilon\frac{\omega^2}{1+\omega^2}$ $\forall\omega\in\mathbb{R}\cup\{\infty\}$ when $\varepsilon\in(0,\frac{30}{289}]$; $\frac{2\omega^2(15+\omega^2)}{\omega^4+30\omega^2+289}\geq\varepsilon\frac{\omega^2}{1+\omega^4}$ $\forall\omega\in\mathbb{R}\cup\{\infty\}$ when $\varepsilon\in(0,\frac{30}{289}]$; and $\frac{2\omega^2(15+\omega^2)}{\omega^4+30\omega^2+289}>0$ $\forall\omega\in(0,\infty)$.

Example 2: Let $M_2(s)=\frac{1}{s^2+s+1}$. Then, $j\omega[M_2(j\omega)-M_2(j\omega)^*]=\frac{2\omega^2}{\omega^4-\omega^2+1}$ and $\bar{M}_2(j\omega)^*\bar{M}_2(j\omega)=\frac{1}{\omega^4-\omega^2+1}$ $\forall\omega\in\mathbb{R}\cup\{\infty\}$. $M_2(s)$ belongs to area 2 of the Venn diagram (see Fig. 2) because $\frac{2\omega^2}{\omega^4-\omega^2+1}\geq\frac{\delta\omega^2}{\omega^4-\omega^2+1}$ $\forall\omega\in\mathbb{R}\cup\{\infty\}$ when $\delta\in(0,2]$; $\frac{2\omega^2}{\omega^4-\omega^2+1}\geq\varepsilon\frac{\omega^2}{1+\omega^2}$ $\forall\omega\in\mathbb{R}\cup\{\infty\}$ only when $\varepsilon=0$; $\frac{2\omega^2}{\omega^4-\omega^2+1}\geq\varepsilon\frac{\omega^2}{1+\omega^4}$ $\forall\omega\in\mathbb{R}\cup\{\infty\}$ when $\varepsilon\in(0,2]$; and $\frac{2\omega^2}{\omega^4-\omega^2+1}>0$ $\forall\omega\in(0,\infty)$.

Example 3: Consider the transfer function $M_3(s)=\frac{s^2+12.5}{s^4+s^3+42.5s^2+12.5s+150}$. Then, long and tedious algebraic manipulations give $j\omega[M_3(j\omega)-M_3(j\omega)^*]=\frac{2\omega^2(12.5-\omega^2)^2}{(\omega^4-42.5\omega^2+150)^2+\omega^2(12.5-\omega^2)^2}$ and $\bar{M}_3(j\omega)^*\bar{M}_3(j\omega)=\frac{(12.5-\omega^2)^2}{(\omega^4-42.5\omega^2+150)^2+\omega^2(12.5-\omega^2)^2}$ $\forall\omega\in\mathbb{R}\cup\{\infty\}$. $M_3(s)$ belongs to area 3 of the Venn diagram (see Fig. 2) since $j\omega[M_3(j\omega)-M_3(j\omega)^*]\geq\delta\omega^2\bar{M}_3(j\omega)^*\bar{M}_3(j\omega)$ for all $\omega\in\mathbb{R}\cup\{\infty\}$ when $\delta\in(0,2]$; $j\omega[M_3(j\omega)-M_3(j\omega)^*]\geq\varepsilon\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}$ $\forall\omega\in\mathbb{R}\cup\{\infty\}$ only when $\varepsilon=0$ as $j\omega[M_3(j\omega)-M_3(j\omega)^*]=0$ at $\omega=\pm\sqrt{12.5}$ rad/s whereas $\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}>0$ at $\omega=\pm\sqrt{12.5}$ rad/s; and at $\omega=\sqrt{12.5}$ rad/s, $j\omega[M_3(j\omega)-M_3(j\omega)^*]\not>0$.

Example 4: Let $M_4(s)=\frac{2s+1}{(s+1)^2}$. Then, $j\omega[M_4(j\omega)-M_4(j\omega)^*]=\frac{4\omega^4}{(1+\omega^2)^2}$ and $\bar{M}_4(j\omega)^*\bar{M}_4(j\omega)=\frac{1+4\omega^2}{(1+\omega^2)^2}$ $\forall\omega\in\mathbb{R}\cup\{\infty\}$. $M_4(s)$ belongs to the area 4 of the Venn diagram (see Fig. 2) because $\frac{4\omega^4}{(1+\omega^2)^2}\geq\delta\omega^2\frac{(1+4\omega^2)}{(1+\omega^2)^2}$ $\forall\omega\in\mathbb{R}\cup\{\infty\}$ only when $\delta=0$; $\frac{4\omega^4}{(1+\omega^2)^2}\geq\varepsilon\frac{\omega^2}{1+\omega^4}$ $\forall\omega\in\mathbb{R}\cup\{\infty\}$ only when $\varepsilon=0$; $\frac{4\omega^4}{(1+\omega^2)^2}\geq\varepsilon\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}$ $\forall\omega\in\mathbb{R}\cup\{\infty\}$ when $\alpha=1$, $\beta=2$ and $\varepsilon\in(0,2]$; and $\frac{4\omega^4}{(1+\omega^2)^2}>0$ $\forall\omega\in(0,\infty)$.

Example 5: The transfer function $M_5(s)=\frac{2s^2+s+1}{(s+1)(2s+1)(s^2+2s+5)}$ belongs to area 5 (i.e., simply a stable NI system without any form of strictness) of the Venn diagram because $j\omega[M_5(j\omega)-M_5(j\omega)^*]=0$ at $\omega=1$ rad/s and $\bar{M}_5(j\omega)^*\bar{M}_5(j\omega)=0.01$ at $\omega=1$ rad/s, which imply that (11) can only be satisfied with $\delta=0$ and $\varepsilon=0$.

Example 6: The transfer functions $M_{6a}(s)=\frac{1}{s}$, $M_{6b}(s)=\frac{1}{s^2+1}$ and $M_{6c}(s)=\frac{1}{s^2}$ belong to area 6 of the Venn diagram (hence do not belong to the IONI class) since they are not asymptotically stable.

V. Connections Between IONI Systems and Dissipativity

Section V-A derives a stable spectral factor of a transfer function associated with the filter term in (11) for usage in the subsequent sections. Section V-B extends the classical notion of dissipativity to include supply rates that involve the time derivative of the system's output taking inspiration from [5], [25], [33] and introduces a new time-domain dissipative framework for characterizing the class of stable IONI systems, including its strict subclasses. In Section V-C, IONI systems are characterized in an equivalent frequency-domain framework with respect to a $(Q(\omega),S(\omega),R(\omega))$-dissipative supply rate.

A. Analysis of the Filter Term Used in Definition 6

In order to establish the connections between the IONI system property (11) and dissipative theory, a new supply rate $w(u,\bar{u},\dot{\bar{y}})$ will be proposed in the sequel to characterize IONI systems in a time-domain dissipative framework. This supply rate involves the input to a physical system (u), an auxiliary input ($\bar{u}$) which is a filtered version of u, and the time-derivative of an auxiliary output ($\dot{\bar{y}}$) where the auxiliary output $\bar{y}=y-M(\infty)u$. In order to obtain $\bar{u}$, a bandpass filter has to be constructed as the stable spectral factor of

$$f(s)=\frac{(-s)^\beta s^\beta}{1+(-s)^{(\alpha+\beta-1)}\,s^{(\alpha+\beta-1)}} \tag{13}$$

where $\alpha\in\mathbb{N}$ and $\beta\in\mathbb{N}$. Note that when $s=j\omega$, $f(j\omega)=\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}$ which is the frequency response function within the last term of (11) associated with ε.

Lemma 13: Let $f(s)=\frac{(-s)^\beta s^\beta}{1+(-s)^{(\alpha+\beta-1)}\,s^{(\alpha+\beta-1)}}$ with $\alpha\in\mathbb{N}$ and $\beta\in\mathbb{N}$. Then, $f(s)$ can be spectral factorized as $f(s)=f_s^\sim(s)f_s(s)$ where $f_s(s)\in\mathscr{RH}_\infty$ is

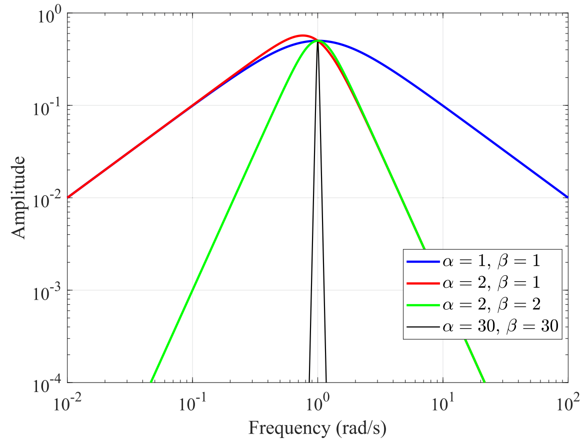

Fig. 3. Log–log frequency plot of the filter function $\bar{f}(\omega) = \frac{1}{\omega^{2\alpha-1}+\frac{1}{\omega^{2\beta-1}}}$ for four different combinations of α and β.

given by

$$f_s(s) = \begin{cases} \dfrac{s}{s+1} \text{ when } \alpha = \beta = 1, \\ \dfrac{s^\beta}{\prod_{i=0}^{\left(\frac{\alpha+\beta-1}{2}-1\right)} \left(s^2 + 2\sin\left[\frac{(2i+1)\pi}{2(\alpha+\beta-1)}\right] s + 1\right)} \\ \text{when } \alpha+\beta \text{ is odd,} \\ \dfrac{s^\beta}{(s+1)\prod_{i=0}^{\left(\frac{\alpha+\beta}{2}-2\right)} \left(s^2 + 2\sin\left[\frac{(2i+1)\pi}{2(\alpha+\beta-1)}\right] s + 1\right)} \\ \text{when } \alpha+\beta \text{ is even and } \alpha+\beta > 2. \end{cases} \quad (14)$$

Proof: Applying the rules of stable spectral factorization [29], [30] of single variable frequency-domain functions in $s \in \mathbb{C}$, $f(s)$ can be factorized as $f(s) = f_s^{\sim}(s) f_s(s)$ where $f_s(s)$ represents the stable spectral factor and $f_s^{\sim}(s) = f_s(-s)$ denotes the corresponding antistable spectral factor. ∎

Fig. 3 shows the frequency plot of the filter function $\bar{f}(\omega) = \dfrac{1}{\omega^{2\alpha-1}+\frac{1}{\omega^{2\beta-1}}} = \dfrac{1}{\omega} f(j\omega)$, on log–log axes, for four different combinations of α and β. The definition of $\bar{f}(\omega)$ guarantees that $\bar{f}(0) = 0$ and $\bar{f}(\infty) = 0$ for any $\alpha, \beta \in \mathbb{N}$. The interpretation of this filter $\bar{f}(\omega)$ [and correspondingly of $f(j\omega)$] is as follows. First, the term $\frac{1}{2\pi}\int_{-\infty}^{\infty} U(j\omega)^*[\varepsilon f(j\omega) I_m] U(j\omega)\, d\omega$ quantifies the input energy dissipation, and hence, it signifies the level of input strictness of an IONI system. Fig. 3 reveals that ISNI systems must have an imaginary-Hermitian frequency response, which is strictly less than zero for all $\omega \in (0, \infty)$ and can only become zero at $\omega = 0$ and $\omega = \infty$. Second, the arrival rate at $\omega = \infty$ and the departure rate at $\omega = 0$ are governed by the parameters α and β, respectively. The arrival (resp. departure) rate at $\omega = \infty$ (resp. $\omega = 0$) is the decay (resp. growth) rate of the imaginary-Hermitian frequency response toward (resp. away from) the real axis near $\omega = \infty$ (resp. $\omega = 0$).

B. IONI Systems in a Time-Domain Dissipative Framework

In this section, we will establish that for an initially relaxed IONI system with a controllable state-space, there always exists a positive semidefinite storage function $V(x)$ such that the system satisfies the dissipation inequality (2) with a particular time-domain supply rate $w(u, \bar{u}, \dot{\bar{y}}) = 2\dot{\bar{y}}^\top u - \delta \dot{\bar{y}}^\top \dot{\bar{y}} - \varepsilon \bar{u}^\top \bar{u}$ for some $\delta \geq 0$ and $\varepsilon \geq 0$, where $\bar{y} = y - M(\infty)u$ is defined as an auxiliary output of M and $\bar{u}$ is a filtered auxiliary input chosen as the inverse Laplace of $\bar{U}(s) = [f_s(s) I_m] U(s)$ where $U(s) = \mathscr{L}[u(t)]$ and $f_s(s) \in \mathscr{RH}_\infty$ is defined in (14). Note that in this section, the admissible inputs u are considered to be in the space $\mathbb{L}_2^m$ along with sufficient smoothness properties such that a unique solution of the state trajectory $x(t)$ exists forward in time $t \geq 0$ and also (since A will be assumed Hurwitz) $x \in \mathbb{L}_2^n$. Hence, $\dot{\bar{y}}(t) = C\dot{x}(t) = CAx(t) + CBu(t)$ also exists for all $t \geq 0$ and $\dot{\bar{y}} \in \mathbb{L}_2^m$. Furthermore, $\bar{u} \in \mathbb{L}_2^m$ since $f_s(s) \in \mathscr{RH}_\infty$ and since $u \in \mathbb{L}_2^m$ by assumption.

Theorem 2: Let M be a finite-dimensional, causal and square system given by the minimal state-space equations $\dot{x} = Ax + Bu$ and $y = Cx + Du$ with zero initial condition. Let the associated transfer function matrix be $M(s) \in \mathscr{RH}_\infty^{m\times m}$. Define $\bar{y} = y - Du$ and $\bar{u} = \mathscr{L}^{-1}[f_s(s) I_m] \star u$ where $f_s(s) \in \mathscr{RH}_\infty$ is defined in (14). Let $\delta \geq 0$, $\varepsilon \geq 0$, $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$. Then, $D = D^\top$ and M is dissipative with respect to the supply rate $w(u, \bar{u}, \dot{\bar{y}}) = 2\dot{\bar{y}}^\top u - \delta \dot{\bar{y}}^\top \dot{\bar{y}} - \varepsilon \bar{u}^\top \bar{u}$ if and only if $M(s)$ is IONI$_{(\delta,\varepsilon,\alpha,\beta)}$.

Proof: The proof has been divided into the sufficiency and necessary parts as follows.

(*Sufficiency*) First note that $M(s)$ is IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ implies that $M(s)$ is stable NI, which in turn implies $D = D^\top$ [1]. To show that an IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ system M is dissipative with respect to the supply rate $w(u, \bar{u}, \dot{\bar{y}}) = 2\dot{\bar{y}}^\top u - \delta \dot{\bar{y}}^\top \dot{\bar{y}} - \varepsilon \bar{u}^\top \bar{u}$, we have to establish that there exists a storage function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that M satisfies the dissipation inequality (2). Since the state space is assumed to be completely controllable, there exists an admissible input $u(t)$ defined as

$$u(t) = \begin{cases} 0 & \text{when } t < t_{-1}, \\ \tilde{u}(t) & \text{when } t_{-1} \leq t \leq 0, \\ 0 & \text{when } t > 0, \end{cases}$$

which steers the system from $x(t_{-1}) = 0$ to any $x(0) \in \mathbb{R}^n$. Let $y(t)$ be the corresponding output and $Y(j\omega)$, $\bar{Y}(j\omega)$, $U(j\omega)$ and $\bar{U}(j\omega)$ denote, respectively, the Fourier transform of the real-valued time-domain signals $y(t)$, $\bar{y}(t)$, $u(t)$ and $\bar{u}(t)$. Also, $\bar{Y}(j\omega) = Y(j\omega) - DU(j\omega) = \bar{M}(j\omega)U(j\omega)$, where $\bar{M}(j\omega) = M(j\omega) - D$ and $\bar{U}(j\omega) = [f_s(j\omega) I_m] U(j\omega)$. Now,

$$\begin{aligned} \int_{t_{-1}}^{0} w(u, \bar{u}, \dot{\bar{y}})\, dt &= \int_{t_{-1}}^{0} (2\dot{\bar{y}}^\top u - \delta \dot{\bar{y}}^\top \dot{\bar{y}} - \varepsilon \bar{u}^\top \bar{u})\, dt \\ &= \int_{-\infty}^{\infty} (2\dot{\bar{y}}^\top u - \delta \dot{\bar{y}}^\top \dot{\bar{y}} - \varepsilon \bar{u}^\top \bar{u})\, dt + \delta \int_{0}^{\infty} \dot{\bar{y}}^\top \dot{\bar{y}}\, dt \\ &\quad + \varepsilon \int_{0}^{\infty} \bar{u}^\top \bar{u}\, dt \quad [\text{since } M \text{ is causal and time-invariant}] \\ &\geq \int_{-\infty}^{\infty} (2\dot{\bar{y}}^\top u - \delta \dot{\bar{y}}^\top \dot{\bar{y}} - \varepsilon \bar{u}^\top \bar{u})\, dt \quad [\text{since } \delta \geq 0 \text{ and } \varepsilon \geq 0] \end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\big(j\omega\bar{Y}(j\omega)\big)^* U(j\omega) + U(j\omega)^*\big(j\omega\bar{Y}(j\omega)\big) - \delta\,\omega^2\,\bar{Y}(j\omega)^*\bar{Y}(j\omega) - \varepsilon\,\bar{U}(j\omega)^*\bar{U}(j\omega)\Big]\,\mathrm{d}\omega$$

[since A is Hurwitz and applying Parseval's theorem [4]]

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}U(j\omega)^*\Big[j\omega\{M(j\omega)-M(j\omega)^*\} - \delta\,\omega^2\bar{M}(j\omega)^* \times \bar{M}(j\omega) - \varepsilon\left(\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}\right)I_m\Big]U(j\omega)\,\mathrm{d}\omega$$

[since $M(\infty)=M(\infty)^\top$ is implied by (11)]

≥ 0 [using Definition 6].

Hence, for arbitrary $t_{-1}\leq 0$ and $x(t_{-1})=0$, we have $\int_{t_{-1}}^{0} w(u,\bar{u},\dot{\bar{y}})\,\mathrm{d}t \geq 0$. We now construct the required supply function as $V_r(x) = \inf_{\substack{0=x^*\to x\\ u(\cdot),\, t_{-1}\leq 0}} \int_{t_{-1}}^{0} w(u,\bar{u},\dot{\bar{y}})\,\mathrm{d}t \geq 0$, where the origin is the point of minimum storage (i.e., $x^*=0$). Thus, $V_r(x)$ can be considered as a storage function candidate associated with the $\mathrm{IONI}_{(\delta,\varepsilon,\alpha,\beta)}$ system M [33].

It remains to be shown that $V_r(x)$ satisfies the dissipation inequality (2). Note that in taking the system from $x=0$ at $t=0$ to $x_1\in\mathbb{R}^n$ at $t=t_1$, we could first take it to $x_0\in\mathbb{R}^n$ at time t_0 while minimizing the energy and then take it to x_1 at time t_1 along the path for which the dissipation inequality is to be evaluated. This is possible since M is a causal and time-invariant system. As $V_r(x_1)$ represents the infimum amount of energy required to reach x_1 at $t=t_1$ from $x=0$ at $t=0$, the energy required to reach the same destination x_1 from the same starting point $x=0$ via any other path will be greater than or equal to $V_r(x_1)$. Therefore, $V_r(x_0)+\int_{t_0}^{t_1} w(u,\bar{u},\dot{\bar{y}})\,\mathrm{d}t \geq V_r(x_1)$ follows. It can, hence, be concluded that the $\mathrm{IONI}_{(\delta,\varepsilon,\alpha,\beta)}$ system M is dissipative with respect to the supply rate $w(u,\bar{u},\dot{\bar{y}}) = 2\dot{\bar{y}}^\top u - \delta\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\bar{u}^\top\bar{u}$ for the same δ, ε, α and β.

(*Necessity*) This part proceeds through a sequence of implications, where frequency-domain integrals with limits from $-\infty$ to ∞ are considered taking inspiration from similar arguments used in [19] and [27]. For the same choice of $\delta\geq 0$, $\varepsilon\geq 0$, and $\alpha,\ \beta\in\mathbb{N}$, and since $M(s)\in\mathscr{RH}_\infty^{m\times m}$,

M is dissipative with respect to

$$w(u,\bar{u},\dot{\bar{y}}) = 2\dot{\bar{y}}^\top u - \delta\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\bar{u}^\top\bar{u}$$

$\Leftrightarrow \exists\, V:\mathbb{R}^n\to\mathbb{R}_{\geq 0}$ such that

$$\int_0^\top \left(2\dot{\bar{y}}^\top u - \delta\,\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\,\bar{u}^\top\bar{u}\right)\mathrm{d}t \geq V(x(T)) - V(x(0)) \quad \forall T\in[0,\infty) \text{ and } \forall u\in\mathbb{L}_2^m$$

$$\Rightarrow \int_0^\infty \left(2\dot{\bar{y}}^\top u - \delta\,\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\,\bar{u}^\top\bar{u}\right)\mathrm{d}t \geq 0 \qquad \forall u\in\mathbb{L}_2^m$$

[since M is stable and $V(x(\infty))=V(0)=V(x(0))$]

$$\Leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\big(j\omega\bar{Y}(j\omega)\big)^* U(j\omega) + U(j\omega)^*\big(j\omega\bar{Y}(j\omega)\big) - \delta\,\omega^2\,\bar{Y}(j\omega)^*\bar{Y}(j\omega) - \varepsilon\,\bar{U}(j\omega)^*\bar{U}(j\omega)\Big]\,\mathrm{d}\omega \geq 0$$

$\forall U\in\mathscr{L}_2^m(j\mathbb{R})$ [via Parseval's theorem [4]]

$$\Leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}U(j\omega)^*\Big[j\omega[M(j\omega)-M(j\omega)^*] - \delta\,\omega^2\bar{M}(j\omega)^* \times \bar{M}(j\omega) - \varepsilon\left(\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}\right)I_m\Big]U(j\omega)\,\mathrm{d}\omega \geq 0 \quad \forall U\in\mathscr{L}_2^m(j\mathbb{R}) \tag{15}$$

$$\Leftrightarrow \Big[j\omega[M(j\omega)-M(j\omega)^*] - \delta\,\omega^2\bar{M}(j\omega)^*\bar{M}(j\omega) - \varepsilon\left(\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}\right)\Big]I_m \geq 0 \quad \forall\omega\in\mathbb{R}\cup\{\infty\}. \tag{16}$$

The equivalence between (15) and (16) is outlined here: (16) $\Rightarrow$ (15) is straightforward and (15) $\Rightarrow$ (16) follows, for example, from the necessity proof of [4, Th. 2.6]. ■

The following corollary is an immediate consequence of Theorem 2 and establishes the time-domain dissipativity of all stable NI systems and also the strict subclasses (e.g., ISNI, $\mathrm{SSNI}_{(\alpha=1,\beta=1)}$, $\mathrm{SSNI}_{(\alpha=2,\beta=1)}$, VSNI, OSNI) under the NI systems class.

Corollary 1: Let M be a finite-dimensional, causal and square system given by the minimal state-space equations $\dot{x}=Ax+Bu$ and $y=Cx+Du$ with $x(0)=0$. Let the associated transfer function matrix be $M(s)\in\mathscr{RH}_\infty^{m\times m}$. Define $\bar{y}=y-Du$ and $\bar{u}=\mathscr{L}^{-1}[f_s(s)I_m]\star u$ where $f_s(s)\in\mathscr{RH}_\infty$ is defined in (14). Then,

1) M is stable NI if and only if $M(\infty)=M(\infty)^\top$ and M is dissipative with respect to $w(u,\dot{\bar{y}}) = 2\dot{\bar{y}}^\top u$;
2) M is ISNI if and only if $M(\infty)=M(\infty)^\top$ and there exist $\varepsilon>0$, $\alpha\in\mathbb{N}$ and $\beta\in\mathbb{N}$ such that M is dissipative with respect to $w(u,\bar{u},\dot{\bar{y}}) = 2\dot{\bar{y}}^\top u - \varepsilon\bar{u}^\top\bar{u}$;
3) M is $\mathrm{SSNI}_{(\alpha=1,\beta=1)}$ if and only if $M(\infty)=M(\infty)^\top$ and there exists $\varepsilon>0$ such that M is dissipative with respect to $w(u,\bar{u},\dot{\bar{y}}) = 2\dot{\bar{y}}^\top u - \varepsilon\bar{u}^\top\bar{u}$ with $\alpha=1$ and $\beta=1$;
4) M is $\mathrm{SSNI}_{(\alpha=2,\beta=1)}$ if and only if $M(\infty)=M(\infty)^\top$ and there exists $\varepsilon>0$ such that M is dissipative with respect to $w(u,\bar{u},\dot{\bar{y}}) = 2\dot{\bar{y}}^\top u - \varepsilon\bar{u}^\top\bar{u}$ with $\alpha=2$ and $\beta=1$;
5) M is VSNI if and only if $M(\infty)=M(\infty)^\top$ and there exist $\delta>0$, $\varepsilon>0$, $\alpha\in\mathbb{N}$ and $\beta\in\mathbb{N}$ such that M is dissipative with respect to $w(u,\bar{u},\dot{\bar{y}}) = 2\dot{\bar{y}}^\top u - \delta\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\bar{u}^\top\bar{u}$; and
6) M is OSNI if and only if $[M(s)-M^\sim(s)]$ has full normal rank, $M(\infty)=M(\infty)^\top$ and there exists $\delta>0$ such that M is dissipative with respect to $w(u,\dot{\bar{y}}) = 2\dot{\bar{y}}^\top u - \delta\dot{\bar{y}}^\top\dot{\bar{y}}$.

Proof: Trivial restriction of Theorem 2 by setting appropriate choices of the parameters $\delta\geq 0$, $\varepsilon\geq 0$, $\alpha\in\mathbb{N}$ and $\beta\in\mathbb{N}$. ■

The following lemma gives a necessary and sufficient condition for checking time-domain dissipativity of an $\mathrm{IONI}_{(\delta,\varepsilon,\alpha,\beta)}$ system without involving a storage function.

Lemma 14: Let $M(s)\in\mathscr{RH}_\infty^{m\times m}$ be the transfer function matrix of a finite-dimensional, causal and initially relaxed system M. Let $y=\mathscr{L}^{-1}[M(s)]\star u$ with $u\in\mathbb{L}_2^m$. Define $\bar{y}=y-M(\infty)u$ and $\bar{u}=\mathscr{L}^{-1}[f_s(s)I_m]\star u$ where $f_s(s)\in\mathscr{RH}_\infty$ is defined in (14). Let $\delta\geq 0$, $\varepsilon\geq 0$, $\alpha\in\mathbb{N}$ and $\beta\in\mathbb{N}$. Then, $M(s)$ is $\mathrm{IONI}_{(\delta,\varepsilon,\alpha,\beta)}$ if and only if $M(\infty)=M(\infty)^\top$ and

$$\int_0^\infty \left(2\dot{\bar{y}}^\top u - \delta\,\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\,\bar{u}^\top\bar{u}\right)\mathrm{d}t \geq 0 \qquad \forall u\in\mathbb{L}_2^m.$$

Proof: The proof readily follows from the necessity part of the proof of Theorem 2 and Definition 6. ■

C. IONI Systems in a Frequency-Domain Dissipative Framework

In this section, $\mathrm{IONI}_{(\delta,\varepsilon,\alpha,\beta)}$ systems are characterized in a frequency-domain dissipative framework with respect to a $(Q(\omega), S(\omega), R(\omega))$-dissipative supply rate that has a strong connection with the time-domain supply rate $w(u,\bar{u},\dot{\bar{y}}) = 2\dot{\bar{y}}^\top u - \delta\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\bar{u}^\top\bar{u}$ introduced in Section V-B. Here, the matrices $Q(\omega) \in \mathbb{R}^{m\times m}$, $S(\omega) \in \mathbb{C}^{m\times m}$ and $R(\omega) \in \mathbb{R}^{m\times m}$ $\forall\omega \in \mathbb{R}$ can be viewed as frequency-domain operators.

Theorem 3: Let $M(s) \in \mathscr{RH}_\infty^{m\times m}$ be the transfer function matrix of a finite-dimensional, causal and initially relaxed system M. Define $D = M(\infty)$. Let $\delta \geq 0$, $\varepsilon \geq 0$, $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$. Then, $M(s)$ is $\mathrm{IONI}_{(\delta,\varepsilon,\alpha,\beta)}$ if and only if $D = D^\top$ and M is $(Q(\omega), S(\omega), R(\omega))$-dissipative with $Q(\omega) = -\delta\omega^2 I_m$, $S(\omega) = -j\omega I_m + \delta\omega^2 D$ and $R(\omega) = -\delta\omega^2 D^\top D - \varepsilon\left(\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}\right) I_m$ $\forall\omega \in \mathbb{R}$.

Proof: First note that $M(s) \in \mathrm{IONI}_{(\delta,\varepsilon,\alpha,\beta)}$ implies that $M(s)$ is stable NI which, in turn, implies $D = D^\top$ [1]. Let $f_s(s)$ be defined as in (14), $y = \mathscr{L}^{-1}[M(s)] \star u$, $\bar{y} = y - Du$ and $\bar{u} = \mathscr{L}^{-1}[f_s(s)I_m] \star u$ with $u \in \mathbb{L}_2^m$. Then,

$$\begin{aligned}
&M(s) \text{ is } \mathrm{IONI}_{(\delta,\varepsilon,\alpha,\beta)}\\
&\Leftrightarrow \int_0^\infty \left(2\dot{\bar{y}}^\top u - \delta\,\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\,\bar{u}^\top\bar{u}\right)\mathrm{d}t \geq 0 \qquad \forall u \in \mathbb{L}_2^m\\
&\qquad \text{[via Lemma 14]}\\
&\Leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[(j\omega\bar{Y}(j\omega))^*U(j\omega) + U(j\omega)^*(j\omega\bar{Y}(j\omega))\\
&\quad - \delta\,(j\omega\bar{Y}(j\omega))^*(j\omega\bar{Y}(j\omega)) - \varepsilon\,\bar{U}(j\omega)^*\bar{U}(j\omega)\Big]\mathrm{d}\omega \geq 0\\
&\quad \forall U \in \mathscr{L}_2^m(j\mathbb{R}) \quad \text{[by applying Parseval's theorem [4]]}\\
&\Leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[Y(j\omega)^*(-\delta\omega^2 I_m)Y(j\omega) + Y(j\omega)^*(-j\omega I_m\\
&\quad + \delta\omega^2 D)U(j\omega) + U(j\omega)^*(j\omega I_m + \delta\omega^2 D^\top)Y(j\omega)\\
&\quad + U(j\omega)^*\Big\{-\delta\omega^2 D^\top D - \varepsilon\left(\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}\right) I_m\Big\}U(j\omega)\Big]\mathrm{d}\omega\\
&\quad \geq 0 \ \forall U \in \mathscr{L}_2^m(j\mathbb{R})\\
&\quad \text{[substituting } \bar{Y}(j\omega) = Y(j\omega) - DU(j\omega) \text{ and } \bar{U}(j\omega)\\
&\quad = (f_s(j\omega)I_m)U(j\omega)\text{]}\\
&\Leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[Y(j\omega)^*Q(\omega)Y(j\omega) + Y(j\omega)^*S(\omega)U(j\omega)\\
&\quad + U(j\omega)^*S(\omega)^*Y(j\omega) + U(j\omega)^*R(\omega)U(j\omega)\Big]\mathrm{d}\omega \geq 0\\
&\quad \forall U \in \mathscr{L}_2^m(j\mathbb{R})\\
&\Leftrightarrow M \text{ is } (Q(\omega), S(\omega), R(\omega))\text{-dissipative.}
\end{aligned}$$

■

The following corollary is an immediate consequence of Theorem 3 and shows that all stable NI systems and also the strict subclasses (i.e., ISNI, $\mathrm{SSNI}_{(\alpha=1,\beta=1)}$, $\mathrm{SSNI}_{(\alpha=2,\beta=1)}$, VSNI, OSNI) exhibit frequency-domain $(Q(\omega), S(\omega), R(\omega))$-dissipativity.

Corollary 2: Let $M(s) \in \mathscr{RH}_\infty^{m\times m}$ be the transfer function matrix of a finite-dimensional, causal and initially relaxed system M. Define $D = M(\infty)$. Then,

1) M is stable NI if and only if $D = D^\top$ and M is $(0, -j\omega I_m, 0)$-dissipative;
2) M is ISNI if and only if $D = D^\top$ and there exist $\varepsilon > 0$, $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$ such that M is $(0, -j\omega I_m, -\varepsilon\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}} I_m)$-dissipative;
3) M is $\mathrm{SSNI}_{(\alpha=1,\beta=1)}$ if and only if $D = D^\top$ and there exists $\varepsilon > 0$ such that M is $(0, -j\omega I_m, -\varepsilon\frac{\omega^2}{1+\omega^2} I_m)$-dissipative;
4) M is $\mathrm{SSNI}_{(\alpha=2,\beta=1)}$ if and only if $D = D^\top$ and there exists $\varepsilon > 0$ such that M is $(0, -j\omega I_m, -\varepsilon\frac{\omega^2}{1+\omega^4} I_m)$-dissipative;
5) M is VSNI if and only if $D = D^\top$ and there exist $\delta > 0$, $\varepsilon > 0$, $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$ such that M is $(-\delta\omega^2 I_m, -j\omega I_m + \delta\omega^2 D, -\delta\omega^2 D^\top D - \varepsilon\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}} I_m)$-dissipative; and
6) M is OSNI if and only if $[M(s) - M^\sim(s)]$ has full normal rank, $D = D^\top$ and there exists $\delta > 0$ such that M is $(-\delta\omega^2 I_m, -j\omega I_m + \delta\omega^2 D, -\delta\omega^2 D^\top D)$-dissipative.

Proof: Trivial restriction of Theorem 3 for appropriate choices of $\delta \geq 0$, $\varepsilon \geq 0$, $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$. ■

Remark 2: In [19]–[21], a frequency-domain dissipation inequality $\frac{1}{2\pi}\int_{-\infty}^{\infty}[Y(j\omega)^*(-j\omega I_m)U(j\omega) + U(j\omega)^*(j\omega I_m) \times Y(j\omega) - U(j\omega)^*(\varepsilon I_m)U(j\omega)]\mathrm{d}\omega \geq 0$ with $\varepsilon \geq 0$ was used which is equivalent to satisfying the time-domain dissipation inequality $\int_0^\top(2\dot{y}^\top u - \varepsilon u^\top u)\,\mathrm{d}t \geq 0$. However, Bhowmick and Patra [18] showed that there are no asymptotically stable systems satisfying the above dissipation inequality with $\varepsilon > 0$. In order to resolve this issue, in this article, a filtered version of the input has been chosen as $\bar{U}(j\omega) = [f_s(j\omega)I_m]U(j\omega)$ to construct the ISNI supply rate where $f_s(s)$ is a stable spectral factor of the transfer function $\frac{(-s)^\beta s^\beta}{1+(-s)^{(\alpha+\beta-1)}\,s^{(\alpha+\beta-1)}}$ as defined in (14). This leads to a new time-domain dissipation inequality $\int_0^\top(2\dot{\bar{y}}^\top u - \varepsilon\bar{u}^\top\bar{u})\,\mathrm{d}t \geq 0$ different from [19]–[21] to characterize ISNI systems [see Example 4 for an example system that satisfies the new dissipation inequality].

The following corollary provides the connection between the time-domain and frequency-domain dissipativity.

Corollary 3: Let the suppositions of Theorem 2 hold and $D = D^\top$. Then, M is dissipative with respect to the supply rate $w(u,\bar{u},\dot{\bar{y}}) = 2\dot{\bar{y}}^\top u - \delta\,\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\,\bar{u}^\top\bar{u}$ if and only if M is $(Q(\omega), S(\omega), R(\omega))$-dissipative with $Q(\omega) = -\delta\omega^2 I_m$, $S(\omega) = -j\omega I_m + \delta\omega^2 D$ and $R(\omega) = -\delta\omega^2 D^\top D - \varepsilon\left(\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}\right) I_m$ $\forall\omega \in \mathbb{R}$.

Proof: Trivial from Theorems 2 and 3. ■

VI. State-Space Characterization of IONI Systems

In this section, we provide a state-space characterization of the full class of IONI systems. The state-space realizations in the results of this section are not required to be minimal.

Theorem 4 (IONI Lemma): Let $M(s) \in \mathscr{RH}_\infty^{m\times m}$ have an arbitrary state-space representation (A, B, C, D) with $\det[j\omega I - A] \neq 0\ \forall \omega \in \mathbb{R}$. Let $\delta \geq 0$, $\varepsilon \geq 0$, $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$. Let (A_f, B_f, C_f, D_f) be an arbitrary state-space representation of $f_s(s) \in \mathscr{RH}_\infty$, as defined in (14), with $\det[j\omega I - A_f] \neq 0\ \forall \omega \in \mathbb{R}$. Let $\begin{bmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C}_1 & \mathscr{D}_1 \\ \mathscr{C}_2 & \mathscr{D}_2 \end{bmatrix} = \left[\begin{array}{cc:c} A & 0 & B \\ 0 & I_m \otimes A_f & I_m \otimes B_f \\ \hdashline CA & 0 & CB \\ \hdashline 0 & I_m \otimes C_f & I_m \otimes D_f \end{array}\right]$ and $(\mathscr{A}, \mathscr{B})$ be controllable. Then, $M(s)$ is $\text{IONI}_{(\delta,\varepsilon,\alpha,\beta)}$ if and only if $D = D^\top$ and there exists $X = X^\top$ such that

$$\text{sym}\begin{bmatrix} X\mathscr{A} & X\mathscr{B} \\ \mathscr{C}_1 & \mathscr{D}_1 \end{bmatrix} \geq \delta\, \text{sqr}\,[\mathscr{C}_1 \quad \mathscr{D}_1] + \varepsilon\, \text{sqr}\,[\mathscr{C}_2 \quad \mathscr{D}_2]. \quad (17)$$

Proof: Let $\bar{M}(s) = M(s) - D$. Then,

$$M(s) \text{ is } \text{IONI}_{(\delta,\varepsilon,\alpha,\beta)}$$

$$\Leftrightarrow \Big[j\omega[M(j\omega) - M(j\omega)^*] - \delta\omega^2 \bar{M}(j\omega)^*\bar{M}(j\omega) - \varepsilon f_s(j\omega)^* f_s(j\omega) I_m \Big] \geq 0 \quad \forall \omega \in \mathbb{R} \cup \{\infty\}$$

$\Leftrightarrow D = D^\top$ and

$$\Big[\big(j\omega\bar{M}(j\omega)\big) + \big(j\omega\bar{M}(j\omega)\big)^* - \delta\big(j\omega\bar{M}(j\omega)\big)^*\big(j\omega\bar{M}(j\omega)\big) - \varepsilon f_s(j\omega)^* f_s(j\omega) I_m \Big] \geq 0 \ \ \forall \omega \in \mathbb{R} \cup \{\infty\}$$

$\Leftrightarrow D = D^\top$ and

$$\begin{bmatrix} j\omega\bar{M}(j\omega) \\ f_s(j\omega) I_m \\ I_m \end{bmatrix}^* \begin{bmatrix} -\delta I_m & 0 & I_m \\ 0 & -\varepsilon I_m & 0 \\ I_m & 0 & 0 \end{bmatrix} \times \begin{bmatrix} j\omega\bar{M}(j\omega) \\ f_s(j\omega) I_m \\ I_m \end{bmatrix} \geq 0 \ \ \forall \omega \in \mathbb{R} \cup \{\infty\}$$

$\Leftrightarrow D = D^\top$ and

$$\begin{bmatrix} (j\omega I - \mathscr{A})^{-1}\mathscr{B} \\ I_m \end{bmatrix}^* \begin{bmatrix} \mathscr{C}_1^\top & \mathscr{C}_2^\top & 0 \\ \mathscr{D}_1^\top & \mathscr{D}_2^\top & I_m \end{bmatrix} \times \begin{bmatrix} -\delta I_m & 0 & I_m \\ 0 & -\varepsilon I_m & 0 \\ I_m & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathscr{C}_1 & \mathscr{D}_1 \\ \mathscr{C}_2 & \mathscr{D}_2 \\ 0 & I_m \end{bmatrix} \times \begin{bmatrix} (j\omega I - \mathscr{A})^{-1}\mathscr{B} \\ I_m \end{bmatrix} \geq 0 \quad \forall \omega \in \mathbb{R} \cup \{\infty\}$$

$\Leftrightarrow D = D^\top$ and $\exists\ X = X^\top$ such that

$$\begin{bmatrix} X\mathscr{A} + \mathscr{A}^\top X & X\mathscr{B} \\ \mathscr{B}^\top X & 0 \end{bmatrix} + \begin{bmatrix} 0 & \mathscr{C}_1^\top \\ \mathscr{C}_1 & \mathscr{D}_1 + \mathscr{D}_1^\top \end{bmatrix} + \begin{bmatrix} -\delta\,\mathscr{C}_1^\top\mathscr{C}_1 - \varepsilon\,\mathscr{C}_2^\top\mathscr{C}_2 & -\delta\,\mathscr{C}_1^\top\mathscr{D}_1 - \varepsilon\,\mathscr{C}_2^\top\mathscr{D}_2 \\ -\delta\,\mathscr{D}_1^\top\mathscr{C}_1 - \varepsilon\,\mathscr{D}_2^\top\mathscr{C}_2 & -\delta\,\mathscr{D}_1^\top\mathscr{D}_1 - \varepsilon\,\mathscr{D}_2^\top\mathscr{D}_2 \end{bmatrix} \geq 0$$

$\Leftrightarrow D = D^\top$ and $\exists\ X = X^\top$ such that

$$\begin{bmatrix} X\mathscr{A} + \mathscr{A}^\top X & \mathscr{C}_1^\top + X\mathscr{B} \\ \mathscr{C}_1 + \mathscr{B}^\top X & \mathscr{D}_1 + \mathscr{D}_1^\top \end{bmatrix} \geq \delta \begin{bmatrix} \mathscr{C}_1^\top \\ \mathscr{D}_1^\top \end{bmatrix} [\mathscr{C}_1 \quad \mathscr{D}_1] + \varepsilon \begin{bmatrix} \mathscr{C}_2^\top \\ \mathscr{D}_2^\top \end{bmatrix} [\mathscr{C}_2 \quad \mathscr{D}_2].$$ ■

Next, we will specialize the IONI lemma into the important subclasses, viz., ISNI, $\text{SSNI}_{(\alpha=1,\beta=1)}$, $\text{SSNI}_{(\alpha=2,\beta=1)}$, VSNI, and OSNI.

Corollary 4 (ISNI Lemma): Let $M(s) \in \mathscr{RH}_\infty^{m\times m}$ have an arbitrary state-space representation (A, B, C, D) with $\det[j\omega I - A] \neq 0\ \forall \omega \in \mathbb{R}$. Then, $M(s)$ is ISNI if and only if $D = D^\top$ and there exist $\varepsilon > 0$, $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}$ and $X = X^\top$ such that

$$\text{sym}\begin{bmatrix} X\mathscr{A} & X\mathscr{B} & 0 \\ \mathscr{C}_1 & \mathscr{D}_1 & 0 \\ \mathscr{C}_2 & \mathscr{D}_2 & \frac{1}{2\varepsilon} I_m \end{bmatrix} \geq 0 \quad (18)$$

where (A_f, B_f, C_f, D_f) is an arbitrary state-space representation of $f_s(s) \in \mathscr{RH}_\infty$, as defined in (14), with $\det[j\omega I - A_f] \neq 0\ \forall \omega \in \mathbb{R}$ and $\begin{bmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C}_1 & \mathscr{D}_1 \\ \mathscr{C}_2 & \mathscr{D}_2 \end{bmatrix} = \left[\begin{array}{cc:c} A & 0 & B \\ 0 & I_m \otimes A_f & I_m \otimes B_f \\ \hdashline CA & 0 & CB \\ \hdashline 0 & I_m \otimes C_f & I_m \otimes D_f \end{array}\right]$ with $(\mathscr{A}, \mathscr{B})$ controllable.

Proof: Trivial application of Theorem 4 with $\delta = 0$ and taking a Schur complement. Note that ISNI systems require arbitrary $\varepsilon > 0$, $\alpha \in \mathbb{N}$, and $\beta \in \mathbb{N}$. ■

*Corollary 5 (**$\text{SSNI}_{(\alpha=1,\beta=1)}$** Lemma):* Let $M(s) \in \mathscr{RH}_\infty^{m\times m}$ have a controllable state-space representation (A, B, C, D) with $\det[j\omega I - A] \neq 0\ \forall \omega \in \mathbb{R}$ and $\det[I + A] \neq 0$. Let $\begin{bmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C}_1 & \mathscr{D}_1 \\ \mathscr{C}_2 & \mathscr{D}_2 \end{bmatrix} = \left[\begin{array}{cc:c} A & 0 & B \\ 0 & -I_m & I_m \\ \hdashline CA & 0 & CB \\ \hdashline 0 & -I_m & I_m \end{array}\right]$. Then, $M(s)$ is $\text{SSNI}_{(\alpha=1,\beta=1)}$ if and only if $D = D^\top$ and there exist $\varepsilon > 0$ and $X = X^\top$ such that

$$\text{sym}\begin{bmatrix} X\mathscr{A} & X\mathscr{B} \\ \mathscr{C}_1 & \mathscr{D}_1 \end{bmatrix} \geq \varepsilon\, \text{sqr}\,[\mathscr{C}_2 \quad \mathscr{D}_2]. \quad (19)$$

Proof: Trivial application of Theorem 4 with $\delta = 0$, $\alpha = 1$ and $\beta = 1$ on noting that in this case $f_s(s) = \frac{s}{s+1}$ and, hence, $A_f = -1$, $B_f = 1$, $C_f = -1$, $D_f = 1$. Also, note that (A, B) is controllable and $\det[I + A] \neq 0$ are equivalent to $\left(\begin{bmatrix} A & 0 \\ 0 & -I_m \end{bmatrix}, \begin{bmatrix} B \\ I_m \end{bmatrix}\right)$ is controllable. ■

*Corollary 6 (**$\text{SSNI}_{(\alpha=2,\beta=1)}$** Lemma):* Let $M(s) \in \mathscr{RH}_\infty^{m\times m}$ have an arbitrary state-space representation (A, B, C, D) with $\det[j\omega I - A] \neq 0\ \forall \omega \in \mathbb{R}$. Let (A_f, B_f, C_f, D_f) be an arbitrary state-space representation of $f_s(s) = \frac{s}{s^2 + \sqrt{2}s + 1}$ with $\det[j\omega I - A_f] \neq 0\ \forall \omega \in \mathbb{R}$. Let

$$\begin{bmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C}_1 & \mathscr{D}_1 \\ \mathscr{C}_2 & \mathscr{D}_2 \end{bmatrix} = \left[\begin{array}{cc:c} A & 0 & B \\ 0 & I_m \otimes A_f & I_m \otimes B_f \\ \hdashline CA & 0 & CB \\ \hdashline 0 & I_m \otimes C_f & I_m \otimes D_f \end{array}\right]$$ and $(\mathscr{A}, \mathscr{B})$ be controllable. Then, $M(s)$ is $\text{SSNI}_{(\alpha=2,\beta=1)}$ if and only if $D = D^\top$ and there exist $\varepsilon > 0$ and $X = X^\top$ such that

$$\text{sym}\begin{bmatrix} X\mathscr{A} & X\mathscr{B} & 0 \\ \mathscr{C}_1 & \mathscr{D}_1 & 0 \\ \mathscr{C}_2 & \mathscr{D}_2 & \frac{1}{2\varepsilon} I_m \end{bmatrix} \geq 0. \tag{20}$$

Proof: Trivial application of Theorem 4 with $\delta = 0$, $\alpha = 2$ and $\beta = 1$. The LMI is just a Schur complement rearrangement. ■

*Corollary 7 (*VSNI Lemma*):* Let $M(s) \in \mathscr{RH}_\infty^{m\times m}$ have an arbitrary state-space representation (A, B, C, D) with $\det[j\omega I - A] \neq 0\ \forall \omega \in \mathbb{R}$. Then, $M(s)$ is VSNI if and only if $D = D^\top$ and there exist $\delta > 0$, $\varepsilon > 0$, $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}$ and $X = X^\top$ such that

$$\text{sym}\begin{bmatrix} X\mathscr{A} & X\mathscr{B} & 0 & 0 \\ \mathscr{C}_1 & \mathscr{D}_1 & 0 & 0 \\ \mathscr{C}_1 & \mathscr{D}_1 & \frac{1}{2\delta} I_m & 0 \\ \mathscr{C}_2 & \mathscr{D}_2 & 0 & \frac{1}{2\varepsilon} I_m \end{bmatrix} \geq 0 \tag{21}$$

where (A_f, B_f, C_f, D_f) is an arbitrary state-space representation of $f_s(s) \in \mathscr{RH}_\infty$, as defined in (14), with $\det[j\omega I - A_f] \neq 0\ \forall \omega \in \mathbb{R}$ and $$\begin{bmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C}_1 & \mathscr{D}_1 \\ \mathscr{C}_2 & \mathscr{D}_2 \end{bmatrix} = \left[\begin{array}{cc:c} A & 0 & B \\ 0 & I_m \otimes A_f & I_m \otimes B_f \\ \hdashline CA & 0 & CB \\ \hdashline 0 & I_m \otimes C_f & I_m \otimes D_f \end{array}\right]$$ with $(\mathscr{A}, \mathscr{B})$ controllable.

Proof: Trivial application of Theorem 4 and using a Schur complement. ■

Corollary 8 (OSNI Lemma): Let $M(s) \in \mathscr{RH}_\infty^{m\times m}$ have a controllable state-space representation (A, B, C, D) with $\det[j\omega I - A] \neq 0\ \forall \omega \in \mathbb{R}$. Then, $M(s)$ is OSNI if and only if $[M(s) - M^\sim(s)]$ has full normal rank, $D = D^\top$ and there exist $\delta > 0$ and $X = X^\top$ such that

$$\begin{bmatrix} XA + A^\top X & XB + A^\top C^\top & A^\top C^\top \\ B^\top X + CA & CB + B^\top C^\top & B^\top C^\top \\ CA & CB & \frac{1}{\delta} I_m \end{bmatrix} \geq 0. \tag{22}$$

Proof: Trivial application of Theorem 4 with $\varepsilon = 0$ and by removing the states associated with $f_s(s)$. The LMI is just a Schur complement rearrangement. ■

VII. OSNI Systems in a Dissipative Framework

As with other strict subclasses (e.g., ISNI, $\text{SSNI}_{(\alpha=1,\beta=1)}$, $\text{SSNI}_{(\alpha=2,\beta=1)}$, VSNI) under the IONI class, the pointwise frequency-domain condition (11) defines the OSNI subclass when $\delta > 0$. The OSNI class was originally proposed in [18] and generalized later in [8]. OSNI systems exhibit several interesting properties and also obey a simple internal stability condition when interconnected with a (not necessarily stable) NI system in a positive feedback loop. This section is dedicated solely to the OSNI class of systems to 1) develop a minimal state-space characterization for OSNI systems, 2) describe OSNI systems both in the time-domain and the frequency-domain dissipative frameworks, and 3) establish the equivalence between the OSNI lemma conditions and the time-domain dissipative characterization of OSNI systems.

A. Specialized OSNI Lemma for Minimal Systems

We first show the connection between OSNI and OSP systems.

Lemma 15: Let $M(s) \in \mathscr{RH}_\infty^{m\times m}$ and $\bar{M}(s) = M(s) - M(\infty)$. Then, the following statements hold:

1) $M(s)$ is OSNI if and only if $F(s) = s\bar{M}(s)$ is OSP and $M(\infty) = M(\infty)^\top$;
2) $M(s)$ is stable NI if and only if $F(s) = s\bar{M}(s)$ is passive and $M(\infty) = M(\infty)^\top$.

Proof: Since the identity $F(j\omega) + F(j\omega)^* - \delta F(j\omega)^* F(j\omega) = (j\omega\bar{M}(j\omega)) + (j\omega\bar{M}(j\omega))^* - \delta(j\omega\bar{M}(j\omega))^*(j\omega\bar{M}(j\omega)) = j\omega[M(j\omega) - M(j\omega)^*] - \delta\omega^2\bar{M}(j\omega)^*\bar{M}(j\omega)$ holds for all $\omega \in \mathbb{R} \cup \{\infty\}$ when $M(\infty) = M(\infty)^\top$, the equivalence in Part 1) [resp. Part 2)] simply follows on choosing $\delta > 0$ [resp. $\delta = 0$]. ■

The following lemma provides a necessary and sufficient condition for a system given by a minimal state-space realization to be OSNI and is a generalization of [18, Lemma 6].

Lemma 16 (OSNI Lemma: Specialized): Let $M(s) \in \mathscr{RH}_\infty^{m\times m}$ have a minimal state-space realization (A, B, C, D). Then, $M(s)$ is OSNI if and only if $[M(s) - M^\sim(s)]$ has full normal rank, $D = D^\top$ and there exist $\delta > 0$ and $Y = Y^\top > 0$ such that

$$AY + YA^T + \delta(CAY)^\top CAY \leq 0 \ \text{ and } \ B + AYC^T = 0. \tag{23}$$

Proof: Since the realization is minimal, A is Hurwitz and hence nonsingular. Then, LMI (22) in Corollary 8 is equivalent to

$$\begin{bmatrix} XA + A^\top X & XB + A^\top C^\top \\ B^\top X + CA & CB + B^\top C^\top \end{bmatrix} - \delta \begin{bmatrix} A^\top C^\top \\ B^\top C^\top \end{bmatrix} [CA \quad CB] \geq 0 \tag{24}$$

by taking a Schur complement with respect to $\frac{1}{\delta} I_m$. This condition then implies that $X < 0$ via $XA + A^\top X \geq \delta A^\top C^\top CA$. Let $Y = -X^{-1}$. Then (24) is equivalent to

$$\begin{bmatrix} \begin{matrix}(-AY - YA^T \\ -\delta\,(CAY)^T(CAY))\end{matrix} & \begin{matrix}(B - YA^TC^T \\ +\delta\, YA^TC^TCB)\end{matrix} \\ \begin{matrix}(B - YA^TC^T \\ +\delta\, YA^TC^TCB)^\top\end{matrix} & \begin{matrix}(CB + B^TC^T \\ -\delta\, B^TC^TCB)\end{matrix} \end{bmatrix} \geq 0$$

which is, in turn, equivalent to

$$AY + YA^T + \delta(CAY)^\top(CAY) \leq 0 \text{ and } B + AYC^\top = 0$$

via a simple congruence transformation, that is, premultiplying with $\begin{pmatrix} I & 0 \\ -C & I \end{pmatrix}$ and postmultiplying with $\begin{pmatrix} I & 0 \\ -C & I \end{pmatrix}^\top$. ■

Note that the matrix inequality in (23) is not in LMI form but can be readily converted into an LMI by applying the Schur complement lemma [4, Appendix A.61].

B. Equivalence Between Time-Domain Dissipativity and State-Space Characterization of OSNI Systems

We have already established that OSNI systems are dissipative with respect to the time-domain supply rate $w(u, \dot{\bar{y}}) = 2\dot{\bar{y}}^\top u - \delta\,\dot{\bar{y}}^\top \dot{\bar{y}}$ with $\delta > 0$ where $\bar{y} = y - Du$ is selected as an auxiliary output of the system. In this subsection, we will show that for a stable LTI system with a minimal state-space realization, the conditions in OSNI Lemma 16 are equivalent to time-domain dissipativity with respect to the proposed supply rate $w(u,\dot{\bar{y}})$ and a specific storage function given by $V(x)=x^\top Y^{-1}x$ with $Y=Y^\top>0$.

Lemma 17: Let M be a causal, square, finite-dimensional, LTI system given by the minimal state-space equations $\dot{x}=Ax+Bu$ and $y=Cx+Du$ with $x(0)=0$, A being Hurwitz and $D=D^\top$. Let the associated transfer function matrix be $M(s)$ and define $\bar{y}=y-Du$. Let $[M(s)-M^\sim(s)]$ have full normal rank. Then, the following statements are equivalent:

1) $M(s)$ is OSNI;
2) there exist $\delta>0$ and $Y=Y^\top>0$ such that
$$AY+YA^\top+\delta(CAY)^\top(CAY)\le 0 \text{ and } B=-AYC^\top;$$
3) there exists $\delta>0$ such that M is dissipative with respect to the supply rate $w(u,\dot{\bar{y}})=2\dot{\bar{y}}^\top u-\delta\,\dot{\bar{y}}^\top\dot{\bar{y}}$.

The storage function in part 3) can be chosen as $V(x)=x^\top Y^{-1}x$ with $Y>0$ from part 2).

Proof: 1) $\Leftrightarrow$ 2) This equivalence is due to OSNI Lemma 16.

1) $\Leftrightarrow$ 3) This equivalence is due to Corollary 1, Part 6). $V(x)=x^\top Y^{-1}x$ is an appropriate storage function because the proof of Lemma 16 shows that (23) is equivalent to (24) with $X=-Y^{-1}$, and the conclusion trivially follows by multiplying (24) with $\begin{bmatrix}x\\u\end{bmatrix}$. ■

C. Frequency-Domain Dissipativity of OSNI Systems

Part 6) of Corollary 2 establishes the connection between OSNI systems property and $(Q(\omega),S(\omega),R(\omega))$-dissipativity. In [19], systems with "mixed" negative imaginary and finite-gain properties were defined using a frequency-domain dissipative framework which can be specialized to purely OSNI systems satisfying the inequality $\frac{1}{2\pi}\int_{-\infty}^{\infty}[U(j\omega)^*\{j\omega(M(j\omega)-M(j\omega)^*)-\delta\,\omega^2M(j\omega)^*M(j\omega)\}U(j\omega)]\,d\omega\ge 0\ \ \forall U\in\mathscr{L}_2^m(j\mathbb{R})$ and for some $\delta>0$. This frequency-domain criterion is equivalent to $\int_0^\top(2\dot{y}^\top u-\delta\dot{y}^\top\dot{y})\,\mathrm{d}t\ge 0\ \forall u\in\mathbb{L}_2^m$ and $\forall T\in[0,\infty)$ via Parseval's theorem [4]. Note, however, that when attempting to apply the characterization of [19] to biproper systems (i.e., $D\neq 0$), the term $\omega^2M(j\omega)^*M(j\omega)$ becomes infinite as $\omega\to\infty$ and, hence, the integral does not converge. Thus, the framework of [19] has to be restricted to strictly proper systems. Similar issues are also detected in the results of [20] and [21], which are built on [19]. To resolve this matter, in this article, the new dissipation inequality $\int_0^\top(2\dot{\bar{y}}^\top u-\delta\dot{\bar{y}}^\top\dot{\bar{y}})\,\mathrm{d}t\ge 0$ was proposed for characterizing OSNI systems by defining an auxiliary output $\bar{y}=y-Du$ of the system. This is equivalent to $\frac{1}{2\pi}\int_{-\infty}^{\infty}[U(j\omega)^*\{j\omega(M(j\omega)-M(j\omega)^*)-\delta\,\omega^2\bar{M}(j\omega)^*\bar{M}(j\omega)\}U(j\omega)]\,d\omega\ge 0$ via Parseval's theorem where $\bar{M}(j\omega)=M(j\omega)-M(\infty)$. Since $\bar{M}(s)$ is always strictly proper, the term $\omega^2\ \bar{M}(j\omega)^*\bar{M}(j\omega)$ does not become improper even when $M(s)$ is biproper and, hence, resolves the aforementioned issue. An illustrative Example 7 is given next to highlight this fact.

Example 7: Consider $M(s)=\frac{s+4}{s+2}$. In this case, $j\omega[M(j\omega)-M(j\omega)^*]-\delta\,\omega^2\bar{M}(j\omega)^*\bar{M}(j\omega)$ gives upon simplification $\frac{4\omega^2}{4+\omega^2}-\delta\frac{4\omega^2}{4+\omega^2}=\frac{4\omega^2}{4+\omega^2}(1-\delta)\ge 0$ for all $\omega\in\mathbb{R}\cup\{\infty\}$ and all $\delta\in(0,1]$. Thus, $M(s)$ is a biproper OSNI system. Note that if this frequency-domain condition was instead $j\omega[M(j\omega)-M(j\omega)^*]-\delta\,\omega^2M(j\omega)^*M(j\omega)$ in accordance with [19, Def. 1], then in the present example, the term $\omega^2M(j\omega)^*M(j\omega)$ would become $\frac{\omega^2(16+\omega^2)}{(4+\omega^2)}$ which tends to infinity as $\omega\to\infty$. Hence, [19, Def. 1] and the corresponding frequency-domain supply rate $(Q(\omega)=-\delta\omega^2I_m, S(\omega)=-j\omega I_m, R(\omega)=0)$ cannot capture biproper OSNI systems.

D. Internal Stability Condition of an NI-OSNI Interconnection

This section deals with internal stability of a positive feedback interconnection of NI and OSNI systems, as shown in Fig. 1. In order to prove the internal stability theorem of the NI-OSNI interconnection, we need the following technical lemma first.

Lemma 18: Let $M(s)$ be a (not necessarily stable) NI system without poles at the origin and $N(s)$ be an OSNI system. Let $[I-M(s)N(s)]$ have full normal rank. Let $\Omega=\{\omega\in(0,\infty): s=j\omega \text{ is not a pole of } M(s)\}$ and let
$$j\,[N(j\omega_0)-N(j\omega_0)^*]>0\qquad\forall\omega_0\in(0,\infty)\backslash\Omega. \tag{25}$$
Finally, let there exist no $\omega\in\Omega$ such that $\det[M(j\omega)-M(j\omega)^*]=0$ and $\det[N(j\omega)-N(j\omega)^*]=0$. Then, $[I-M(s)N(s)]$ does not have any transmission zero on the $j\omega$-axis for any $\omega\in(0,\infty)$.

Proof (Case I): Suppose $M(s)$ has $K\in\mathbb{N}$ nonrepeated pole pairs on the $j\omega$-axis. The rest of the proof in this case has been divided into two parts: Part A proves the result for the set of frequencies $\omega\in(0,\infty)\backslash\Omega$, while Part B establishes the desired result for all $\omega\in\Omega$.

Part A: Let $s=j\omega_0$ with $\omega_0\in(0,\infty)\backslash\Omega$ be a simple pole of $M(s)$. Then, $M(s)$ can be factorized as $M_0(s)+\frac{A_0}{s-j\omega_0}$ with $M_0(s)$ being analytic in the neighbourhood of $s=j\omega_0$ and $A_0=\lim_{s\to j\omega_0}(s-j\omega_0)M(s)$. Now, the residue property of NI systems (see Definition 4) gives $Z_0=\lim_{s\to j\omega_0}j(s-j\omega_0)M(s)=Z_0^*\ge 0$ which implies $Z_0=jA_0$ with
$$A_0=-A_0^*. \tag{26}$$
Now, choose a sufficiently small $\gamma>0$. Since $M(s)$ is NI,
$$j[M(j\omega)-M(j\omega)^*]\ge 0\quad\forall\omega\in\{\omega:0<|\omega-\omega_0|<\gamma\}$$
$$\Leftrightarrow j\left[\left(M_0(j\omega)+\frac{A_0}{j(\omega-\omega_0)}\right)-\left(M_0(j\omega)+\frac{A_0}{j(\omega-\omega_0)}\right)^*\right]\ge 0\quad\forall\omega\in\{\omega:0<|\omega-\omega_0|<\gamma\}$$
$$\Leftrightarrow j[M_0(j\omega)-M_0(j\omega)^*]+\frac{A_0+A_0^*}{(\omega-\omega_0)}\ge 0\quad\forall\omega\in\{\omega:0<|\omega-\omega_0|<\gamma\}$$
$$\Rightarrow j[M_0(j\omega_0)-M_0(j\omega_0)^*]\ge 0 \tag{27}$$
because $A_0+A_0^*=0$ from (26). Now the assumption $j[N(j\omega_0)-N(j\omega_0)^*]>0$ for all $\omega_0\in(0,\infty)\backslash\Omega$ implies $\det[N(j\omega_0)]\neq 0$ via Lemma 3 and ensures
$$(jN(j\omega_0))^{-1}+(jN(j\omega_0))^{-*}>0. \tag{28}$$

Inequalities (27) and (28) together yield

$$[(jN(j\omega_0))^{-1} + (jM_0(j\omega_0))] + [(jN(j\omega_0))^{-1} + (jM_0(j\omega_0))]^* > 0. \quad (29)$$

However, we need to show that $[I - M(s)N(s)]$ has no transmission zero at $s = j\omega_0$, which is equivalent to $[jN(s)]^{-1} + [jM(s)]$ having no transmission zero at $s = j\omega_0$. We show this via contradiction. Suppose $s = j\omega_0$ is a transmission zero of $[jN(s)]^{-1} + [jM(s)]$. Then there exists a nonzero vector $y_0 \in \mathbb{C}^m$ such that $[(jN(s))^{-1} + (jM(s))]y_0 = 0$ at $s = j\omega_0$.

Now, for all $\omega \in \{\omega : 0 < |\omega - \omega_0| < \gamma\}$, $(jM(j\omega)) + (jM(j\omega))^* = (jM_0(j\omega)) + (jM_0(j\omega))^*$ because $M(j\omega) = M_0(j\omega) + \dfrac{A_0}{j(\omega - \omega_0)}$ and $[A_0 + A_0^*] = 0$. Thus

$$\begin{aligned} & y_0^* \left[(jN(j\omega_0))^{-1} + (jM_0(j\omega_0))\right] y_0 \\ & \qquad + y_0^* \left[(jN(j\omega_0))^{-1} + (jM_0(j\omega_0))\right]^* y_0 \\ &= \lim_{\omega \to \omega_0} y_0^* \left[(jN(j\omega))^{-1} + (jM(j\omega))\right] y_0 \\ & \qquad + \lim_{\omega \to \omega_0} y_0^* \left[(jN(j\omega))^{-1} + (jM(j\omega))\right]^* y_0 \\ &= 0. \end{aligned}$$

However, this contradicts (29), which implies that there does not exist any nonzero vector $y_0 \in \mathbb{C}^m$ such that $[(jN(s))^{-1} + (jM(s))]y_0 = 0$. Therefore, $[I - M(s)N(s)]$ does not have any transmission zero at $s = j\omega_0$ for any $\omega_0 \in (0, \infty)\backslash\Omega$.

Part B: For convenience, define the following three auxiliary sets of frequencies:

$$\begin{aligned} \Omega_m &= \{\omega \in \Omega : \det\left[M(j\omega) - M(j\omega)^*\right] \neq 0\}, \\ \Omega_n &= \{\omega \in \Omega : \det\left[N(j\omega) - N(j\omega)^*\right] \neq 0\}, \\ \Omega_{\overline{mn}} &= \{\omega \in \Omega : \det\left[M(j\omega) - M(j\omega)^*\right] = 0 \text{ and} \\ & \qquad \det\left[N(j\omega) - N(j\omega)^*\right] = 0\}. \end{aligned}$$

It is clear that $\Omega = \Omega_m \cup \Omega_n \cup \Omega_{\overline{mn}}$. The assumption that there does not exist any $\omega \in \Omega$ such that $\det[M(j\omega) - M(j\omega)^*] = 0$ and $\det[N(j\omega) - N(j\omega)^*] = 0$ signifies that $\Omega_{\overline{mn}} = \emptyset$. Since $\det[M(j\omega) - M(j\omega)^*] \neq 0 \ \forall\omega \in \Omega_m$, it implies $j[M(j\omega) - M(j\omega)^*] > 0 \ \forall\omega \in \Omega_m$ (due to $M(s)$ being NI) which in turn implies $\det[M(j\omega)] \neq 0 \ \forall\omega \in \Omega_m$ via Lemma 3. Therefore, for all $\omega \in \Omega_m$, we have

$$\det[(jM(j\omega))] \det[(jM(j\omega))^{-1} + jN(j\omega)] \neq 0, \quad (30)$$

which implies

$$\det[I - M(j\omega)N(j\omega)] \neq 0 \qquad \forall\omega \in \Omega_m. \quad (31)$$

Similarly, it can be established that $j[N(j\omega) - N(j\omega)^*] > 0 \ \forall\omega \in \Omega_n$, which implies that $\det[N(j\omega)] \neq 0 \ \forall\omega \in \Omega_n$ via Lemma 3. Therefore, for all $\omega \in \Omega_n$, we have

$$\det[(jN(j\omega))] \det[(jN(j\omega))^{-1} + jM(j\omega)] \neq 0, \quad (32)$$

which implies

$$\det[I - M(j\omega)N(j\omega)] \neq 0 \qquad \forall\omega \in \Omega_n. \quad (33)$$

Combining the results (31) and (33), it can be concluded that $\det[I - M(j\omega)N(j\omega)] \neq 0 \ \forall\omega \in \Omega$ which is equivalent to $[I - M(j\omega)N(j\omega)]$ having no transmission zero on the $j\omega$-axis for any $\omega \in \Omega$ using [40, Corollary 3.30].

Hence, Parts A and B jointly prove that $[I - M(s)N(s)]$ does not have any transmission zero at $s = j\omega$ for any $\omega \in (0, \infty)$.

(*Case II*) Suppose now that $M(s) \in \mathscr{RH}_\infty^{m\times m}$. It can readily be established that $[I - M(s)N(s)]$ does not have any transmission zero on the $j\omega$-axis for any $\omega \in (0, \infty)$ via an argument similar to Part B of the proof of Case I upon noticing that in this case $\Omega = (0, \infty)$.

Cases I and *II* complement each other to establish the lemma by addressing all possible combinations of the systems $M(s)$ and $N(s)$. ■

We are now ready to state the feedback stability result for the positive feedback interconnection (see Fig. 1) of an NI system without poles at the origin and an OSNI system.

Theorem 5: Let $M(s)$ be a (not necessarily stable) NI system without poles at the origin and $N(s)$ be an OSNI system. Let $\Omega = \{\omega \in (0, \infty) : s = j\omega \text{ is not a pole of } M(s)\}$ and let $j[N(j\omega_0) - N(j\omega_0)^*] > 0 \ \forall\omega_0 \in (0, \infty)\backslash\Omega$. Suppose there exists no $\omega \in \Omega$ such that $\det[M(j\omega) - M(j\omega)^*] = 0$ and $\det[N(j\omega) - N(j\omega)^*] = 0$. Then, the positive feedback interconnection of $M(s)$ and $N(s)$, as shown in Fig. 1, is internally stable if and only if

$$\det[I - M(\infty)N(\infty)] \neq 0, \quad (34a)$$

$$\lambda_{\max}\left[(I - M(\infty)N(\infty))^{-1}(M(\infty)N(0) - I)\right] < 0, \quad (34b)$$

$$\lambda_{\max}\left[(I - N(0)M(\infty))^{-1}(N(0)M(0) - I)\right] < 0. \quad (34c)$$

Proof: Let $M(s)$ and $N(s)$ have minimal state-space realizations (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2), respectively, given by

$$M : \begin{cases} \dot{x}_1 = A_1 x_1 + B_1 u_1 \\ y_1 = C_1 x_1 + D_1 u_1 \end{cases} \text{ and } N : \begin{cases} \dot{x}_2 = A_2 x_2 + B_2 u_2 \\ y_2 = C_2 x_2 + D_2 u_2 \end{cases}$$

where $D_1 = D_1^\top$, $D_2 = D_2^\top$, $\det[A_1] \neq 0$ and A_2 is Hurwitz.

(*Case I*) Suppose $M(s)$ has $K \in \mathbb{N}$ nonrepeated pole pairs on the $j\omega$-axis. The assumption $j[N(j\omega_0) - N(j\omega_0)^*] > 0 \ \forall\omega_0 \in (0, \infty)\backslash\Omega$ implies $\det[N(j\omega_0)] \neq 0$ via Lemma 3 which in turn implies that for all $\omega_0 \in (0, \infty)\backslash\Omega$, $s = j\omega_0$ is not a transmission zero of $N(s)$. This, hence, prevents any pole-zero cancelation of $M(s)N(s)$ at $s = j\omega_0$ for all $\omega_0 \in (0, \infty)\backslash\Omega$ since $N(s)$ has no poles nor transmission zeros at $s = j\omega_0$ for all $\omega_0 \in (0, \infty)\backslash\Omega$. For the rest of the frequencies $\omega \in \Omega$, no pole-zero cancelation occurs since $N(s) \in \mathscr{RH}_\infty^{m\times m}$ and $M(s)$ does not have any pole at $s = j\omega$ for all $\omega \in \Omega$. Furthermore, for all $\{s \in \mathbb{C} : \Re[s] > 0\} \cup \{0\}$, no pole-zero cancelation can occur in $M(s)N(s)$ as $N(s) \in \mathscr{RH}_\infty$ and $M(s)$ has no poles in $\{s \in \mathbb{C} : \Re[s] > 0\} \cup \{0\}$. Hence, $M(s)N(s)$ has no pole-zero cancelation in the entire closed right-half plane (RHP).

Since $M(s)$ is NI without poles at the origin and $N(s)$ is OSNI, there exist real symmetric matrices $Y_1 > 0$ and $Y_2 > 0$ such that $M(s)$ satisfies $A_1 Y_1 + Y_1 A_1^\top \leq 0$ and $B_1 + A_1 Y_1 C_1^\top = 0$ [1], [41], while $N(s)$ satisfies $A_2 Y_2 + Y_2 A_2^\top + \delta_2 (C_2 A_2 Y_2)^\top (C_2 A_2 Y_2) \leq 0$ with some $\delta_2 > 0$ and $B_2 + A_2 Y_2 C_2^\top = 0$ via Lemma 16. The second inequality implies $A_2 Y_2 + Y_2 A_2^\top \leq 0$ since $\delta_2 > 0$. Define $U = I - D_1 D_2$, $V = I - D_2 D_1$, $\Phi = \begin{bmatrix} A_1 Y_1 & 0 \\ 0 & A_2 Y_2 \end{bmatrix}$ and $\mathcal{T} = \begin{bmatrix} Y_1^{-1} - C_1^\top D_2 U^{-1} C_1 & -C_1^\top V^{-1} C_2 \\ -C_2^\top U^{-1} C_1 & Y_2^{-1} - C_2^\top U^{-1} D_1 C_2 \end{bmatrix}$. The positive feedback interconnection of $M(s)$ and $N(s)$ is internally stable if and only if $\det[I - M(\infty)N(\infty)] \neq 0$ and $[I -$

$M(s)N(s)]^{-1} \in \mathscr{RH}_\infty$ since $M(s)N(s)$ has no pole-zero cancelation in the entire closed RHP [40, Th. 5.7]. It is easy to construct a stabilizable and detectable state-space realization for $[I - M(s)N(s)]^{-1}$ and, hence, $[I - M(s)N(s)]^{-1} \in \mathscr{RH}_\infty$ if and only if the corresponding system matrix A_{cl} is Hurwitz. As in [35], it can be shown that $A_{cl} = \Phi\mathcal{T}$. Then, following the proof of [35, Th. 9], except that Lemma 18 must be used instead of [35, Lemma 6] to take into account the fact that $N(s)$ is an OSNI system here instead of an SNI system, internal stability is equivalent to $\det[I - M(\infty)N(\infty)] \neq 0$ and $\mathcal{T} > 0$ which is equivalent to conditions (34a)–(34c).

(*Case II*) Suppose $M(s) \in \mathscr{RH}_\infty^{m\times m}$. Since both $M(s)$ and $N(s)$ belong to $\mathscr{RH}_\infty$, no pole-zero cancelation of the loop transfer function $M(s)N(s)$ can occur in the entire closed RHP. The rest of the proof follows Case I.

Cases I and *II* together complete the proof. ∎

The following corollary specializes Theorem 5 when the systems satisfy additional constraints at infinite frequency.

Corollary 9: Let $M(s)$ be a (not necessarily stable) NI system without poles at the origin and $N(s)$ be an OSNI system. Let $\Omega = \{\omega \in (0,\infty) : s = j\omega \text{ is not a pole of } M(s)\}$ and let $j[N(j\omega_0) - N(j\omega_0)^*] > 0\ \forall \omega_0 \in (0,\infty)\backslash\Omega$. Let either $M(s)$ be strictly proper, or else, $M(\infty)N(\infty) = 0$ and $N(\infty) \geq 0$. Suppose there exists no $\omega \in \Omega$ such that $\det[M(j\omega) - M(j\omega)^*] = 0$ and $\det[N(j\omega) - N(j\omega)^*] = 0$. Then, the positive feedback interconnection of $M(s)$ and $N(s)$, as shown in Fig. 1, is internally stable if and only if $\lambda_{\max}[M(0)N(0)] < 1$.

Proof: The proof readily follows from Theorem 5 by imposing the additional constraints that either $M(s)$ is strictly proper, or else, $M(\infty)N(\infty) = 0$ and $N(\infty) \geq 0$. ∎

The following numerical example illustrates the applicability of Lemma 18, Theorem 5 and Corollary 9.

Example 8: Both Theorem 5 and Corollary 9 can capture the internal stability of the NI system $M(s) = \begin{bmatrix} \frac{(s^2+s+6)}{2(s+2)(s^2+4)} & 0 \\ 0 & \frac{s^2+12.5}{s^4+s^3+42.5s^2+12.5s+150} \end{bmatrix}$ in positive feedback with the OSNI system $N(s) = \frac{(s^2+1)}{(s+1)^4}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ because the only pole of $M(s)$ on the $j\omega$-axis restricted to $\omega \in (0,\infty)$ is $s = j2$ but $j[N(j2) - N(j2)^*] = \begin{bmatrix} 0.46 & 0.23 \\ 0.23 & 0.23 \end{bmatrix} > 0$, and the only frequency $\omega \in (0,\infty)\backslash\{2\}$ where $\det[M(j\omega) - M(j\omega)^*] = 0$ is $\omega = \sqrt{12.5}$ rad/s whereas the only frequency $\omega \in (0,\infty)\backslash\{2\}$ where $\det[N(j\omega) - N(j\omega)^*] = 0$ is $\omega = 1$ rad/s. It is easy to verify that $[I - M(s)N(s)]$ does not have any transmission zeros on the $j\omega$-axis for $\omega \in (0,\infty)$, as stated by Lemma 18, and it is also easy to verify that $M(s)[I - N(s)M(s)]^{-1}$ has only stable poles, as stated by Theorem 5 or Corollary 9 (since in these results $\lambda_{\max}[M(0)N(0)] = 0.79 < 1$).

VIII. Conclusion

In this article, we define the class of stable IONI systems. This new IONI class captures all stable NI systems and includes within it the existing strict subclasses (e.g., SSNI [22], SSNI [23], and OSNI [8], [18]). It also creates two new strict subclasses: ISNI and VSNI. This article also establishes the missing link between NI theory and classical dissipativity in the sense of Willems [9]. A new time-domain dissipative supply rate $w(u, \bar{u}, \dot{\bar{y}})$ is introduced to characterize the full class of IONI systems which involves the system's input (u), an auxiliary filtered version of the input ($\bar{u}$) and the time-derivative of an auxiliary output of the system ($\dot{\bar{y}}$). This article also proves that IONI systems belong to a class of dissipative systems defined with respect to the particular supply rate $w(u, \bar{u}, \dot{\bar{y}})$. In addition to the time-domain dissipative framework, a $(Q(\omega), S(\omega), R(\omega))$-dissipative supply rate is also developed to characterize IONI systems. Most importantly, all these characterizations are shown to be equivalent and they are also consistent with the original pointwise frequency-domain definition of stable NI systems. Furthermore, necessary and sufficient state-space conditions are derived in LMI form to check whether a given system is $\text{IONI}_{(\delta,\varepsilon,\alpha,\beta)}$ or ISNI or OSNI or VSNI or $\text{SSNI}_{(\alpha=1,\beta=1)}$ or $\text{SSNI}_{(\alpha=2,\beta=1)}$. Finally, a necessary and sufficient internal stability condition is also presented for a positive feedback interconnection of an NI system with an OSNI system when the NI system may contain poles on the $j\omega$-axis except at the origin.

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