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An Engineer's Brief Introduction to Microwave Quantum Optics and a Single-Port State-Space Representation

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ABSTRACT Classical microwave circuit theory is incapable of representing some phenomena at the quantum level. To include quantum statistical effects, various theoretical treatments can be employed. Quantum input-output network (QION) theory is one such treatment. Another formalism, called *SLH* theory, incorporates scattering matrices (*S*), coupling vectors (*L*), and system Hamiltonians (*H*). These theoretical treatments require a reformulation of classical microwave theory. To make these topics comprehensible to an electrical engineer, we demonstrate some underpinnings of microwave quantum optics in terms of microwave engineering. For instance, we equate traveling-wave phasors in a transmission line (V_0^+) directly to bosonic field operators. Furthermore, we extend QION to include a state-space representation and a transfer function for a single port quantum network. This serves as a case study to highlight how microwave methodologies can be applied in open quantum systems. Although the same conclusion could be found from a full *SLH* theory treatment, our method was derived directly from first principles of QION.

INDEX TERMS Microwave quantum optics, quantum input–output network theory, quantum networking, state-space representation, transmission lines.

I. INTRODUCTION

Although classical circuit theory has been the forefront of microwave engineering for many decades, the theory is incapable of explaining quantum noise from single-photon detection, long distance communication, phase amplifiers, etc. [1]. With growing interest to utilize microwave networks for quantum communication [1], quantum computing [2], quantum information [3], and quantum networking [4], a second quantization of circuit components has provided a successful transition from classical to quantum treatment for microwave networks. Such models include QION theory [5]–[9] and *SLH* theory [10]–[12], which *SLH* has success in applying classical methodologies to quantum mechanics including a state-space representation.

The difficulty of these models arises from the need to accurately describe the dynamics of an open quantum system. In particular, a popular treatment in both optical and microwave networks is to express bosonic operators in terms of quantum input/output operators (Gaussian noise increments). From here, a quantum stochastic differential equation can fully describe the dynamics of a quantum network [5]. This is the fundamental principle behind QION and *SLH* theory, where

SLH expands on this idea to include multiport networks. This makes QION theory a useful tool to solve simple single-port quantum networks. However, in literature, QION formalism was dominated by the perspective of quantum optics with no clear transition for microwave theory. This becomes a concerning problem when describing microwave systems using QION since the research field diverges from standardized notation depending on one's interest. In addition, there has been little attempt to connect QION to classical methodologies, which can potentially expand already established microwave techniques to quantum networks.

This article was developed to outline the importance of microwave formalism for a simple single port circuit connected to a transmission line. In Section II, we introduced classical normal modes of a lumped *LC* oscillator to later compare classical methodology to quantum circuit theory. In Section III, we derived quantum *LC* operators of nodal charge and nodal flux in the form of annihilation operators. We formulated a comprehensible definition of a bosonic wave operator(s) for a one-dimensional transmission line and showed how to map between classical microwave formalism to quantum electric circuit theory. Finally, in Section IV, we

apply these definitions to create a state-space representation and transfer function for a single-port lumped LC oscillator coupled to a transmission line.

Although the methodologies applied in this article can be used to solve nonlinear networks such as Josephson junctions and qubits [13], [14], the approximations made in nonlinear circuits do not affect the overall treatment of QION compared to a linear network. To illustrate the analogies between classical and quantum methodologies, this article focuses on a simple single-port lumped LC example. This approach can be easily extended to a multiport system and/or more complicated systems.

II. CLASSICAL NORMAL MODES OF THE LUMPED LC OSCILLATOR

We first consider a lumped LC oscillator, consisting of an inductor with inductance L and capacitor with capacitance C . The equations of motion for LC circuit are given by [15]

$$\begin{aligned} \frac{dI}{dt} &= -\frac{1}{L}V \\ \frac{dV}{dt} &= \frac{1}{C}I \end{aligned} \quad (1)$$

which, through linear combinations, yields the normal mode equations

$$\begin{aligned} a &= \frac{1}{2}\sqrt{L}(I + i\omega CV) \\ a^* &= \frac{1}{2}\sqrt{L}(I - i\omega CV) \end{aligned} \quad (2)$$

where i is the imaginary unit and ω is the natural oscillation frequency $\omega = \frac{1}{\sqrt{LC}}$. The normalization provides that the square of the amplitudes represents the energy stored in the modes, and the sum of the squares of the modes gives the total energy of the system, $E = |a(t)|^2 + |a^*(t)|^2 = \frac{1}{2}[CV^2(t) + LI^2(t)]$.

This understanding of normal modes in the classical sense helps us understand the form necessary to introduce bosonic operators.

III. QUANTIZATION OF ELECTRIC CIRCUITS

A. QUANTIZATION OF THE LUMPED LC OSCILLATOR

To quantize the lumped LC circuit, we can follow the same conventions from [15], starting with the same equations of motion as (1). However, since most quantum engineering systems involve Josephson Junctions, it will be more convenient to consider the equations of motion in terms of node flux and charge. We define node flux as $\Phi(t) = \int^t dt' V(t')$ such that voltage is given by $V(t) = \frac{d\Phi}{dt}$; we can also define current to be $I(t) = -\frac{\Phi}{L}(t)$. Using the fact that node charge is given by $Q(t) = CV(t)$, we can rewrite (1) in terms of flux and charge

$$\frac{d\Phi}{dt} = \frac{Q}{C}$$

$$\frac{dQ}{dt} = -\frac{\Phi}{L}. \quad (3)$$

This same result can be obtained by finding the Lagrangian and Hamiltonian of the system, where flux Φ is the analog to displacement, and charge Q is its momentum conjugate. They can then be made quantum operators that follow the canonical commutation relation

$$[\hat{\Phi}, \hat{Q}] = i\hbar. \quad (4)$$

We find the quantized normal mode operators to be [16], [17]

$$\begin{aligned} \hat{a} &= \frac{\hat{\Phi}}{\sqrt{2L\hbar\omega}} + i\frac{\hat{Q}}{\sqrt{2C\hbar\omega}}, \\ \hat{a}^\dagger &= \frac{\hat{\Phi}}{\sqrt{2L\hbar\omega}} - i\frac{\hat{Q}}{\sqrt{2C\hbar\omega}} \end{aligned} \quad (5)$$

where ω is the resonant frequency given by $\omega = \frac{1}{\sqrt{LC}}$. Substituting $\Phi = LI$ and $Q = CV$ into these quantized normal modes returns (2) normalized by $(\sqrt{2\hbar\omega})^{-1}$, which ensures that the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ holds and that our operators are dimensionless, rather than having units of square root energy, like the classical normal modes of the lumped oscillator.

By rearranging (5), one can write flux, charge, current, and voltage in terms of bosonic operators \hat{a} and \hat{a}^\dagger

$$\begin{aligned} \hat{\Phi} &= L\hat{I} = \sqrt{\frac{L\hbar\omega}{2}}(\hat{a} + \hat{a}^\dagger), \\ \hat{Q} &= C\hat{V} = -i\sqrt{\frac{C\hbar\omega}{2}}(\hat{a} - \hat{a}^\dagger). \end{aligned} \quad (6)$$

Additionally, the Hamiltonian is given by $\mathcal{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$.

B. QUANTIZATION OF THE LOSSLESS TRANSMISSION LINE

We now turn to consideration of the lossless transmission line, beginning with the familiar Telegrapher's equations [18]–[21]

$$\begin{aligned} \frac{\partial}{\partial z}V(z, t) &= -L'\frac{\partial}{\partial t}I(z, t) \\ \frac{\partial}{\partial z}I(z, t) &= -C'\frac{\partial}{\partial t}V(z, t) \end{aligned} \quad (7)$$

where L' and C' are inductance and capacitance per unit length, respectively. Again, like with the quantization of the lumped oscillator, it will be convenient to consider the system in terms of node flux and node charge. We relate voltage as $\frac{\partial}{\partial t}\Phi(z, t) = V(z, t)$ and $I = -\frac{1}{L'}\frac{\partial}{\partial z}\Phi(z, t)$. Then, substituting our relations, we obtain 1-d wave equations in terms of flux

$$v_p^2 \frac{\partial^2}{\partial z^2}\Phi(z, t) = \frac{\partial^2}{\partial t^2}\Phi(z, t) \quad (8)$$

where v_p is the phase velocity given by $v_p = \frac{1}{\sqrt{L'C'}}$. We note that (8) is the same result obtained by considering the

Lagrangian of the system and finding the Euler–Lagrange equation of motion, where node flux is the analog to the displacement done by [9], [16], [17], and [22]. To remain consistent, we will consider solutions to (8). It can be shown that the charge density q (charge per unit length) is the conjugate momentum to flux, with the commutation relation given by

$$[\hat{\Phi}(z, t), \hat{q}(z, t)] = i\hbar\delta(z - z'). \quad (9)$$

We can then present normal modes for the transmission line given by [16]

$$\begin{aligned} \hat{b}_k &= \sqrt{\frac{\omega_k C' \ell}{2\hbar}} \hat{\Phi}_k + i\sqrt{\frac{1}{2 C' \ell \hbar \omega_k}} \hat{Q}_k \\ \hat{b}_k^\dagger &= \sqrt{\frac{\omega_k C' \ell}{2\hbar}} \hat{\Phi}_k - i\sqrt{\frac{1}{2 C' \ell \hbar \omega_k}} \hat{Q}_k \end{aligned} \quad (10)$$

where ℓ is a length along the transmission line, typically where the transmission line becomes periodic, k is the wave number, and $\omega_k = v_p k$. It can be shown that the commutation relation is given by $[\hat{b}_k, \hat{b}_k^\dagger] = 1$, ensured by the normalization. \hat{b}_k and \hat{b}_k^\dagger are the familiar Bose lowering and raising operators, respectively. The Hamiltonian is then given in the familiar form of $\mathcal{H} = \sum_k \hbar\omega_k (\hat{b}_k^\dagger \hat{b}_k + \frac{1}{2})$ for a quantized simple harmonic oscillator. Equation (10) yields the node flux and node charge at a specified mode k given by

$$\begin{aligned} \hat{\Phi}_k &= \sqrt{\frac{\hbar}{2\omega_k C' \ell}} (\hat{b}_k + \hat{b}_k^\dagger) \\ \hat{Q}_k &= -i\sqrt{\frac{C' \ell \hbar \omega_k}{2}} (\hat{b}_k - \hat{b}_k^\dagger) \end{aligned} \quad (11)$$

and they can be considered as traveling waves of flux and charge, respectively [17], [22]

$$\begin{aligned} \hat{\Phi}_R(z, t) &= \sum_{k>0} \sqrt{\frac{\hbar}{2\omega_k C' \ell}} (\hat{b}_k e^{-i(\omega_k t - kz)} + \hat{b}_k^\dagger e^{i(\omega_k t - kz)}) \\ \hat{Q}_R(t) &= -i \sum_{k>0} \sqrt{\frac{C' \ell \hbar \omega_k}{2}} (\hat{b}_k e^{-i(\omega_k t - kz)} - \hat{b}_k^\dagger e^{i(\omega_k t - kz)}). \end{aligned} \quad (12)$$

Using the fact that $Q = C' \ell V$ and $-L' \ell I = \Phi$, or by considering time derivatives of (12), one can find the right-moving voltage and current waves to be

$$\begin{aligned} \hat{V}_R(z, t) &= -i \sum_{k>0} \sqrt{\frac{\hbar \omega_k}{2 C' \ell}} (\hat{b}_k e^{-i(\omega_k t - kz)} - \hat{b}_k^\dagger e^{i(\omega_k t - kz)}) \\ \hat{I}_R(z, t) &= \frac{-1}{L' \ell} \sum_{k>0} \sqrt{\frac{\hbar}{2\omega_k C' \ell}} (\hat{b}_k e^{-i(\omega_k t - kz)} + \hat{b}_k^\dagger e^{i(\omega_k t - kz)}). \end{aligned} \quad (13)$$

It can be shown that $[\hat{I}_R, \hat{V}_R] \neq 0$ and, similarly, for the left-moving voltage and current waves, which contributes to zero

point fluctuations in power. We note that voltage and current, described in (13), is offset by a phase $\phi = \frac{\pi}{2}$ in comparison to microwave theory. This is due to the fact that the choice of origin is arbitrary and we chose flux to be our position coordinate and charge to be momentum. To match notation in microwave theory, we can introduce a phase offset of $\pi/2$ which maps $\hat{b}'_k = \hat{b}_k e^{-i\pi/2} = -i\hat{b}_k$ and $\hat{b}'_k{}^\dagger = \hat{b}_k^\dagger e^{i\pi/2} = i\hat{b}_k^\dagger$. Substituting this into (13), we obtain voltage as a sum of the Bose operators that is real

$$\hat{V}_R(z, t) = \sum_{k>0} \sqrt{\frac{\hbar \omega_k}{2 C' \ell}} (\hat{b}'_k e^{-i(\omega_k t - kz)} + \hat{b}'_k{}^\dagger e^{i(\omega_k t - kz)}). \quad (14)$$

Similarly, current is given by

$$\begin{aligned} \hat{I}_R(z, t) &= \frac{i}{L' \ell} \sum_{k>0} \sqrt{\frac{\hbar}{2\omega_k C' \ell}} (\hat{b}'_k e^{i(\omega_k t - kz)} - \hat{b}'_k{}^\dagger e^{-i(\omega_k t - kz)}). \end{aligned} \quad (15)$$

Following [9], [17], and [22], we can extend (14) to the continuum limit, where $\ell \rightarrow \infty$

$$\begin{aligned} \hat{V}_R(z, t) &= \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \sqrt{\frac{\hbar \omega Z_0}{2}} (\hat{b}_R(\omega) e^{-i(\omega t - kz)} + h.c.) \\ \hat{V}_L(z, t) &= \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \sqrt{\frac{\hbar \omega Z_0}{2}} (\hat{b}_L(\omega) e^{-i(\omega t + kz)} + h.c.) \end{aligned} \quad (16)$$

where Z_0 is the characteristic impedance given by $Z_0 = \sqrt{\frac{L'}{C'}}$, and

$$\begin{aligned} \hat{b}_R(\omega) &= \sqrt{2\pi} (L' C' \ell^2)^{-1/4} \sum_{k>0} \hat{b}'_k \delta(\omega - \omega_k), \\ \hat{b}_L(\omega) &= \sqrt{2\pi} (L' C' \ell^2)^{-1/4} \sum_{k<0} \hat{b}'_k \delta(\omega - \omega_k). \end{aligned} \quad (17)$$

The resulting commutation relation is given by

$$[\hat{b}_R(\omega), \hat{b}_R^\dagger(\omega')] = [\hat{b}_L(\omega), \hat{b}_L^\dagger(\omega')] = 2\pi \delta(\omega - \omega'). \quad (18)$$

The normalization of (17) ensures that the commutation relations of (9) and (18) hold. Equation (9) is important because it depicts the uncertainty relationship between charge and flux, which are canonical conjugates. Equation (18) is important because we recover boson commutation relations.

C. MAPPING QUANTUM CIRCUITS BY WAY OF MICROWAVE ENGINEERING

We now consider a classical, single-mode voltage traveling-wave, moving to the right, with instantaneous solution $V_R(z, t) = |V_0^+| \cos(\omega t - kz + \phi)$. Using sinusoidal steady-state analysis to provide phasor notation, $V_R(t) = \text{Re}\{V_0^+ \exp[i(\omega t - kz + \phi)]\}$. Note that the Bose operators could also be written as $\hat{b}_k = e^{-i\hat{\Phi}} \sqrt{\hat{N}_k}$ and

TABLE 1. Quantum Engineering Rosetta Stone of Generalized Scattering Parameters in Microwave Engineering language [19], With Bosonic Operators in a Quantized Transmission Line. The Characteristic Impedance is Given by $Z_0 = \sqrt{\frac{L'}{C'}}$ and Phase Velocity Given by $v_p = \frac{1}{\sqrt{L'C'}}$

Microwave Engineering	Quantum Engineering
V_0^+	$\sqrt{\frac{2\hbar\omega_k}{C'\ell}} \hat{b}_k^\dagger e^{-i\pi/2}, (k > 0)$
V_0^-	$\sqrt{\frac{2\hbar\omega_k}{C'\ell}} \hat{b}_k^\dagger e^{-i\pi/2}, (k < 0)$
$\sqrt{\frac{C'\ell}{2\hbar\omega_k}} V_0^+ e^{i\pi/2}$	\hat{b}_k^\dagger
$a = \frac{V_0^+}{\sqrt{Z_0}}$	$\sqrt{\frac{2\hbar\omega_k v_p}{\ell}} \hat{b}_k^\dagger e^{-i\pi/2}, (k > 0)$
$b = \frac{V_0^-}{\sqrt{Z_0}}$	$\sqrt{\frac{2\hbar\omega_k v_p}{\ell}} \hat{b}_k^\dagger e^{-i\pi/2}, (k < 0)$

$\hat{b}_k^\dagger = \sqrt{\hat{N}_k} e^{i\hat{\phi}}$ [23], where $\hat{\phi}$ is a *hypothetical* phase operator, defined uniquely *only* for the interval $[0, 2\pi]$ [24], [25]. We also note that $\hat{b}_k \hat{b}_k^\dagger = \hat{N}_k + 1$ and $\hat{b}_k^\dagger \hat{b}_k = \hat{N}_k$, which ensures that the commutation relation $[\hat{b}_k, \hat{b}_k^\dagger] = 1$ holds. We can consider these quantum-mechanical expressions in a semiclassical limit, where we propose bosonic operators now to be downgraded to the role of simple scalars. In this limit, we can equate our prior expression for traveling-wave voltage with a standard, plane-wave engineering expression. In this case, we recover the following relationships to standard microwave expressions [19]–[21]

$$V_0^+ = |V_0^+| e^{i\hat{\phi}} = \sqrt{\frac{2\hbar\omega_k}{C'\ell}} \hat{b}_k^\dagger e^{-i\pi/2}, (k > 0)$$

$$\hat{b}_k^\dagger = \sqrt{\hat{N}_k} e^{i\hat{\phi}} = \sqrt{\frac{C'\ell}{2\hbar\omega_k}} V_0^+ e^{i\pi/2}, (k > 0). \quad (19)$$

A relationship between V_0^- and \hat{b}_k^\dagger can be found by considering the left-moving voltages instead. Using the relationship between V_0^+ and \hat{b}_k^\dagger , one can relate Pozar's formalism for generalized scattering parameters, $a = \frac{V_0^+}{\sqrt{Z_0}}$ with \hat{b}_k^\dagger . A comprehensive translation—possibly an interdisciplinary Rosetta Stone—between formalisms can be seen in Table 1.

It is tempting to speculate how one could upgrade parameters V_0^+ and V_0^- to quantum operators, \hat{V}_0^+ and \hat{V}_0^- , and derive open quantum systems in terms of these operators, rather than with using traditional bosonic operators.

IV. HEISENBERG EQUATION OF MOTION, INPUT-OUTPUT RELATION, AND STATE-SPACE REPRESENTATION

Assume a transmission line coupled to a quantum lumped LC oscillator (system) by a coupling capacitor with capacitance C_c ; see Fig. 1 (the capacitance and inductance of the system are denoted as L and C , respectively). This creates an open

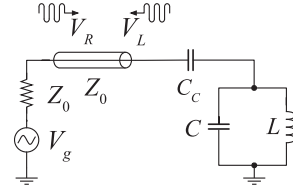


FIGURE 1. Lumped LC parallel tank circuit is coupled to a transmission line through a coupling capacitor. Although thermal photons in a semi-infinite transmission line could also be considered, here, we implicitly assume a generator to be matched at the source of the transmission line.

quantum system

$$H = H_{\text{sys}} + H_{\text{int}} + H_{\text{bath}} \quad (20)$$

where H_{sys} is the Hamiltonian of lumped LC, H_{int} is the Hamiltonian of the interaction between the lumped LC and bosonic bath operator (voltage wave), and H_{bath} is the Hamiltonian of the bosonic field. We assume that the transmission line is semi-infinite, where $z = 0$ and $t = 0$, the right-propagating wave \hat{V}_R enters the lumped LC oscillator at $t = 0$, and, at a later time t , the left propagating wave \hat{V}_L is reflected back. This results in the bath Hamiltonian H_{bath} defined in Section III-B where the right-moving bosonic operators are the input bosonic operators $b_{\text{in}}(\omega) = \hat{b}_R(\omega)$ and, similarly, the output bosonic operator are the left-moving bosonic operators. The system Hamiltonian H_{sys} is the Hamiltonian of the lumped LC oscillator described in Section III-A using the values in Fig. 1 (right side).

We assume the interaction Hamiltonian between the input voltage and system H_{int} to be the same as the energy stored in the coupling capacitor E_{C_c} , where $H_{\text{int}} = E_{C_c} = \frac{1}{2} C_c (\hat{V}_{\text{bath}} - \hat{V}_{\text{sys}})^2$ (with $C \gg C_c$). We assume that $\frac{1}{2} C_c \hat{V}_{\text{bath}}^2$ and $\frac{1}{2} C_c \hat{V}_{\text{sys}}^2$ are negligible compared to the bath and system Hamiltonian terms. This is because the impedance across the coupling capacitor will be much greater than the impedance across the system capacitor C . Therefore, more energy will flow across the system capacitor than the coupling capacitor ($H \gg \frac{1}{2} C_c \hat{V}_{\text{sys}}^2$). A similar argument can be used for $\frac{1}{2} C_c \hat{V}_{\text{bath}}^2$. Then interaction Hamiltonian reduces to $H_{\text{int}} \approx -C_c \hat{V}_{\text{sys}} \hat{V}_{\text{bath}}$.

Each Hamiltonian can be defined as

$$H_{\text{sys}} = \hbar \Delta a^\dagger a, \quad (21a)$$

$$H_{\text{int}} = i \frac{\hbar C_c}{2} \sqrt{\frac{Z_0 \omega_R}{2\pi C}} \int_0^\infty \sqrt{\omega} (a - a^\dagger) (\hat{b}(\omega) + \hat{b}^\dagger(\omega)) d\omega, \quad (21b)$$

$$H_{\text{bath}} = \hbar \int_0^\infty \omega \hat{b}^\dagger(\omega) \hat{b}(\omega) d\omega \quad (21c)$$

where $\Delta = \omega - \omega_R$ is the detuning frequency of the lumped LC, the resonant frequency for the circuit is $\omega_R = (LC)^{-\frac{1}{2}}$, \hat{V}_{sys} is the voltage operator across a capacitor defined in Section III-A, Z_0 is the characteristic impedance of the transmission line, and an implied tensor product between $(b(\omega) + b^\dagger(\omega))$ and $(a - a^\dagger)$. We also denote the identity matrix in

the bosonic operator Hilbert space as \mathbf{I}_{bath} , and the identity matrix in the system's Hilbert space as \mathbf{I}_{sys} .

Before we solve the dynamics of this system, we would want to change the basis of the interaction Hamiltonian to interaction (Dirac) picture in the perspective of the system [5]. This is done by considering a bare Hamiltonian $H_0 = H_{\text{bath}} + H_{\text{int}}$ with $H_{\text{int}}(t) = e^{\frac{i}{\hbar}H_0 t} H_{\text{int}} e^{-\frac{i}{\hbar}H_0 t}$ such that

$$H_{\text{int}}(t) = i\hbar \frac{C_c}{2} \sqrt{\frac{Z_0 \omega_R}{2\pi C}} \int_0^\infty \sqrt{\omega} (a(t) - a^\dagger(t)) (\hat{b}(\omega) + \hat{b}^\dagger(\omega)) d\omega \quad (22)$$

where $a(t) = ae^{-i\Delta t}$. Commutation relations still hold $[a(t), a^\dagger(t')] = \delta(t-t')$ and $[a^\dagger(t), a^\dagger(t')] = [a(t), a(t')] = 0$. In addition, the system operator in the interaction frame is modified to $H_{\text{sys}} = \hbar\Delta a^\dagger(t)a(t)$.

Let us consider a wave at frequency Ω . At large frequencies away from Ω , the field has very little interaction with the system states [17]. In consequence, we can integrate both the bath and interaction Hamiltonian on the interval $(-\infty, \infty)$. This means for both the interaction and bath Hamiltonian, the limits of integration can change as $\int_0^\infty \rightarrow \int_{-\infty}^\infty$.

For weak coupling between the system and bath, the interaction Hamiltonian, as it is now, does not conserve energy with counter-rotating terms being nonconservative $\hat{b}^\dagger(\omega)a^\dagger(t)$ and $\hat{b}(\omega)a(t)$ [7]. To fix this, we introduce the rotating wave approximation to get rid of these terms in the interaction Hamiltonian [4]–[7], [26].

We assume that the rate of change in frequency is minus-ule around Ω such that $\omega \approx \Omega$. This means that we can utilize the Markovian approximation $\sqrt{\frac{\kappa}{2\pi}} \approx \frac{C_c}{2} \left(\frac{Z_0 \omega_R}{2\pi C}\right)^{\frac{1}{2}} \sqrt{\Omega}$, where κ is the photon rate around resonance [5]–[7]. This approximation, $\sqrt{\kappa}$, describes how strong bosons interact with the network for every unit $(\text{time})^{-\frac{1}{2}}$. The interaction Hamiltonian yields

$$H_{\text{int}}(t) = i\hbar \sqrt{\frac{\kappa}{2\pi}} \int_{-\infty}^\infty (\hat{b}^\dagger(\omega)a(t) - a^\dagger(t)\hat{b}(\omega)) d\omega \quad (23)$$

with implied outer products between $a(t)$ and $\hat{b}(\omega)$ operators. Shifting the lab frame to Ω modifies the bath Hamiltonian as $H_{\text{bath}} = \hbar \int_{-\infty}^\infty (\omega - \Omega) \hat{b}^\dagger(\omega) \hat{b}(\omega) d\omega$ [4]. Often, the Markovian approximation is written as $\kappa(\omega) \approx \sqrt{\frac{\kappa}{2\pi}}$, which is the normal formalism in quantum optics as it connects κ with how much photons are entering the network [27] (units of $(\text{time})^{-1}$). However, this nomenclature may not be as insightful since the coupling between bath and system goes by units of $(\text{time})^{-\frac{1}{2}}$, where the meaning of the dimensional unit analysis is lost. Therefore, it is tempting to create a representation where the Markovian approximation, $\kappa(\omega) \rightarrow \frac{\kappa'}{\sqrt{2\pi}}$ $\kappa' = \sqrt{\kappa}$.

A. HEISENBERG–LANGEVIN EQUATION

We want to solve the equation of motion for the bosonic field operators. Specifically, we want to develop an input

bosonic operator. To do this, we will have to solve the Heisenberg equation of motion in the bosonic field frame. In the interaction frame of the bosonic field, the dynamics only depend on the interaction and bath Hamiltonian such that $\dot{\hat{b}}(\omega) = -\frac{i}{\hbar} [\hat{b}(\omega), H_{\text{int}} + H_{\text{bath}}]$ to describe all the dynamics of $\hat{b}(\omega)$ from t_0 to t . The solutions to this equation yields

$$\hat{b}(\omega) = \hat{b}(\omega, t = t_0) e^{-i(\omega - \Omega)(t - t_0)} + \frac{1}{\sqrt{2\pi}} \int_{t_0}^t \sqrt{\kappa} a(t') e^{-i(\omega - \Omega)(t' - t)} dt'. \quad (24)$$

We can find the evolution of the system operator $a(t)$ from t_0 to t in the Heisenberg picture (and interaction picture) just as we did with $\hat{b}(\omega)$. This equation is also known as the Heisenberg–Langevin equation [4]–[6], [9] is given as

$$\begin{aligned} \dot{a}(t) &= -\frac{i}{\hbar} [a(t), H_{\text{sys}} + H_{\text{int}}], \\ &= -i[a(t), \Delta a^\dagger(t)a(t)] \\ &\quad + [a(t), \sqrt{\frac{\kappa}{2\pi}} \int_{-\infty}^\infty (\hat{b}^\dagger(\omega)a(t) - a^\dagger(t)\hat{b}(\omega)) d\omega] \end{aligned} \quad (25)$$

where there is an implied inner product with identity matrix of the bosonic field $(a(t) = a(t) \otimes \mathbf{I}_{\text{bath}})$.

Here, we assume that the system state cannot evolve until the right propagating wave enters the network and, for a finite field between the times t and $t + dt$, the field interacts with the system changing its state. In a similar manner, the wave reflected from the lumped LC cannot change (or be created) until the system state changes. This means that the bosonic field operators are independent (commutes) with system operator at integrated time t' being bigger than the present time t . Therefore, $[\hat{b}(\omega), a(t')] = [\hat{b}^\dagger(\omega), a(t')] = [\hat{b}(\omega), a^\dagger(t')] = [\hat{b}^\dagger(\omega), a^\dagger(t')] = 0$ when $t' > t$. Equation (26) yields

$$\begin{aligned} \dot{a}(t) &= -i\Delta a(t) + \sqrt{\frac{\kappa}{2\pi}} \int_{-\infty}^\infty \hat{b}^\dagger(\omega) [a(t), a(t)] d\omega \\ &\quad - \sqrt{\frac{\kappa}{2\pi}} \int_{-\infty}^\infty [a(t), a^\dagger(t)] \hat{b}(\omega) d\omega. \end{aligned} \quad (27)$$

We substitute the commutation relations

$$\dot{a}(t) = -i\Delta a(t) - \sqrt{\frac{\kappa}{2\pi}} \int_{-\infty}^\infty \hat{b}(\omega) d\omega. \quad (28)$$

We define the bosonic field (at $z = 0$) entering the lumped LC at $t_0 = 0$ as the input bosonic field

$$b_{\text{in}}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \hat{b}(\omega, t = 0) e^{-i(\omega - \Omega)t} d\omega \quad (29)$$

where $\hat{b}(\omega, t = 0)$ is the initial field entering the system in frequency domain. The output operator is derived in a similar manner but with $t_0 = t_1$, where t_1 is the first “infinitesimal slice” of the wave changed by the system state.

Plugging (24) into (28)

$$\dot{a}(t) = -i\Delta a(t) - \sqrt{\kappa} (b_{\text{in}}(t))$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^t \sqrt{\kappa} a(t') e^{-i(\omega-\Omega)(t'-t)} dt' d\omega). \quad (30)$$

Using properties in [5] where $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(\omega-\Omega)(t-t')} d\omega = \delta(t-t')$ and $2 \int_{t_0}^{\infty} \delta(t-t') a(t') dt' = \text{sgn}(t-t_0) a(t)$, we integrate between t and $t+dt$. For this network, as we have solved the Heisenberg–Langevin equation based on input–output theory [5], [9]

$$\dot{a}(t) = -\left(i\Delta + \frac{\kappa}{2}\right) a(t) - \sqrt{\kappa} b_{\text{in}}(t) \quad (31)$$

where $a(0) = a \otimes \mathbf{I}_{\text{bath}}$.

B. INPUT-OUTPUT RELATION

The most important result in input–output theory is the map between an input field to its output field. The work [5] derived a general form of this based on the negligible time delay between the input port and output port. Using this assumption, the solution of (24) for $t_0 = 0$ (the input field) equals the solution of the wave function for the output field ($t_0 = t_1$) at some τ . This yields

$$\begin{aligned} \hat{b}(\omega, t=0) e^{-i(\omega-\Omega)t} + \frac{1}{\sqrt{2\pi}} \int_0^{\tau} \sqrt{\kappa} a(t') e^{-i(\omega-\Omega)(t'-t)} dt' \\ = b(\omega, t=t_1) e^{-i(\omega-\Omega)(t-t_1)} \\ + \frac{1}{\sqrt{2\pi}} \int_{t_1}^{\tau} \sqrt{\kappa} a(t') e^{-i(\omega-\Omega)(t'-t)} dt'. \end{aligned} \quad (32)$$

Integrating from ω to $\omega + d\omega$ in (32) on both sides gives

$$\begin{aligned} \sqrt{2\pi} b_{\text{in}}(t) + \frac{1}{\sqrt{2\pi}} \int_0^{\tau} \sqrt{\kappa} a(t') 2\pi \delta(t-t') dt' \\ = \sqrt{2\pi} b_{\text{out}}(t) - \frac{1}{\sqrt{2\pi}} \int_{\tau}^{t_1} \sqrt{\kappa} a(t') 2\pi \delta(t-t') dt' \quad (33) \\ \sqrt{2\pi} b_{\text{in}}(t) + \frac{\sqrt{2\pi}\kappa}{2} a(t) = \sqrt{2\pi} b_{\text{out}}(t) - \frac{\sqrt{2\pi}\kappa}{2} a(t). \end{aligned} \quad (34)$$

Divide (34) by $\sqrt{2\pi}$ and rearrange the equation so that $b_{\text{out}}(t)$ is by itself on the left-hand side. This yields the input–output relation (originally derived by Gardiner and Collett [5])

$$b_{\text{out}}(t) = \sqrt{\kappa} a(t) + b_{\text{in}}(t) \quad (35)$$

where $b_{\text{in/out}}(t)$ is denoted as $\mathbf{I}_{\text{sys}} \otimes b_{\text{in/out}}(t)$ and $a(t)$ as $a(t) \otimes \mathbf{I}_{\text{bath}}$.

C. STATE-SPACE REPRESENTATION

The equations of motion derived from the Heisenberg–Langevin equation and input–output theorem can be combined into a system of equations; also known as a state-space representation if the equations describe the entire dynamics of the network [4], [28]–[32]. The system operator $a(t)$ is a ladder operator with quadratic degrees of freedom, which can be treated as a state variable. The bosonic field operators are input and output vectors. This means solutions to the Heisenberg–Langevin equation and input–output relation

can fully describe the dynamics of a quantum network. Here, we can connect the dynamics to a state-space representation

$$\dot{a}(t) = \mathbf{A} a(t) + \mathbf{B} b_{\text{in}}(t) \quad (36a)$$

$$b_{\text{out}}(t) = \mathbf{C} a(t) + \mathbf{D} b_{\text{in}}(t). \quad (36b)$$

Since the solutions to (31) and (35) can completely describe the dynamics of the LC system operator $a(t)$ and the reflected (output) wave $b_{\text{out}}(t)$ given an input operator $b_{\text{in}}(t)$, then $\mathbf{A} = -(i\Delta + \frac{\kappa}{2})$, $\mathbf{B} = -\sqrt{\kappa}$, $\mathbf{C} = \sqrt{\kappa}$, and $\mathbf{D} = 1$. In this example, the $ABCD$ matrices are 1×1 dimensional since this is a 1-port network which corresponds to the number of the ports for the quantum LC oscillator.

As in classical state-space representation, the $ABCD$ formulation for linear quantum systems is the dynamical equation of the system operator and the output wave function from 0 to t . The lumped LC circuit described in (36a)–(36b) is an appropriate description of a passive network since the matrix of coefficients— $ABCD$ matrices—only scale for a non-Hermitian conjugate form. This would be different for active networks since nonconservative energy terms can arise from Hermitian conjugate of either the system operators or the coupling operators in H_{int} [33]–[36]. A linear map in the spirit of Bogoliubov formulation is normally used to solve realization theory and control theory for both passive and active networks with many quadratic degrees of freedom [34], [37]; however, we will arrive at the same conclusion without taking into account this linear mapping.

D. TRANSFER FUNCTION

In classical systems, the transfer function $\mathbf{H}(s)$ (or transfer matrix in multiport networks) serves as a way to model characteristic solutions of time-invariant input–output problems in the Laplace domain [31], [32]. Similarly, a transfer function for linear quantum networks with many (usually quadratic) degrees of freedom can be constructed from a state-space representation [37]–[43] using (36a) and (36b). This is done by taking the Laplace transform of (36a) and (36b), assuming that the initial state $a(0)$ in Laplace domain is zero

$$b_{\text{out}}(s) = (\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D})b_{\text{in}}(s) \quad (37)$$

where s is a complex variable, \mathbf{I} is the identity matrix, and $(s\mathbf{I} - \mathbf{A})$ is invertible.

The transfer function is defined as a ratio of output over input in Laplace domain, $\mathbf{H}(s) = \frac{b_{\text{out}}(s)}{b_{\text{in}}(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$. The transfer function for the lumped LC circuit results in

$$\mathbf{H}(s) = \frac{(s - \frac{\kappa}{2}) + i\Delta}{(s + \frac{\kappa}{2}) + i\Delta} \quad (38)$$

for $\text{Re}\{s\} > 0$. For this derivation, we have treated $ABCD$ matrices and $\mathbf{H}(s)$ as matrices when they are scalars for a one-port network like the lumped LC problem we developed in this article. This notation is used to impart the idea that these quantities are matrices with dimensions corresponding to the number of input/output ports a network has.

Throughout our derivation, we implicitly explained that the operators can be expressed as a Wiener noise probability distribution. This can be seen in (24), where the bosonic wave is dependent on the convolution of the system operator. This means that we could have represented the system and bosonic dynamics using Itô calculus with the bosonic field as a quantum Wiener noise operator [5], [6].

V. CONCLUSION

In this article, we demonstrated how to construct a simple single-port quantum circuit from first principles of QION theory. A lumped LC circuit coupled to a transmission line can be easily modeled as an open quantum system, where the dynamical solutions can be turned into a state-space representation and a corresponding transfer function. We then provided a formalism of quantum circuits in terms of microwave engineering, resulting in an easier understanding to solving microwave systems. Our results imply that we can construct quantum networks based on fundamental theorems of QION; however, as we generalize to multiport systems, QION modeling becomes increasingly difficult. Although SLH theory is needed to accurately model multiport networks, QION can do this for a simple single port.

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