

Received October 22, 2020; accepted November 24, 2020; date of publication December 11, 2020; date of current version January 11, 2021.

Digital Object Identifier 10.1109/TQE.2020.3044179

Protocols for Creating and Distilling Multipartite GHZ States With Bell Pairs

SEBASTIAN DE BONE^{1,2} , RUNSHENG OUYANG^{3,4} ,
KENNETH GOODENOUGH¹ , AND DAVID ELKOUSS¹  (Member, IEEE)

¹QuTech, Delft University of Technology, 2628 CJ, Delft, The Netherlands

²QuSoft, Centrum Wiskunde & Informatica, 1098 XG, Amsterdam, The Netherlands

³Department of Physics, Tsinghua University, Beijing 100084, China

⁴Department of Information Technology and Electrical Engineering, ETH Zürich, 8092 Zürich, Switzerland

Corresponding author: David Elkouss (d.elkousscoronas@tudelft.nl).

This work was supported by the Netherlands Organization for Scientific Research (NWO/OCW), as part of the Quantum Software Consortium Program under Project 024.003.037/3368. The work of R. Ouyang was supported by Tsinghua Xuetang Talents Program. This article was presented in part at the Quantum Resource Estimation Workshop as part of the International Symposium on Computer Architecture, Valencia, Spain, May 2020.

ABSTRACT The distribution of high-quality Greenberger–Horne–Zeilinger (GHZ) states is at the heart of many quantum communication tasks, ranging from extending the baseline of telescopes to secret sharing. They also play an important role in error-correction architectures for distributed quantum computation, where Bell pairs can be leveraged to create an entangled network of quantum computers. We investigate the creation and distillation of GHZ states out of nonperfect Bell pairs over quantum networks. In particular, we introduce a heuristic dynamic programming algorithm to optimize over a large class of protocols that create and purify GHZ states. All protocols considered use a common framework based on measurements of nonlocal stabilizer operators of the target state (i.e., the GHZ state), where each nonlocal measurement consumes another (nonperfect) entangled state as a resource. The new protocols outperform previous proposals for scenarios without decoherence and local gate noise. Furthermore, the algorithms can be applied for finding protocols for any number of parties and any number of entangled pairs involved.

INDEX TERMS Distributed quantum computation, entanglement distillation, Greenberger–Horne–Zeilinger (GHZ) states.

I. INTRODUCTION

Quantum computation promises a computational advantage for algorithmic problems in the fields of cryptography, database searching, simulations of atoms and molecules, and solving linear equations. There are several approaches and technologies concurrently investigated for scaling the near term quantum devices to full-fledged quantum computers. One of the approaches is *distributed* or *networked* quantum computing [1], [2]. In this approach, multiple computers holding a small number of qubits are connected via entanglement [3].

The promise of distributed quantum computing is the possibility of building a quantum computer without the difficulty of engineering a large multiqubit device. In exchange, the feasibility of such a networked device critically lies in the availability of high-fidelity entanglement. This is because entanglement is required for the realization of multiqubit operations between different quantum computers. In particular, entangled states are necessary for performing error detection

measurements in error-correction codes executed with distributed quantum computers.

Using error correction for fault-tolerant quantum computation relies on encoded data. To correct or track the errors on the encoded data, it is necessary to periodically perform joint measurements on different qubits. If the whole encoded state lies in a single quantum device, these joint measurements can be performed by applying the appropriate multi-qubit operations and measuring an ancilla qubit. However, in distributed implementations, the joint measurements become nonlocal. The ingredient that enables the joint measurements is Greenberger–Horne–Zeilinger (GHZ) states. By consuming an n -qubit GHZ state, it is possible to perform a nonlocal measurement between n parties. The challenge of distributed quantum computation is to produce GHZ states at a fast enough rate and with high enough fidelity to enable fault-tolerant quantum computation.

Creating GHZ states is experimentally challenging. A simple protocol for creating an n -qubit GHZ state consists of

fusing $n - 1$ Bell pairs. However, the fidelity of the GHZ state degrades exponentially with n . This problem can be overcome by more complicated protocols that *distill* or *purify* either the input Bell states or any of the intermediate states of a protocol. This generally improves the fidelity of the final GHZ state, but comes at the price of consuming a larger number of Bell pairs.

Several physical systems can process quantum information and have a coherent optical interface for generating remote entanglement [4]. Some examples are nitrogen-vacancy centers [5]–[10], silicon-vacancy centers [11]–[14], and ion traps [15], [16]. Some of these platforms have already demonstrated the generation of long-lived remote entanglement [17] and even distillation [8]. However, the rate at which entanglement can be produced is slower than the gate times. In consequence, the rate at which GHZ states are produced becomes the bottleneck for the performance of distributed quantum computer implementations [18]. Moreover, while our motivation stems from distributed quantum computation, efficient GHZ generation has direct application in several other applications, including secret sharing [19], anonymous transmission [20], clock synchronization [21], and extending the baseline of telescopes [22].

The goal of this article is to minimize the number of Bell pairs necessary to produce high-fidelity GHZ states. We do this by searching the protocol space for creating GHZ states out of Bell pairs. The difficulty of the problem is that given a number of parties and a number of input Bell pairs, the number of possible protocols is very large. In fact, it grows superexponentially with these parameters. Our approach to deal with the large number of protocols is therefore to take the heuristic approximation that optimal protocols for some number of copies of a GHZ state are composed of optimal protocols for a smaller number of copies or parties. This heuristic leads to a *dynamic program*.

Distillation is better understood in the bipartite case [23]–[26] than in the multipartite case [27]–[35]. In the bipartite case, it is even known that some protocols achieve an optimal tradeoff between rate and fidelity [36].

In the context of a distributed implementation of the surface code, Nickerson *et al.* [37] optimized a family of protocols for generating four-partite GHZ states out of noisy Bell pairs. To facilitate experimental feasibility, the GHZ distillation protocols require three qubits per node. The number of possible protocols in this family, while large, is still brute-force tractable. Subsequent work optimized a similar family of protocols in the presence of loss [18]. We leave the extension of our approach to more realistic settings, including loss for future work. In contrast with Nickerson *et al.* [37], we are interested in more general protocols that minimize the number of Bell pairs consumed independently of the size of the required quantum register. This different ansatz is justified by recent experimental progress with multiqubit registers [9].

In Sections II and III, we introduce the formalism and building blocks of the GHZ generation protocols considered.

TABLE 1. Each Column Shows From Left to Right: The Basis States $|\phi^{s_1 s_2 s_3}\rangle$ of the Three-Qubit GHZ Basis, Their Representations in the Computational Basis, and the Stabilizers of the State

	Computational basis	$X_1 X_2 X_3$	$Z_1 Z_2$	$Z_2 Z_3$
$ \phi^{+++}\rangle$	$(000\rangle + 111\rangle)/\sqrt{2}$	+1	+1	+1
$ \phi^{++-}\rangle$	$(001\rangle + 110\rangle)/\sqrt{2}$	+1	+1	-1
$ \phi^{+-+}\rangle$	$(011\rangle + 100\rangle)/\sqrt{2}$	+1	-1	+1
$ \phi^{+--}\rangle$	$(010\rangle + 101\rangle)/\sqrt{2}$	+1	-1	-1
$ \phi^{-++}\rangle$	$(000\rangle - 111\rangle)/\sqrt{2}$	-1	+1	+1
$ \phi^{-+-}\rangle$	$(001\rangle - 110\rangle)/\sqrt{2}$	-1	+1	-1
$ \phi^{--+}\rangle$	$(011\rangle - 100\rangle)/\sqrt{2}$	-1	-1	+1
$ \phi^{---}\rangle$	$(010\rangle - 101\rangle)/\sqrt{2}$	-1	-1	-1

Note: The signs $s_1, s_2,$ and s_3 describe the relation of the basis state $|\phi^{s_1 s_2 s_3}\rangle$ with stabilizer generators $s_1 X_1 X_2 X_3, s_2 Z_1 Z_2,$ and $s_3 Z_2 Z_3.$

In Section IV, we show that existing GHZ generation protocols are included in our search space. In Section V, we present our dynamic program. In Section VI, we show the performance of the best GHZ creation protocols found. Finally, we draw our conclusions in Section VII.

II. BELL AND GHZ DIAGONAL STATES

Here, we introduce notation and definitions used in the rest of this article together with our model for states.

We describe nonperfect Bell and GHZ states in the *stabilizer* formalism [38]. A *stabilizer operator* or *stabilizer* of a quantum state $|\psi\rangle$ is an operator O that verifies $O|\psi\rangle = |\psi\rangle$, i.e., $|\psi\rangle$ is an eigenvector of O with eigenvalue +1 and, in consequence, leaves $|\psi\rangle$ invariant. An n -qubit pure quantum state has 2^n stabilizer operators. These 2^n operators form the *stabilizer group* of the state, which is generated by a subset of n operators. An n -qubit GHZ state $(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})/\sqrt{2}$ is described by the stabilizer group generated by the operators $\{X_1 X_2 \dots X_n, Z_1 Z_2, Z_2 Z_3, \dots, Z_{n-1} Z_n\}$. The stabilizer group includes the identity \mathbb{I} . We call the $2^n - 1$ operators in this group that are not the identity \mathbb{I} *nontrivial* stabilizers of the state.

We use the stabilizer formalism to define a basis for a general n -qubit system. In analogy with the Bell basis, we call this basis the *GHZ basis*. It is also known as the *cat basis* [28]. The basis states of the n -qubit GHZ basis are defined as the 2^n states $|\phi^{s_1 s_2 s_3 \dots s_n}\rangle$ with stabilizer generators $s_1 X_1 X_2 \dots X_n, s_2 Z_1 Z_2, s_3 Z_2 Z_3, \dots, s_n Z_{n-1} Z_n$, where $s_i \in \{+1, -1\}$ for all $i \in \{1, 2, \dots, n\}$. As an example, we show the eight basis states of the three-qubit GHZ basis in Table 1. For any n , the basis state $|\phi^{++++\dots}\rangle$ is the n -qubit GHZ state $(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})/\sqrt{2}$. We use the capital Φ symbol to denote the density matrix corresponding to a basis state—i.e., for a general state in the n -qubit GHZ basis $\Phi^{s_1 s_2 \dots s_n} \equiv |\phi^{s_1 s_2 \dots s_n}\rangle\langle\phi^{s_1 s_2 \dots s_n}|$.

For $n = 2$, the GHZ basis reduces to the Bell basis. We restrict our attention to states that do not contain off-diagonal elements in the Bell basis, which we call *Bell diagonal states*. This restriction does not reduce the applicability of our methods, because any bipartite qubit state can be transformed to a

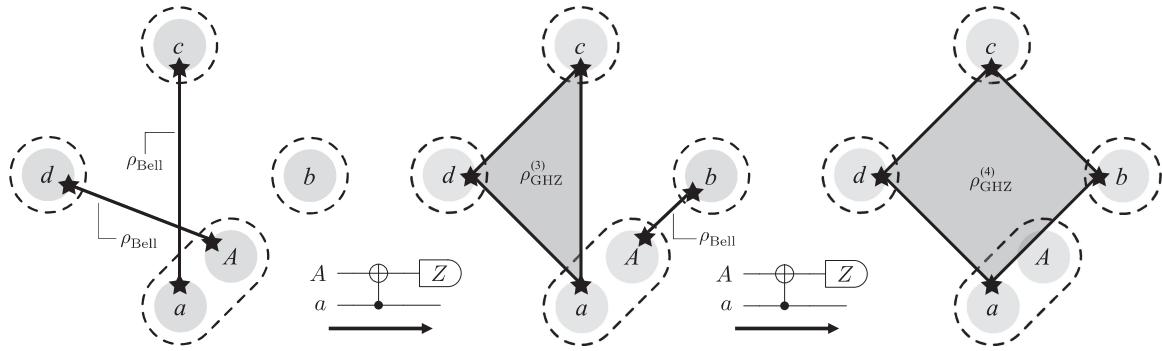


FIGURE 1. Fusion operation allows to merge Bell diagonal states and GHZ diagonal states, which overlap at one of the network parties. This party applies the operation depicted on top of the arrow, followed by local Pauli gate corrections that depend on the measurement outcome (not depicted).

Bell diagonal state with the same fidelity via *twirling* [39], a procedure that relies on local operations and classical communication.

In the general case, we also restrict our attention to diagonal states in the GHZ basis. This is justified because the operations introduced in Section III take GHZ diagonal states as input states. Therefore, to track the state of a distillation protocol with input Bell diagonal states and composed of these operations, it is sufficient to consider GHZ diagonal states. For n parties p_1, p_2, \dots, p_n , we write these states as

$$\rho_{p_1 p_2 \dots p_n} = \sum_{(s_1, s_2, \dots, s_n) \in \{+1, -1\}^n} A_{s_1 s_2 \dots s_n} \Phi^{s_1 s_2 \dots s_n}. \quad (1)$$

Unless otherwise stated, in the remainder, we use the term *state* to denote both Bell diagonal states and GHZ diagonal states, and call the $A_{s_1 s_2 \dots s_n}$ elements in (1) the *coefficients* of the state.

We use the shorthand $|\phi_n^+\rangle$ for $|\phi^{++++}\rangle$ when we need to explicitly denote the number of qubits of the state. We define the *fidelity* as $F \equiv \langle \phi_n^+ | \rho | \phi_n^+ \rangle$; it provides a measure of the closeness between a general n -qubit state ρ and the pure state $|\phi_n^+\rangle$. For diagonal density matrices in the Bell or the GHZ basis, the fidelity equals the value of the A_{++++} coefficient. We define F_{Bell} as the fidelity of a Bell diagonal state $|\phi_2^+\rangle$ and F_{GHZ} or $F_{\text{GHZ}}^{(n)}$ as the fidelity of an n -qubit GHZ diagonal state $|\phi_n^+\rangle$.

III. OPERATIONS ON BELL AND GHZ DIAGONAL STATES

This section discusses two operations on Bell and GHZ diagonal states: fusion operations, which merge two states, and distillation operations, which consume one state to improve the fidelity of another state.

A. FUSION

The *fusion* operation merges two states. The operation takes an n_1 -qubit state and an n_2 -qubit state that overlap in one network node, in the sense that this node holds (at least) one qubit of each state. The fusion operation consists of a CNOT gate between one qubit of each state, a measurement in the Z basis of the target qubit (see Fig. 1), and local Pauli gate corrections to the qubits of the other state. If the two qubits

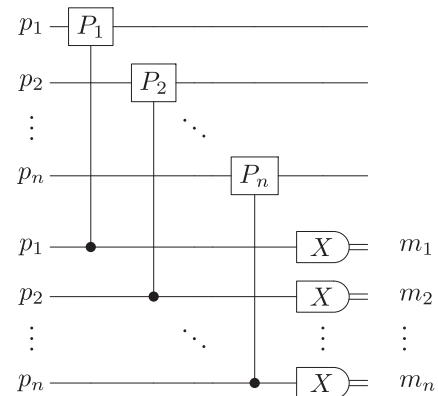


FIGURE 2. Quantum circuit to measure the nonlocal n -qubit Pauli operator $P_1 P_2 \dots P_n$ on a general n -qubit quantum state ρ_m (the *main state*). The n -qubit state ρ_m is distributed over n parties p_1, p_2, \dots, p_n in such a way that every party has one qubit of ρ_m . The n -qubit state $|\phi_n^+\rangle_a$ (the *ancillary state*), shared over the same n network parties, enables measuring $P_1 P_2 \dots P_n$ on ρ_m . The parties find the measurement outcome $m = \prod_{i=1}^n m_i$ combining their individual measurements $(m_1, m_2, \dots, m_n) \in \{+1, -1\}^n$.

involved are qubit i of the n_1 -qubit state and qubit j of the n_2 -qubit, we say that we are fusing the n_1 -qubit state at qubit i with the n_2 -qubit state at qubit j . This results in a new $(n_1 + n_2 - 1)$ -qubit state. The fusion operation is deterministic.

B. NONLOCAL STABILIZER MEASUREMENTS

A nonlocal stabilizer measurement also involves two states: a *main state* and an *ancillary state*. The ancillary state is consumed to measure a stabilizer operator on the main state.

In the context of a distillation scheme, the stabilizer operator is one of the stabilizers of some target state. Then, the nonlocal stabilizer measurement can be understood as an error-detection scheme. A $+1$ outcome projects the state into the corresponding eigenspace compatible with the target state, whereas a -1 outcome projects the state into the corresponding eigenspace orthogonal to the target state. For this reason, the state is kept when the measurement outcome is $+1$ and discarded otherwise.

In Fig. 2, we see a quantum circuit that measures a joint Pauli operator $P_1 P_2 \dots P_n$ with the aid of an n -qubit state $|\phi_n^+\rangle = (|0\rangle^{\otimes n} + |1\rangle^{\otimes n})/\sqrt{2}$. The qubits of the ancillary state are measured out individually in the X basis, and the network parties use classical communication to calculate the full measurement outcome, whereas the nonlocal measurement of the stabilizer in Fig. 2 is perfect, in practical situations, the ancillary state is noisy and the operation is only carried out approximately.

The n -qubit GHZ state $|\phi_n^+\rangle$ has three different types of stabilizers: ZZ stabilizers of weight 2, combinations of ZZ operators of weight m (where $m \leq n$ is always an even number), and operators of weight n consisting of combinations of $X_1 X_2 \dots X_n$ and any number of the ZZ operators. Measuring the latter type, nonlocally requires another n -qubit ancillary state. The ZZ operators can be measured with a Bell state as ancillary state. For combinations of ZZ operators of higher weight $m \leq n$, a GHZ state of weight m is required.

IV. GHZ DISTILLATION PROTOCOLS

Many GHZ creation protocols can be described by combining the operations from Section III. In particular, fusion operations create larger multipartite states and nonlocal stabilizer measurements can be used to increase the fidelity of the main state. As an example, in the following, we describe two protocols from Nickerson *et al.* [37] for creating and distilling a GHZ state shared by $n = 4$ network parties: one that uses $k = 22$ Bell states (*expedient*), and one that uses $k = 42$ Bell states (*stringent*).

Expedient and stringent are the result of a brute-force search over protocols following a concrete multistep structure [37]. The first step consists of nonlocal measurements involving all qubits on two opposite sides of the network (of the type shown between qubits a and b , and between qubits c and d in Fig. 3). After that, the two purified Bell pairs are stored. In the second step, the protocols continue with several rounds of nonlocal measurements that increase the fidelity of the two Bell pairs in the other direction (between qubits B and C , and between qubits A and D in Fig. 3). In the third step, the four Bell pairs are fused into a four-qubit GHZ state. The fourth step is analogous to the second one: two Bell pairs are purified between qubits B and C , and between qubits A and D . Finally, the purified Bell pairs are used to perform two ZZ nonlocal measurements to the four-qubit GHZ state.

Fig. 3 shows a schematic representation of both protocols. The expedient protocol is depicted in the top part of Fig. 3 and consumes 22 Bell states. Steps 1 and 2 create purified Bell pairs between qubits a and b , and between qubits c and d . Steps 3 and 4 create purified Bell pairs between qubits B and C , and between qubits A and D . In step 5, the GHZ state is created from these Bell pairs. Steps 6 and 7 are repetitions of steps 3 and 4, and step 8 consists of two ZZ nonlocal measurements on the GHZ state.

The stringent protocol has the same main structure as the expedient protocol, but consumes 42 Bell states. The protocol is depicted in the bottom of Fig. 3. In comparison with

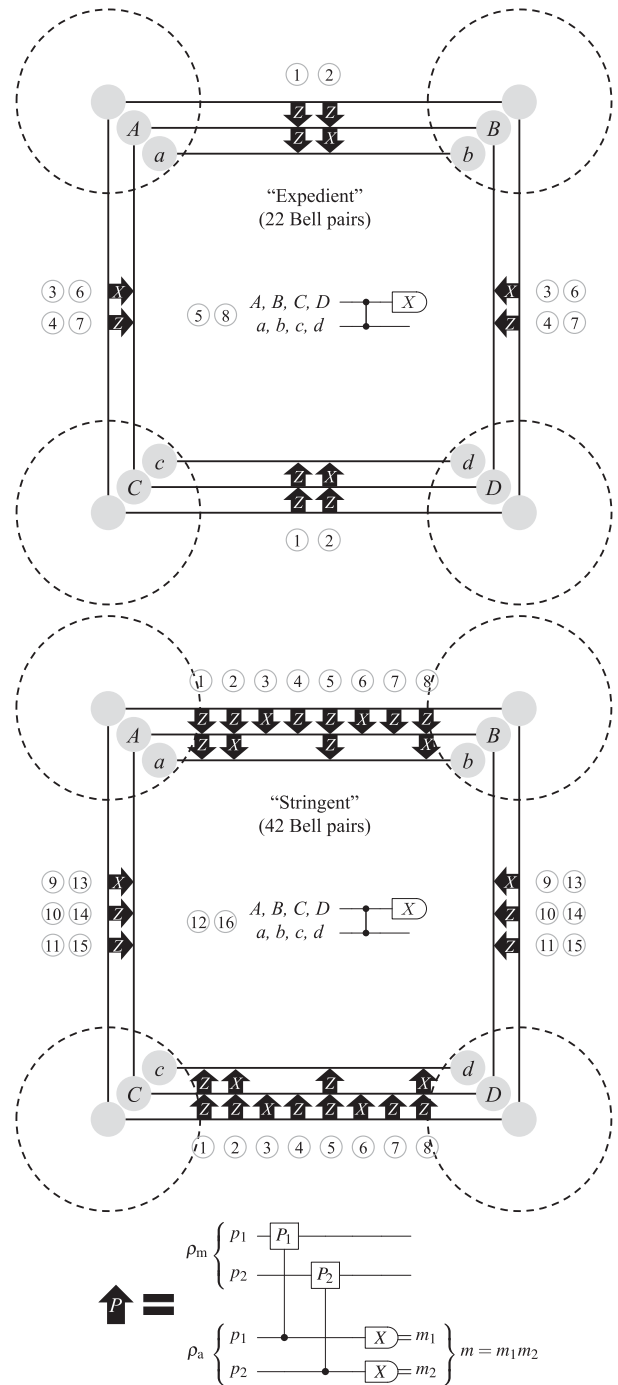


FIGURE 3. Two protocols for creating and purifying a four-qubit GHZ diagonal state out of Bell diagonal states shared between four network parties [37]. For both protocols, each node (dotted circles) requires three qubits (solid gray dots). The numbers denote the order in which individual steps are carried out. Solid lines depict Bell pairs shared between qubits. (Black arrows) Consumption of a Bell pair—the Bell pairs at the beginning of the arrow need to be regenerated for the next step.

expedient, stringent consumes an additional number of Bell states to increase the fidelity of the states that take part in the fusion step; analogously a larger number of Bell states is consumed to create the states to perform the final ZZ nonlocal measurements.

V. DYNAMIC PROGRAMS TO OPTIMIZE GHZ GENERATION

In this section, we discuss several algorithms to optimize GHZ generation. In particular, we consider the problem of probabilistically generating an n -qubit GHZ state with maximum fidelity F_{GHZ} starting from k isotropic Bell pairs with fidelity F_{Bell}

$$\rho_{AB} = F_{\text{Bell}} \Phi^{++} + \sum_{(s_1, s_2) \neq (+, +)} \frac{1 - F_{\text{Bell}}}{3} \Phi^{s_1 s_2}. \quad (2)$$

We do not restrict the number of qubits that the n nodes need to hold, and we also do not restrict their connectivity.

Ideally, we would consider all possible ways of distributing the Bell pairs over the network parties, and all possible combinations of the operations from Section III. Unfortunately, the number of these combinations grows super-exponentially in n (see [40] for a similar argument). This makes a brute-force approach infeasible for relevant values of (n, k) —in particular, for the protocols described in Section IV: $(n, k) = (4, 22)$ and $(n, k) = (4, 42)$. For this reason, similar to the approaches in [40] and [41] for Bell pair distribution in the context of quantum repeater chains, we propose heuristic *dynamic programs* for optimizing the distribution of GHZ states.

The dynamic programs reduce the complexity of the optimization and enable finding good protocols for large values of n and k . However, the output is not necessarily optimal. In the following, we first describe a simple dynamic program for optimizing GHZ generation. Next, we build on this description to present a randomized version of the dynamic program.

A. BASE DYNAMIC PROGRAM

The dynamic program takes as input the problem parameters n_{max} , k_{max} , F_{Bell} , and a buffer size b . The parameters specify the size of the final n -qubit GHZ state, the number of Bell pairs k , their fidelity F_{Bell} , and the number of protocols b to store at each intermediate step of the algorithm—i.e., for each combination of the number of Bell pairs and GHZ state size. The pseudocode of this algorithm can be found in Algorithm 1.

The algorithm begins with $(n, k) = (2, 1)$ and proceeds iteratively combining the solutions for smaller values of n and k until $(n, k) = (n_{\text{max}}, k_{\text{max}})$. More precisely, for each value of (n, k) , the algorithm combines the protocols found for smaller values of (n, k) in all possible ways to perform either a nonlocal measurement or to fuse the states, evaluates the fidelity of the resulting state for each combination, and stores the b protocols that achieve the largest fidelity. For $(n, k) = (2, 1)$, the algorithm stores a Bell pair with fidelity F_{Bell} . Subsequently, the algorithm increases k to $(n, k) = (2, 2)$ and it continues increasing k , until $(n, k) = (2, k_{\text{max}})$. At that point, n is increased by one and k is reset to $n - 1$. This process continues until n_{max} and k_{max} are reached.

We would like to stress that this is a heuristic approach and, in general, it does not lead to the optimal algorithms.

Algorithm 1: Base Dynamic Program to Optimize GHZ Generation.

Input : n_{max} : number of qubits of final GHZ state
 k_{max} : number of isotropic Bell pairs
 F_{Bell} : fidelity of isotropic Bell pairs
 b : number of protocols to store per step

Output: Protocol to create an n -qubit GHZ state out of k isotropic Bell pairs.

```

for  $\{(n, k) \mid n \leq n_{\text{max}}, k \leq k_{\text{max}}, k \geq n - 1\}$  do
    # Try all non-local measurement combinations.
    for  $stabilizer \in \{stabilizers\ of \ |\phi_n^+\}$  do
         $n' \leftarrow$  weight of  $stabilizer$ 
        for  $k' \in [n' - 1, k - n + 1]$  do
            Measure  $stabilizer$  on stored  $(n, k - k')$  state
            consuming the stored  $(n', k')$  state.
            Store new state if it has higher fidelity than
            one the existing  $(n, k)$  states.
            If more than  $b$   $(n, k)$  states are stored,
            remove the one with lowest fidelity.
        end
    end
    # Try all fusion combinations.
    for  $n_2 \in [2, n - 1]$  do
         $n_1 \leftarrow n - n_2 + 1$ 
        for  $k_2 \in [n_2 - 1, k - n + 1]$  do
             $k_1 \leftarrow k - k_2$ 
            for  $(i, j) \in [0, n_1 - 1] \times [0, n_2 - 1]$  do
                Fuse stored states  $(n_1, k_1)$  at qubit  $i$  and
                 $(n_2, k_2)$  at qubit  $j$ .
                Store new state if it has higher fidelity
                than the existing  $(n, k)$  state.
                If more than  $b$   $(n, k)$  states are stored,
                remove the one with lowest fidelity.
            end
        end
    end
end
return stored  $(n_{\text{max}}, k_{\text{max}})$  state with highest fidelity
    
```

This can be observed in Fig. 4. We plot the fidelity of the produced F_{GHZ} with respect to the fidelity F_{Bell} of the input Bell pairs. The fidelity of the produced F_{GHZ} is not always monotonically increasing. Moreover, as we increase the size of the buffer b , the program tends to find better protocols. However, lines with different values of b cross, highlighting the suboptimality of the output.

B. RANDOMIZED VERSION OF THE DYNAMIC PROGRAM

The base dynamic algorithm allows optimizing GHZ generation for moderate sizes. Unfortunately, even if faster than brute force, it still scales exponentially with the size of the GHZ state, and, for fixed GHZ size, it scales quadratically with the size of the buffer. In this section, we discuss a randomized version of the base dynamic program. This algorithm scales to larger GHZ sizes and, in practice, finds better protocols.

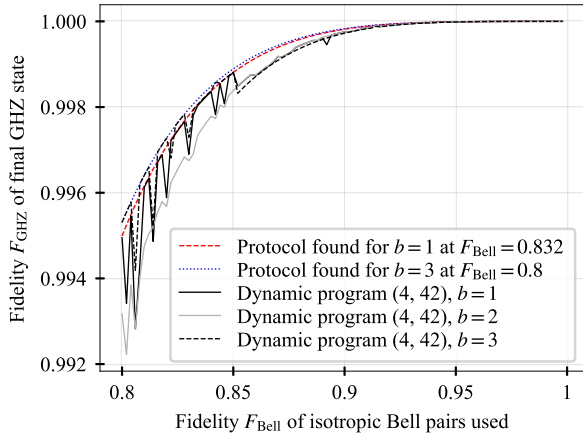


FIGURE 4. Fidelity of the final four-qubit GHZ as a function of the input Bell pair fidelity for a fixed number of input Bell pairs $k = 42$. We plot (black solid line, gray solid line, and dashed black line) the results of the base dynamic program with a buffer size $b = 1, 2, 3$ for each input Bell pair fidelity. We compare the results to the fidelity achieved by two fixed protocols for all input Bell pair fidelities (dashed red line and dotted blue line). The fixed protocols correspond with the protocols that the base dynamic program found for $b = 1, F_{\text{Bell}} = 0.832$ and $b = 3, F_{\text{Bell}} = 0.8$. They achieve approximately the convex hull of the individual protocols found by the base dynamic program with $b = 1$ and $b = 3$, respectively.

The randomized algorithm takes an additional parameter compared to the base dynamic algorithm. This is the temperature T that is used to decide whether or not to keep intermediate protocols. The randomized algorithm has the same two loops over (n, k) as the base one. It has an outer loop over the GHZ state size n starting from $n = 1$ to n_{max} and an inner loop over the number of Bell pairs starting from $k = n - 1$ to k_{max} . The two algorithms differ in how they construct the pool of protocols for (n, k) .

Similar to the base algorithm, the randomized algorithm fills a pool of b protocols for each combination of (n, k) . For $1 \leq i \leq b$, the algorithm chooses between nonlocal measurement or fusion with probability one half. Then, it selects uniformly at random a stabilizer to perform a nonlocal measurement or a fusion scheme to implement. Both the nonlocal measurement or the fusion scheme combine two smaller states. The parameters are singled out by the scheme choice: $(n, k - k')$ and (n, k') for the nonlocal measurement or $(n - n' + 1, k - k')$ and (n, k') for fusion. Finally, the two states are chosen uniformly at random from the b states stored.

The i th slot of the pool of b states is filled with certainty with the new state if either it is the first state (i.e., $i = 1$) or it achieves a higher fidelity than the previous protocol. If these conditions are not met, the new state can still probabilistically be accepted with probability $e^{\Delta F_{\text{GHZ}}/T}$, where ΔF_{GHZ} is the fidelity resulting from the new protocol minus the fidelity resulting from the previous protocol. If the value for T is set at a high value, states with a lower fidelity than the fidelity of the previous protocol are more likely to be accepted. If the new protocol is not stored, the i th slot is filled with the $(i - 1)$ th state. This approach makes that protocols

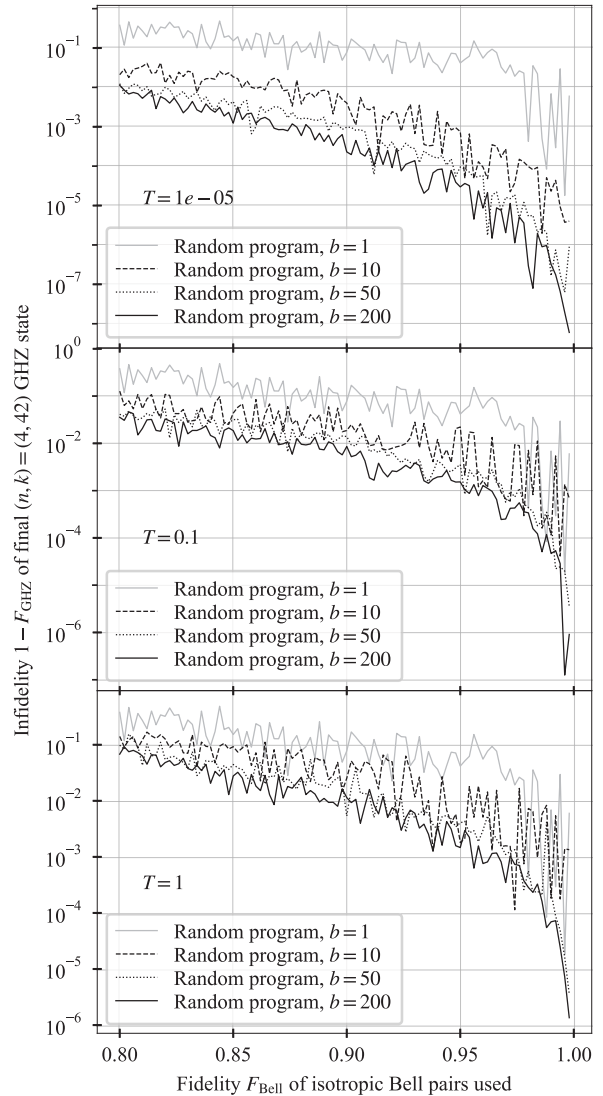


FIGURE 5. Performance of the randomized dynamic program as a function of the configuration parameters, for different fixed temperatures; from top to bottom, $T = 0.00001, T = 0.1$, and $T = 1$.

that lead to a high F_{GHZ} end up in multiple buffer slots, and are therefore more likely to be randomly selected at larger values of (n, k) .

For the randomized algorithm, the value for the buffer size b is usually set significantly larger than the value for b in the standard algorithm. This can be understood by realizing that the standard version of the dynamic algorithm visits all allowed combinations of states at smaller (n, k) , whereas the randomized version visits exactly b combinations at each (n, k) .

In the following, we investigate the effect of the configuration parameters, the buffer size b and the temperature T , on the performance of the algorithm. In particular, we fix $(n, k) = (4, 42)$ and evaluate the fidelity of the final GHZ state as a function of the input Bell pair fidelity. We show the results in three plots in Fig. 5. From top to bottom,

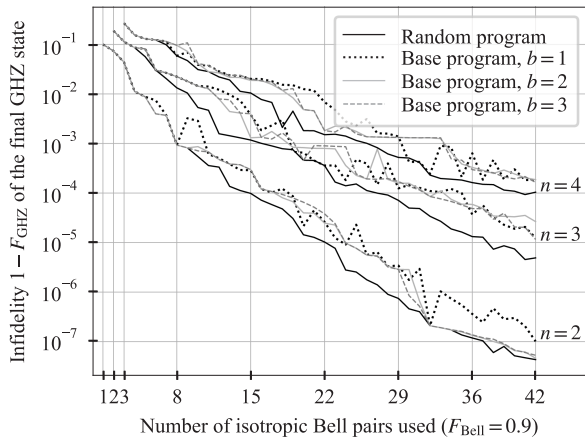


FIGURE 6. Fidelity achieved by the protocols from the base and random dynamic programs as a function of the number of Bell pairs. The fidelity of the input Bell pairs is fixed to $F_{\text{Bell}} = 0.9$. From top to bottom, each set of four lines corresponds to GHZ size $n = 4, 3, 2$. The lines labeled “base program” indicate the best protocols found by the base dynamic algorithm for different buffer sizes b . The black solid line indicates the best protocol found by the random variant. For the base algorithm, we used buffer sizes $b \in \{1, 3\}$. For the random algorithm, we ran the algorithm 44 times, with 18 different temperatures (between $T = 0.00001$ and $T = 0.0009$) per iteration and $b = 200$.

the temperatures are fixed to $T = 0.00001$, $T = 0.1$, and $T = 1$. In each plot, we show four lines corresponding to four different buffer sizes $b \in \{1, 10, 50, 200\}$. We see that, in general, a lower temperature gives better results. This can be understood by realizing that for low temperatures, states with higher fidelity are more likely to be stored in many slots of the buffer. This also indicates that fidelity is a good measure for determining the quality of a protocol as a building block for a larger protocol.

C. COMPARISON BETWEEN THE DYNAMIC PROGRAMS

We end the discussion of the dynamic programs by comparing the output of the base dynamic program against the randomized version. In Fig. 6, we compare the best results with $(n, k) = (4, 42)$ of the base dynamic program against the randomized program. For the parameters chosen, the randomized dynamic program outperforms the base dynamic program.

VI. RESULTS

In this section, we use the dynamic algorithms to find good GHZ creation protocols. First, we investigate how the different variants of the dynamic program discussed in Section V compare with each other and what the optimal parameter configurations are. Then, we use the programs to investigate scenarios of interest. First, given the importance of the surface code, we study the distribution of four-qubit GHZ states. Second, we explore how the quality of the GHZ state for the best protocols scales with the number of parties n .

The source code of the dynamic algorithm can be found online [42].

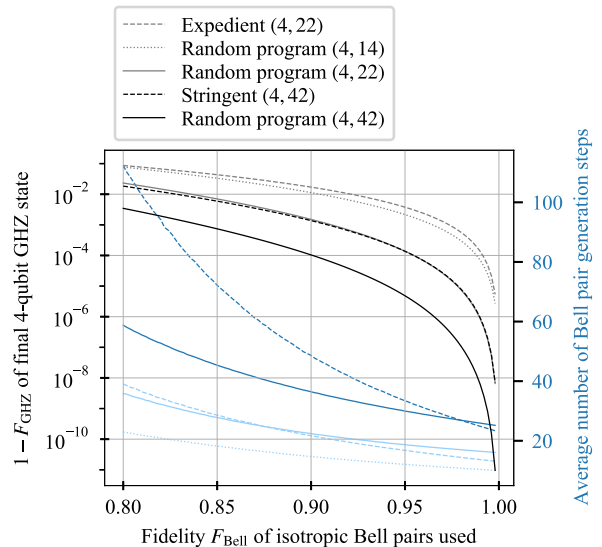


FIGURE 7. Comparison between the *expedient* and *stringent* protocols [37] (see Fig. 3), and the best algorithms found with the dynamic programs for $(n, k) = (4, 14)$, $(n, k) = (4, 22)$, and $(n, k) = (4, 42)$. The protocols are found with the randomized version of the dynamic program, using the settings and parameters discussed in Fig. 6. The blue lines show the average number of Bell pair generation steps for each of the protocols (right y-axis ticks). To calculate this metric, we take the creation of one Bell pair as one time step, and neglect the duration of all other elements of the protocols. The averages are calculated by executing the protocols 100 000 times for each fidelity.

A. COMPARISON WITH EXISTING PROTOCOLS FOR FOUR PARTIES

First, we investigate protocols for $n = 4$. In Fig. 7, we compare the best protocols that our dynamic algorithms find for parameters $(n, k) = (4, 14)$, $(n, k) = (4, 22)$, and $(n, k) = (4, 42)$ with the expedient and stringent protocols from Nickerson et al. [37]. The figure shows the infidelity (one minus the fidelity) of the output GHZ state as a function of the fidelity of the input Bell pairs. We see that under the conditions considered here, the new protocols create higher quality GHZ states with the same or even with a smaller number of Bell states. On the other hand, while expedient and stringent require three qubits per node, these protocols typically require more qubits per node. For example, the $(n, k) = (4, 14)$ protocol found with our dynamic program requires that two of the nodes have four qubits, as can be seen in Fig. 8. The best $(n, k) = (4, 22)$ protocol found by the random dynamic program can be performed if all four nodes have four qubits (see Fig. 8), and the best $(n, k) = (4, 42)$ found can be achieved with five qubits per node (see Fig. 8).

Let us now investigate to which degree the new protocols achieve these higher fidelities consuming a smaller amount of resources. We note that k represents the *minimum* number of Bell pairs needed to generate a GHZ state. These protocols are probabilistic, and the success probability depends on the fidelity of the states, going from one if the states are perfect to zero if the measured state is a minus one eigenstate of one of the measurement operators. If the protocol fails at some step, the step needs to be run again from the beginning. This

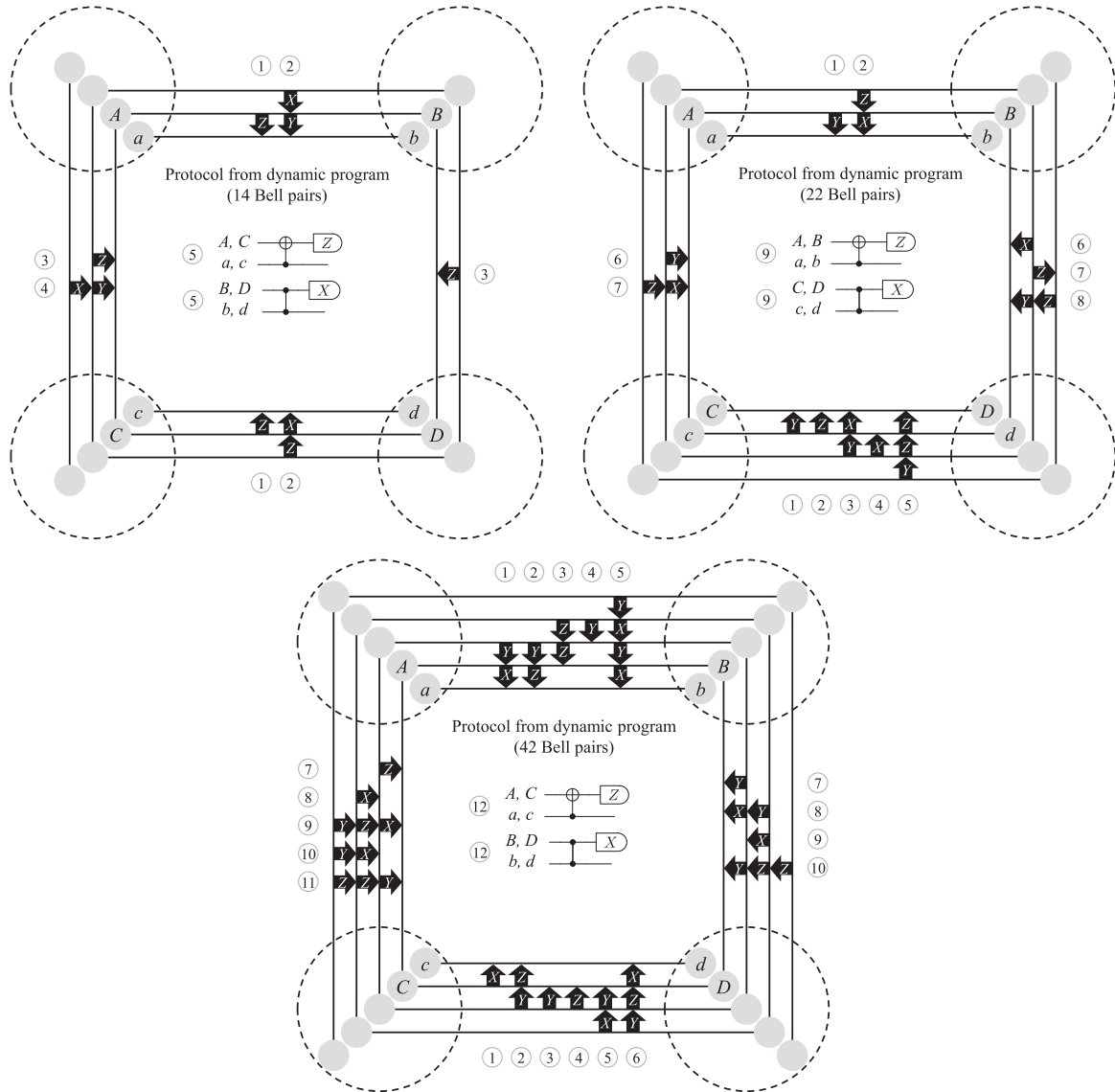


FIGURE 8. Protocols for creating and purifying a four-qubit GHZ diagonal state out of 14, 22, and 42 Bell diagonal states shared between four network parties, found with the randomized version of the dynamic program presented in this article. See Fig. 3 for more information about the notation.

means that the average number of Bell pairs might be very different from k .

A related figure of merit that might be more relevant in practice is the average number of entanglement generation steps. For this, we assume that network nodes can generate a Bell pair deterministically with one other node over some unit time step. Hence, different pairs of nodes can generate entanglement in parallel over some unit time step. For instance, the two left nodes in Fig. 8 can generate entanglement in parallel to the right nodes. If the duration of the time step is qualitatively larger than the gate time, the number of entanglement generation time steps represents to first-degree approximation the duration of the protocol.

We find that k is a good proxy for the average number of entanglement generation steps. We see in Fig. 7 that lower

values of k correspond with a lower average number of generation steps. Moreover, minimizing k leads to a reduction in the average number of generation steps. Interestingly, the average number of generation steps of stringent and expedient cross with some of the new protocols. In particular, stringent—a (4,42) protocol—crosses with the new (4,42) protocol, and expedient—a (4,22) protocol—crosses with the new (4,22) protocol. The reason for this is the higher symmetry in the structure of expedient (see Fig. 3) and stringent (see Fig. 3). In particular, they use the exact same number of Bell pairs at opposite sites of the network, whereas the new protocols contain small asymmetries in this respect. For very high input fidelities, the success probabilities of all distillation steps are close to one, leading to a lower average number of entanglement generation steps.

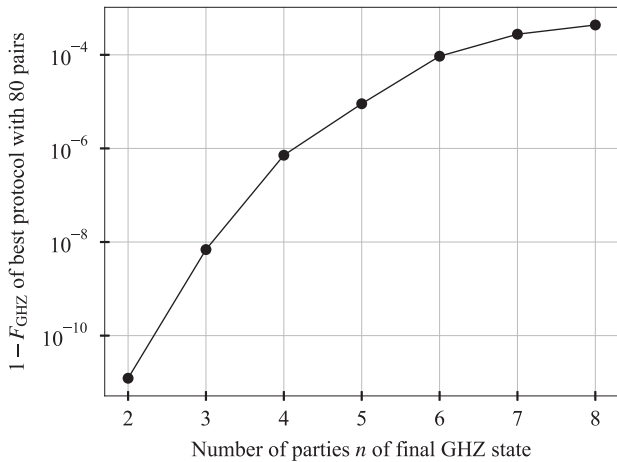


FIGURE 9. Infidelity ($1 - F_{\text{GHZ}}$) of the best protocols found for an n -qubit GHZ as a function of n and input Bell pair fidelity $F_{\text{Bell}} = 0.9$ and $k = 80$. The protocols are found with the randomized version of the dynamic program, using the settings and parameters discussed in Fig. 6.

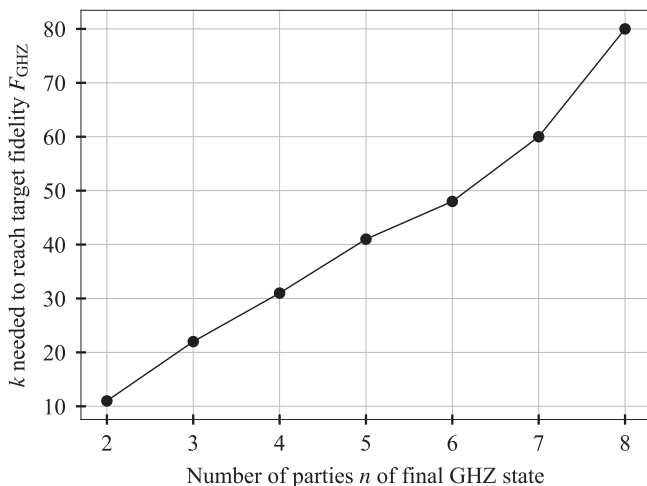


FIGURE 10. Number k of isotropic Bell pairs with $F_{\text{Bell}} = 0.9$ needed to make an n -qubit GHZ state with $F_{\text{GHZ}} \geq 0.999565$ as a function of n . This is the best F_{GHZ} found for an (8,80) GHZ state with the random dynamic program in all our attempts. The protocols are found with the randomized version of the dynamic program, using the settings and parameters discussed in Fig. 6.

B. RESULTS FOR LARGE NUMBER OF PARTIES

Here, we investigate the tradeoffs between the number of parties and the number of Bell pairs for $n > 4$.

First, for a fixed number of Bell pairs, we investigate how the GHZ fidelity drops as we increase the number of parties. In particular, we fix $k = 80$ and vary n from two to eight. In Fig. 9, we show the fidelity F_{GHZ} of the final n -qubit GHZ state as a function of n for a fixed Bell pair fidelity $F_{\text{Bell}} = 0.9$. We see that even for this high number of input pairs, the output fidelity drops sharply with the number of parties.

Second, we invert the question and we investigate how many Bell pairs are necessary to achieve a fixed target GHZ fidelity $F_{\text{GHZ}} = 0.999565$ for a different target number of parties. This is the best F_{GHZ} found for an (8,80) GHZ state

with the random dynamic program in all our attempts. In Fig. 10, we show the number of Bell pairs k with fidelity $F_{\text{Bell}} = 0.9$ needed to create an n -qubit GHZ state with fidelity $F_{\text{GHZ}} \geq 0.999565$. We observe that for the available data points, the number of pairs scales roughly linearly with the number of parties.

VII. CONCLUSION

In this article, we searched for protocols that generate high-fidelity GHZ states out of nonperfect Bell states. The goal was to minimize the number of Bell pairs for creating a high-quality GHZ state. We did this by using a dynamic program to search the protocol space, allowing two types of operations: fusion and distillation.

We found protocols that distill GHZ states with higher fidelity compared to previously known protocols. Compared to previous research [37], the protocols found require roughly half the number of pairs to achieve a similar fidelity. Out of the different algorithm variants that we implemented, the randomized version found the best protocols in most regimes. For $n = 2$ to $n = 8$ parties involved, we investigated how the fidelity of the final GHZ state decreases by increasing the number of parties with a fixed number of pairs ($k = 80$), and calculated how many Bell pairs are needed for the distribution of an n -qubit GHZ state of a fixed fidelity. Our programs can be used to find protocols for an arbitrary number of parties and entangled states involved.

GHZ states are required for implementing error-correction codes in distributed quantum computing. The distributed implementation of codes beyond the surface code has not been thoroughly explored. However, the case for the surface code in distributed implementations is weaker [43]. Richer connectivities than direct neighbors can be achieved with relative ease. Our tools open the door to implementations of alternative quantum error correcting codes that require high-quality GHZ states of weights different than four.

While these results are promising, future work should quantify the precise effect on the noise threshold for distributed implementations of the surface code. For this, a starting step will be to evaluate how the new protocols behave in more realistic scenarios. In particular, it would be interesting to perform a follow-up of the search for GHZ protocols, including noise and loss, whereas for the situation considered here, increasing the number of Bell pairs to perform additional distillation steps always increases the fidelity of the final states, this will no longer be the case for scenarios that include decoherence or gate noise. Hence, for noisy settings, the quality of the memory will determine whether including more distillation steps actually leads to an increase in the fidelity of the final state.

ACKNOWLEDGMENT

The authors would like to thank T. Coopmans, S. Jeffery, T. Taminau, and F. Rozpędek for helpful discussions and feedback.

REFERENCES

- [1] J. I. Cirac, A. K. Ekert, S. F. Huelga, and C. Macchiavello, "Distributed quantum computation over noisy channels," *Phys. Rev. A*, vol. 59, pp. 4249–4254, Jun. 1999, doi: [10.1103/PhysRevA.59.4249](https://doi.org/10.1103/PhysRevA.59.4249).
- [2] L. K. Grover, "Quantum teleportation," 1997, *arXiv:quant-ph/9704012*.
- [3] R. Van Meter and S. J. Devitt, "The path to scalable distributed quantum computing," *Computer*, vol. 49, pp. 31–42, Sep. 2016, doi: [10.1109/MC.2016.291](https://doi.org/10.1109/MC.2016.291).
- [4] D. D. Awschalom, R. Hanson, J. Wrachtrup, and B. B. Zhou, "Quantum technologies with optically interfaced solid-state spins," *Nat. Photon.*, vol. 12, pp. 516–527, Sep. 2018, doi: [10.1038/s41566-018-0232-2](https://doi.org/10.1038/s41566-018-0232-2).
- [5] T. H. Taminiau, J. Cramer, T. Van Der Sar, V. V. Dobrovitski, and R. Hanson, "Universal control and error correction in multi-qubit spin registers in diamond," *Nat. Nanotechnol.*, vol. 9, no. 3, pp. 171–176, 2014, doi: [10.1038/nnano.2014.2](https://doi.org/10.1038/nnano.2014.2).
- [6] J. Cramer *et al.*, "Repeated quantum error correction on a continuously encoded qubit by real-time feedback," *Nat. Commun.*, vol. 7, May 2016, Art. no. 11526, doi: [10.1038/ncomms11526](https://doi.org/10.1038/ncomms11526).
- [7] A. Reiserer *et al.*, "Robust quantum-network memory using decoherence-protected subspaces of nuclear spins," *Phys. Rev. X*, vol. 6, Jun. 2016, Art. no. 021040, doi: [10.1103/PhysRevX.6.021040](https://doi.org/10.1103/PhysRevX.6.021040).
- [8] N. Kalb *et al.*, "Entanglement distillation between solid-state quantum network nodes," *Science*, vol. 356, pp. 928–932, Jun. 2017, doi: [10.1126/science.aan0070](https://doi.org/10.1126/science.aan0070).
- [9] C. E. Bradley *et al.*, "A ten-qubit solid-state spin register with quantum memory up to one minute," *Phys. Rev. X*, vol. 9, Sep. 2019, Art. no. 031045, doi: [10.1103/PhysRevX.9.031045](https://doi.org/10.1103/PhysRevX.9.031045).
- [10] M. H. Aboeih *et al.*, "Atomic-scale imaging of a 27-nuclear-spin cluster using a quantum sensor," *Nature*, vol. 576, pp. 411–415, Dec. 2019, doi: [10.1038/s41586-019-1834-7](https://doi.org/10.1038/s41586-019-1834-7).
- [11] A. Sipahigil *et al.*, "An integrated diamond nanophotonics platform for quantum-optical networks," *Science*, vol. 354, pp. 847–850, Nov. 2016, doi: [10.1126/science.aah6875](https://doi.org/10.1126/science.aah6875).
- [12] D. D. Sukachev *et al.*, "Silicon-vacancy spin qubit in diamond: A quantum memory exceeding 10 ms with single-shot state readout," *Phys. Rev. Lett.*, vol. 119, Nov. 2017, Art. no. 223602, doi: [10.1103/PhysRevLett.119.223602](https://doi.org/10.1103/PhysRevLett.119.223602).
- [13] C. T. Nguyen *et al.*, "Quantum network nodes based on diamond qubits with an efficient nanophotonic interface," *Phys. Rev. Lett.*, vol. 123, Oct. 2019, Art. no. 183602, doi: [10.1103/PhysRevLett.123.183602](https://doi.org/10.1103/PhysRevLett.123.183602).
- [14] C. T. Nguyen *et al.*, "An integrated nanophotonic quantum register based on silicon-vacancy spins in diamond," *Phys. Rev. B*, vol. 100, Oct. 2019, Art. no. 165428, doi: [10.1103/PhysRevB.100.165428](https://doi.org/10.1103/PhysRevB.100.165428).
- [15] D. Hucul *et al.*, "Modular entanglement of atomic qubits using photons and phonons," *Nat. Phys.*, vol. 11, pp. 37–42, Jan. 2015, doi: [10.1038/nphys3150](https://doi.org/10.1038/nphys3150).
- [16] R. Nigmatullin, C. J. Ballance, N. de Beaudrap, and S. C. Benjamin, "Minimally complex ion traps as modules for quantum communication and computing," *New J. Phys.*, vol. 18, Oct. 2016, Art. no. 103028, doi: [10.1088/1367-2630/18/10/103028](https://doi.org/10.1088/1367-2630/18/10/103028).
- [17] P. C. Humphreys *et al.*, "Deterministic delivery of remote entanglement on a quantum network," *Nature*, vol. 558, pp. 268–273, Jun. 2018, doi: [10.1038/s41586-018-0200-5](https://doi.org/10.1038/s41586-018-0200-5).
- [18] N. H. Nickerson, J. F. Fitzsimons, and S. C. Benjamin, "Freely scalable quantum technologies using cells of 5-to-50 qubits with very lossy and noisy photonic links," *Phys. Rev. X*, vol. 4, Dec. 2014, Art. no. 041041, doi: [10.1103/PhysRevX.4.041041](https://doi.org/10.1103/PhysRevX.4.041041).
- [19] M. Hillery, V. Bužek, and A. Berthiaume, "Quantum secret sharing," *Phys. Rev. A*, vol. 59, pp. 1829–1834, Mar. 1999, doi: [10.1103/PhysRevA.59.1829](https://doi.org/10.1103/PhysRevA.59.1829).
- [20] M. Christandl and S. Wehner, "Quantum anonymous transmissions," in *Proc. 26th Ann. Int. Conf. Theory Appl. Cryptol. Inf. Secur.*, 2005, pp. 217–235, doi: [10.1007/11593447_12](https://doi.org/10.1007/11593447_12).
- [21] P. Kómár *et al.*, "A quantum network of clocks," *Nat. Phys.*, vol. 10, pp. 582–587, Aug. 2014, doi: [10.1038/nphys3000](https://doi.org/10.1038/nphys3000).
- [22] E. T. Khabiboulline, J. Borregaard, K. De Greve, and M. D. Lukin, "Optical interferometry with quantum networks," *Phys. Rev. Lett.*, vol. 123, Aug. 2019, Art. no. 070504, doi: [10.1103/PhysRevLett.123.070504](https://doi.org/10.1103/PhysRevLett.123.070504).
- [23] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, "Purification of noisy entanglement and faithful teleportation via noisy channels," *Phys. Rev. Lett.*, vol. 76, pp. 722–725, Jan. 1996, doi: [10.1103/PhysRevLett.76.722](https://doi.org/10.1103/PhysRevLett.76.722).
- [24] D. Deutsch, A. Ekert, R. Jozsa, C. Macchiavello, S. Popescu, and A. Sanpera, "Quantum privacy amplification and the security of quantum cryptography over noisy channels," *Phys. Rev. Lett.*, vol. 77, no. 13, pp. 2818–2821, 1996, doi: [10.1103/PhysRevLett.77.2818](https://doi.org/10.1103/PhysRevLett.77.2818).
- [25] K. Fujii and K. Yamamoto, "Entanglement purification with double selection," *Phys. Rev. A*, vol. 80, Oct. 2009, Art. no. 042308, doi: [10.1103/PhysRevA.80.042308](https://doi.org/10.1103/PhysRevA.80.042308).
- [26] S. Krastanov, V. V. Albert, and L. Jiang, "Optimized entanglement purification," *Quantum*, vol. 3, Feb. 2019, Art. no. 123, doi: [10.22331/q-2019-02-18-123](https://doi.org/10.22331/q-2019-02-18-123).
- [27] M. Muraö, M. B. Plenio, S. Popescu, V. Vedral, and P. L. Knight, "Multiparticle entanglement purification protocols," *Phys. Rev. A, At., Mol., Opt. Phys.*, vol. 57, no. 6, pp. R4075–R4078, 1998, doi: [10.1103/PhysRevA.57.R4075](https://doi.org/10.1103/PhysRevA.57.R4075).
- [28] E. N. Maneva and J. A. Smolin, "Improved two-party and multi-party purification protocols," *Contemporary Math.*, vol. 305, pp. 203–212, 2002, doi: [10.1090/conm/305/05220](https://doi.org/10.1090/conm/305/05220).
- [29] W. Dür, H. Aschauer, and H. J. Briegel, "Multiparticle entanglement purification for graph states," *Phys. Rev. Lett.*, vol. 91, Sep. 2003, Art. no. 107903, doi: [10.1103/PhysRevLett.91.107903](https://doi.org/10.1103/PhysRevLett.91.107903).
- [30] K. H. Ho and H. F. Chau, "Purifying Greenberger-Horne-Zeilinger states using degenerate quantum codes," *Phys. Rev. A*, vol. 78, Oct. 2008, Art. no. 042329, doi: [10.1103/PhysRevA.78.042329](https://doi.org/10.1103/PhysRevA.78.042329).
- [31] S. Glancy, E. Knill, and H. M. Vasconcelos, "Entanglement purification of any stabilizer state," *Phys. Rev. A*, vol. 74, Sep. 2006, Art. no. 032319, doi: [10.1103/PhysRevA.74.032319](https://doi.org/10.1103/PhysRevA.74.032319).
- [32] C. Kruszynska, A. Miyake, H. J. Briegel, and W. Dür, "Entanglement purification protocols for all graph states," *Phys. Rev. A*, vol. 74, Nov. 2006, Art. no. 052316, doi: [10.1103/PhysRevA.74.052316](https://doi.org/10.1103/PhysRevA.74.052316).
- [33] M. Huber and M. Plesch, "Purification of genuine multipartite entanglement," *Phys. Rev. A*, vol. 83, Jun. 2011, Art. no. 062321, doi: [10.1103/PhysRevA.83.062321](https://doi.org/10.1103/PhysRevA.83.062321).
- [34] E. Hostens, J. Dehaene, and B. De Moor, "Hashing protocol for distilling multipartite Calderbank-Shor-Steane states," *Phys. Rev. A*, vol. 73, Apr. 2006, Art. no. 042316, doi: [10.1103/PhysRevA.73.042316](https://doi.org/10.1103/PhysRevA.73.042316).
- [35] E. Hostens, J. Dehaene, and B. De Moor, "Stabilizer state breeding," *Phys. Rev. A*, vol. 74, Dec. 2006, Art. no. 062318, doi: [10.1103/PhysRevA.74.062318](https://doi.org/10.1103/PhysRevA.74.062318).
- [36] F. Rozpedek, T. Schiet, L. P. Thinh, D. Elkouss, A. C. Doherty, and S. Wehner, "Optimizing practical entanglement distillation," *Phys. Rev. A*, vol. 97, Jun. 2018, Art. no. 062333, doi: [10.1103/PhysRevA.97.062333](https://doi.org/10.1103/PhysRevA.97.062333).
- [37] N. H. Nickerson, Y. Li, and S. C. Benjamin, "Topological quantum computing with a very noisy network and local error rates approaching one percent," *Nat. Commun.*, vol. 4, 2013, Art. no. 1756, doi: [10.1038/ncomms2773](https://doi.org/10.1038/ncomms2773).
- [38] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*. Cambridge, U.K.: Cambridge Univ. Press, 2000.
- [39] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, "Mixed-state entanglement and quantum error correction," *Phys. Rev. A, At., Mol., Opt. Phys.*, vol. 54, no. 5, pp. 3824–3851, 1996, doi: [10.1103/PhysRevA.54.3824](https://doi.org/10.1103/PhysRevA.54.3824).
- [40] K. Goodenough, D. Elkouss, and S. Wehner, "Optimising repeater schemes for the quantum internet," 2020, *arXiv:2006.12221*.
- [41] L. Jiang, J. Taylor, N. Khaneja, and M. Lukin, "Optimal approach to quantum communication using dynamic programming," in *Proc. Nat. Acad. Sci. USA*, vol. 104, pp. 17291–17296, Oct. 2007, doi: [10.1073/pnas.0703284104](https://doi.org/10.1073/pnas.0703284104).
- [42] S. W. de Bone, R. Ouyang, K. Goodenough, and D. Elkouss, "GHZ_prot_I: Software used for the paper 'Protocols for creating and distilling multipartite GHZ states with Bell pairs,'" Sep. 2020. [Online]. Available: https://github.com/sebastiandebone/GHZ_prot_I
- [43] E. T. Campbell, B. M. Terhal, and C. Vuillot, "Roads towards fault-tolerant universal quantum computation," *Nature*, vol. 549, pp. 172–179, Sep. 2017, doi: [10.1038/nature23460](https://doi.org/10.1038/nature23460).