Comments and Corrections

Correction to Section V-B of "Contraction Theory and the Master Stability Function: Linking Two Approaches to Study Synchronization in Complex Networks"

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***Abstract*—This document contains corrections to the paper entitled "Contraction Theory and the Master Stability Function: Linking Two Approaches to Study Synchronization in Complex Networks." The results reported here should substitute those included in Section V-B therein.**

As shown in [2], we can associate to a generic connected network of the form:

$$\dot{X} = F(X) - \alpha(L \otimes I_n)X \tag{1}$$

the following auxiliary virtual system:

$$\dot{Y} = F(Y) - \alpha(L \otimes I_n)Y - (1_{N\times N} \otimes K_0)(Y - X) \tag{2}$$

where $Y := [y_1^T, \ldots, y_N^T]^T$ is the set of virtual state variables, K_0 is some symmetric positive definite matrix, and $1_{N\times N}$ is the $N \times N$ matrix whose elements are all equal to 1. (For further details on the notation and terminology used here the reader is referred to [1], [2].)

System (2) has $Y = X$ as a solution and admits the particular solution $y_1 = \cdots = y_N = y_\infty$, with y_∞ such that [2]

$$\dot{y}_\infty = f(y_\infty) - nK_0 y_\infty + K_0 \sum_{j=1}^{N} x_j(t).$$

Thus, contraction of system (2) in the Euclidean norm immediately implies that network (1) achieves synchronization (see, e.g., [2]). We can then state the following result.

Theorem 1: Consider a network with N identical nodes. If 1) the network is connected, 2) the coupling functions are linear and increasing, and 3) the auxiliary system (2) is contracting in the Euclidean norm w.r.t. Y for all X for some range of the coupling strength, $\alpha \in A$; then, the master stability function (MSF) of (1) will be negative in A, i.e., the network locally synchronizes for $\alpha \in A$.

Proof: From the hypotheses, we have that system (2) is contracting in the Euclidean norm, i.e., the symmetric part of its Jacobian, e.g., J_s, given by

$$J_s := \left[\frac{\partial F}{\partial y}\right]_s - \alpha(L \otimes I_n) - 1_{N\times N} \otimes K_0$$

is negative definite $\forall \alpha \in A$.

Let $J_r = -\alpha(L \otimes I_n) - 1_{N\times N} \otimes K_0$. Its maximum eigenvalue can be computed, as shown in [2], using the Courant–Fischer theorem (see, e.g., [3]) as

$$\lambda_{\max}(J_r) = -\lambda_{m+1}\left(\alpha(L \otimes I_n)\right).$$

with λ_{m+1} being the first eigenvalue associated to the transversal dynamics. Thus, if the auxiliary system is contracting, then

$$\lambda_{m+1}\left(\alpha(L \otimes I_n)\right) > \max_i \lambda_{\max}\left(\left[\frac{\partial f}{\partial y_i}\right]_s\right) \tag{3}$$

$\forall \alpha \in A$. We can then conclude that the matrix

$$\frac{\partial F}{\partial y} - \alpha(L \otimes I_n) \tag{4}$$

is negative definite $\forall \alpha \in A$. Hence, the linear system

$$\dot{\xi} = \left(\frac{\partial F}{\partial y} - \alpha(L \otimes I_n)\right)\xi$$

is contracting. The proof is then concluded by noticing that the dynamics of the previously mentioned system around the synchronization manifold yield the variational equation used, according to the MSF approach [4], to calculate the Lyapunov exponents. Now, since the system is contracting, then its Lyapunov exponents will be negative, as shown in [1], immediately implying negativity of the MSF. ■

References

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