# A Simple Ladder Realization of Maximally Flat Allpass Fractional Delay Filters

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***Abstract*—This brief proposes a new ladder structure for the Thiran fractional delay filter (i.e., maximally flat allpass fractional delay filter given by the Thiran approximation). The proposed ladder structure is based on a continued fraction representation. Although there exists a similar approach that was proposed by Tassart and Depalle, their structure is not realizable because of delay-free loops. On the other hand, we show that the proposed method avoids generating delay-free loops and thus successfully yields a realizable ladder structure for the Thiran fractional delay filter in a very simple form.**

***Index Terms*—Continued fraction, delay-free loop, ladder structure, Thiran fractional delay filter.**

## I. Introduction

FRACTIONAL delay digital filters have been extensively studied in the literature on digital filter design and applications, and until recently, many elegant results have been reported regarding this field [1]–[14]. In this brief, we pay attention to the infinite impulse response (IIR) allpass fractional delay filter with maximally flat group delay at the zero frequency, which is obtained by means of the Thiran approximation [15]. This fractional delay filter has the following transfer function with the closed-form coefficients [1]:

$$H_{\mathrm{FD}}(z) = \frac{a_N + a_{N-1}z^{-1} + \cdots + z^{-N}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$
$$a_k = (-1)^k \binom{N}{k} \prod_{n=0}^{N} \frac{D-N+n}{D-N+k+n}, \quad 1 \le k \le N \tag{1}$$

where $N$ is the order of the filter, and $D$ is the desired delay. It was shown in [2] that the filter stability is ensured for any $D$ satisfying $D > N-1$. The advantage of this allpass fractional delay filter lies in the closed-form coefficients, which allow us to easily obtain the transfer function with the desired group delay. This design method was extended to the synthesis of variable fractional delay filters [3]–[6], [10]. Moreover, in [7], a high-accuracy half-sample delay IIR filter was proposed, which led to many practical applications such as in IIR Simpson integrators and half-band and diamond-shaped filters, and in digital image magnification. Other practical applications were also developed, such as in designing a sample-rate converter [4] and modeling of musical instruments [5], [8].

The purpose of this brief is to present a new simple structure for the transfer function (1) of the Thiran fractional delay filter. Since most of the existing methods related to the Thiran fractional delay filter rely on the direct-form structure or the cascaded structure, it would be worthwhile and interesting from an implementation point of view to provide some other possible structures. Such an attempt was made by Tassart and Depalle in [9]. They found that the transfer function of the Thiran fractional delay filter can be obtained by a continued fraction expansion of $(1+x)^y$ and they tried to apply this result to the synthesis of a ladder-type structure for the Thiran fractional delay filter. However, unfortunately, this approach did not work because of the existence of delay-free loops in the resultant structure; thus, they concluded that this approach seems to be usable only from a theoretical point of view. Hence, construction of an explicit structure (other than the direct-form structure) for the Thiran fractional delay filter has still been an open problem.

In this brief, we overcome the aforementioned problem of [9] and propose a simple method for constructing a ladder structure for the Thiran fractional delay filter. To this end, we first present a modified formulation of the conventional continued fraction expansion in [9]. Then, we prepare a "prototype" ladder structure for this modified continued fraction. This prototype structure is obtained by extending the scheme of Mitra and Sherwood [16]. Finally, by means of a slight modification of this structure, we derive the desired ladder structure for the Thiran fractional delay filter.

The organization of this brief is as follows. In Section II we review the theory given in [9] and address the details of the aforementioned problem. Section III presents our ladder structure. Section IV gives the conclusion.

## II. Preliminaries

Here, we first review the relationship between the Thiran fractional delay filter and the continued fraction. Throughout this brief, we will use the following Abramowitz notation [17] for the continued fraction:

$$\cfrac{B_1}{A_1 + \cfrac{B_2}{A_2 + \cfrac{B_3}{A_3 + \ddots}}} \equiv \frac{B_1}{A_1+}\frac{B_2}{A_2+}\frac{B_3}{A_3+}\cdots. \tag{2}$$

In [9], it was found that the Thiran fractional delay filter (1) can be formulated in terms of the continued fraction. In order

Manuscript received May 27, 2013; revised August 23, 2013; accepted November 29, 2013. Date of publication January 21, 2014; date of current version March 14, 2014. This brief was recommended by Associate Editor Y. Yu.

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Digital Object Identifier 10.1109/TCSII.2013.2296131

to derive this result, a continued fraction expansion of $(1+x)^y$ is first introduced as

$$\begin{aligned} f(x,y) \equiv& (1+x)^y \\ =& 1 + \frac{yx}{1-}\frac{(y-1)x}{2+}\frac{(y+1)x}{3-}\frac{(y-2)x}{2+} \\ & \frac{(y+2)x}{5-}\frac{(y-3)x}{2+}\cdots. \end{aligned} \tag{3}$$

Substituting $z^{-1}-1$ into $x$ and $D$ into $y$ in (3), we have

$$\begin{aligned} & f(z^{-1}-1, D) \\ &= z^{-D} \\ &= 1 + \frac{D(z^{-1}-1)}{1-}\frac{(D-1)(z^{-1}-1)}{2+} \\ & \quad \frac{(D+1)(z^{-1}-1)}{3-}\frac{(D-2)(z^{-1}-1)}{2+}\cdots \\ & \quad \frac{(D+k-1)(z^{-1}-1)}{(2k-1)-}\frac{(D-k)(z^{-1}-1)}{2+}\cdots. \end{aligned} \tag{4}$$

This equation clearly shows that the ideal fractional delay $z^{-D}$ can be expressed in terms of the infinite continued fraction as given. In addition, it was observed in [9] that the $N$th-order Thiran fractional delay filter (1) is equal to the $(2N+1)$st approximant of (4), i.e.,

$$\begin{aligned} H_{\mathrm{FD}}(z) =& 1 + \frac{D(z^{-1}-1)}{1-}\frac{(D-1)(z^{-1}-1)}{2+} \\ & \frac{(D+1)(z^{-1}-1)}{3-}\frac{(D-2)(z^{-1}-1)}{2+}\cdots \\ & \frac{(D+N-1)(z^{-1}-1)}{(2N-1)-}\frac{(D-N)(z^{-1}-1)}{2}. \end{aligned} \tag{5}$$

This provides a mathematical formulation of the $N$th-order Thiran fractional delay filter in terms of the continued fraction. The rigorous proof of this fact was given in [18].

We then address the problem of the strategy in [9] for constructing a filter structure for the representation of (5). In [9], it was first shown that the transfer function given by the following continued fraction:

$$H(z) = \frac{p_1 z^{-1}}{r_1-}\frac{q_1 z^{-1}}{s_1-}\frac{p_2 z^{-1}}{r_2-}\frac{q_2 z^{-1}}{s_2-}\cdots\frac{p_N z^{-1}}{r_N-}\frac{q_N z^{-1}}{s_N} \tag{6}$$

can be realized by a ladder structure (as in Figs. 2 and 4 [9]). Then, the following substitutions into (6) were performed:

$$\begin{aligned} z^{-1} &\leftarrow z^{-1}-1 \\ (p_k,\, q_k,\, r_k,\, s_k) &\leftarrow (1-k-D,\, D-k,\, 2k-1,\, 2) \end{aligned} \tag{7}$$

where $1 \le k \le N$. Now, it immediately follows that the transfer function given by the given substitutions yields the fractional part of (5), i.e., the following relationship holds:

$$1 - H_{\mathrm{FD}}(z) = \widehat{H}(z) \tag{8}$$

where $\widehat{H}(z)$ is the transfer function that is obtained through the substitutions (7) in (6). Therefore, it seems from this result that the Thiran fractional delay filter can be realized by a ladder structure if we perform the substitutions (7) on the block diagram of the ladder structure of (6). As stated in [9], however, this strategy fails to provide feasible filter structures. The reason for this failure lies in the substitution $z^{-1} \leftarrow z^{-1}-1$. If we carry out this substitution on the block diagram of (6), the resultant block diagram yields delay-free loops; thus, the resultant filter structure is not realizable.

In the following, we propose a new approach that overcomes this problem.

## III. Main Result

Although the proposed method also makes use of a continued fraction, the continued fraction to be used here is different from (3). Instead of (3), we introduce the continued fraction expansion of $x^y$ as follows [19]:

$$\begin{aligned} x^y = & \frac{1}{1-}\frac{y}{\frac{x}{x-1}+}\frac{y-1}{2-}\frac{y+1}{3\frac{x}{x-1}+}\frac{y-2}{2-}\cdots \\ & \frac{y+k-1}{(2k-1)\frac{x}{x-1}+}\frac{y-k}{2-}\cdots. \end{aligned} \tag{9}$$

The details of the derivation of (9) is given in Appendix A. In the proposed method, we start with the reciprocal of (9), i.e.,

$$\begin{aligned} g(x,y) \equiv& \frac{1}{x^y} \\ =& 1 - \frac{y}{\frac{x}{x-1}+}\frac{y-1}{2-}\frac{y+1}{3\frac{x}{x-1}+}\frac{y-2}{2-}\cdots \\ & \frac{y+k-1}{(2k-1)\frac{x}{x-1}+}\frac{y-k}{2-}\cdots. \end{aligned} \tag{10}$$

Letting $x = z^{-1}$ and $y = -D$ in (10), we obtain the formulation of the ideal fractional delay $z^{-D}$ as the following infinite continued fraction:

$$\begin{aligned} z^{-D} =& g(z^{-1}, -D) \\ =& 1 - \frac{-D}{\frac{z^{-1}}{z^{-1}-1}+}\frac{-D-1}{2-}\frac{-D+1}{3\frac{z^{-1}}{z^{-1}-1}+}\frac{-D-2}{2-}\cdots \\ & \frac{-D+k-1}{(2k-1)\frac{z^{-1}}{z^{-1}-1}+}\frac{-D-k}{2-}\cdots \\ =& 1 + \frac{D}{(-1)\frac{z^{-1}}{1-z^{-1}}+}\frac{-(D+1)}{2+}\frac{D-1}{(-3)\frac{z^{-1}}{1-z^{-1}}+}\frac{-(D+2)}{2+} \\ & \cdots\frac{D-k+1}{(-2k+1)\frac{z^{-1}}{1-z^{-1}}+}\frac{-(D+k)}{2+}\cdots. \end{aligned} \tag{11}$$

Considering the $(2N+1)$st approximant of (11), we now have the following transfer function:

$$\begin{aligned} G_{\mathrm{FD}}(z) =& 1 + \frac{D}{(-1)\frac{z^{-1}}{1-z^{-1}}+}\frac{-(D+1)}{2+}\frac{D-1}{(-3)\frac{z^{-1}}{1-z^{-1}}+} \\ & \frac{-(D+2)}{2+}\cdots\frac{D-N+1}{(-2N+1)\frac{z^{-1}}{1-z^{-1}}+}\frac{-(D+N)}{2}. \end{aligned} \tag{12}$$

It should be noted that the continued fraction given by (12) coincides with the right-hand side of (5). (The proof of this fact is given in Appendix B.) Hence, it follows that the transfer function $G_{\mathrm{FD}}(z)$ is identical to the $N$th order Thiran fractional delay filter $H_{\mathrm{FD}}(z)$. In other words, the transfer function $G_{\mathrm{FD}}(z)$ gives a new formulation of the Thiran fractional delay filter.

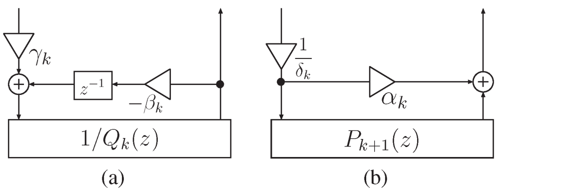


Fig. 1. Basic building blocks for $G_p(z)$. (a) Block diagram of $P_k(z)$. (b) Block diagram of $1/Q_k(z)$.

We will then present a new ladder structure using (12). To this end, we first introduce the following prototype transfer function $G_p(z)$:

$$G_p(z) = \alpha_0 + \frac{\gamma_1}{\beta_1 z^{-1}+} \frac{\delta_1}{\alpha_1+} \frac{\gamma_2}{\beta_2 z^{-1}+} \frac{\delta_2}{\alpha_2+} \cdots \frac{\gamma_N}{\beta_N z^{-1}+} \frac{\delta_N}{\alpha_N}. \quad (13)$$

From (13) and (12), we know that the Thiran fractional delay filter based on our formulation $G_{\mathrm{FD}}(z)$ is obtained by making the following substitutions in the prototype filter $G_p(z)$:

$$\begin{aligned} \alpha_0 &\leftarrow 1 \\ (\alpha_k, \beta_k, \gamma_k, \delta_k) &\leftarrow (2, -2k+1, D-k+1, -(D+k)) \\ z^{-1} &\leftarrow \frac{z^{-1}}{1-z^{-1}} \end{aligned} \quad (14)$$

where $1 \le k \le N$.

An important point to be noted here is that we can find a direct correspondence between the prototype transfer function (13) and a ladder structure. In other words, we can easily construct a ladder structure for the prototype filter $G_p(z)$ [16]. Therefore, the substitutions (14) on this prototype filter will yield a new ladder structure for the Thiran fractional delay filter.

The proposed method consists of the following two steps to construct a ladder structure for the Thiran fractional delay filter.

1) Construct a ladder structure for the prototype filter $G_p(z)$.
2) Perform the substitutions (14) on the prototype filter and obtain the desired ladder structure for the Thiran fractional delay filter $G_{\mathrm{FD}}(z)$.

As will be shown later, this strategy does not yield delay-free loops in the resultant structure.

We now give the details of the first step. The ladder structure for the prototype filter $G_p(z)$ can be obtained by slightly modifying the type IB ladder form of [16]. Since the type IB ladder form of [16] corresponds to the transfer function of (13) with $\gamma_k = \delta_k = 1$ for all $k$, it is necessary to modify the type IB ladder form in such a manner that the ladder can correspond to (13) with $\gamma_k \ne 1$ and $\delta_k \ne 1$. Using $G_p(z)$ given by (13), we first define the following transfer functions:

$$\begin{aligned} P_k(z) &= \frac{\gamma_k}{\beta_k z^{-1}+} \frac{\delta_k}{\alpha_k+} \frac{\gamma_{k+1}}{\beta_{k+1} z^{-1}+} \frac{\delta_{k+1}}{\alpha_{k+1}+} \cdots \frac{\gamma_N}{\beta_N z^{-1}+} \frac{\delta_N}{\alpha_N} \\ Q_k(z) &= \frac{\delta_k}{\alpha_k+} \frac{\gamma_{k+1}}{\beta_{k+1} z^{-1}+} \frac{\delta_{k+1}}{\alpha_{k+1}+} \cdots \frac{\gamma_N}{\beta_N z^{-1}+} \frac{\delta_N}{\alpha_N} \end{aligned} \quad (15)$$

from which it readily follows that

$$P_k(z) = \frac{\gamma_k}{\beta_k z^{-1} + Q_k(z)} \quad Q_k(z) = \frac{\delta_k}{\alpha_k + P_{k+1}(z)}. \quad (16)$$

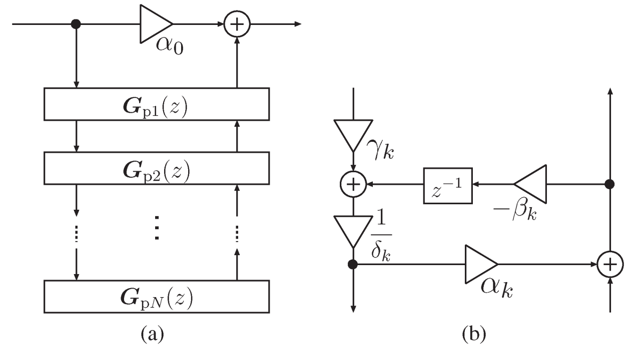


Fig. 2. Ladder structure for the prototype filter. (a) Block diagram of $G_p(z)$. (b) Block diagram of $\boldsymbol{G}_{pk}(z)$.

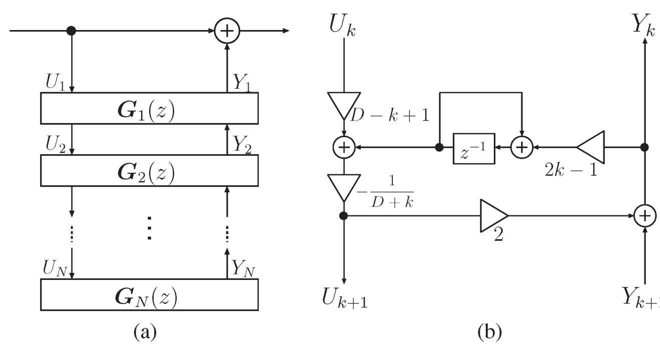


Fig. 3. Proposed ladder structure for the $N$th-order Thiran fractional delay filter. (a) block diagram of $G_{\mathrm{FD}}(z)$. (b) Block diagram of $\boldsymbol{G}_k(z)$.

We next rewrite these transfer functions as

$$\begin{aligned} P_k(z) &= \frac{1}{Q_k(z)} \cdot \frac{\gamma_k}{1 + \beta_k z^{-1}/Q_k(z)} \\ \frac{1}{Q_k(z)} &= \frac{1}{\delta_k} \left(\alpha_k + P_{k+1}(z)\right). \end{aligned} \quad (17)$$

Then, we can easily obtain the block diagrams corresponding to (17), as in Fig. 1. It is now easy to see that we can obtain a ladder structure for the prototype filter $G_p(z)$ by using the building blocks in Fig. 1, with $P_{N+1}(z) = 0$. The resultant ladder structure is shown in Fig. 2, in which the prototype filter $G_p(z)$ is constructed by $N$ ladder sections and each ladder section $\boldsymbol{G}_{pk}(z)$ consists of the aforementioned two building blocks $P_k(z)$ and $1/Q_k(z)$.

The second step is simply to apply the substitutions (14) to the ladder structure of $G_p(z)$. As a result, we finally obtain the proposed ladder structure for the $N$th-order Thiran fractional delay filter, as in Fig. 3. We emphasize that the proposed structure does not include any delay-free loop; thus, the proposed method successfully overcomes the problem of [9]. The avoidance of delay-free loops is due to the substitution $z^{-1} \leftarrow z^{-1}/(1 - z^{-1})$. Since the system $z^{-1}/(1 - z^{-1})$ does not have a direct feedthrough term, the replacement of $z^{-1}$ with $z^{-1}/(1 - z^{-1})$ does not generate any delay-free loop. On the other hand, the method of [9] attempted to replace $z^{-1}$ with $z^{-1} - 1$. This system included the direct feedthrough term $-1$ that resulted in the unrealizable ladder structure with delay-free loops.

In addition to the avoidance of delay-free loops, the proposed method has a very simple structure. Each ladder section shown in Fig. 3(b) consists of very simple coefficients that are easily calculated from $D$ and $k$. Moreover, the two integer coefficients (2 and $2k-1$) can be implemented by shifters and adders; thus, each ladder section requires only two multipliers.

We then discuss the stability (i.e., the bounded-input–bounded-output stability) with respect to the proposed method. It should be noted that the system $z^{-1}/(1-z^{-1})$ that is used in the substitution (14) is not stable because the pole of $z^{-1}/(1-z^{-1})$ is $z=1$ and lies on the unit circle. However, this fact does not affect the stability issue and the proposed method always yields stable filters. The reason for this guaranteed stability is apparent from the fact that the transfer function $G_{\mathrm{FD}}(z)$ given by (12) coincides with the original description of the Thiran fractional delay filter (1).

The guaranteed stability with respect to the proposed method can be shown from another point of view. Looking at the ladder section $\boldsymbol{G}_k(z)$, as in Fig. 3(b), this ladder section can be described by the following relationship:

$$\begin{pmatrix} Y_k(z) \\ U_{k+1}(z) \end{pmatrix} = \boldsymbol{G}_k(z) \begin{pmatrix} U_k(z) \\ Y_{k+1}(z) \end{pmatrix} \tag{18}$$

where the two-input–two-output transfer function $\boldsymbol{G}_k(z)$ is found to be

$$\boldsymbol{G}_k(z) = -\frac{1}{(D+k)\left(1-\frac{D-3k+2}{D+k}z^{-1}\right)} \times \begin{pmatrix} 2(D-k+1)(1-z^{-1}) & -(D+k)(1-z^{-1}) \\ (D-k+1)(1-z^{-1}) & (2k-1)z^{-1} \end{pmatrix}. \tag{19}$$

In the case of $k=N$, $Y_{N+1}(z)$ in (18) becomes zero and the transfer function of the $N$th ladder section reduces to the following single-input–single-output system:

$$\boldsymbol{G}_N(z) = -\frac{2(D-N+1)(1-z^{-1})}{(D+N)\left(1-\frac{D-3N+2}{D+N}z^{-1}\right)}. \tag{20}$$

Now it readily follows that, for $1 \le k \le N$, each ladder section $\boldsymbol{G}_k(z)$ has the single pole $z=(D-3k+2)/(D+k)$. In addition, it can be proven that $|(D-3k+2)/(D+k)|<1$ for all $k$, i.e., all of these poles are inside the unit circle. The proof of $|(D-3k+2)/(D+k)|<1$ is given as follows. By considering

$$(D+k)^2-(D-3k+2)^2 = 4(2k-1)(D-k+1) \tag{21}$$

and the fact that $D > N-1 \ge k-1 \ge 0$, we know that (21) is always positive; thus, $(D-3k+2)^2/(D+k)^2<1$. Therefore, $|(D-3k+2)/(D+k)|<1$ holds, completing the proof. Based on this discussion, we conclude that all of the $N$ ladder sections are stable; thus, the entire system given in Fig. 3(a) is stable.

*Remarks 1:* It is important to note that the prototype filter without the substitution $z^{-1} \leftarrow z^{-1}/(1-z^{-1})$ is not necessarily stable. In other words, the prototype filter given in Fig. 2 with $(\alpha_k, \beta_k, \gamma_k, \delta_k) = (2, -2k+1, D-k+1, -(D+k))$ may have unstable poles. In this filter, the poles of the subsections $\boldsymbol{G}_{pk}(z)$ given in Fig. 2(b) are calculated to be $z=-2(2k-1)/(D+k)$, which does not necessarily satisfy $|z|<1$ (for example, setting $D=1.1$ and $k=2$ results in an unstable pole). From this fact, we intuitively see that the substitution $z^{-1} \leftarrow z^{-1}/(1-z^{-1})$ on this filter maps such unstable poles into the region inside the unit circle and results in the stable Thiran filter. It should be also noted that the substitution $z^{-1} \leftarrow z^{-1}/(1-z^{-1})$ also changes the frequency characteristics of the prototype filter. That is, the prototype filter without the substitution $z^{-1} \leftarrow z^{-1}/(1-z^{-1})$ does not show the allpass characteristic, and this substitution converts the prototype filter into the desired allpass filter with maximally flat fractional delay.

We finally discuss the expected impact of the proposed ladder structure from the viewpoint of the finite wordlength problem. As is well known, the finite wordlength problem must be taken into account in order to implement high-accuracy digital filters on a finite wordlength processor. It was shown that a ladder structure is a promising candidate for such a high-accuracy structure [20], [21], and the same was proven in the field of analog filters [22]. Although theoretical analysis of the performance of the proposed ladder structure is beyond the scope of this brief, we have experimentally confirmed that the proposed structure has a useful property in common with the conventional ladder structures. In the proposed ladder structure, we have found that the state variables (i.e., outputs of delay elements) are mutually orthogonal for white noise input. As was stated in [21] and [22], this orthogonal property leads to the high-accuracy structure and may be also useful in adaptive signal processing. Hence, for the proposed ladder structure, we are currently investigating the theoretical proof and a practical application of the orthogonal property.

## IV. Conclusion

This brief has presented a new ladder structure for the Thiran fractional delay filter. We have derived the proposed ladder structure by using the continued fraction expansion of $x^y$ and by applying appropriate substitutions to the continued fraction. This strategy has successfully overcome the problem of the existing method [9]. The ladder structure given by [9] was unrealizable because of the existence of delay-free loops, whereas the structure proposed in this brief avoids the generation of delay-free loops. In addition, it has been shown that the proposed ladder structure consists of very simple ladder sections that require only two multipliers. Therefore, the proposed structure is very suitable to hardware implementation.

## Appendix A
## Derivation of (9)

Applying the following equivalence transformation [17]:

$$\frac{B_1}{A_1+}\frac{B_2}{A_2+}\frac{B_3}{A_3+}\cdots = \frac{C_1B_1}{C_1A_1+}\frac{C_1C_2B_2}{C_2A_2+}\frac{C_2C_3B_3}{C_3A_3+}\cdots \tag{22}$$

to (3) with $C_{2k-1}=1/x$ and $C_{2k}=1$ for all $k$, we obtain

$$(1+x)^y = 1+\frac{y}{\frac{1}{x}-}\frac{y-1}{2+}\frac{y+1}{\frac{3}{x}-}\frac{y-2}{2+}\cdots \tag{23}$$

from which it readily follows that

$$\frac{1}{(1+x)^y} = \frac{1}{1-}\frac{y}{-\frac{1}{x}+}\frac{y-1}{2-}\frac{y+1}{-\frac{3}{x}+}\frac{y-2}{2-}\cdots. \quad (24)$$

Now, applying the following relationship:

$$(1+x)^y = \frac{1}{\left(1+\frac{-x}{x+1}\right)^y} \quad (25)$$

to (23) and (24), we have

$$\begin{aligned}(1+x)^y &= \left.\frac{1}{(1+x)^y}\right|_{x\leftarrow -x/(x+1)} \\ &= \frac{1}{1-}\frac{y}{\frac{x+1}{x}+}\frac{y-1}{2-}\frac{y+1}{3\frac{x+1}{x}+}\frac{y-2}{2-}\cdots. \quad (26)\end{aligned}$$

Performing the substitution $x \leftarrow x-1$ on (26) yields (9).

## Appendix B
## Proof of the Equivalence of (5) and (12)

First, recall the following relationship called the even part of a continued fraction [23]:

$$\frac{B_1}{A_1+}\frac{B_2}{A_2+}\cdots\frac{B_{2N-1}}{A_{2N-1}+}\frac{B_{2N}}{A_{2N}+}\cdots = \frac{B'_1}{A'_1-}\cdots\frac{B'_N}{A'_N-}\cdots \quad (27)$$

where

$$\begin{aligned}B'_1 &= B_1A_2, A'_1 = A_1A_2 + B_2, B'_2 = B_2B_3A_4 \\ B'_k &= B_{2k-2}B_{2k-1}A_{2k-4}A_{2k}\,(k\ge 3) \\ A'_l &= A_{2l}(A_{2l-2}A_{2l-1} + B_{2l-1}) + A_{2l-2}B_{2l}\,(l\ge 2). \quad (28)\end{aligned}$$

Applying this relationship to (5), we can rewrite $H_{\mathrm{FD}}(z)$ as the following representation with $N$ fractional terms:

$$H_{\mathrm{FD}}(z) = 1 + \frac{\overline{B}_1}{\overline{A}_1-}\frac{\overline{B}_2}{\overline{A}_2-}\cdots\frac{\overline{B}_N}{\overline{A}_N} \quad (29)$$

where

$$\begin{aligned}\overline{B}_1 &= 2D(z^{-1}-1), \overline{A}_1 = -(D-1)z^{-1} + (D+1) \\ \overline{B}_2 &= -2(D-1)(D+1)(1-z^{-1})^2 \\ \overline{B}_k &= -4(D-k+1)(D+k-1)(1-z^{-1})^2\,(k\ge 3) \\ \overline{A}_l &= 2(2l-1)(1+z^{-1})\,(l\ge 2). \quad (30)\end{aligned}$$

Similarly, (12) can be rewritten as

$$G_{\mathrm{FD}}(z) = 1 + \frac{\widehat{B}_1}{\widehat{A}_1-}\frac{\widehat{B}_2}{\widehat{A}_2-}\cdots\frac{\widehat{B}_N}{\widehat{A}_N} \quad (31)$$

where

$$\begin{aligned}\widehat{B}_1 &= \frac{1}{z^{-1}-1}\overline{B}_1, \quad \widehat{B}_k = \frac{1}{(z^{-1}-1)^2}\overline{B}_k\,(k\ge 2) \\ \widehat{A}_l &= \frac{1}{z^{-1}-1}\overline{A}_l\,(l\ge 1). \quad (32)\end{aligned}$$

Hence, we find that (31) is given by applying (22) with $C_1 = \cdots = C_N = 1/(z^{-1}-1)$ to (29). This fact shows that (5) coincides with (12) and completes the proof.

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