Reaching Law Approach to the Sliding Mode Control of Periodic Review Inventory Systems

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***Abstract*—In this paper, a discrete-time sliding mode inventory management strategy based on a novel non-switching type reaching law is introduced. The proposed reaching law eliminates undesirable chattering, and ensures that the sliding variable rate of change is upper bounded by a design parameter which does not depend on the system initial conditions. This approach guarantees fast convergence with non-negative, upper limited supply orders, and ensures that the maximum stock level may be specified *a priori* by the system designer. Furthermore, a sufficient condition for 100% customers' demand satisfaction is derived. The inventory replenishment system considered in this paper involves multiple suppliers with different lead times and different transportation losses in the delivery channels.**

***Note to Practitioners*—This paper presents a new periodic review inventory management strategy which prevents from exceeding the available storage capacity, ensures smooth order evolution and helps attenuate the bullwhip effect. The strategy is scalable, computationally efficient, and easy to implement in any typical inventory replenishment system. The strategy explicitly accounts for transportation losses and different lead times of commodity suppliers.**

***Index Terms*—Digital control, sliding mode control.**

I. Introduction

CONTINUOUS time variable structure and sliding mode control systems were originally introduced about 60 years ago in Russia [15], [40]. Their exceptional robustness [13] and good computational efficiency, have immediately gained them much interest and many advocates in the control engineering community [12], [14], [18]. A few years later, discrete-time sliding mode control systems have also been proposed [32], [41] and then analyzed in numerous significant studies [1]–[3], [5], [10], [11], [16], [17], [19], [20], [23]–[26], [28]–[31], [33], [35], [42]–[44].

Both discrete and continuous time sliding mode controllers push the system representative point (state vector) onto a predetermined hypersurface in the state space. This can be achieved in two different ways. Either assuming a certain control algorithm and demonstrating that this algorithm guarantees stability of the sliding motion on the hypersurface, or applying the reaching law approach. In the latter case the desirable evolution of the sliding variable is first specified, and then a controller which ensures that the variable changes according to the specification is determined. The reaching law approach was first introduced by Gao and Hung for continuous time systems [18]. In that paper, constant, constant plus proportional, and power rate reaching laws were considered. Then, in paper [19] (see also [4] for further comments), the idea of constant plus proportional rate reaching law has been extended to discrete-time systems. Since then the reaching law approach to the control of discrete-time systems has been used by many researchers [9], [20], [22], [28], [31], [33], [34], [38]. Even though much research in this field has already been done, the original approach proposed in [19] is still very popular. Therefore, in this paper, we extend the results of [19] in order to obtain a non-switching discrete-time sliding mode controller [3], [5] and to ensure faster convergence of the controlled system without increasing the magnitude of the control signal. The first of the two objectives is accomplished with the application of the quasi-sliding mode definition proposed in [5], and the latter one is achieved by the introduction of a variable, state dependent convergence rate factor in the proposed reaching law. In the second part of the paper, we apply the proposed reaching law to design a new periodic review inventory replenishment strategy [8], [21], [22], [27], [36], [37], [39] for a warehouse with multiple remote suppliers and delivery channels characterized by different commodity loss factors. We demonstrate favorable properties of the designed strategy which could not be achieved with the application of the original constant plus proportional reaching law. In particular, we show that our reaching law ensures non-negative upper bounded supply orders which do not depend on the warehouse capacity, and therefore are fairly desirable in the considered system. Furthermore, we demonstrate that our reaching law-based controller eliminates the risk of exceeding warehouse capacity and may ensure full customers' demand satisfaction. The work presented in this paper differs from our earlier results [22] in the following three aspects. First, a totally new reaching law appropriate for any dynamic system is proposed in this paper, second, the supply chain model considered here explicitly takes into account transportation losses which was not the case in [22], and finally a novel feasible order allocation among various remote providers is introduced.

II. Non-Switching Reaching Law

Let us consider the following discrete-time system:

$$\boldsymbol{x}\left[(k+1)T\right] = \boldsymbol{A}\boldsymbol{x}(kT) + \Delta\boldsymbol{A}\boldsymbol{x}(kT) + \boldsymbol{b}u(kT) + \boldsymbol{f}(kT) \quad (1)$$

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where $\boldsymbol{x}(k)$ is the state vector $(\dim(\boldsymbol{x}) = n \times 1)$, $\boldsymbol{A}$ is the state matrix, $\Delta\boldsymbol{A}$ is the model uncertainty matrix, $\boldsymbol{b}$ is the input vector, $u(kT)$ is a scalar input, and $\boldsymbol{f}(kT)$ is a disturbance vector. We denote the demand state vector by $\boldsymbol{x_d}$, and define the closed-loop system error as $\boldsymbol{e}(kT) = \boldsymbol{x_d} - \boldsymbol{x}(kT)$. Then, we select the sliding variable as

$$s(kT) = \boldsymbol{c}^T\boldsymbol{e}(kT). \quad (2)$$

With this choice of variable s, equation $s(kT) = 0$ determines the sliding hyperplane. The elements $c_1, c_2, \ldots c_n$ of vector $\boldsymbol{c}$ are selected in such a way that $\boldsymbol{c}^T\boldsymbol{b} \neq 0$ and that the closed-loop system exhibits the desired performance. This can be done in a few ways including quadratic optimization [26], pole placement method [19], dead-beat design [6], [23], etc.

In this paper, the quasi-sliding mode is defined similarly as in [5], i.e., it is such a motion of the system that its representative point (state) remains in a given band around sliding hyperplane $s(kT) = 0$, where $s(kT)$ is defined by (2). According to this definition, the representative point (state of the system) in the quasi-sliding mode is confined to a specified vicinity of the hyperplane. Contrary to the definition introduced in [19], in this paper, crossing the hyperplane is allowed but not required.

Let us now consider the following reaching law

$$s[(k+1)T] = \{1 - q[s(kT)]\}\, s(kT) - \tilde{S}(kT) - \tilde{F}(kT) + F_1 + S_1 \quad (3)$$

where

$$\tilde{S}(kT) = \tilde{S}[\boldsymbol{x}(kT)] = \boldsymbol{c}^T\Delta\boldsymbol{A}\boldsymbol{x}(kT) \quad (4)$$

represents the influence of the model uncertainty on the sliding variable evolution and

$$\tilde{F}(kT) = \boldsymbol{c}^T\boldsymbol{f}(kT) \quad (5)$$

denotes the effect of disturbance on this variable. Furthermore, S_1 and F_1 are the mean values of $\tilde{S}$ and $\tilde{F}$, namely

$$S_1 = \frac{S_U + S_L}{2}, \quad F_1 = \frac{F_U + F_L}{2} \quad (6)$$

where S_U, S_L are upper and lower bounds of $\tilde{S}$, and F_U, F_L are upper and lower bounds of $\tilde{F}$, i.e.,

$$S_L \leq \tilde{S} \leq S_U, \quad F_L \leq \tilde{F} \leq F_U. \quad (7)$$

The notation used in (4)–(7) is adopted from [19].

Convergence rate factor $q[s(kT)]$ in (3) is given by

$$q[s(kT)] = \frac{s_0}{s_0 + |s(kT)|} \quad (8)$$

where s_0 is a design constant. The constant is chosen so that $s_0 > S_2 + F_2$, where S_2 and F_2 represent the greatest possible deviation of $\tilde{S}$ and $\tilde{F}$ from their mean values S_1, F_1

$$S_2 = \frac{S_U - S_L}{2}, \quad F_2 = \frac{F_U - F_L}{2}. \quad (9)$$

Appropriate choice of s_0 allows to find a satisfactory compromise between excessive magnitudes of the control signal generated in the system, and sluggish convergence to the vicinity of $s(kT) = 0$. The proposed reaching law has two major advantages over the one presented in [19]. First, it does not contain a discontinuous term, so it does not lead to chattering. Second, since $q[s(kT)]$ increases with the decrease of $s(kT)$, our reaching law results in faster convergence and better robustness with the same bounds on the control signal magnitude.

In order to find the control signal $u(kT)$ which ensures that the sliding variable evolution is indeed described by (3), we use (1) to rewrite (2) as follows:

$$s[(k+1)T] = \boldsymbol{c}^T\boldsymbol{x_d} - \boldsymbol{c}^T[\boldsymbol{A}\boldsymbol{x}(kT) + \Delta\boldsymbol{A}\boldsymbol{x}(kT) + \boldsymbol{b}u(kT) + \boldsymbol{f}(kT)]. \quad (10)$$

Then, comparing (3) and (10), we obtain

$$u(kT) = -(\boldsymbol{c}^T\boldsymbol{b})^{-1}\left\{[1 - q(s(kT))]\, s(kT) + \boldsymbol{c}^T\boldsymbol{A}\boldsymbol{x}(kT) + F_1 + S_1 - \boldsymbol{c}^T\boldsymbol{x}_d\right\}. \quad (11)$$

Since all terms in (11) are either constants, or variables which do not depend on unknown terms $\Delta\boldsymbol{A}$ or $\boldsymbol{f}(kT)$, this is a feasible control signal which can actually be implemented in the system considered in this paper.

In the next two theorems, we demonstrate that once the representative point of system (1) has reached a band around the sliding hyperplane $s(kT) = 0$, it remains inside the band, and also that the proposed reaching law makes the point always move towards this band.

Theorem 1: If the following inequality:

$$|s(kT)| \leq \frac{s_0(S_2 + F_2)}{s_0 - (S_2 + F_2)} \quad (12)$$

is satisfied at some instant $k = k_0$, then it is also true for any $k > k_0$.

Proof: From (3), we observe that $|s[(k+1)T]|$ increases with the increase of $|s(kT)|$. Therefore, even assuming the most disadvantageous possible influence of the disturbance and model uncertainty, if (12) is satisfied for some k, then from (3), we obtain

$$\begin{aligned}|s[(k+1)T]| &\leq \frac{\frac{s_0^2(S_2+F_2)^2}{[s_0-(S_2+F_2)]^2}}{s_0(S_2+F_2)/[s_0-(S_2+F_2)]+s_0} + (S_2+F_2)\\ &= (S_2+F_2)^2/[s_0-(S_2+F_2)] + (S_2+F_2)\\ &= \frac{s_0(S_2+F_2)}{[s_0-(S_2+F_2)]}. \end{aligned} \quad (13)$$

Using this observation and assumption (12) by virtue of the principle of mathematical induction, we conclude that (12) indeed holds for all $k > k_0$.

Theorem 2: If the absolute value of $s(kT)$ is greater than the right hand side of (12), then $s(kT)$ converges, at least asymptotically, to the band specified by (12).

Proof: In the proof, we will consider two cases, namely, the positive and negative values of $s(kT)$.

Case 1: If

$$s(kT) = \frac{s_0(S_2 + F_2)}{s_0 - (S_2 + F_2)} + \delta > \frac{s_0(S_2 + F_2)}{s_0 - (S_2 + F_2)} = s^* > 0 \quad (14)$$

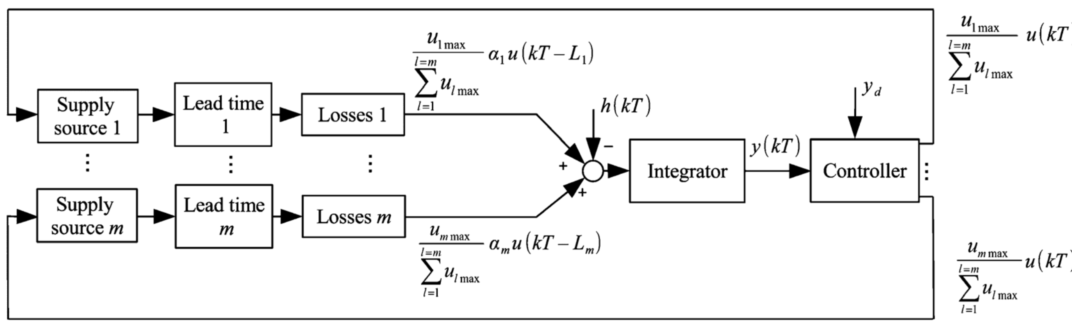

Fig. 1. Inventory supply model.

then using (3), we obtain

$$\begin{aligned} s\left[(k+1)T\right] - s(kT) &= -\frac{s_0(s^*+\delta)}{s_0+s^*+\delta} - \tilde{S} - \tilde{F} + S_1 + F_1 \\ &= \frac{-s_0^2(S_2+F_2+\tilde{S}+\tilde{F}-S_1-F_1)}{s_0^2+\delta\left[s_0-(S_2+F_2)\right]} \\ &\quad - \frac{\delta(s_0-S_2-F_2)(s_0+\tilde{S}+\tilde{F}-S_1-F_1)}{s_0^2+\delta\left[s_0-(S_2+F_2)\right]} \\ &\leq -\frac{\delta(s_0-S_2-F_2)(s_0+\tilde{S}+\tilde{F}-S_1-F_1)}{s_0^2+\delta\left[s_0-(S_2+F_2)\right]}. \end{aligned} \tag{15}$$

Since $s_0 > S_2 + F_2 \geq \tilde{S} + \tilde{F} - S_1 - F_1$, the difference $s(k+1) - s(k)$ is negative and it approaches zero only if δ tends to zero. This proves that if initial value of $s(k)$ is positive, then it asymptotically converges to the band specified by relation (12).

Case 2: Similarly if

$$s(kT) = -\frac{s_0(S_2+F_2)}{s_0-(S_2+F_2)} - \delta < -\frac{s_0(S_2+F_2)}{s_0-(S_2+F_2)} = -s^* \tag{16}$$

then again, using (3), we obtain

$$s\left[(k+1)T\right] - s(kT) \geq \frac{\delta(s_0-S_2-F_2)(s_0+\tilde{S}+\tilde{F}-S_1-F_1)}{s_0^2+\delta\left[s_0-(S_2+F_2)\right]}. \tag{17}$$

Inequality (17) shows, that if (16) is satisfied, then difference $s(k+1) - s(k)$ is positive and it approaches zero only if δ tends to zero.

Taking into account the conclusions of both cases, we find that if $s(kT)$ lies outside the band around $s(kT) = 0$ specified by (12), then it asymptotically converges to this band.

III. Inventory Supply Model

In this section, we consider an inventory management system with m remote providers. The transportation channel between the pth provider $(p = 1, \ldots, m)$ and the warehouse is characterized by its lead time L_p and commodity loss factor $\alpha_p \in (0, 1]$. Furthermore, each of the providers has its own maximum admissible supply rate, i.e., the greatest amount of goods that it can send during one review period. This amount, for the pth provider is denoted by $u_{\max p}$.

The commodities obtained from the providers are used to satisfy an *a priori* unknown consumers' demand. The orders for the commodities are generated by the controller located at the distribution center. The control signal u determines the total amount of supplies requested from all of the providers. This value is distributed among the providers, proportionally to the maximum amount of goods they can send, i.e., supplier p receives an order equal to $u \cdot u_{\max p} / \sum_{i=1}^{i=m} u_{\max i}$. The block diagram of the periodic review inventory system considered in this section is shown in Fig. 1. It is assumed, that each lead time L_p is a multiple of the review period T, i.e., $L_p = \mu_p T$, where μ_p is a positive integer. In fact, this is a well justified assumption, since even if the actual order procurement time is a non-integer multiple of the review period, still the arrival of goods at the warehouse is detected by the enterprise management system only at discrete-time instants. Therefore, this time is actually rounded up to the nearest integer multiple of T. The warehouse stock level at time kT is denoted by $y(kT)$. The consumers' demand is modeled by an *a priori* unknown function of time $d(kT)$, bounded by a known constant $d_{\max}$. If the amount of stored goods is insufficient, the demand cannot be fully covered. Therefore, an additional function $h(kT)$ is introduced, which represents the amount of goods actually sold to the customers. For any $k \geq 0$, the following inequalities hold:

$$0 \leq h(kT) \leq d(kT) \leq d_{\max}. \tag{18}$$

The warehouse is assumed to be empty prior to the beginning of the control process, i.e., $y(kT < 0) = 0$. Moreover, the first order is sent at $kT = 0$, i.e., $u(kT < 0) = 0$. The inventory stock level for $kT > 0$ can therefore be obtained as the difference between all acquired and sold goods

$$y(kT) = \sum_{p=1}^{m} \sum_{j=0}^{k-1} \alpha_p \gamma_p u(jT - L_p) - \sum_{j=0}^{k-1} h(jT) \tag{19}$$

where $\gamma_p = u_{\max p} / \sum_{i=1}^{m} u_{\max i}$.

We can represent all providers with equal lead times as a single "aggregate" supplier, so as to get a simplified, equivalent version of the system model. The amount of commodities that arrive at the distribution center from this "aggregate" supplier is equal to $a_i u$, where

$$a_i = \frac{\sum\limits_{p:m_p=i} \alpha_p u_{\max p}}{\sum\limits_{i=1}^{m} u_{\max i}} \tag{20}$$

for $i = 1, \ldots, n-1$ and $n = \max(\mu_p) + 1$. Naturally, if no provider has the lead time iT, then the corresponding coefficient $a_i = 0$. Now, we can express the stock level as follows:

$$y(kT) = \sum_{i=1}^{n-1}\sum_{j=0}^{k-1} a_i u\left[(j-i)T\right] - \sum_{j=0}^{k-1} h(jT). \tag{21}$$

We can also represent the above relation in the standard state space form

$$\begin{aligned} \boldsymbol{x}\left[(k+1)T\right] &= \boldsymbol{A}\boldsymbol{x}(kT) + \boldsymbol{b}u(kT) + \boldsymbol{w}h(kT) \\ y(kT) &= \boldsymbol{r}^T\boldsymbol{x}(kT) \end{aligned} \tag{22}$$

where $\boldsymbol{x}(kT) = [x_1(kT)\ x_2(kT)\ \ldots\ x_n(kT)]^T$ is the state vector, $y(kT) = x_1(kT)$ is the on-hand stock level. The remaining state variables are the delayed values of the control signal, i.e.,

$$x_i(kT) = u\left[(k-n+i-1)T\right] \tag{23}$$

for $i = 2, \ldots, n$. $\boldsymbol{A}$ is $n \times n$ state matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & a_{n-1} & a_{n-2} & & a_1 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & & 0 \end{bmatrix} \tag{24}$$

and $\boldsymbol{b}$, $\boldsymbol{w}$, and $\boldsymbol{r}$ are $n \times 1$ vectors

$$\boldsymbol{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \boldsymbol{r} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{25}$$

The desired state of the system is $\boldsymbol{x_d} = [y_d 0 \ldots 0]$, where y_d is the demand level of the on-hand stock. In general, the bigger parameter y_d is chosen, the greater amount of goods has to be accommodated in the warehouse and the better demand satisfaction is achieved. Still, the effect of this parameter on the overall system performance will be more precisely analyzed in the next section.

Closer analysis of the system described in this paper reveals that its total control signal is limited by the following constraint: $u(kT) \in [0, \sum_{p=1}^{p=m} u_{\max p}]$. This precludes the use of a linear controller, as the magnitude of its output would strongly depend on the initial conditions of the system and could not satisfy the constraint. Therefore, in the next section, we propose a nonlinear sliding mode controller.

IV. Controller Design

In this section, we will develop a sliding mode controller that ensures the desired sliding variable evolution. We begin by selecting the elements of vector $\boldsymbol{c}$, which describe the sliding hyperplane, so that the closed-loop system exhibits dead-beat characteristics. First, we calculate the control signal needed to satisfy $s[(k+1)T] = 0$ as

$$u(kT) = \left(\boldsymbol{c}^T\boldsymbol{b}\right)^{-1}\boldsymbol{c}^T\left[\boldsymbol{x}_d - \boldsymbol{A}\boldsymbol{x}(kT)\right] \tag{26}$$

and substitute it into (22). This results in the following state matrix of the closed-loop system:

$$\boldsymbol{A_c} = \left[\boldsymbol{I_n} - \boldsymbol{b}\left(\boldsymbol{c}^T\boldsymbol{b}\right)^{-1}\boldsymbol{c}^T\right]\boldsymbol{A} = \begin{bmatrix} 1 & a_{n-1} & a_{n-2} & & a_1 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\frac{c_1}{c_n} & -\frac{a_{n-1}c_1}{c_n} & -\frac{a_{n-2}c_1+c_2}{c_n} & & -\frac{a_1c_1+c_{n-1}}{c_n} \end{bmatrix} \tag{27}$$

which has the following characteristic polynomial

$$\det(z\boldsymbol{I_n} - \boldsymbol{A_c}) = z^n + z^{n-1}\left(\frac{c_1a_1 + c_{n-1} - c_n}{c_n}\right) + \cdots + \left(\frac{c_1a_{n-1} - c_2}{c_n}\right)z. \tag{28}$$

We have already assumed that $\boldsymbol{c}^T\boldsymbol{b} \neq 0$. This condition and relation (25), imply $c_n \neq 0$. A linear discrete-time system is asymptotically stable if and only if all of its eigenvalues lie inside a unit circle on the z-plane. Moreover, to obtain finite time error convergence to zero the characteristic polynomial (28) should have the following form:

$$\det(z\boldsymbol{I_n} - \boldsymbol{A_c}) = z^n. \tag{29}$$

We find that (28) reduces to (29) when vector $\boldsymbol{c}$ is chosen as follows:

$$\begin{cases} c_1 = 1 \\ c_i = \sum_{j=1}^{i-1} a_{n-j}, \quad \text{for } i = 2, \cdots, n \end{cases} \tag{30}$$

We observe, that with this choice of vector $\boldsymbol{c}$ the variable $s(kT)$ has a clear physical meaning, i.e., it represents the difference between the desired warehouse stock level and the sum of amounts of goods in the warehouse and currently in transit.

We will now implement the proposed reaching law to derive the controller that steers the representative point of the system to the proximity of the sliding hyperplane $\boldsymbol{c}^T\boldsymbol{e}(kT) = 0$, where $\boldsymbol{c}$ is given by (30). For the considered periodic review inventory system, the perturbations of the sliding variable caused by the model uncertainty and disturbance are

$$S_1 = S_2 = 0, \quad F_1 = -\frac{d_{\max}}{2}, \quad F_2 = \frac{d_{\max}}{2}. \tag{31}$$

Using (2) and (31) with (11), we obtain

$$\begin{aligned} u(kT) &= -\left(\boldsymbol{c}^T\boldsymbol{b}\right)^{-1}\left\{\{1 - q\left[s(kT)\right]\}s(kT) + c^T\boldsymbol{A}\boldsymbol{x}(kT) - \boldsymbol{c}^T\boldsymbol{x}_d - d_{\max}/2\right\} \\ &= \left(c^T\boldsymbol{b}\right)^{-1}\left\{q\left[s(kT)\right]s(kT) - \boldsymbol{c}^T(\boldsymbol{A} - \boldsymbol{I_n})\boldsymbol{x}(kT) + d_{\max}/2\right\}. \end{aligned} \tag{32}$$

One can easily notice, that by selecting $\boldsymbol{c}$ according to (30), we have obtained $\boldsymbol{c}^T(\boldsymbol{A} - \boldsymbol{I}_n) = [0 \ldots 0]$. Therefore, using (25) and

(30), we can obtain the control signal, which ensures the desired sliding variable evolution as

$$u(kT) = \frac{q\left[s(kT)\right]s(kT) + d_{\max}/2}{\sum_{i=1}^{n-1} a_i}. \quad (33)$$

Let us notice at this point, that any other choice of vector $\boldsymbol{c}$ would lead to a more convoluted expression determining $u(kT)$ and less computationally efficient controller. Moreover, it is worth to point out that application of other hyperplane design methods (pole placement or quadratic optimization) in conjunction with the reaching law approach is redundant as both the reaching law approach and these methods are used primarily to satisfy input and state constraints of the controlled systems. These observations justify the choice of the sliding hyperplane determined by (30).

In the remainder of this section, important properties of the proposed control strategy will be stated in three theorems and proved. In the first one, we will demonstrate, that control signal (33) is always non-negative and upper bounded by an *a priori* known constant. Since this signal directly corresponds to the amounts of goods sent by the providers, both of these features are essential for the practical application of the proposed strategy.

Theorem 3: For any $k \geq 0$, control signal (33) satisfies the following two inequalities:

$$0 \leq u(kT) \leq \frac{\frac{s_0 y_d}{y_d + s_0} + \frac{d_{\max}}{2}}{\sum_{i=1}^{n-1} a_i}. \quad (34)$$

Proof: As shown in Theorems 1 and 2, sliding variable $s(kT)$ will start at some initial value $s(0)$, and in each consecutive step its absolute value will decrease unless (12) is satisfied. Moreover, once (12) becomes true, it will hold for the rest of the control process. For the system under consideration

$$s(0) = \boldsymbol{c}^T \boldsymbol{x}_d = y_d. \quad (35)$$

Using (12), (31), and (35), we observe that

$$s(kT) \in \left[-\frac{s_0 d_{\max}}{2s_0 - d_{\max}}, y_d\right] \quad (36)$$

for all $k \geq 0$.

We now notice, that the value of the control signal (33) is always increasing with the increase of $s(kT)$. Therefore, its minimum value will be generated for the smallest possible $s(kT)$, and the maximum value for the greatest $s(kT)$. This observation allows us to simply substitute the limits of interval (36) into (33) and conclude that (34) indeed holds.

We have assumed that the requests for goods are distributed among the providers according to the maximum amount of goods that they can deliver. Therefore, if s_0 is selected in such a way that the right-hand side of (34) is equal to $\sum_{p=1}^{p=m} u_{\max p}$, then no supplier will be requested to send more goods, than it is actually able to provide.

Any successful inventory management strategy should guarantee that all of the incoming goods can be stored in the distribution center. In the following theorem, we will determine the upper bound of the on-hand stock. Therefore, if warehouse capacity equal to or greater than this bound is secured, then there will be no risk of hiring (usually very costly) emergency storage space.

Theorem 4: With the application of the proposed control strategy, for every $k \geq 0$, the inventory stock level will satisfy the following condition:

$$y(kT) \leq y_d + \frac{s_0 d_{\max}}{2s_0 - d_{\max}}. \quad (37)$$

Proof: From (36), we obtain

$$s(kT) \geq -\frac{s_0 d_{\max}}{2s_0 - d_{\max}} \quad (38)$$

for any $k \geq 0$. Using (2) and (23), we can rewrite (38) as

$$y(kT) \leq y_d + \frac{s_0 d_{\max}}{2s_0 - d_{\max}} - \sum_{i=2}^{n} c_i u\left[(k - n + i - 1)T\right]. \quad (39)$$

We have already proven that the control signal is always non-negative. Therefore, we conclude that (39) implies (37).

In order to obtain the greatest possible profit, one may wish to eliminate lost sales risk. Therefore, it is reasonable to establish conditions ensuring that the consumers' demand is always fully satisfied. For that purpose, in the last theorem, we determine the smallest value of the demand inventory stock level ensuring that after some initial time, the warehouse will not be empty. We can notice from (21) that this implies full satisfaction of the customers' demand.

Theorem 5: If the following condition is satisfied:

$$y_d > \frac{d_{\max} \sum_{i=1}^{n-1} i a_i}{\sum_{i=1}^{n-1} a_i} + \frac{s_0 d_{\max}}{2s_0 - d_{\max}} \quad (40)$$

then $y(kT) > 0$ for any $k \geq k_0 + n - 1$, where k_0 is the first time instant when (12) is true.

Proof: Using (12), for any $k \geq k_0$, we obtain

$$y(kT) \geq y_d - \sum_{i=2}^{n} c_i u\left[(k - n + i - 1)T\right] - \frac{s_0 d_{\max}}{2s_0 - d_{\max}}. \quad (41)$$

Moreover, substituting (12) into (33), we get

$$u(kT) \leq \frac{d_{\max}}{\sum_{i=1}^{n-1} a_i} \quad (42)$$

which is also true for any $k \geq k_0$. By combining (41) and (42), we arrive at

$$y(kT) \geq y_d - \frac{d_{\max} \sum_{i=1}^{n-1} i a_i}{\sum_{i=1}^{n-1} a_i} - \frac{s_0 d_{\max}}{2s_0 - d_{\max}} \quad (43)$$

for any $k \geq k_0 + n - 1$. Therefore, if (40) holds, then the right-hand side of the above inequality is always strictly positive.

TABLE I
SIMULATION PARAMETERS

p	L_p [days]	α_p	$u_{\max}$ [items]
1	6	0.98	20
2	8	0.99	10
3	11	0.96	35

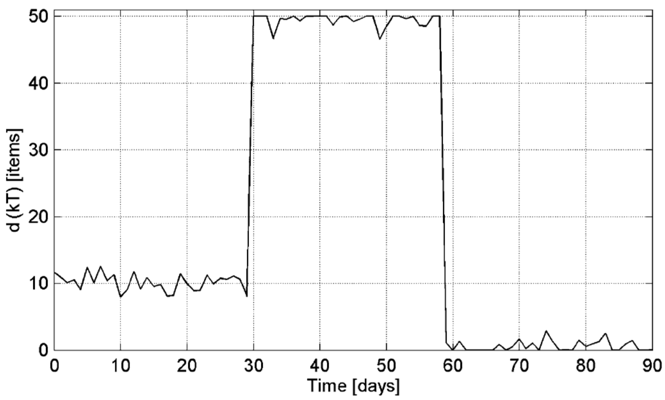

Fig. 2. Consumers' demand.

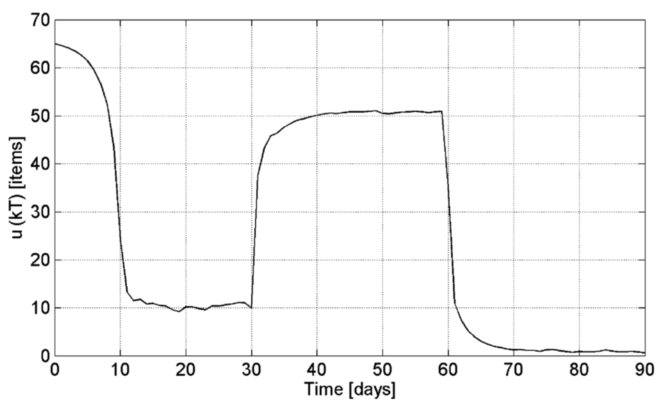

Fig. 3. Control signal.

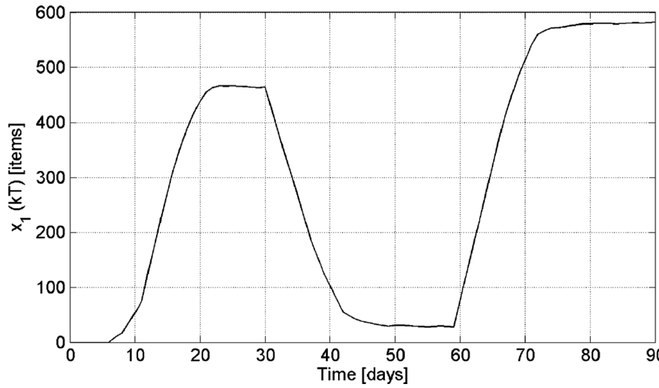

Fig. 4. On-hand stock level.

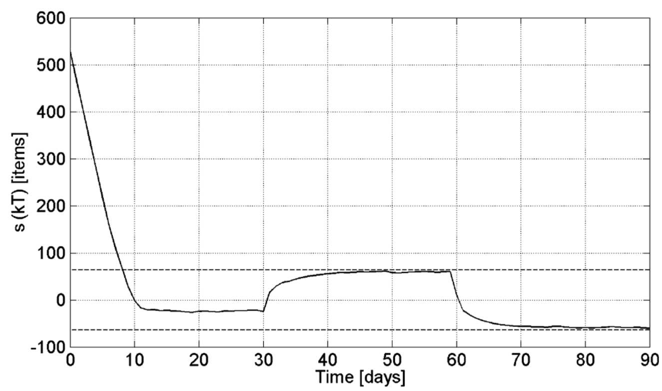

Fig. 5. Sliding variable evolution.

V. SIMULATION RESULTS

In order to verify the properties of the proposed control law computer simulations of an inventory system with three suppliers were performed. The review period $T = 1$ day and the parameters of the suppliers and transportation channels are shown in Table I. The greatest lead time equals 11 days, which implies $\max(\mu_p) = 11$, and $n = 12$. The elements a_i of the first row of matrix $\boldsymbol{A}$ are $a_6 = 0.302$, $a_8 = 0.152$, $a_{11} = 0.517$, and the remaining elements a_i are equal to zero. The maximum daily consumers' demand $d_{\max} = 50 \text{ items}$. The actual evolution of the demand in the simulation example is shown in Fig. 2. The demand exhibits abrupt changes between small and large values, which reflect the most adverse conditions that can appear in the considered system. The total amount of goods, which can be provided by all suppliers on a single day is $\sum_{p=1}^{p=m} u_{\max p} = 65 \text{ items}$. Therefore, in order to ensure that the control signal never exceeds this value, we select $s_0 = 41.03 \text{ items}$. As–according to Theorem 5–the minimum demand value of the stock level that ensures full consumers' demand satisfaction is 513 items, we select y_d equal to 530 items. The simulation results are presented in Figs. 3– 5. The control signal is shown in Fig. 3. One can easily see that it is always non-negative and smaller than or equal to $\sum_{p=1}^{p=m} u_{\max p} = 65 \text{ items}$. Therefore, no supplier is at any time expected to provide more commodities than it is really able to. Furthermore, comparing Figs. 2 and 3, we notice that the supply orders are significantly smoother than the consumers' demand is. This shows that our strategy attenuates the highly undesirable bullwhip effect. The inventory stock level is illustrated in Fig. 4. As stated in Theorems 4 and 5 it never exceeds 594 items, and after some initial time it does not drop to zero any more. This means, that the risk of hiring costly emergency storage is eliminated, and full consumers' demand satisfaction is ensured. The evolution of the sliding variable is shown in Fig. 5. As determined by Theorems 1 and 2 the variable converges to the band $|s(kT)| \leq 63.9$ (the band limits are shown by dashed lines), and after reaching the band, the variable does not leave it for the rest of the control process.

We also consider the performance of the system when all its parameters, and the consumers' demand transient are the same as in the first scenario, but initially the warehouse is not empty. We analyze two cases: the first one when the initial stock level $x_1(0) = 297 \text{ items}$ is equal to the half of its maximum value, and the second one when the level $x_1(0) = 594 \text{ items}$ equals its maximum value. Since the initial on-hand stock affects only the beginning of the control process, in Figs. 6– 8, we depict the transients until the 25th day. This is justified by the fact, that later on the transients are almost identical to each other and also to those shown in Figs. 3–5.

As we can observe from Figs. 6–8 all of the advantageous properties of the proposed controller mentioned before, hold

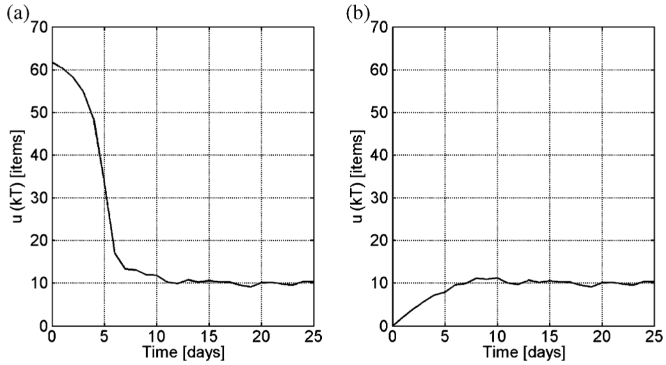

Fig. 6. Control signal for: (a) $x_1(0) = 297$ items and (b) $x_1(0) = 594$ items.

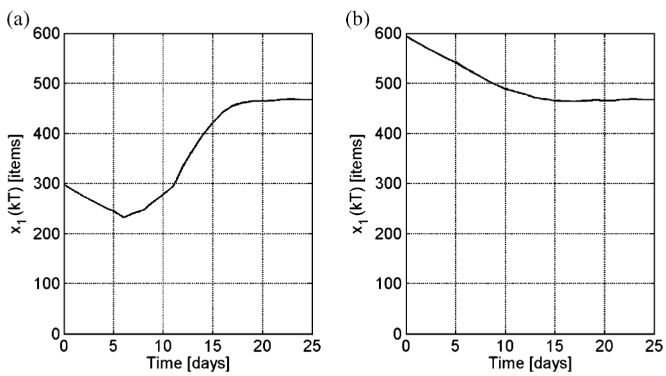

Fig. 7. On-hand stock level for: (a) $x_1(0) = 297$ items and (b) $x_1(0) = 594$ items.

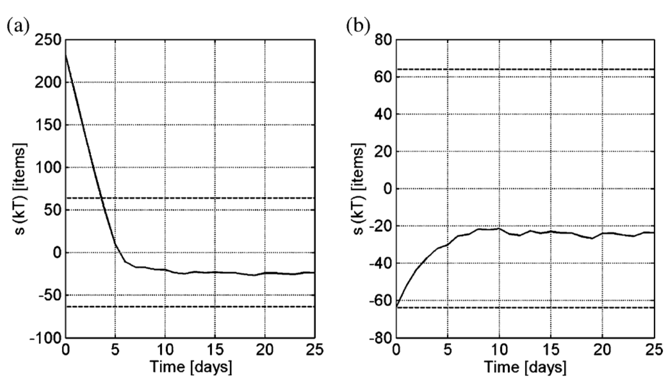

Fig. 8. Sliding variable evolution for: (a) $x_1(0) = 297$ items and (b) $x_1(0) = 594$ items.

also for nonzero initial stock values. Comparing Fig. 3 and Fig. 6(a), we can notice that the initial value of the control signal only marginally depends on the initial conditions, in fact much less as compared to a linear controller. Furthermore, we can observe from Fig. 7 that if the initial stock level is sufficiently high, then it will not drop to zero, even before the start of the quasi-sliding motion, and full consumer demand satisfaction will be ensured for any $k \geq 0$.

VI. Conclusion

In this work, a new reaching law for discrete-time sliding mode control of dynamic plants has been introduced and applied to design a feasible management strategy for periodic review inventory systems with multiple suppliers and transportation losses. The reaching law introduced in this paper, as opposed to the ones previously proposed in literature does not require crossing the sliding hyperplane in each successive control step of the quasi-sliding motion. This property eliminates chattering and allows fast convergence with limited control signal. The proposed reaching law based inventory management strategy ensures non-negative and upper bounded supply orders, eliminates the risk of costly emergency storage, and may ensure that no business opportunities are lost. These important properties have been proved analytically and verified by computer simulations. Our further research efforts will be focused on inventory systems with different customer classes, i.e., customers whose orders can be backordered and those whose orders must be rejected if they cannot be satisfied immediately from on-hand inventory [7]. We will also employ the reaching law proposed in this paper for other applications. This is possible as our reaching law is not customized, but may be directly used with all other types of hyperplanes or even nonlinear hypersufaces.

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