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Generalized Mittag-Leffler Input Stability of the Fractional-Order Electrical Circuits

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ABSTRACT This article addresses new applications of the generalized Mittag-Leffler input stability to the fractional-order electrical circuits. We consider the fractional-order electrical circuits in the context of the generalized Caputo-Liouville derivative. We propose the Lyapunov characterizations of the fractional differential equations. A new numerical discretization, including the fractional differential equations represented by the generalized Caputo derivative, has been successfully applied to the fractional electrical circuits. To support the results, we have proposed the graphics generated by our numerical discretization. The graphics of the solutions have been analyzed and interpreted in the context of generalized Mittag-Leffler input stability and the generalized Mittag-Leffler stability. The generalized Mittag-Leffler input stability is a new stability notion for the fractional differential equations recently introduced in the literature.

INDEX TERMS Generalized Caputo-Liouville operators, fractional order electrical circuits, generalized Mittag-Leffler input stability.

I. INTRODUCTION

FRACTIONAL calculus is a new arena focusing in many domains, like physics [16], [19], [23], [35], [37], mechanics [23], fluid models [27], science and engineering [16], [24], mathematical modeling in biology [36], mathematical physics [16], [17], [20], [30], mathematical modeling [3], [31], [32], [33] and others [25], [29], [35]. The existence of many fractional operators constitutes the primary importance of this field. There exist fractional operators with singular kernels and fractional operators with no singular kernels. The Caputo and the Riemann-Liouville derivatives [18] were the first derivatives introduced in this field. These derivatives find many applications and continue to attract many researchers. Later, to correct the inconvenience of the Riemann-Liouville derivative, regarding it unphysical initial condition, the Caputo-Fabrizio operator [2] and the Atnagana-Baleanu operator [1] have been proposed. For the applications of the fractional operators with exponential kernel and the Mittag-Leffler kernel, we advise readers to check the following works [3], [4]. These recent years, modeling electrical circuits using fractional operators have been introduced in the literature. The motivations of these introductions are due to the fact the fractional operators describe more realistically the real-world problems, and another reason is the fractional operators take into account the memory effect. In other words, the next behaviors of the model are explained by the past behavior of the model. Presently, the deterministic *RL*, *RC*, *LC*, and *RLC* electrical circuits have their fractional versions, and all the fractional operators have been utilized in the mathematical modeling of these circuits [7], [9], [21].

The literature of fractional order electrical circuits is rich. We recall some of them. In [5], Elwakil proposes a study on the electrical circuits in the context of the fractional-order derivative; he experiments the numerical simulations of these models in the context of fractional order operators. In [6], Petras introduces in fractional calculus the fractional-order Chua's electrical circuit where the elements are electrical components. In [7], Sarafraz and Tavazoei propose an investigation on the realization of fractional-order functions by passive electrical networks constituted by the fractional capacitor and the *RLC* components. In [8], Rawdan offers a stability analysis of the $RL_\beta C_\alpha$ system in the context of the fractional operator. In [9], Gomez *et al.* have proposed work on the *RLC* electrical circuit with Caputo derivative and have suggested the analytical solutions of the considered models. In [10], Sarafraz and Tavazoei investigate on *RLC* circuit by studying the passive realization of the fractional impedances. In [11], Barbosa *et al.* in conference article propose work concerning the fractional Van der Pol oscillator represented by a fractional operator with singularity as the Riemann-Liouville derivative, the Caputo operator. In [12], Mitkowski and Skruch, in their article, study the super-capacitors represented by fractional operators with RC components. In [13], Calik and Sirin analyze the charge variation in the LC circuit represented by fractional derivative as the Caputo-Liouville derivative. In [21], Sene and Gomez-Aguilar have proposed the analytical solution of the fractional-order *RL*, *RC*, *LC*, and *RLC* electrical circuits by considering different types of fractional operators. For more investigations related to electrical circuits in context of fractional order derivative see in [48], [49], [50], [51], [52].

The stability analysis of the fractional differential equations occupies an essential place in control theory. In the context of fractional calculus, there exist many stability notions as asymptotic stability, global asymptotic stability, local stability, Mittag-Leffler stability, fractional input stability, Mittag-Leffler input stability, and others. The Mittag-Leffler stability of the equilibrium points of the fractional differential equations is one of the most important stability notions in fractional calculus, recently introduced in the literature. The Mittag-Leffler stability has received many investigations in these recent years. We cite some of them. In [39], [40], Li *et al.* have introduced in fractional calculus the Mittag-Leffler stability and have presented as well its Lyapunov characterization. In [38], Ren *et al.* have proposed the Mittag-Leffler stability and its generalization of the fractional-order gene regulatory networks. In [41], Wyrwas *et al.* have discussed and analyzed the Mittag-Leffler stability of the fractional difference equations. Modeling the electrical circuits with the recent and new fractional operators has been addressed in this article. We mainly consider the fractional-order *RL*, *RC*, *LC*, and *RLC* electrical circuits represented by the generalized Caputo-Liouville derivative. The main objective is to analyze the generalized Mittag-Leffler input stability of the considered fractional-order electrical circuits. The numerical discretization of the fractional-order differential equations represents an important part of fractional calculus, due to the fact many models used in fractional calculus are complex and determining the analytical solutions are not simple problems. There exist many investigations related to the numerical schemes of the fractional differential equations. In [44], Deng and Li give a review of the existing numerical schemes for the fractional differential equations. In [45], Li and Zeng present the difference methods for fractional differential equations. In [24], Sene proposes the numerical scheme of the fractional diffusion equations. For more investigations related to the numerical schemes, see in the following paper and book [46], [47]. For novelty, we offer a novel numerical scheme to approach the solutions of the new fractional-order electrical circuits. The discretization is based on the discretization of the fractional Riemann-Liouville integral and Volterra equation. To support our results, we represent graphically the solutions generated by the proposed numerical scheme. And we analyze the behaviors of the solutions in the context of the generalized Mittag-Leffler input stability. Note that there exist two methods to focus on the generalized Mittag-Leffler input stability, the trajectories, and the Lyapunov direct method. The importance of the Lyapunov direct approach can be explained by the fact the analytical solutions of the fractional differential equations are not all time trivial, and the Lyapunov functions give alternative ways to study the stability notions.

II. BASIC DEFINITIONS OF FRACTIONAL OPERATORS

In this section, we recall the generalized fractional derivatives operators utilized in fractional calculus, namely the fractional integral in Riemann-Liouville sense, the general form of the Caputo-Liouville derivative, and its associated Laplace transform.

Definition 1 [14], [15]: The generalization of the Riemann-Liouville integral of order α with $\kappa > 0$ of a continuous function $k : [0, +\infty[\longrightarrow \mathbb{R}$ is described by the following relationship

$$\left(I^{\alpha,\kappa} k\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\kappa - s^\kappa}{\kappa}\right)^{\alpha-1} k(s) \frac{ds}{s^{1-\kappa}}, \qquad (1)$$

where the function $\Gamma(\cdots)$ represents the Gamma function, for all $t > 0$, and $0 < \alpha < 1$.

Definition 2 [14], [15]: The generalization of the Riemann-Liouville fractional operator of order α with $\kappa > 0$ of a continuous function $k : [0, +\infty[\longrightarrow \mathbb{R}$ is described by the following relationship

$$\left(D^{\alpha,\kappa} k\right)(t) = \left(t^{1-\kappa} \frac{d}{dt}\right)\left(I^{1-\alpha,\kappa} k\right)(t), \qquad (2)$$

where the function $\Gamma(\cdots)$ denotes the Gamma function, for all $t > 0$, and $0 < \alpha < 1$.

Definition 3 [15]: The generalization of the Caputo-Liouville fractional operator of order α with $\kappa > 0$ of a continuous function $k : [0, +\infty[\longrightarrow \mathbb{R}$ is described by the following relationship

$$\left(D_c^{\alpha,\kappa} k\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \left(\frac{t^\kappa - s^\kappa}{\kappa}\right)^{-\alpha} k'(s) ds, \qquad (3)$$

where the function $\Gamma(\cdots)$ denotes the Gamma Euler function, for all $t > 0$, and the fractional order satisfies $0 < \alpha < 1$.

We will use the Laplace transform to solve the fractional differential equations. We define the Laplace transform of the Caputo-Liouville generalized operator in the following definition [14], [15].

Definition 4 [15]: The Laplace transform of the Caputo-Liouville generalized fractional operator of a continuous function $k : [0, +\infty[\longrightarrow \mathbb{R}$ is represented as the form

$$\mathcal{L}_\kappa\left\{\left(D_c^{\alpha,\kappa} k\right)(t)\right\} = s^\alpha \mathcal{L}_\kappa\{k(t)\} - s^{\alpha-1} k(0), \qquad (4)$$

with the Laplace transform of the function $k : [0, +\infty[\longrightarrow \mathbb{R}$ expressed as the following description

$$\mathcal{L}_\rho\{k(t)\}(s) = \int_0^\infty e^{-s\frac{t^\rho}{\kappa}} k(t) \frac{dt}{t^{1-\kappa}}. \quad (5)$$

Definition 5 [14], [15]: The Laplace transform of the Riemann-Liouville generalized fractional operator of an function $k : [0, +\infty[\longrightarrow \mathbb{R}$ is given by the relationship

$$\mathcal{L}_\rho\left\{\left(D^{\alpha,\kappa}k\right)(t)\right\} = s^\alpha \mathcal{L}_\rho\{k(t)\} - \left(I^{1-\alpha,\kappa}k\right)(0). \quad (6)$$

Definition 6 [15], [25]: In the following expression, we give the definition of the Mittag-Leffler function with two parameter, we have the following expression

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (7)$$

with $\alpha > 0$, $\beta \in \mathbb{R}$ and $z \in \mathbb{C}$. The exponential function is obtained when the orders satisfy the condition represented by $\alpha = \beta = 1$.

III. GENERALIZED MITTAG-LEFFLER INPUT STABILITY THEORY

In this section, we recall the theory related to the generalized Mittag-Leffler input stability of the fractional differential equations. We consider the class of the fractional differential equations defined by

$$D_c^{\alpha,\kappa} z = f(t, z, u), \quad (8)$$

with the initial condition defined by the expression

$$z(0) = z_0 = \xi. \quad (9)$$

where $z \in \mathbb{R}^n$. The function $f : [0, +\infty[\times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is supposed to be locally Lipschitz and continuous. This assumption ensures the existence of the solution of the fractional differential equation defined by Eq. (8). For simplification in the rest of this article the solution starting at the initial condition z_0 and with the input u will be represented by the function $z(t) = z(t, z_0, u)$. The function u is called exogenous input, and its role is to take into account all not declared phenomena, error terms in confectioning the differential equations, or disturbances. It is also be called the perturbation term. As we will notice, the solution of the fractional differential equation defined by Eq. (8) depends on the values of the input u.

The generalized Mittag-Leffler input stability aims to take into account the following properties: the solution of the Eq. (8) is bounded when the input is bounded and when the input converges, the solution of the differential equation converges too. This properties can be rewritten using comparison functions, that is

$$\|z(t)\| \leq \beta\left(\|\xi\|, t^\kappa\right) + \gamma(\|u\|). \quad (10)$$

Note that the function β should be expressed by using the Mittag-Leffler function which belongs to the class $\mathcal{KL}$ functions [25], [26]. That is when we fix the first argument, the function β decreases according to time, when t^κ tends to infinity. And when we fix the time, the function β is zero at $z = 0$ and converge to infinity as z tends to infinity. We summarize in the following sentence: $\beta(s, .)$ is non increasing function and tends to zero as its arguments tend to infinity and $\beta(., t^\kappa)$ is an increasing positive definite function.

The function γ belongs to a class $\mathcal{K}_\infty$ function [25], [26], that is $\gamma(0) = 0$ and γ tends to infinity when the s tends to infinity.

We can observe these two functions describe as well the properties contained in the generalized Mittag-Leffler input stability. We notice when the input converges to zero, the function γ converges to zero, which in particular implies the state z admits as upper bound the function $\beta(., t^\kappa)$. Thus, the state z will converge as well because the function $\beta(s, .)$ is non increasing function and tends to zero as its arguments tend to infinity. The described property is called converging input converging state (*CICS*) for the rest of this article. The same explanation is obtained when the input u is bounded. In other words, bounded input generates a bounded state, and we will represent it as the following abbreviation *BIBS* property. Another important remark is when the input is null, then when Eq. (8) admits $z = 0$ as a trivial equilibrium point this point is automatically global asymptotically stable [25], [26], that is

$$\|z(t)\| \leq \beta\left(\|\xi\|, t^\kappa\right). \quad (11)$$

We summarize the generalized Mittag-Leffler input stability notion in the following definition.

Definition 7: The fractional-order differential equation (8) described by the Caputo fractional derivative is said to be generalized Mittag-Leffler input stable if, there exists a class $\gamma \in \mathcal{K}_\infty$ function such that for any initial condition $\|z_0\| = \|\xi\|$, its solution satisfies the following condition

$$\|z(t)\| \leq \left[m(\|\xi\|)E_\alpha\left(-\eta\left(\frac{t^\kappa}{\kappa}\right)^\alpha\right)\right]^b + \gamma(\|u\|), \quad (12)$$

where the constant $b > 0$, $\eta > 0$ and m is locally Lipschitz with the condition $m(0) = 0$.

The Mittag-Leffler input stability can be extended with the Caputo derivative, see in [25]. It also can be defined using the Riemann-Liouville fractional derivative see in [28]. The first term of Eq. (12) depends on the used fractional operator. For the Caputo derivative the first term can be represented as $\beta(\|\xi\|, t) = [m(\|\xi\|)E_\alpha(-\eta t^\alpha)]^b$. We notice Eq. (12) is a generalization of the Mittag-Leffler input stability described by Caputo derivative, see in [22]. In the context of the Riemann-Liouville fractional derivative, the first term of Eq. (12) take the form $\beta(\|\xi\|, t) = [m(\|\xi\|)t^{\alpha-1}E_{\alpha,\alpha}(-\eta t^\alpha)]^b$. We can notice the Mittag-Leffler function is multiplied by a function $t^{\alpha-1}$. In conclusion, the definition of the Mittag-Leffler input stability depends on the used fractional operator and can be extended with all types of fractional operators.

For an understanding of this new stability notion, we illustrate it by a simple linear example. Let the function

$f(t, z, u) = Cz + Bu$, where $C \in \mathbb{R}^{n\times n}$ is Hurwitz matrix and $B \in \mathbb{R}^{n\times m}$, then we obtain the following

$$D_c^{\alpha,\kappa} z = Cz + Bu. \tag{13}$$

The first procedure is to get the analytical solution of the fractional differential equation. Utilizing the Laplace transform to both sides of Eq. (13) we obtain the following relationship

$$\begin{aligned} s^{\alpha}\tilde{z}(s) - \tilde{z}(0) &= C\tilde{z}(s) + B\tilde{u}(s), \\ \left(s^{\alpha}I - C\right)\tilde{z}(s) &= \xi + B\tilde{u}(s), \\ \left(s^{\alpha}I - C\right)^{-1}\xi + \left(s^{\alpha}I - C\right)^{-1}B\tilde{u}(s) &= \tilde{z}(s). \end{aligned} \tag{14}$$

The inverse of the Laplace transform gives the solution of the fractional differential equation (13) described by the relation

$$z(t) = \xi E_{\alpha}\left(-C\left(\frac{t^{\kappa}}{\kappa}\right)^{\alpha}\right) + g(t, u) \tag{15}$$

where

$$g(t, u) = \int_0^t \left(\frac{s^{\kappa} - t^{\kappa}}{\kappa}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-C\left(\frac{t^{\kappa}}{\kappa}\right)^{\alpha}\right)Bu(s)ds.$$

The second step in the procedure is to apply the Euclidean norm to both sides of Eq. (15), we follow the sketch described in the following lines

$$\|z(t)\| \le \|\xi\|\left\|E_{\alpha}\left(-C\left(\frac{t^{\kappa}}{\kappa}\right)^{\alpha}\right)\right\| + \|B\|\|u\|\|I\|, \tag{16}$$

where

$$I = \int_0^t \left(\frac{s^{\kappa} - t^{\kappa}}{\kappa}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-C\left(\frac{t^{\kappa}}{\kappa}\right)^{\alpha}\right)ds.$$

The second term I is bounded when the matrix C is stable or verify Matignon criterion, therefore we can rewrite Eq. (16) as the following form

$$\|z(t)\| \le \|\xi\|\left\|E_{\alpha}\left(-C\left(\frac{t^{\kappa}}{\kappa}\right)^{\alpha}\right)\right\| + \|B\|\|u\|\epsilon, \tag{17}$$

where ϵ is the upper bound of the integral part I. We consider the function $\beta(\|\xi\|, t^{\kappa}) = \|\xi\|\|E_{\alpha}(-C(\frac{t^{\kappa}}{\kappa})^{\alpha})\|$, we can observe when the constant $\|\xi\|$ is fixed, then the function β is a nonincreasing function and converges to zero when the time t tends to infinity. The second, when we fix the time, the function β is a positive definite function. Thus, we can conclude the function β belongs to the class $\mathcal{KL}$ functions. Let's $\gamma(\|u\|) = \|B\|\|u\|\epsilon$ is clearly a class $\mathcal{K}$ function. Finally, we conclude that when the matrix C satisfies the Matignon criterion, the fractional differential equation (13) is generalized Mittag-Leffler input stable. Thus, Eq. (17) satisfies the *BIBS* and the *CICS* properties. For example, according to the above procedure, it is straightforward to observe the fractional differential equation described by

$$D_c^{\alpha,\kappa} z = -3z + u, \tag{18}$$

is generalized Mittag-Leffler input stable. One and famous remark to take into account is, when *BIBS* property is not verified, then the fractional differential equation will not be generalized Mittag-Leffler input stable. For example, we consider the fractional differential equation defined by

$$D_c^{\alpha,\kappa} z = -3z + zu, \tag{19}$$

does not satisfy the *BIBS* property, for example, when we consider a constant exogenous input $u = 4$, then Eq. (19) can be rewritten as the following form

$$D_c^{\alpha,\kappa} z = z, \tag{20}$$

which explodes when the time tends to infinity. These two examples explain clearly what is the generalized Mittag-Leffler input stability. Since the introduction of this new stability notion, its Lyapunov characterizations have been introduced. We have two methods to characterize the generalized Mittag-Leffler input stability in terms of Lyapunov functions. We recall them in the following lemmas.

Lemma 1 [28]: Assume the existence of positive function $W : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ continuous and differentiable, and three functions belonging to a class $\mathcal{K}_{\infty}$ functions, there are ϖ_1, ϖ_2, and ϖ_3, obeying to the following relationships:

1) $\|z\|^a \le W(t, z) \le \varpi_1(\|z\|)$.
2) If for any $\|z\| \ge \varpi_2((|u|)) \Longrightarrow D_c^{\alpha,\kappa} W(t, z) \le -\varpi_3((\|z\|))$.

where a is non-negative constant. Then the fractional equation (8) under Caputo-Liouville derivative is generalized Mittag-Leffler input stable.

But the utilization of the Lemma 1 is not all time possible. Alternatively, another Lyapunov characterization was proposed to solve this issue; we recall this second characterization in the following Lemma.

Lemma 2 [28]: Assume the existence of positive function $W : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ continuous and differentiable, and two functions belonging to a class $\mathcal{K}_{\infty}$ of functions, there are the functions ϖ_1, the function ζ, satisfying the following assumptions:

1) $\|z\|^a \le W(t, z) \le \varpi_1(\|z\|)$.
2) $D_c^{\alpha,\kappa} W(t, z) \le -kW(z, t) + \zeta(\|u\|)$.

where a and k are non negative constants. Then fractional equation (8) under generalized Caputo-Liouville derivative is generalized Mittag-Leffler input stable.

Note that for quadratic functions the following identity will have much importance in the use of the Lyapunov functions, we have [25], [28], [41], [43]

$$D_c^{\alpha,\kappa} z^2 \le z D_c^{\alpha,\kappa} z. \tag{21}$$

The illustrations of these two Lemmas will be done when we investigate on the generalized Mittag-Leffler input stability of the *RL*, *LC*, *RC*, and *RLC* electrical circuits. Note that the use of the Lyapunov function comes from the inconveniences to get the analytical solutions of many classes of fractional differential equations. The inconvenience of these two lemmas are the construction of the Lyapunov functions, which are not trivial in many circumstances.

IV. APPLICATIONS TO THE FRACTIONAL-ORDER *RL*, *RC*, *LC* AND *RLC* ELECTRICAL CIRCUITS

The generalized Mittag-Leffler input stability is a new stability notion and can be applied in the electrical circuits, due to the fact the *RL*, *RC*, *LC*, and *RLC* electrical circuits can be expressed like Eq. (14). In this section, we apply the generalized Mittag-Leffler input stability and the generalized Mittag-Leffler stability to the fractional-order *RL*, *RC*, *LC*, and *RLC* electrical circuits using Lyapunov direct method. In other words, we construct an appropriate Lyapunov function for a considered fractional differential equation. We begin this section by the fractional-order *RL* circuit represented by the equation

$$D_c^{\alpha,\kappa} z = -\frac{R}{L} z + u, \tag{22}$$

where R denotes the resistance, and L is the inductance. Here z measures the intensity across the inductor into the electrical circuit. We will use two methods; the first will permit to prove the generalized Mittag-Leffler input stability using the characterization in Lemma 1. The method particularly called θ-method. The second method will consist of proving the generalized Mittag-Leffler stability using Lemma 2. We will first experiment with the θ-method. Let's the Lyapunov candidate function defined by $V(z) = z^2/2$. Applying Lemma 1, and calculating the generalized Caputo-Liouville derivative of the Lyapunov function along the trajectories, we get the following form

$$\begin{aligned} D_c^{\alpha,\kappa} V(z) &\leq z D_t^{\alpha,\kappa} z, \\ &= z\left[-\frac{R}{L} z + u\right], \\ &= -\frac{R}{L} z^2 + zu, \\ &\leq -\frac{R}{L}(1-\theta) z^2 - \frac{R}{L}\theta z^2 + zu, \end{aligned} \tag{23}$$

where $\theta \in (0, 1)$. From Eq. (24), we have the following relationship

$$-\frac{R}{L}\theta z^2 + zu \leq 0 \Longrightarrow D_t^{\alpha,\kappa} V(z) \leq -\frac{R}{L}(1-\theta) z^2.$$

Applying the Euclidean norm, we arrive at the following, when $\|z\| \geq \frac{L\theta\|u\|}{R}$ then $D_t^{\alpha,\kappa} V(z) \leq -\frac{R}{L}(1-\theta) z^2$. Under Lemma 1, we conclude the fractional-order *RL* electrical circuit described by Eq. (22) is generalized Mittag-Leffler input stable. To see this conclusion, more precisely, the readers can follow the following procedure. Let $\lambda = \frac{R}{2L}(1-\theta)$, thus we obtain the following

$$\|z\| \geq L\theta\|u\|/R \Longrightarrow D^{\alpha,\kappa} V(z) = -\lambda V(x) - m(t), \tag{24}$$

where m is a positive and continuous function. Applying the Laplace transform, we obtain the following relationship

$$\begin{aligned} \|z\| &\geq \frac{L\theta\|u\|}{R} \Longrightarrow s^\alpha \tilde{V}(s) - s^\alpha V(0) = -\lambda \tilde{V}(s) - \tilde{m}(s), \\ \|z\| &\geq \frac{L\theta\|u\|}{R} \Longrightarrow \tilde{V}(s) = -\frac{V(0)}{s^\alpha + \lambda} - \frac{\tilde{m}(s)}{s^\alpha + \lambda}. \end{aligned} \tag{25}$$

Applying the inverse of the Laplace transform to the left hand of Eq. (25), we get the following relationship by neglecting the negative term we get

$$\begin{aligned} \|z\| &\geq \frac{L\theta\|u\|}{R} \Longrightarrow z^a(t) \leq V(t) \leq V(0) E_\alpha\left(-\lambda\left(\frac{t^\kappa}{\kappa}\right)^\alpha\right), \\ \|z\| &\geq \frac{L\theta\|u\|}{R} \Longrightarrow z^a(t) \leq V(0) E_\alpha\left(-\lambda\left(\frac{t^\kappa}{\kappa}\right)^\alpha\right), \end{aligned} \tag{26}$$

where a is a positive constant number, Eq. (26) is another representation of the generalized Mittag-Leffler input stability of the fractional-order *RL* electrical circuit. The second Lemma can be utilized using the following procedure; we have that

$$\begin{aligned} D_c^{\alpha,\kappa} V(z) &\leq -\frac{R}{L} z^2 + zu, \\ &= -\frac{R}{L} z^2 + \frac{z^2}{2} + \frac{u^2}{2}, \\ &= -\left[\frac{R}{L} - \frac{1}{2}\right] z^2 + \frac{u^2}{2}. \end{aligned} \tag{27}$$

Under the condition, $\frac{R}{L} > \frac{1}{2}$, then using Lemma 2, the fractional-order *RL* electrical circuit is generalized Mittag-Leffler input stable. We observe from Eq. (27), the exogenous input $u = 0$, the Lyapunov characterization is given by

$$D_c^{\alpha,\kappa} V(z) \leq -\frac{R}{L} z^2, \tag{28}$$

which corresponds to the Lyapunov characterization of the generalized Mittag-Leffler stability of the trivial equilibrium point $z = 0$. The same procedure can be applied for the fractional-order *RC* electrical circuit. We continue our investigations with the fractional-order *RC* electrical circuit represented by the equation

$$D_c^{\alpha,\kappa} z = -\frac{1}{RC} z + u, \tag{29}$$

where C denotes the capacitance, and R is the resistance. Here z measures the voltage across the capacitor.We consider the Lyapunov candidate function defined by $V(z) = z^2/2$. We apply the identity (21) again, and we calculate the generalized Caputo derivative to the Lyapunov function along the trajectories of (29), we get the following form

$$\begin{aligned} D_c^{\alpha,\kappa} V(z) &\leq z D_t^{\alpha,\kappa} z, \\ &= z\left[-\frac{1}{RC} z + u\right], \\ &= -\frac{1}{RC} z^2 + zu, \\ &\leq -\frac{1}{RC}(1-\theta) z^2 - \frac{1}{RC}\theta z^2 + zu. \end{aligned} \tag{30}$$

Using Eq. (30), we have the following relationship

$$-\frac{1}{RC}\theta z^2 + zu \leq 0 \Longrightarrow D_t^{\alpha,\kappa} V(z) \leq -\frac{1}{RC}(1-\theta) z^2.$$

We apply the Euclidean norm, we arrive at the following, when $\|z\| \geq RC\theta\|u\|$ then $D_t^{\alpha,\kappa} V(z) \leq -\frac{1}{RC}(1-\theta) z^2$. Under

Lemma 2, we conclude the fractional-order RC electrical circuit described by Eq. (29) is generalized Mittag-Leffler input stable. To see this conclusion, readers can follow the following procedure. Let $\lambda = \frac{1}{2RC}(1-\theta)$, thus we obtain

$$\|z\| \geq RC\theta\|u\| \Longrightarrow D^{\alpha,\kappa}V(z) = -\lambda V(x) - m(t), \quad (31)$$

where m is a positive and continuous function. Applying the Laplace transform, we obtain the following relationship

$$\|z\| \geq RC\theta\|u\| \Longrightarrow s^{\alpha}\tilde{V}(s) - s^{\alpha}V(0) = -\lambda\tilde{V}(s) - \tilde{m}(s),$$
$$\|z\| \geq RC\theta\|u\| \Longrightarrow \tilde{V}(s) = -\frac{V(0)}{s^{\alpha}+\lambda} - \frac{\tilde{m}(s)}{s^{\alpha}+\lambda}. \quad (32)$$

Applying the inverse of the Laplace transform to the left hand of Eq. (32), we get the following relationship by neglecting the negative term we get

$$\|z\| \geq RC\theta\|u\| \Longrightarrow z^{a}(t) \leq V(t) \leq V(0)E_{\alpha}\left(-\lambda\left(\frac{t^{\kappa}}{\kappa}\right)^{\alpha}\right),$$
$$\|z\| \geq RC\theta\|u\| \Longrightarrow z^{a}(t) \leq V(0)E_{\alpha}\left(-\lambda\left(\frac{t^{\kappa}}{\kappa}\right)^{\alpha}\right), \quad (33)$$

where a is a positive constant number, Eq. (33) is another representation of the generalized Mittag-Leffler input stability of the fractional-order RC electrical circuit. We continue with the fractional order LC circuit, but we use the trajectory to prove the generalized Mittag-Leffler input stability. Let's the fractional-order LC electrical circuit defined by

$$D_{c}^{\alpha,\kappa}z = -\frac{1}{\sqrt{LC}}z + u, \quad (34)$$

where C denotes the capacitance, and L represents the inductance. Here z measures the intensity across the inductor. For simplification, we replace in Eq. (13) the matrix $Cr = -\frac{1}{\sqrt{LC}}$ and $B = 1$, we obtain the following analytical solution

$$z(t) = \xi E_{\alpha}\left(-\frac{1}{\sqrt{LC}}\left(\frac{t^{\kappa}}{\kappa}\right)^{\alpha}\right) + g(t,u). \quad (35)$$

where

$$g(t,u) = \int_{0}^{t}\left(\frac{s^{\kappa}-t^{\kappa}}{\kappa}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{1}{\sqrt{LC}}\left(\frac{t^{\kappa}}{\kappa}\right)^{\alpha}\right)u(s)ds.$$

The second step in the procedure consists of applying the Euclidean norm to both sides of Eq. (35), we follow the sketch described in the following lines

$$\|z(t)\| \leq \|\xi\|\left\|E_{\alpha}\left(-\frac{1}{\sqrt{LC}}\left(\frac{t^{\kappa}}{\kappa}\right)^{\alpha}\right)\right\| + \|u\|\|I\|, \quad (36)$$

where

$$I = \int_{0}^{t}\left(\frac{s^{\kappa}-t^{\kappa}}{\kappa}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{1}{\sqrt{LC}}\left(\frac{t^{\kappa}}{\kappa}\right)^{\alpha}\right)ds.$$

The second term of the integral is bounded when the matrix $-\frac{1}{\sqrt{LC}}$ is stable or satisfies the Matignon criterion. Therefore, we can rewrite Eq. (36) as the following form

$$\|z(t)\| \leq \|\xi\|\left\|E_{\alpha}\left(-\frac{1}{\sqrt{LC}}\left(\frac{t^{\kappa}}{\kappa}\right)^{\alpha}\right)\right\| + \|u\|\epsilon, \quad (37)$$

where ϵ is the upper bound of the integral part of Eq. (36). Let the function $\beta(\|\xi\|, t^{\kappa}) = \|\xi\|\|E_{\alpha}(-\frac{1}{\sqrt{LC}}(\frac{t^{\kappa}}{\kappa})^{\alpha})\|$. Thus, we can conclude the function β belongs to a class $\mathcal{KL}$ functions. Let $\gamma(\|u\|) = \|u\|\epsilon$ is clearly a class $\mathcal{K}_{\infty}$ functions. Finally, we conclude that when the matrix Cr satisfies Matignon criterion, then the fractional-order LC electrical circuit equation (34) is generalized Mittag-Leffler input stable.

We finish this section by providing conditions under which the fractional-order RLC electrical circuit is generalized Mittag-Leffler input stable. Let's the fractional differential equation defined by the equation

$$D_{c}^{\alpha,\kappa}z_1 = \frac{z_2}{L},$$
$$D_{c}^{\alpha,\kappa}z_2 = -\frac{z_1}{C} - \frac{R}{L}z_2 + u, \quad (38)$$

Let's the Lyapunov candidate function defined by $V(z) = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2$. The Caputo-Liouville generalized derivative of the Lyapunov candidate function along the trajectories gives

$$D_{c}^{\alpha,\kappa}V(z) \leq zD_{t}^{\alpha,\kappa}z,$$
$$\leq \frac{z_1z_2}{L} - \frac{z_1z_2}{C} - \frac{R}{L}z_2^2 + z_2u. \quad (39)$$

The first assumption is when the values $L = C$, the Lyapunov characterization in Eq. (39) can be represented by the following inequality

$$D_{c}^{\alpha,\kappa}V(z) \leq -\frac{R}{L}z_2^2 + z_2u,$$
$$\leq -\left[\frac{R}{L} - \frac{1}{2}\right]z_2^2 + \frac{u^2}{2}. \quad (40)$$

Using Lemma 2, we observe when $\frac{R}{L} > \frac{1}{2}$ and, in addition $L = C$, then the fractional-order RLC electrical circuit (38) is generalized Mittag-Leffler input stable.

In our last example, we consider more complex linear fractional differential equation defined by the following equation

$$D_{c}^{\alpha,\kappa}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = A\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + B\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad (41)$$

where the matrix A is given by the following form

$$A = \begin{pmatrix} -\frac{R_1+R_2}{L_1} & \frac{R_2}{L_1} & 0 \\ \frac{R_2}{L_2} & -\frac{R_2+R_3}{L_2} & \frac{R_3}{L_2} \\ 0 & \frac{R_3}{L_3} & \frac{R_3}{L_2} \end{pmatrix}, \quad (42)$$

the matrix B is given by the following form

$$B = \begin{pmatrix} \frac{1}{L_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{L_3} \end{pmatrix}, \quad (43)$$

and the exogenous input is represented by the form

$$u = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}. \quad (44)$$

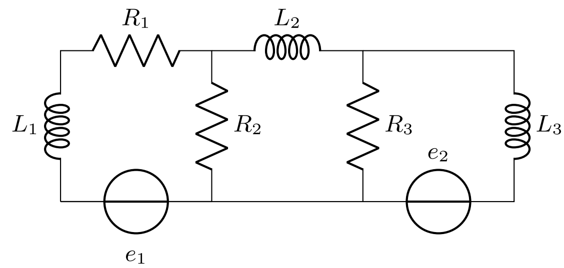

FIGURE 1. Circuit diagram.

The diagram of the circuit is represented in Figure 1 and can be found in [53]. To do not lost generality, we suppose the values of the resistances R_1,R_2, and R_3 are given, and the matrix A satisfies the Matignon criterion. We consider L_2 and L_2 are constants. For more simplification, we pose $u = (u_1, u_2)$ and $z = (z_1, z_2, z_3)$. Thus, Eq. (41) can be written as the following form

$$D_c^{\alpha,\kappa} z = Az + Bu. \tag{45}$$

Let the Lyapunov candidate function defined by $V(z) = z^T Pz$, where $A^T P + PA = -Q$ and P is positive, symmetric, square definite matrix. Applying Caputo generalized fractional derivative along the trajectory of Eq. (45), we have

$$\begin{aligned} D_c^{\alpha,\kappa} V(z) &\leq z^T P D_c^{\alpha,\kappa} z, \\ &= [Az + Bu]^T Pz + z^T P[Az + Bu], \\ &= z^T \left(A^T P + PA\right) z + 2u^T B^T z, \\ &\leq -\lambda_{min}(Q)\|z\|^2 + 2\lambda_{max}(P)\|u\|\|B\|\|z\|, \end{aligned} \tag{46}$$

where $\lambda_{min}(Q)$ is minimum eigenvalue of the matrix Q and $\lambda_{max}(P)$ is maximum eigenvalue of the matrix P. We set $\theta \in (0, 1)$ and $k = \frac{2\lambda_{max}(P)\|B\|}{\lambda_{min}(Q)-\theta}$ and $\gamma(r) = kr$. Thus, if $\|z\| \geq \gamma(\|u\|)$, then

$$D_c^{\alpha,\kappa} V \leq -\theta\|z\|^2.$$

From Lemma 1, we conclude the fractional differential equation (45) is generalized Mittag-Leffler input stable. In conclusion, the necessary and sufficient condition for the generalized Mittag-Leffler input stability of fractional differential equation (45) is the matrix A should satisfy the Matignon criterion given by $|\arg(\lambda(A))| > \alpha\pi/2$ and the input u should be bounded and convergent.

V. NUMERICAL DISCRETIZATION AND THE FIGURES WITH INTERPRETATIONS OF THE FRACTIONAL-ORDER CIRCUITS

In this section, we propose a novel numerical discretization of the fractional-order differential equations described by the generalized Caputo-Liouville derivative. To support our results, we aim the graphical representations. This issue will help us to avoid the problem of getting the analytical solutions of the electrical circuits. The generalized Mittag-Leffler input stability and the generalized Mittag-Leffler stability will be illustrated by the convergence of the solutions to the trivial equilibrium points. The first remark is to observe the resolution of the fractional differential equation in terms of the generalized Riemann-Liouville integral can be obtained as the following form

$$z(t) = \xi + I^{\alpha,\kappa} f(t, z, u). \tag{47}$$

The second step is to evaluate the above Eq. (47) at the point t_n and by simple manipulation, and neglecting the eventual typo errors in writing the expression, we get the following relationship

$$z(t_n) = \xi + I^{\alpha,\kappa} f(t_n, z, u). \tag{48}$$

The main idea in the rest of the discretization is the use the discretization of the generalized Riemann-Liouville integral $I = I^{\alpha,\kappa} f(t_n, z, u)$ given by the following expression

$$\begin{aligned} I &= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left(\frac{t_n^\kappa - s^\kappa}{\kappa}\right)^{\alpha-1} f(s, z(s), u(s)) \frac{ds}{s^{1-\kappa}}, \\ &= \frac{\kappa^{1-\alpha}}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \frac{s^{1-\kappa}}{\left(t_n^\kappa - s^\kappa\right)^{1-\alpha}} f\left(s, z_j, u_j\right) ds, \\ &= \frac{\kappa^{1-\alpha}}{\Gamma(\alpha)} \sum_{j=0}^{n-1} f\left(s, z_j, u_j\right) \int_{t_j}^{t_{j+1}} \frac{s^{1-\kappa}}{\left(t_n^\kappa - s^\kappa\right)^{1-\alpha}} ds, \\ &= \frac{\kappa^{1-\alpha}}{\alpha\Gamma(\alpha)} \sum_{j=0}^{n-1} \left[\left(t_n^\kappa - t_j^\kappa\right)^\alpha - \left(t_n^\kappa - t_{j+1}^\kappa\right)^\alpha\right] f\left(t_j, z_j, u_j\right). \end{aligned} \tag{49}$$

Using the grid point $t_n = (nh)^{1/\kappa}$, where h denotes a constant step-size. After simple calculations, we arrive at

$$I^{\alpha,\kappa} f(t_n, z, u) = h^\alpha \sum_{j=1}^{n} c_{n-j} f\left(t_j, z_j\right), \tag{50}$$

where $c_{n-j} = ((n - j + 1)^\alpha - (n - j)^\alpha) / \frac{\kappa^{1-\alpha}}{\Gamma(1+\alpha)}$ and $c_n = ((n + 1)^\alpha - (n)^\alpha) / \frac{\kappa^{1-\alpha}}{\Gamma(1+\alpha)}$. The next step is to use the first-order interpolant polynomial of the function $f(\tau)$ given by the following expression

$$f(\tau) = f\left(t_{j+1}, z_{j+1}\right) + \frac{\tau - t_{j+1}}{h}\left[f\left(t_{j+1}, z_{j+1}\right) - f\left(t_j, z_j\right)\right]. \tag{51}$$

The numerical discretization of the generalized fractional integral will be given after recursive calculations by the following relationship

$$I^{\alpha,\kappa} f(t_n, z, u) = h^\alpha \left[\bar{c}_n^{(\alpha)} f(0) + \sum_{j=1}^{n} c_{n-j}^{(\alpha)} f\left(t_j, z_j, u_j\right)\right], \tag{52}$$

where the parameters in the previous equation are described as the following form

$$\bar{c}_n^{(\alpha)} = \frac{(n-1)^\alpha - n^\alpha(n - \alpha - 1)}{\kappa^{\alpha-1}\Gamma(2 + \alpha)}, \tag{53}$$

and when the indices follow the following natural values $n = 1, 2, \ldots,$ the previous parameter is expressed by

$$c_0^{(\alpha)} = \frac{1}{\kappa^{\alpha-1}\Gamma(2+\alpha)}$$
$$\text{and } c_n^{(\alpha)} = \frac{(n-1)^{\alpha+1} - 2n^{\alpha+1} + (n+1)^{\alpha+1}}{\kappa^{\alpha-1}\Gamma(2+\alpha)}. \quad (54)$$

Now replacing Eq. (52) into Eq. (48) we obtain the following implicit discretization, that is

$$z(t_n) = \xi + h^\alpha \left[\bar{c}_n^{(\alpha)} f(0) + \sum_{j=1}^{n} c_{n-j}^{(\alpha)} f\left(t_j, z_j, u_j\right) \right]. \quad (55)$$

Let's the approximate solution $z(t_n)$ of the Eq. (8), and we suppose z_n is the exact solution of the Eq. (8). The error estimation exists in the numerical discretization. Thus the residual function with $\kappa = 1$ for the implicit discretization is given by the following function

$$|z(t_n) - z_n| = \mathcal{O}\left(h^{\min\{\alpha+1,2\}}\right). \quad (56)$$

The implicit discretization converges as well when the parameter h converge to 0. Another remark is the stability of the numerical discretization follows from the fact the functions f is Lipschitz continuous, which is assumed at the beginning of this article. The above discretization will be adapted in the context of the fractional-order *RL*, *LC*, *RC*, and *RLC* electrical circuits. Before continuing the resolution, we will fix the values of the parameters.

Let's the fractional-order *RL* electrical circuit defined by the following equations

$$D_c^{\alpha,\kappa} z = -\frac{R}{L} z + u, \quad (57)$$

where the input u is given by the function $u = \frac{E_0}{L}$. The function $f(t, z, u) = -\frac{R}{L} z + u$ and $z(0) = \xi = 1$. The discretized form of the fractional-order *RL* electrical circuit can be expressed in the following form

$$z(t_n) = \xi + h^\alpha \left[\bar{c}_n^{(\alpha)} f(0) + \sum_{j=1}^{n} c_{n-j}^{(\alpha)} f\left(t_j, z_j, u_j\right) \right], \quad (58)$$

where

$$f\left(t_j, z_j, u_j\right) = -\frac{R}{L} z_j + \frac{E_0}{L}. \quad (59)$$

To analyze the generalized Mittag-Leffler input stability of the fractional-order *RL* electrical circuit, we fix the order to $\alpha = 0.95$. We suppose $R = 50\Omega$, $L = 100H$, and we compute in Figure 2, the solutions with three different values of the exogenous inputs $E_0 = 0$ (yellow line), $E_0 = 5$ (red line) and $E_0 = 10$ (blue line). We notice all the trajectories converge to the trivial equilibrium point, but this convergence is conditioned by the converge of the exogenous input. Thus we note clearly by observing the Figure 2 the fractional-order *RL* electrical circuit respects as well the property *BIBS*. This is the generalized Mittag-Leffler stability of the electrical circuit model (57). Finally, we can

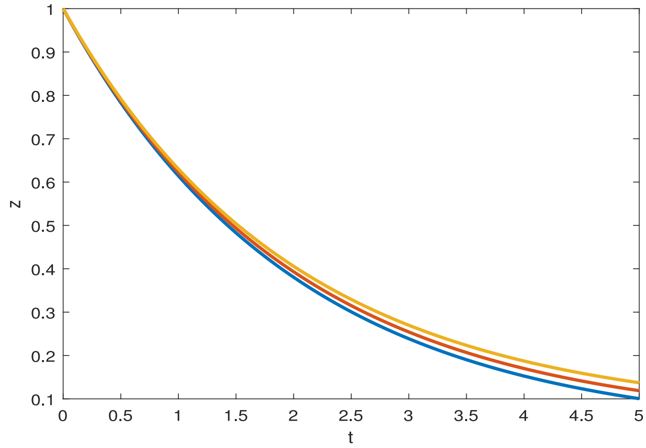

FIGURE 2. Dynamics of the solutions of RL electrical circuit.

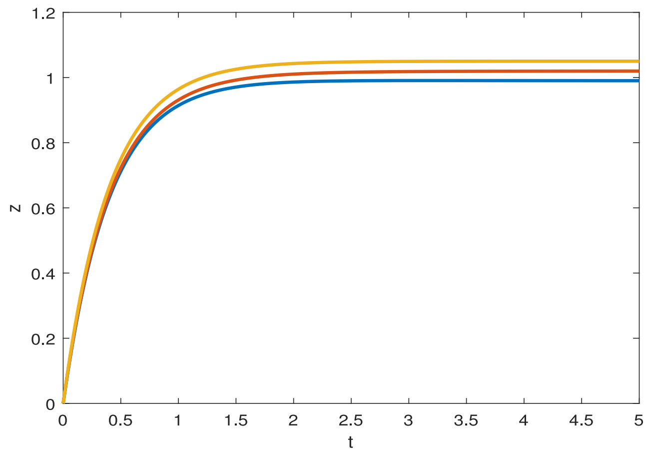

FIGURE 3. Dynamics of the solutions of RC electrical circuit.

conclude the fractional-order *RL* electrical circuit is generalized Mittag-Leffler stable. Furthermore, with $E_0 = 0$, we also observe the trivial equilibrium $z = 0$ is generalized Mittag-Leffler stable.

Let's the fractional-order *RC* electrical circuit defined by the following fractional differential equation

$$D_c^{\alpha,\kappa} z = -\frac{1}{RC} z + u, \quad (60)$$

where the input u is given by the function $u = \frac{E_0}{RC}$. The function $f(t, z, u) = -\frac{1}{RC} z + u$ and $z(0) = \xi = 0$. The discretized form of the fractional-order *RC* electrical circuit (60) can be expressed in the following form

$$z(t_n) = \xi + h^\alpha \left[\bar{c}_n^{(\alpha)} f(0) + \sum_{j=1}^{n} c_{n-j}^{(\alpha)} \left(-\frac{1}{RC} z_j + \frac{E_0}{RC} \right) \right]. \quad (61)$$

We fix the order $\alpha = 0.95$. We assume $R = 0.8\Omega$, $C = 0.5F$, and we consider in Figure 3, the graphical representations with different values of the inputs which are $E_0 = 1$ (yellow line), $E_0 = 5$ (red line) and $E_0 = 10$ (blue line). We notice all the solutions converge to a constant number $\gamma(u) = E_0$.

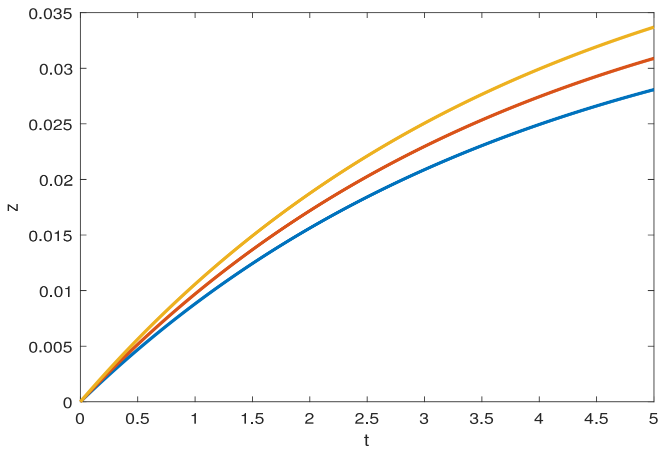

FIGURE 4. Dynamics of the solutions of LC electrical circuit.

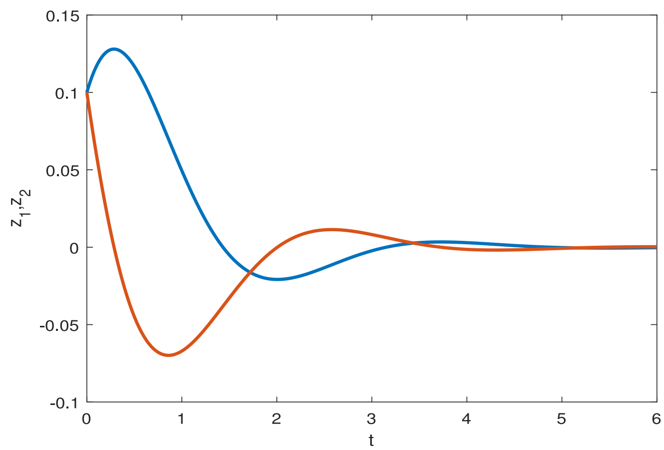

FIGURE 5. Dynamics of the solutions of RLC electrical circuit.

Let's the fractional-order LC electrical circuit defined by the following fractional differential equation

$$D_c^{\alpha,\kappa} z = -\frac{1}{\sqrt{LC}} z + u, \tag{62}$$

where the input u is given by the function $u = \frac{E_0}{L}$. The function $f(t, z, u) = -\frac{1}{\sqrt{LC}} z + u$ and $z(0) = \xi = 0$. The discretized form of the fractional-order LC electrical circuit (62) can be expressed in the following form

$$z(t_n) = \xi + h^\alpha \left[\bar{c}_n^{(\alpha)} f(0) + \sum_{j=1}^{n} c_{n-j}^{(\alpha)} \left(-\frac{1}{\sqrt{LC}} z_j + \frac{E_0}{L} \right) \right]. \tag{63}$$

We fix the order $\alpha = 0.95$. We assume $L = 100mH$, $C = 0.15F$, and we consider in Figure 4, the graphical representations with different values of the inputs which are $E_0 = 1$ (yellow line), $E_0 = 5$ (red line) and $E_0 = 10$ (blue line). We notice all the solutions increase and converge to the constant number $\gamma(u) = E_0 C$ when the time tends to infinity.

We finish by the fractional-order RLC circuit under generalized Caputo-Liouville derivative defined by the double fractional differential equations

$$D_c^{\alpha,\kappa} z_1 = \frac{z_1}{L},$$
$$D_c^{\alpha,\kappa} z_2 = -\frac{z_1}{C} - \frac{R}{L} z_2 + u, \tag{64}$$

where the input u is given by the function $u = \frac{E_0}{RC}$. Here we consider two different functions $f(t, z, u) = \frac{z_1}{L}$ and $g(t, z, u) = -\frac{z_1}{C} - \frac{R}{L} z_2 + u$. Using the numerical discretizations, we obtain the following schemes

$$z_1(t_n) = \xi_1 + h^\alpha \left[\bar{c}_n^{(\alpha)} f(0) + \sum_{j=1}^{n} c_{n-j}^{(\alpha)} \left(\frac{1}{L} z_{1j} \right) \right],$$

$$z_2(t_n) = \xi_1 + h^\alpha \left[\bar{c}_n^{(\alpha)} g(0) + \sum_{j=1}^{n} c_{n-j}^{(\alpha)} \left(-\frac{z_{1j}}{C} - \frac{R z_{2j}}{L} + u_j \right) \right].$$

We fix the order to $\alpha = 0.95$. We assume $R = 1$, $L = 0.45mH$, $C = 0.45F$, and we consider in Figure 5, the graphical representation with the value of the input given by $E_0 = 0$. We depict the solution in two dimensional space.

The trajectories oscillate around the equilibrium point, but for a long time, we observe these oscillations converge to zero. We conclude finally, the trivial equilibrium of the fractional-order the RLC electrical circuit (64) is generalized Mittag-Leffler stable.

VI. CONCLUSION

This article has focussed on the generalized Mittag-Leffler input stability and the numerical discretization of the fractional-order electrical circuits. The generalized Mittag-Leffler input stability has been provided using the trajectory method and by constructing the adequate Lyapunov function. The generalized Caputo-Liouville fractional operator was used in the investigations. To support the results of this article, the graphical representations have been proposed.

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