IEEE Open Journal of
Circuits and Systems

Received 29 June 2020; revised 30 August 2020; accepted 24 September 2020. Date of publication 7 October 2020; date of current version 10 December 2020.

Digital Object Identifier 10.1109/OJCAS.2020.3029254

Nonlinear Fractional-Order Circuits and Systems: Motivation, A Brief Overview, and Some Future Directions

MOHAMMAD SALEH TAVAZOEI [1] (Member, IEEE), MAHSAN TAVAKOLI-KAKHKI [2], AND FEDERICO BIZZARRI [3,4] (Senior Member, IEEE)

[1] Electrical Engineering Department, Sharif University of Technology, Tehran 11155-4363, Iran

[2] Electrical Engineering Department, K. N. Toosi University of Technology, Tehran 16317-14191, Iran

[3] DEIB, Politecnico di Milano, 20133 Milan, Italy

[4] Advanced Research Center on Electronic Systems "E. De Castro" (ARCES), University of Bologna, 40126 Bologna, Italy

This article was recommended by Associate Editor R. Caponetto.

CORRESPONDING AUTHOR: M. S. TAVAZOEI (e-mail: tavazoei@sharif.edu)

The work of the Mohammad Saleh Tavazoei and Mahsan Tavakoli-Kakhki was supported by Iran National Science Foundation (INSF).

ABSTRACT In recent years, fractional-order differential operators, and the dynamic models constructed based on these generalized operators have been widely considered in design and practical implementation of electrical circuits and systems. Simultaneously, facing with fractional-order dynamics and the nonlinear ones in electrical circuits and systems enforces us to use more advanced tools (in comparison to those commonly used in design and analysis of linear fractional-order/nonlinear integer-order circuits and systems) for their analysis, design, and implementation. Discussing on such a motivation, this tutorial paper aims to provide an overview on the recent achievements in proposing effective tools for analysis and design of nonlinear fractional-order circuits and systems. Moreover, some open problems, which can specify future directions for continuing research works on the aforementioned subject, are discussed.

INDEX TERMS Fractional-order circuits and systems, nonlinear circuits and systems, fractional-order electrical elements, stability analysis, dynamical behavior analysis.

I. INTRODUCTION

The possibility of the appearance of fractional-order differential operators in the dynamic models of electrical circuits is a very long-known fact (e.g., the appearance of the fractional-order operator $1/\sqrt{s}$ in the impedance function of an infinite RC network). Even though, the first coherent attempts to recognize the applicability of fractional-order dynamics to the circuits and systems society were devoted to proposing the innovative methods for approximating the fractional-order operators by rational filters (see for instance [1], [2], and [3]). The need to approximate the fractional-order differential operators by rational filters, for using them in circuit implementation had caused that such operators were marginally considered in design of electrical networks, in a relatively long period of time. But the recent advances in fabrication of fractional-order electrical elements, such as fractional-order capacitors and inductors, and also proposing effective techniques for successful emulation of these electrical elements, lead to significant attention for profiting from the great potential of fractional-order dynamics in circuits and systems design [4]. On the other hand, nonlinear behaviors cannot be ignored in many real-world circuits and systems. Motivated by these facts, this article provides an overview on nonlinear fractional-order circuits and systems. In this overview, some recent relevant research works are reviewed. Also, on the basis of the reviewed topics, some challenges, which can specify future directions for continuing the research on the aforementioned topic and invite further research works, are discussed.

The paper is organized as follows. In Section II, some related mathematical concepts are briefly introduced. Section III is devoted to discussing on fractional-order electrical elements and their fabrication methods. In Section IV, it is explained why study on nonlinear fractional-order circuits and systems is necessary from the viewpoint of electrical engineering perspective. Sections V and VI respectively deal with reviewing the recent works on stability study and oscillatory behavior analysis in nonlinear fractional-order circuits and systems. Also, Section VII is devoted to control systems modeled by nonlinear fractional-order dynamics. Moreover, some challenges and open problems on the topics reviewed in the previous sections are described in Section VIII. Finally, the paper is closed by conclusions in Section IX.

II. A BRIEF MATHEMATICAL BACKGROUND

Fractional calculus [5], as a sub-branch of mathematical analysis, is a generalization for the well-known traditional calculus. The convenience and ease of use of traditional calculus had caused that the fractional calculus was marginally considered in engineering applications over the long years from introducing the basic foundations of this mathematical tool [6]. Even though, in recent decades by revealing the great potential of fractional calculus to provide more effective solutions for various engineering problems, considerable efforts have been done to apply this mathematical tool in different engineering fields [7]. The key-point in the generalization offered by fractional calculus is to extend the differential operators such that they exhibit non-integer orders. For instance, inspired by the Cauchy formula for the repeated integrals, the integral of function $f(t)$ of order $\alpha \in \mathbb{R}^+$ (with the lower terminal 0 and the upper terminal t) is defined as

$$ {}_0I_t^{\alpha} f(t) \triangleq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \tag{1} $$

where

$$ \Gamma(\alpha) \triangleq \int_0^{\infty} e^{-t} t^{\alpha-1} dt. \tag{2} $$

The interpretation of this generalized operator in the Laplace domain is similar to that of ordinary integral operators, i.e., it can be shown that

$$ \mathbb{L}\left\{ {}_0I_t^{\alpha} f(t) \right\} = \frac{1}{s^{\alpha}} \mathbb{L}\{f(t)\}. \tag{3} $$

On the basis of the fractional-order integral operator, fractional derivative can be defined in two forms. The first form for defining the fractional derivative operator of order $\alpha \in \mathbb{R}^+$, known as the Riemann–Liouville derivative, is given by

$$ {}_0^{RL}D_t^{\alpha} f(t) \triangleq \frac{d^m}{dt^m} \left\{ {}_0I_t^{m-\alpha} f(t) \right\}, \tag{4} $$

where $m = \lceil \alpha \rceil$ ($\lceil \alpha \rceil$ denotes the smallest integer, which is not less than α). The Caputo derivative is the second form of fractional derivative operators, and it is directly defined through the fractional-order integral operator. The Caputo derivative of order $\alpha \in \mathbb{R}^+ - \mathbb{N}$ is defined as

$$ {}_0^{C}D_t^{\alpha} f(t) \triangleq {}_0I_t^{m-\alpha} \left\{ \frac{d^m}{dt^m} f(t) \right\}, \tag{5} $$

where $m = \lceil \alpha \rceil$. In the rest of the paper, for briefness in the notations, ${}_0^{C}D_t^{\alpha}$ with $\alpha = m \in \mathbb{N}$ denotes the traditional operator $\frac{d^m}{dt^m}$. Also, the general notation ${}_0D_t^{\alpha}$ specifies either Riemann–Liouville or Caputo derivative operators.

The Mittag-Leffler functions, as a general form of the exponential ones, play a fundamental role in describing the eigenfunctions of fractional derivative operators. The two-parameter Mittag-Leffler function is defined as

$$ E_{\alpha,\beta}(z) \triangleq \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{6} $$

where $\alpha, \beta > 0$. In the special case $\alpha = \beta = 1$, (6) converts to the exponential function e^z.

III. FRACTIONAL-ORDER ELECTRICAL ELEMENTS

A. FRACTIONAL-ORDER CAPACITORS AND INDUCTORS

The current-voltage relation in electrical capacitors, as the most common electrical passive energy-storage element, is simply described by using a first-order derivative operator. In fractional-order capacitors, as generalized elements in the viewpoint of using fractional calculus concepts, such a relation is described by a fractional-order derivative operator. Namely, if $i(t)$ and $v(t)$ respectively denote the current and voltage of a fractional capacitor, then

$$ i(t) = C_{\alpha}\ {}_0D_t^{\alpha} v(t), \tag{7} $$

where $\alpha > 0$ is the order of the capacitor and $C_{\alpha} > 0$ denotes its pseudo-capacitance (with unit $Fs^{1-\alpha}$). Also, the current-voltage relation in a fractional-order inductor with the pseudo-inductance L_{β} (unit: $Hs^{1-\beta}$) and order $\beta > 0$ is described by

$$ v(t) = L_{\beta}\ {}_0D_t^{\beta} i(t). \tag{8} $$

In a more general framework, fractional-order capacitors and inductors can be considered as constant phase elements (CPEs) [8], i.e., the components whose impedance/admittance functions possess a constant phase in the frequency domain. A special case of CPEs is the Warburg impedance element [9], which is originally appeared in modeling of semi-infinite diffusion processes[1] and exhibits the constant phase 45° in the frequency domain.

Up to now, different techniques have been proposed for fabrication of fractional capacitors. Solid-state fractional capacitors fabricated by nanocomposite materials [11], [12], solid-state fractional capacitors constructed by ferroelectric polymers and reduced graphene structures [13], fractional capacitors constructed by electrolyte processes [14], fractional capacitors developed by using resistive-dielectric-conductive structures [15], and fractional capacitors fabricated by using copper silicon electrodes and porous film [16], [17], or by using platinized silicon electrodes and porous film [18] are some experimentally implemented

1. In the case of modeling finite-length diffusion processes [9], irrational forms [10] such as $\tanh(\sqrt{\tau s})/\sqrt{\tau s}$ and $\coth(\sqrt{\tau s})/\sqrt{\tau s}$ ($\tau > 0$) are appeared to describe the corresponding impedance/admittance functions.

types of this fractional-order element. In addition, some effective methods have been also proposed for the emulation of fractional-order capacitors and inductors (e.g., emulation of these elements on the basis of using RC/RL networks [19], [20], MOS transistors [21], generalized immittance converters [22], operational transconductance amplifiers [23], [24], and power converters [25]).

B. FRACTIONAL-ORDER MEMRISTORS

In 1971, Chua introduced the *memristor*, as the fourth fundamental two-terminal passive circuit element (along with electrical resistor, inductor, and capacitor), which relates magnetic flux linkage and electric charge [26], [27]. The existence of such a component, as a nonlinear circuit element, in electrical networks implies that applying nonlinear tools for analysis of these networks is unavoidable. Considering the generalization property achieved by using the fractional-order operators, fractionalized versions of the memristor element have been also introduced in literature. For example, in [28] a non-ideal fractional-order memristor described by the current-voltage relation

$$v(t) = R_\alpha(q)\ i(t), \tag{9}$$

where

$$R_\alpha(q) = R_P + 0.5(R_{AP} - R_P)(\mathrm{sgn}(q + Q_0) - \mathrm{sgn}(q - Q_0)), \tag{10}$$

$$q(t) = {}_0I_t^\alpha i(t), \tag{11}$$

and R_P, R_{AP}, and Q_0 are some constant parameters, has been introduced. As another example, in [29] it has been introduced a fractional-order memristor modeled by

$$i(t) = \left(a_0 + \sum_{i=1}^{r} a_i\ {}_0I_t^{\alpha_i} v(t)\right) v(t), \tag{12}$$

in which a_i $(i = 1, \ldots, r)$ are constants and the constants α_i $(i = 1, \ldots, r)$ are between zero and one. Some other models for the fractional-order memristor elements can be found in [30], [31], [32], and [33]. Also, successful analog implementations of these elements have been reported in [29] and [34].

IV. CIRCUITS AND SYSTEMS MODELED BY NONLINEAR FRACTIONAL-ORDER DYNAMICS

In the previous section, the electrical elements with fractional-order differential current-voltage relations have been introduced. Up to now, different electrical structures containing these fractional-order elements have been designed and implemented, e.g., fractional-order low/band/high-pass filters [35], [36], [37], fractional-order oscillators [38], fractional-order passive impedances [39], [40], fractional-order resonators [41], [42] (see Fig. 1), and fractional-order phase-locked loops [43]. For the analysis of an electrical network containing these elements, fractional calculus based tools should be used. Furthermore, considering the nonlinear behavior of such elements (e.g., the nonlinear nature of the fractional-order memristor) or the presence of other nonlinear components in the under-study electrical networks yield in facing a nonlinear fractional-order circuit. For the effective analysis, design, and implementation of such circuits, the theoretical tools developed in the field of nonlinear fractional-order circuits and systems can be useful. Besides this motivation, originated from using fractional-order elements in electrical circuit design and practice, another main motivation can be also mentioned for the study on nonlinear fractional-order circuits and systems in the viewpoint of electrical engineering perspective. This motivation is induced from modeling the involved components by fractional-order dynamics with the aim of achieving a more accurate model. As a simple sample, we can refer to the interesting works of Westerlund in the early 1990s [44], [45]. In these works, with the concluding remark "dead matter has memory!" it has been claimed and experimentally verified that the widespread electrical capacitors have a degree of fractionality. This means that a more precise model for an electrical capacitor may be in the form (7) (some interesting cases, in which using this more precised model yields in offering the better justifications for the relevant phenomena observed in practice, have been discussed in [45] and [46]). Nevertheless, due to insignificance of fractionality nature in common capacitors, i.e., possessing fractional orders very near to 1 (see [45, Tab. 1] and [46, Tab. 3]), the straightforward current-voltage relation $i(t) = Cdv(t)/dt$ has been simply used for describing their dynamical behavior. Some more advanced samples of benefiting from fractional-order dynamics in modeling of electrical components are modeling of on-chip inductors constructed by the Siliconbenzocyclobutene technology [47], voltammetric sensors [48], on-chip interconnects in nanoscale CMOS circuits [49], CMOS metamaterial transmission lines [50], [51], and large three-dimensional RC networks [52].

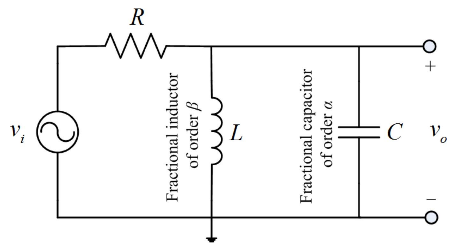

FIGURE 1. A fractional-order electrical network used in [41] as a resonator with the resonance frequency $\omega_r = (LC)^{-1/(\alpha+\beta)}$.

V. STABILITY ANALYSIS

This section deals with an overview of the achievements on stability analysis of fractional-order systems. Section V-A focuses on the special case of linear time invariant (LTI) fractional-order systems. Also, the Lyapunov indirect and direct methods for stability analysis of nonlinear fractional-order systems are reviewed in Sections V-B and V-C.

A. LINEAR TIME-INVARIANT CASE

The Lyapunov indirect method for stability analysis of the equilibrium points of nonlinear systems has been constructed based on the linearization of these systems at the equilibrium points, and then applying the stability analysis tools for linear systems to the obtained linearized models. Consequently, the tools primarily obtained for stability analysis of linear systems may be also useful in stability analysis of the nonlinear ones. A basic theorem on stability analysis of LTI fractional-order systems, which can be used as foundation for the Lyapunov indirect method in fractional-order systems, is as the following.

Theorem 1 [53]: The equilibrium point $x = \underline{0}$ in the LTI system ${}_{0}^{C}D_t^{\alpha} x(t) = Ax(t)$, where $\alpha \in (0, 1)$ and $A \in \mathbb{R}^{n\times n}$, is asymptotically stable if and only if each eigenvalue λ_i of matrix A satisfies condition $|\arg(\lambda_i)| > \alpha\pi/2$.

B. STABILITY ANALYSIS BASED ON LYAPUNOV INDIRECT METHOD

Due to the simplicity and variety of the tools introduced for stability analysis of fractional-order systems, using linearization techniques for stability analysis of equilibrium points in nonlinear fractional-order systems is a common approach. Such an approach has been initially proposed for nonlinear commensurate order systems and incommensurate order ones, in [54, Sec. 3] and [55, Sec. 3], respectively. In this subsection, some more advanced tools in the linearization of nonlinear fractional-order systems are reviewed.

Theorem 2 [56]: Consider the nonlinear fractional-order system

$$ {}_{0}^{C}D_t^{\alpha} x(t) = Ax(t) + f(x(t)), \tag{13} $$

where $\alpha \in (0, 1)$, $A \in \mathbb{R}^{n\times n}$, and the continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz with condition $f(\underline{0}) = \underline{0}$. Define

$$ l_f(r) \triangleq \sup \frac{\|f(x) - f(y)\|}{\|x - y\|}, \tag{14} $$

where the supremum is taken over all non-equal x and y in a closed ball of radius r and center $\underline{0}$ in $\mathbb{R}^n$. If

$$ \lim_{r\to 0} l_f(r) = 0, \tag{15} $$

then the equilibrium point $x = \underline{0}$ of system (13) is asymptotically stable, provided that the condition $|\arg(\lambda_i)| > \alpha\pi/2$ is met for any eigenvalue λ_i of matrix A.

Special forms of the above-mentioned result have been presented in [57] and [58] with the aim of suppression of the chaotic oscillation by stabilization of the equilibrium points in chaotic fractional-order systems. Considering the assumptions on function f in Theorem 2, $x = \underline{0}$ is called a hyperbolic equilibrium point for system (13) if all eigenvalues of matrix A are non-zero and do not lie on the half-lines $\arg(z) = \alpha\pi/2$ in the complex plane [59]. In [59, Th. 3], the topological equivalence of the behaviors of the trajectories of the nonlinear system (13) and its linearization, i.e., ${}_{0}^{C}D_t^{\alpha} x(t) = Ax(t)$, in the neighborhood of the origin is shown.

An extension for Theorem 2, which deals with linearization based stability analysis in time-delay nonlinear fractional-order systems, is as follows.

Theorem 3 [60]: Consider the time-delay system

$$ {}_{0}^{C}D_t^{\alpha} x(t) = Ax(t-\tau) + g(x(t), x(t-\tau)), \tag{16} $$

where $\alpha \in (0, 1)$, $\tau > 0$, $A \in \mathbb{R}^{n\times n}$, $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, and $g(\underline{0}, \underline{0}) = \underline{0}$. Assume that function g is continuous and locally Lipschitz. Define

$$ d_g(r) \triangleq \sup \frac{\|g(x, y) - g(\tilde{x}, \tilde{y})\|}{\|x - \tilde{x}\| + \|y - \tilde{y}\|}, \tag{17} $$

where the supremum is over all $x, \tilde{x}, y, \tilde{y} \in \mathbb{R}^n$ satisfying $||x||, ||\tilde{x}||, ||y||, ||\tilde{y}|| \leq r$ and $(x, y) \neq (\tilde{x}, \tilde{y})$. In this case, the equilibrium point $x = \underline{0}$ is asymptotically stable if $\lim_{r\to 0} d_g(r) = 0$ and all of the eigenvalues of matrix A are in the region

$$ \Omega = \left\{ z \in \mathbb{C} | z \neq 0, |z| < \left(\frac{|\arg(z)| - \alpha\pi/2}{\tau} \right)^{\alpha}, \right. $$
$$ \left. \& |\arg(z)| > \alpha\pi/2 \right\}. $$

C. STABILITY ANALYSIS BASED ON LYAPUNOV DIRECT METHOD

The Lyapunov direct method (also known as the second Lyapunov method) grounds on finding/constructing scalar Lyapunov functions for stability analysis of the equilibrium points. A primary theorem on the Lyapunov direct method in fractional-order systems is as follows.

Theorem 4 [61]: Assume that $x = \underline{0}$ is an equilibrium point for the fractional-order system

$$ {}_{0}^{C}D_t^{\alpha} x(t) = f(x(t)), \tag{18} $$

where $\alpha \in (0, 1)$ and the continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz around the origin. The equilibrium point $x = \underline{0}$ in this system is asymptotically stable if the differentiable convex function $V : \mathbb{R}^n \to \mathbb{R}$ with condition $V(\underline{0}) = 0$ is found such that

$$ k_1 \|x\|^a \leq V(x) \leq k_2 \|x\|^b \tag{19} $$

and

$$ \langle \nabla V, f(x) \rangle \leq -k_3 \|x\|^b, \tag{20} $$

hold in a neighborhood of the origin for some positive constants a, b, k_1, k_2, and k_3, where $\langle ., . \rangle$ denotes the inner product operator in $\mathbb{R}^n$. If condition (20) is replaced by

$$ \langle \nabla V, f(x) \rangle \leq -k_3 \|x\|^c, \tag{21} $$

where $c > b > 0$ and $k_3 > 0$, then the equilibrium point $x = \underline{0}$ is weakly asymptotically stable.[2]

Similar theorems for guaranteeing asymptotic stability, which are based on fractional-order derivative of Lyapunov functions (instead of integer-order derivative of such functions), can be found in [62, Th. 11] and [63, Ths. 5.4 and 6.2].

Generally speaking, exponential stability, as a special form of asymptotic stability, guarantees that the decay rate of the system response is no less than that of a decaying exponential function. Due to the very nature of the eigenfunctions of fractional-order differentiation operators, we deal with *Mittag-Leffler stability* in fractional-order systems rather than with the concept of exponential stability (this means that for fractional-order systems, in comparison to their integer-order counterparts, a slower rate of convergence is expected, e.g., the decay rate similar to $t^{-\alpha}$ in asymptotically stable fractional-order LTI systems of order α [64]). Definition of Mittag-Leffler stability, as a special form of asymptotic stability [63, Remark 4.4], is as follows.

Definition 1 (Mittag-Leffler Stability)[3] *[62]:* The equilibrium point $x = \underline{0}$ of system (18) is called Mittag-Leffler stable if the positive constants λ, b, and r and the positive function $m : \mathbb{R}^n \to \mathbb{R}$, which is locally Lipschitz around the origin and satisfies $m(\underline{0}) = 0$, exist such that

$$\|x(t)\| \leq \left(m(x(0))\ E_{\alpha,1}\left(-\lambda\left(t^{\alpha}\right)\right)\right)^{b}, \tag{22}$$

for all $\|x(0)\| < r$.

The Lyapunov based theorems, introducing sufficient conditions to guarantee the Mittag-Leffler stability of an equilibrium point, can be found in [62, Th. 5], [63, Th. 5.1], and [65, Th. 8]. The general conditions in these theorems, similar to those of Theorem 4, are on the basis of finding Lyapunov functions, whose fractional-order derivatives are negative definite. Nevertheless, this task, due to innovative nature of finding suitable Lyapunov function candidates and the difficulties of working with fractional-order differential operators in the viewpoint of obtaining a closed form for the derivative of typical Lyapunov functions, may not be straightforward. Even though, in some research works, systematic approaches are proposed to construct Lyapunov function candidates for fractional-order systems (see for instance [66]). An effective trick to overcome the aforementioned difficulties is to use the inequalities on fractional-order derivatives of the Lyapunov function candidates that specify the upper bounds for these derivatives with respect to the fractional derivatives of the pseudo-state vector of the system (this idea has been originally proposed in [67], and then extended in other research works). An example for such inequalities is described by

$${}_{0}^{C}D_{t}^{\alpha}V(x(t)) \leq \left(\frac{\partial V}{\partial x}\right)^{T} {}_{0}^{C}D_{t}^{\alpha}x(t), \tag{23}$$

where $\alpha \in (0, 1)$, $x(t) : \mathbb{R}^{\geq 0} \to \mathbb{R}^n$ and $V : \mathbb{R}^n \to \mathbb{R}$ are continuous and differentiable functions, and $V(x)$ is convex over $\mathbb{R}^n$ [68]. In the case $\alpha = 1$, (23) reduces to equality $dv/dx = (\frac{\partial V}{\partial x})^T \dot{x}$. The special forms of inequality (23) have been introduced in [67, Lemma 1] (the special case $V(x) = x^2$, where $x \in \mathbb{R}$), [69, Lemma 4] (the special case $V(x) = x^T P x$, where $x \in \mathbb{R}^n$ and $P \in \mathbb{R}^{n\times n}$ is a symmetric positive-definite matrix), and [70, Lemma 1] (the special case $V(x) = x^{2^p}$, where $p \in \mathbb{N}$ and $x \in \mathbb{R}$). Such inequalities have been considerately used for facilitating Lyapunov based stability analysis (e.g., in global stability analysis in fractional-order neural networks [71] and in time-delay systems [72]) and introducing Lyapunov based control methods (for example, in design of stabilizing static controllers [73] and adaptive ones [74], [75] for nonlinear fractional-order systems). It is worth noting that for using the inequalities in the form (23), the assumption on differentiability of $x(t)$ is required, whereas in the general case the solution of system (18) may be not differentiable (for more details, see [76]). In such a general case, the following inequality

$${}_{0}^{C}D_{t}^{\alpha}V(x(t)) \leq \left\langle \nabla V, {}_{0}^{C}D_{t}^{\alpha}x(t)\right\rangle, \tag{24}$$

where $V : \mathbb{R}^n \to \mathbb{R}$ satisfying $V(\underline{0}) = 0$ is a continuous, differentiable, and convex function on $\mathbb{R}^n$, can be used (See [61, Th. 2]). Another popular approach for constructing Lyapunov functions for fractional-order systems is based on rewriting the system's equations in an equivalent form, which is obtained by considering the frequency distributed model of fractional integrators. For more details about this approach and some of its applications in control systems design, see [77] and [78], [79], respectively.

Stability analysis of incommensurate order systems, in comparison with that of commensurate order ones, is generally more complicated.[4] For instance, considering Theorem 1 and performing a simple linearization based analysis, asymptotic stability of an equilibrium point of the integer-order system $\dot{x}(t) = f(x(t))$ yields in asymptotic stability of such an equilibrium point in the commensurate order system (18) with order $\alpha \in (0, 1)$, whereas this conclusion is not valid for incommensurate order counterparts of system $\dot{x}(t) = f(x(t))$ (for a counterexample, see [80, Eq. (2)]). In [80], it has been studied under which conditions the stability of an incommensurate order system can be inherited from that of its integer-order counterpart. Through this study done by benefiting from the Lyapunov direct method, it has been proved

2. For definition of a weakly asymptotically stable equilibrium point, see [61, Definition 1-(ii)].

3. For a more generalized definition, known as *generalized Mittag-Leffler stability*, see [63, Definition 4.2] and [65, Definition 7].

4. A fractional-order system defined in the pseudo-state space form is called a commensurate order system if all orders involved in the pseudo-state space form of the system are equal (e.g., as that seen in the model (18)). Otherwise, it is called an incommensurate order system.

that the equilibrium point $x = \underline{0} \in \mathbb{R}^n$ in system

$$ {}_{0}^{C}D_t^{\alpha_j} \hat{x}_j(t) = f_j(x(t)), \quad j = 1, \ldots, m, \tag{25} $$

where $\alpha_j \in (0, 1]$, $\hat{x}_j \in \mathbb{R}^{n_j}$, $x = [\hat{x}_1^T, \ldots, \hat{x}_m^T]^T$, $f_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$, and $\sum_{j=1}^{m} n_j = n$, and also in the system $\dot{x}(t) = f(x(t))$ with $f = [f_1^T, \ldots, f_m^T]^T$ is asymptotic stable if the convex functions $V_j : \mathbb{R}^{n_j} \to \mathbb{R}$ $(j = 1, \ldots, m)$, satisfying

$$ \sum_{j=1}^{m} \left(\partial V_j(\hat{x}_j)/\partial \hat{x}_j\right)^T f_j(x) < 0 \tag{26} $$

for $x \neq \underline{0}$, are found.

Some effective methods have been also proposed for stability analysis of time-delay nonlinear fractional-order systems, e.g., using generalized versions of the Gronwall inequality [81], [82], benefiting from the comparison techniques [83], [84], using fractional-order version of the Razumikhin Theorem [85], applying Lyapunov–Krasovskii functionals containing a fractional-order integral part [86], and using Laplace transform based approaches [87].

VI. OSCILLATORY FRACTIONAL-ORDER CIRCUITS AND SYSTEMS

Almost four decades passed from introducing the primary ideas to take the advantage of fractional-order operators in design of linear oscillators [88]. On the basis of marginally stable linear fractional-order dynamics, sinusoidal oscillators [89], [90], [91] and multi-frequency ones [92], [93], which are electrically implementable by using fractional-order inductors and capacitors, have been designed. Also, various forms of nonlinear fractional-order systems generating regular oscillations or irregular (chaotic) ones have been studied in literature. A remarkable distinction exists among regular oscillations generated by a fractional-order system in the form (18) and those generated by the integer-order counterpart $\dot{x}(t) = f(x(t))$. The oscillations generated by the former cannot be periodic [94], [95], [96], whereas the existence of non-constant solutions meeting condition $x(t) = x(t + T)$ (T: a positive constant) for all $t \geq 0$ is quite prevalent in the latter. Nevertheless, in the steady state (after transient time) the oscillatory responses of (18) can behave similarity to periodic signals (i.e., $T > 0$ may be found such that these solutions satisfy condition $x(t) \approx x(t + T)$ where $t \to \infty$) [97]. Considering this point, steady-state oscillatory behaviors of fractional-order systems generating regular oscillations have been analyzed via applying various tools, such as using numerical based methods [98], [99], stability region based analysis approaches [100], [101], harmonic balance techniques [102], homotopy analysis methods [103], [104], and averaging techniques [105].

For a survey on fractional-order systems generating irregular oscillations and their applications, we refer the reader to [106].

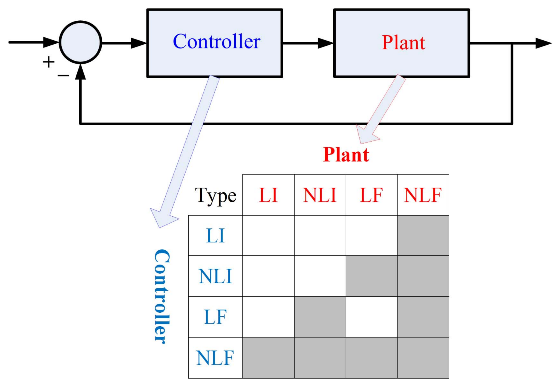

FIGURE 2. A control system in a unity negative feedback structure (The highlighted cases specify the ones in which the closed loop system is described by a nonlinear fractional-order model).

VII. NONLINEAR FRACTIONAL-ORDER CONTROL SYSTEMS

A simple control system structure is shown in Fig. 2. Considering the nonlinearity and/or fractionality nature in modeling of the process in control system of Fig. 2 and/or benefiting from the potential of nonlinear and/or fractional operators in controller design in this control system may cause that the closed loop system of Fig. 2 is a nonlinear fractional-order one. According to different choices for modeling of the process and choosing the controller structure among linear/nonlinear and integer/fractional-order dynamics, i.e., considering the different choices of linear(L)/nonlinear(NL) integer(I)/fractional(F) order process and linear(L)/nonlinear(NL) integer(I)/fractional(F) order controller, 16 distinct combinations result, and there are 9 of them (the highlighted cases in the table of Fig. 2) in which the corresponding control system is described by a nonlinear fractional-order model. Among these 9 cases, control system analysis and design in the case of NLF process and NLF controller, as the most general case, has received more attention than the other ones (some samples of the methods proposed in this case for control of nonlinear fractional-order systems are nonlinear fractional PI control [107], predictive control [108], adaptive sliding mode control [109], adaptive backstepping control [110], adaptive neuro-fuzzy control [111], and adaptive iterative learning control [112]). Nevertheless, the other cases yielding a nonlinear fractional-order control system have been also considered in literature, e.g., LI process and NLF controller in [113], LF process and NLI controller in [114], [115], LF process and NLF controller in [116], NLI process and LF controller in [117], [118], NLI process and NLF controller in [119], NLF process and LI controller in [120], NLF process and LF controller in [121], and NLF process and NLI controller in [122].

Motivated by the fact that fractional-order dynamics can provide a more realistic framework for modeling of the agents and coordinated behavior of them in some situations (to find out some typical situations, see [123, Sec. 1.2]) and also considering the nonlinear behaviors naturally exist in real-world phenomena and processes [124, Ch. 8], nonlinear fractional-order dynamics have been extensively considered in the context of multi-agent systems. Distributed control of nonlinear fractional-order multi-agent systems in different scenarios, such as leader–follower consensus control [125], [126], [127] and leaderless consensus control [128], [129], [130] has been a hot research topic in recent years.

VIII. SOME CHALLENGES INVITING FUTURE RESEARCH WORKS

A. THE NONLINEAR EFFECT OF INTERNAL AND ENVIRONMENTAL FACTORS ON FRACTIONALITY DEGREE OF THE ELECTRICAL ELEMENTS

Dependency of the specifications of the fabricated electrical elements on internal and environmental factors, such as self-heating [131], ambient temperature [132], working voltage [133], and electrical stresses [134], is unavoidable. Such a dependency may be an undesirable phenomenon, when a constancy in behavior is expected from the electrical element. Consequently, in this case effective solution is needed to decrease the sensitivity to the variations of the element specifications. On the other hand, the aforementioned dependency can be beneficial in some cases (especially, in design of electrical sensors). Standing on either side of the issue (reducing the effects of dependency or benefiting from it), awareness of how the element specifications are affected by influential factors is necessary. Motivated by this necessity, various studies have been performed on variation of the specifications of the classic electrical elements with respect to internal and environmental changes. The need to extend such studies to fractional-order elements, due to growing interest for using these elements in circuit design and practice, is inevitable. One of the main aspects, which should be noticed in the future studies, is to investigate the dependency of the fractionality degree of the element (may be quantitatively evaluated by the fractional value of the order of the element) to internal and environmental factors (for a sample research work on fractionality degree of the fractional-order integrators emulated by ladder/nested ladder networks, see [135]).

B. TRAJECTORIES' BEHAVIOR IN THE PSEUDO-STATE PLANE

Investigating the behavior of the trajectories of the second-order systems in the state plane is an effective approach for qualitative analysis of such systems. Consider the well-behaved second-order system

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= f_2(x_1, x_2).\end{aligned} \tag{27}$$

It is known that two different trajectories of this system in the state plane $x_1 - x_2$ either have no intersection point or completely coincide. Consequently, if a non-trivial trajectory of (27) intersects itself, it is a periodic orbit for this system. But due to the non-locality property of fractional-order derivative operators (in contrast to locality feature of the integer-order ones), this result is not generally valid for the fractional-order system

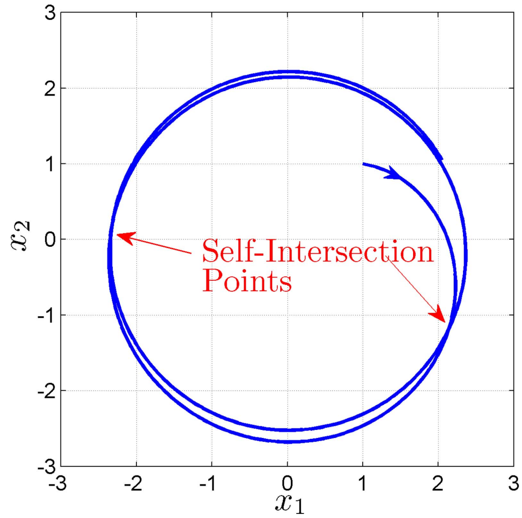

FIGURE 3. A trajectory of system (28) where $\alpha_1 = \alpha_2 = 0.6$, $f_1(x_1, x_2) = \sqrt{2}(\cos(0.3\pi)x_1 + \sin(0.3\pi)x_2)$, and $f_2(x_1, x_2) = \sqrt{2}(-\sin(0.3\pi)x_1 + \cos(0.3\pi)x_2)$, which begins from the initial point $(x_1(0), x_2(0)) = (1, 1)$.

$$\begin{aligned}{}_0^C D_t^{\alpha_1} x_1 &= f_1(x_1, x_2)\\ {}_0^C D_t^{\alpha_2} x_2 &= f_2(x_1, x_2),\end{aligned} \tag{28}$$

where $\alpha_1 \in (0, 1)$ and $\alpha_2 \in (0, 1]$, i.e., a trajectory of system (28) may interest itself, while it is not a periodic orbit for this system (for a sample trajectory, see Fig. 3). For more clarifying the point, notice that the solutions of system (28) satisfy the following Volterra integral equation [136].

$$\begin{aligned}x_1(t) &= x_1(0) + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\tau)^{\alpha_1 - 1} f_1(x_1(\tau), x_2(\tau))d\tau\\ x_2(t) &= x_2(0) + \frac{1}{\Gamma(\alpha_2)}\int_0^t (t-\tau)^{\alpha_2 - 1} f_2(x_1(\tau), x_2(\tau))d\tau\end{aligned} \tag{29}$$

If a trajectory of system (28) intersects itself, and the self-intersection point is related to times $t = t_0$ and $t = t_0 + T$ (i.e., $x_i(t_0) = x_i(t_0 + T)$ for $i = 1, 2$), then from (29) the equality

$$\begin{aligned}&\int_0^{t_0} (t_0 - \tau)^{\alpha_i - 1} f_i(x_1(\tau), x_2(\tau))d\tau\\ &\quad = \int_0^{t_0+T} (t_0 + T - \tau)^{\alpha_i - 1} f_i(x_1(\tau), x_2(\tau))d\tau\end{aligned} \tag{30}$$

should hold for $i = 1, 2$. In the special case $(\alpha_1, \alpha_2) = (1, 1)$ (integer-order system), (30) is reduced to

$$\int_0^T f_i(x_1(\tau), x_2(\tau))d\tau = 0, \tag{31}$$

which yields in $x_i(t) = x_i(t + T)$ $(i = 1, 2)$ for all $t \geq 0$ (periodicity of the solution). But such a deduction is not valid in the case $(\alpha_1, \alpha_2) \neq (1, 1)$ (fractional-order system), because of the effectiveness of the terms $(t_0 - \tau)^{\alpha_i - 1}$ and $(t_0 + T - \tau)^{\alpha_i - 1}$ in the integrals of (30) as a consequence of the non-locality property of fractional-order derivative operators appeared in (28). In fact, considering the existing related literature, our knowledge on behavior analysis of the trajectories of (28) in the pseudo-state[5] plane $x_1 - x_2$ in the viewpoint of the existence/nonexistence of self-intersection points is still incomplete. For example, the conditions on orders α_1 and α_2 and functions f_1 and f_2 yielding in the intersection/non-intersection of a trajectory by itself are not known (For a research work investigating a related problem, see [138]). As another example, the existence of an upper bound on the number of self-intersection points of the trajectories of system (28) in the pseudo-state plane $x_1 - x_2$ is not clear. These two open problems focus on intersection of a trajectory of (28) with itself in the pseudo-state plane. Let us describe another problem focusing on the long-term behavior of trajectories of system (28). According to the Poincaré-Bendixson theorem [139], it is known that the oscillatory trajectories of system (27) either are periodic orbits themselves or tend to such orbits in the steady state. But, from the results of [94], [95], [96], the fractional-order system (28) cannot have any periodic orbit. This means that the limit set of an oscillatory trajectory of system (28), unlike that happens in system (27), is not itself a system solution. As a consequence for this fact, the lack of non-uniqueness of the limit set for the trajectories of fractional-order systems, whose integer-order counterparts have unique limit cycles, is demonstrated in numerical simulation results. For instance, such an observation has been reported for the fractionalized Van der Pol oscillator in [140] (See Fig. 4). Study on the limit sets of the trajectories of system (28) and investigating whether such limit sets are unique or not are interesting topics for future research works.

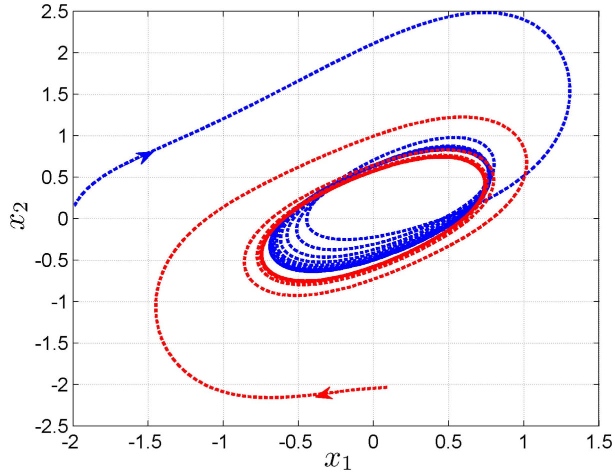

FIGURE 4. Two trajectories of system (28), where $\alpha_1 = 0.6$, $\alpha_2 = 1$, $f_1(x_1, x_2) = x_2$, and $f_2(x_1, x_2) = -x_1 - 0.6x_2(x_1^2 - 1)$ (a fractionalized version of the Van der Pol oscillator [140]). The trajectories not only intersect each other, but also tend to different limit sets (unlike that happens for the traditional Van der Pol oscillator, i.e., in the case $\alpha_1 = \alpha_2 = 1$).

C. ORDER-DEPENDENT LYAPUNOV BASED CONDITIONS GUARANTEEING STABILITY OF THE EQUILIBRIUM POINTS

As discussed in Section V-C, applying the inequalities in the forms (23), (24), and (26) is a common approach to prove stability of an equilibrium point in fractional-order systems. These forms of inequalities are inherently order-independent ones. This means that if an inequality in such forms is satisfied, not only the stability of the equilibrium point in the under-study fractional-order system is proved for all orders between zero and one, but also the stability of this equilibrium point in the integer-order counterpart system is deduced. With the aim of reducing the conservatism induced by order-independent stability analysis approaches, finding order-dependent conditions, whose meeting results in stability of the equilibrium points in nonlinear fractional-order systems, on the basis of the Lyapunov direct method can be considered as a future direction in research on stability analysis of fractional-order systems. It is worth noting that order-dependent conditions for stability analysis of LTI fractional-order system has been previously derived. For instance, according to [141, Th. 9], the necessary and sufficient condition for asymptotic stability of the LTI system ${}_0^C D_t^\alpha x = Ax$, where $\alpha \in (0, 1)$ and $x \in \mathbb{R}^n$, is the existence of the symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ which satisfies the following matrix inequality

$$\left(-(-A)^{\frac{1}{2-\alpha}}\right)^T P + P\left(-(-A)^{\frac{1}{2-\alpha}}\right) \prec 0. \tag{32}$$

Considering the available tools for stability analysis of LTI fractional-order systems, obtaining order-dependent stability conditions based on linearization techniques (Section V-B) is not a difficult task (For a sample, see [142]).

D. REGION OF ATTRACTION IN NONLINEAR FRACTIONAL-ORDER SYSTEMS

Assume that $x = x^*$ is a stable equilibrium point for the integer-order system $\dot{x}(t) = f(x(t))$, where $x \in \mathbb{R}^2$. The region of attraction of the equilibrium point $x = x^*$ is an open invariant set [139, Lemma 8.1]. But, there is no assurance about possessing of these properties for the region of attraction of this equilibrium point in the fractional-order system ${}_0^C D_t^\alpha x(t) = f(x(t))$ (the invarianceness of the regions of attraction in system ${}_0^C D_t^\alpha x(t) = f(x(t))$ is doubted by noticing the short-term behavior of the trajectories of this system (Section VIII-B)). To eliminate this ambiguity by analytically investigating the properties of regions of attractions in fractional-order systems can be a challenging problem, which invites further research works. Also, it is known that the boundary of the region of attraction of the equilibrium point $x = x^*$ in integer-order systems (if exists) is specified by the limit cycles and stable manifolds of unstable equilibrium points of the system [143, Sec. 5.1.5]. The absence

5. The vector $x = [x_1 \ x_2]^T$ is called the pseudo-state vector of system (28). For more details on this point, see [137, Sec. 2].

of limit cycles in system ${}_0^C D_t^\alpha x(t) = f(x(t))$ [94], [95], [96] obscures the specifiers of the boundary of the regions of attractions in fractional-order systems. For more clarifying the point, consider the system

$$\begin{aligned} {}_0^C D_t^\alpha x_1 &= x_2 + x_1\left(x_1^2 + x_2^2 - 1\right) \\ {}_0^C D_t^\alpha x_2 &= -x_1 + x_2\left(x_1^2 + x_2^2 - 1\right), \end{aligned} \tag{33}$$

where $\alpha \in (0, 1]$. The origin is the unique equilibrium point of this system. Linearizing system (33) around $(x_1, x_2) = (0, 0)$ reveals that this equilibrium point is stable. If $\alpha = 1$ (integer-order system), a simple analysis in polar coordinates allows to derive that $x_1^2 + x_2^2 = 1$ is an unstable limit cycle for system (33) [144, Ex. 2.7]. This limit cycle specifies the boundary of the region of attraction of the equilibrium point $(x_1, x_2) = (0, 0)$, where $\alpha = 1$. For $\alpha \in (0, 1)$, system (33) has no limit cycle. Hence, we have no exact information about the boundary of the region of attraction of the origin in this case. Note that such a boundary is not specified by two specifier forms of boundaries in integer-order systems (i.e., unstable limit cycles and stable manifolds of unstable equilibrium points). This example revealed that further analysis for finding specifications which can potentially determine the boundary of regions of attraction in fractional-order systems should be developed. This achievement will be helpful to propose the effective methods for estimation/approximation of the regions of attraction in nonlinear fractional-order systems.

E. CONVERSE LYAPUNOV THEOREMS FOR NONLINEAR FRACTIONAL-ORDER SYSTEMS

The subject of converse of the Lyaounov direct method includes various theorems guaranteeing the existence of Lyapunov functions for stable equilibrium points (for an interesting survey on converse Lyapunov theorems in integer-order systems, see [145]). One of the main applications of these converse theorems is their applicability in stability analysis of perturbed nonlinear systems [146], [147] and design of robust stabilizing controllers [30]. Considering such an applicability, it seems that presenting the converse Lyapunov theorems for fractional-order dynamics can result in significant progresses in robust stability analysis of uncertain nonlinear fractional-order systems and design of robust controller for these systems. Even though initial steps have been taken in this regard [149], further research efforts are still needed to provide a powerful diverse set of converse Lyapunov theorems for nonlinear fractional-order systems.

IX. CONCLUSION

In this article, the motivation of considering nonlinear fractional-order dynamics in the context of electrical circuits and systems was discussed. Also, a brief overview on the studies done on nonlinear fractional-order circuits and systems with focus on stability and oscillatory behavior analysis was presented. Clearly, overview on the works done on the subject of nonlinear fractional-order circuits and systems is not closed by this article, and to prepare more detailed overviews focusing on the other aspects related to this subject can be considered as future review works. Finally, some relevant challenging problems, which can specify some future directions in research on the aforementioned subject, were introduced.

REFERENCES

[1] G. Carlson and C. Halijak, “Approximations of fixed impedances,” *IRE Trans. Circuit Theory*, vol. 9, no. 3, pp. 302–303, Sep. 1962.

[2] G. Carlson and C. Halijak, “Approximation of fractional capacitors $(1/s)^{1/n}$ by a regular Newton process,” *IEEE Trans. Circuit Theory*, vol. 11, no. 2, pp. 210–213, Jun. 1964.

[3] S. D. Roy, “On the realization of a constant-argument immittance or fractional operator,” *IEEE Trans. Circuit Theory*, vol. 14, no. 3, pp. 264–274, Sep. 1967.

[4] A. S. Elwakil, “Fractional-order circuits and systems: An emerging interdisciplinary research area,” *IEEE Circuits Syst. Mag.*, vol. 10, no. 4, pp. 40–50, 4th Quart., 2010.

[5] I. Podlubny, *Fractional Differential Equations*. San Diego, CA, USA: Academic, 1999.

[6] J. Tenreiro Machado, V. Kiryakova, and F. Mainardi, “Recent history of fractional calculus,” *Commun. Nonlinear Sci. Numer. Simulat.*, vol. 16, pp. 1140–1153, Mar. 2011.

[7] D. Cafagna, “Fractional calculus: A mathematical tool from the past for present engineers,” *IEEE Ind. Electron. Mag.*, vol. 1, no. 2, pp. 35–40, Jul. 2007.

[8] M. E. Fouda, A. Allagui, A. S. Elwakil, S. Das, C. Psychalinos, and A. G. Radwan, “Nonlinear charge-voltage relationship in constant phase element,” *AEU Int. J. Electron. Commun.*, vol. 117, Apr. 2020, Art. no. 153104.

[9] B. A. Boukamp, “Derivation of a distribution function of relaxation times for the (fractal) finite length warburg,” *Electrochimica Acta*, vol. 252, pp. 154–163, Oct. 2017.

[10] M. S. Tavazoei, “Fractional/distributed-order systems and irrational transfer functions with monotonic step responses,” *J. Vib. Control*, vol. 20, no. 11, pp. 1697–1706, 2014.

[11] D. A. John, S. Banerjee, and K. Biswas, “Nanocomposite material characterization of a solid-state fractional capacitor,” *IEEE Trans. Electron Devices*, vol. 67, no. 3, pp. 1136–1142, Mar. 2020.

[12] D. A. John, S. Banerjee, G. W. Bohannan, and K. Biswas, “Solid-state fractional capacitor using MWCNT-epoxy nanocomposite,” *Appl. Phys. Lett.*, vol. 110, Apr. 2017, Art. no. 163504.

[13] A. Kartci, A. Agambayev, N. Herencsar, and K. N. Salama, “Series-, parallel-, and inter-connection of solid-state arbitrary fractional-order capacitors: Theoretical study and experimental verification,” *IEEE Access*, vol. 6, pp. 10933–10943, 2018.

[14] I. S. Jesus and J. A. Tenreiro Machado, “Development of fractional order capacitors based on electrolyte processes,” *Nonlinear Dyn.*, vol. 56, pp. 45–55, Apr. 2009.

[15] L. Kadlcik and P. Horský, “A CMOS follower-type voltage regulator with a distributed-element fractional-order control,” *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 65, no. 9, pp. 2753–2763, Sep. 2018.

[16] K. Biswas, S. Sen, and P. K. Dutta, “Realization of a constant phase element and its performance study in a differentiator circuit,” *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 53, no. 9, pp. 802–806, Sep. 2006.

[17] M. S. Krishna, S. Das, K. Biswas, and B. Goswami, “Fabrication of a fractional order capacitor with desired specifications: A study on process identification and characterization,” *IEEE Trans. Electron Devices*, vol. 58, no. 11, pp. 4067–4073, Nov. 2011.

[18] D. Mondal and K. Biswas, “Packaging of single-component fractional order element,” *IEEE Trans. Electron Devices*, vol. 13, no. 1, pp. 73–80, Mar. 2013.

[19] J. Valsa and J. Vlach, “RC models of a constant phase element,” *Int. J. Circuit Theory Appl.*, vol. 41, pp. 59–67, Jan. 2013.

[20] A. Charef, “Analogue realization of fractional order integrator, differentiator and fractional $PI^\lambda D^\mu$ controller,” *IEE Proc. Control Theory Appl.*, vol. 153, no. 6, pp. 714–720, Nov. 2006.

[21] P. Bertsias, C. Psychalinos, A. G. Radwan, and A. S. Elwakil, "High-frequency capacitorless fractional-order cpe and FI emulator," *Circuits Syst. Signal Process.*, vol. 37, pp. 2694–2713, Jul. 2018.

[22] A. Adhikary, S. Choudhary, and S. Sen, "Optimal design for realizing a grounded fractional order inductor using GIC," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 65, no. 8, pp. 2411–2421, Aug. 2018.

[23] S. Kapoulea, G. Tsirimokou, C. Psychalinos, and A. S. Elwakil, "Generalized fully adjustable structure for emulating fractional-order capacitors and inductors of orders less than two," *Circuits Syst. Signal Process.*, vol. 39, pp. 1797–1814, Apr. 2020.

[24] G. Tsirimokou, C. Psychalinos, T. Freeborn, and A. S. Elwakil, "Emulation of current excited fractional-order capacitors and inductors using OTA topologies," *CMicroelectron. J.*, vol. 55, pp. 70–81, Sep. 2016.

[25] Y. Jiang and B. Zhang, "High-power fractional-order capacitor with $1 < \alpha < 2$ based on power converter," *IEEE Trans. Ind. Electron.*, vol. 65, no. 4, pp. 3157–3164, Apr. 2018.

[26] L. Chua, "Memristor—The missing circuit element," *IEEE Trans. Circuit Theory*, vol. 18, no. 5, pp. 507–519, Sep. 1971.

[27] D. B. Strukov, G. S. Snider, D. R. Stewart, and R. S. Williams, "The missing memristor found," *Nature*, vol. 435, pp. 80–83, May 2008.

[28] F. Z. Wang, L. Shi, H. Wu, N. Helian, and L. O. Chua, "Fractional memristor," *Appl. Phys. Lett.*, vol. 111, Dec. 2017, Art. no. 243502.

[29] E. M. Hamed, M. E. Fouda, A. G. Alharbi, and A. G. Radwan, "Experimental verification of triple lobes generation in fractional memristive circuits," *IEEE Access*, vol. 6, pp. 75169–75180, 2018.

[30] J. Tenreiro Machado, "Fractional generalization of memristor and higher order elements," *Commun. Nonlinear Sci. Numer. Simulat.*, vol. 18, pp. 264–275, Feb. 2013.

[31] M. Shi, Y. Yu, and Q. Xu, "Window function for fractional-order HP TiO_2 non-linear memristor model," *IET Circuits Devices Syst.*, vol. 12, no. 4, pp. 447–452, Jul. 2018.

[32] M. E. Fouda and A. G. Radwan, "On the fractional-order memristor model," *J. Fractional Calculus Appl.*, vol. 4, no. 1, pp. 1–7, 2013.

[33] Y. F. Pu and X. Yuan, "Fracmemristor: Fractional-order memristor," *IEEE Access*, vol. 4, pp. 1872–1888, 2016.

[34] Y. F. Pu, X. Yuan, and B. Yu, "Analog circuit implementation of fractional-order memristor: Arbitrary-order lattice scaling fracmemristor," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 65, no. 9, pp. 2903–2916, Sep. 2018.

[35] A. Ali, A. Radwan, and A. Soliman, "Fractional order Butterworth filter: Active and passive realizations," *IEEE J. Emerg. Sel. Topics Circuits Syst.*, vol. 3, no. 3, pp. 346–354, Sep. 2013.

[36] D. Kubanek, T. Freeborn, and J. Kotona, "Fractional-order band-pass filter design using fractional-characteristic specimen functions," *Microelectron. J.*, vol. 86, pp. 77–86, Apr. 2019.

[37] K. Baxevanaki, S. Kapoulea, C. Psychalinos, and A. S. Elwakil, "Electronically tunable fractional-order highpass filter for phantom electroencephalographic system model implementation," *AEU Int. J. Electron. Commun.*, vol. 110, Oct. 2019, Art. no. 152850.

[38] A. S. Elwakil, A. Agambayev, A. Allagui, and K. N. Salama, "Experimental demonstration of fractional-order oscillators of orders 2.6 and 2.7," *Chaos Solitons Fract.*, vol. 96, pp. 160–164, Mar. 2017.

[39] M. S. Sarafraz and M. S. Tavazoei, "Realizability of fractional-order impedances by passive electrical networks composed of a fractional capacitor and RLC components," *IEEE Trans. Circuits Syst. I*, vol. 62, no. 12, pp. 2829–2835, Dec. 2015.

[40] M. S. Sarafraz and M. S. Tavazoei, "Passive realization of fractional-order impedances by a fractional element and RLC components: Conditions and procedure," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 64, no. 3, pp. 585–595, Mar. 2017.

[41] G. Tsirimokou, C. Psychalinos, A. S. Elwakil, and K. N. Salama, "Electronically tunable fully integrated fractional-order resonator," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 65, no. 2, pp. 166–170, Feb. 2018.

[42] A. Adhikary, S. Sen, and K. Biswas, "Practical realization of tunable fractional order parallel resonator and fractional order filters," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 63, no. 8, pp. 1141–1152, Aug. 2016.

[43] M. C. Tripathy, D. Mondal, K. Biswas, and S. Sen, "Design and performance study of phase-locked loop using fractional-order loop filter," *Int. J. Circuit Theory Appl.*, vol. 46, pp. 776–792, Jun. 2015.

[44] S. Westerlund, "Dead matter has memory!" *Physica Scripta*, vol. 43, pp. 174–179, Feb. 1991.

[45] S. Westerlund, "Capacitor theory," *IEEE Trans. Dielectr. Electr. Insul.*, vol. 1, no. 5, pp. 826–839, Oct. 1994.

[46] S. Faraji and M. S. Tavazoei, "The effect of fractionality nature in differences between computer simulation and experimental results of a chaotic circuit," *Central Eur. J. Phys.*, vol. 11, no. 6, pp. 836–844, 2013.

[47] L. Shi, Y. Yuan, J. Gao, L. Zhou, and J. Mao, "Compact fractional-order model of on-chip inductors with BCB on high resistivity silicon," *IEEE Trans. Compon. Packag. Manuf. Technol.*, vol. 10, no. 5, pp. 878–886, May 2020.

[48] S. Kumar and A. Ghosh, "An improved fractional-order circuit model for voltammetric taste sensor system with infused tea as analyte," *IEEE Sensors J.*, vol. 20, no. 14, pp. 7792–7800, Jul. 2020, doi: 10.1109/JSEN.2020.2979994.

[49] M. Al-Daloo, A. Soltan, and A. Yakovlev, "Advance interconnect circuit modeling design using fractional-order elements," *IEEE Trans. Comput.-Aided Design Integr. Circuits Syst.*, vol. 39, no. 10, pp. 2722–2734, Oct. 2020, doi: 10.1109/TCAD.2019.2962779.

[50] C. Yang, H. Yu, Y. Shang, and W. Fei, "Characterization of CMOS metamaterial transmission line by compact fractional-order equivalent circuit model," *IEEE Trans. Electron Devices*, vol. 62, no. 9, pp. 3012–3018, Sep. 2015.

[51] Y. Shang, H. Yu, and W. Fei, "Design and analysis of CMOS-based terahertz integrated circuits by causal fractional-order RLGC transmission line model," *IEEE J. Emerg. Sel. Topics Circuits Syst.*, vol. 3, no. 3, pp. 355–366, Sep. 2013.

[52] R. K. H. Galvao, S. Hadjiloucas, K. H. Kienitz, H. M. Paiva, and R. J. M. Afonso, "Fractional order modeling of large three-dimensional RC networks," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 60, no. 3, pp. 624–637, Mar. 2013.

[53] D. Matignon, "Stability properties for generalized fractional differential systems," *ESAIM Proc.*, vol. 5, pp. 145–158, Aug. 2002.

[54] E. Ahmed, A. M. A. El-Sayed, and H. A. A. El-Saka, "Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models," *J. Math. Anal. Appl.*, vol. 325, pp. 542–553, Jan. 2007.

[55] M. S. Tavazoei and M. Haeri, "Chaotic attractors in incommensurate fractional order systems," *Physica D, Nonlinear Phenom.*, vol. 237, pp. 2628–2637, Oct. 2008.

[56] N. D. Cong, T. S. Doan, S. Siegmund, and H. T. Tuan, "Linearized asymptotic stability for fractional differential equations," *Electron. J. Qual. Theory Differ. Equ.*, vol. 39, pp. 1–13, Jun. 2016.

[57] L. Chen, Y. Chai, R. Wu, and J. Yang, "Stability and stabilization of a class of nonlinear fractional-order systems with caputo derivative," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 59, no. 9, pp. 602–606, Sep. 2012.

[58] X. J. Wen, Z. M. Wu, and J. G. Lu, "Stability analysis of a class of nonlinear fractional-order systems," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 55, no. 11, pp. 1178–1182, Nov. 2008.

[59] C. Li and Y. Ma, "Fractional dynamical system and its linearization theorem," *Nonlinear Dyn.*, vol. 71, pp. 621–633, Mar. 2013.

[60] H. T. Tuan and H. Trinh, "A linearized stability theorem for nonlinear delay fractional differential equations," *IEEE Trans. Autom. Control*, vol. 63, no. 9, pp. 3180–3186, Sep. 2018.

[61] H. T. Tuan and H. Trinh, "Stability of fractional-order nonlinear systems by Lyapunov direct method," *IET Control Theory Appl.*, vol. 12, no. 17, pp. 2417–2422, Nov. 2018.

[62] Y. Li, Y. Q. Chen, and I. Podlubny, "Mittag–Leffler stability of fractional order nonlinear dynamic systems," *Automatica*, vol. 45, pp. 1965–1969, Aug. 2009.

[63] Y. Li, Y. Q. Chen, and I. Podlubny, "Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag–Leffler stability," *Comput. Math. Appl.*, vol. 59, pp. 1810–1821, Mar. 2010.

[64] D. Matignon, "Stability results for fractional differential equations with applications to control processing," in *Proc. IMACS Multiconf. Comput. Eng. Syst. Appl.*, vol. 5, 1996, pp. 963–968.

[65] J. Yu, H. Hu, S. Zhou, and X. Lin, "Generalized Mittag–Leffler stability of multi-variables fractional order nonlinear systems," *Automatica*, vol. 49, pp. 1798–1803, Jun. 2013.

[66] Y. Li, D. Zhao, Y. Q. Chen, I. Podlubny, and C. Zhang, "Finite energy Lyapunov function candidate for fractional order general nonlinear systems," *Commun. Nonlinear Sci. Numer. Simulat.*, vol. 78, Nov. 2019, Art. no. 104886.

[67] N. Aguila-Camacho, M. A. Duarte-Mermoud, and J. A. Gallegos, "Lyapunov functions for fractional order systems," *Commun. Nonlinear Sci. Numer. Simulat.*, vol. 19, pp. 2951–2957, Sep. 2014.

[68] W. Chen, H. Dai, Y. Song, and Z. Zhang, "Convex Lyapunov functions for stability analysis of fractional order systems," *IET Control Theory Appl.*, vol. 11, no. 7, pp. 1070–1074, Apr. 2017.

[69] M. A. Duarte-Mermoud, N. Aguila-Camacho, J. A. Gallegos, and R. Castro-Linares, "Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems," *Commun. Nonlinear Sci. Numer. Simulat.*, vol. 22, pp. 650–659, May 2015.

[70] D. Ding, D. Qi, and Q. Wang, "Non-linear Mittag–Leffler stabilisation of commensurate fractional-order non-linear systems," *IET Control Theory Appl.*, vol. 9, no. 5, pp. 681–690, 2015.

[71] S. Zhang, Y. Yu, and J. Yu, "LMI conditions for global stability of fractional-order neural networks," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 28, no. 10, pp. 2423–2433, Oct. 2017.

[72] B. B. He, H. C. Zhou, Y. Q. Chen, and C. H. Kou, "Asymptotical stability of fractional order systems with time delay via an integral inequality," *IET Control Theory Appl.*, vol. 12, no. 12, pp. 1748–1754, Aug. 2018.

[73] Y. Zhao, Y. Wang, X. Zhang, and H. Li, "Feedback stabilisation control design for fractional order non-linear systems in the lower triangular form," *IET Control Theory Appl.*, vol. 10, no. 9, pp. 1061–1068, Jun. 2016.

[74] Y. X. Li, Q. Y. Wang, and S. Tong, "Fuzzy adaptive fault-tolerant control of fractional-order nonlinear systems," *IEEE Trans. Syst., Man, Cybern., Syst.*, early access, Mar. 4, 2019, doi: 10.1109/TSMC.2019.2894663.

[75] H. Liu, Y. Pan, and J. Cao, "Composite learning adaptive dynamic surface control of fractional-order nonlinear systems," *IEEE Trans. Cybern.*, vol. 50, no. 6, pp. 2557–2567, Jun. 2020.

[76] C. Wu and X. Liu, "Lyapunov and external stability of Caputo fractional order switching systems," *Nonlinear Anal. Hybrid Syst.*, vol. 34, pp. 131–146, Nov. 2019.

[77] J. C. Trigeassou, N. Maamri, J. Sabatier, and A. Oustaloup, "A Lyapunov approach to the stability of fractional differential equations," *Signal Process.*, vol. 91, pp. 437–445, Mar. 2011.

[78] S. Song, J. H. Park, B. Zhang, and X. Song, "Observer-based adaptive hybrid fuzzy resilient control for fractional-order nonlinear systems with time-varying delays and actuator failures," *IEEE Trans. Fuzzy Syst.*, early access, Nov. 22, 2019, doi: 10.1109/TFUZZ.2019.2955051.

[79] R. Sakthivel, K. Raajananthini, F. Alzahrani, and B. Kaviarasan, "Observer-based modified repetitive control for fractional-order nonlinear systems with unknown disturbances," *IET Control Theory Appl.*, vol. 13, no. 18, pp. 3132–3138, Dec. 2019.

[80] V. Badri and M. S. Tavazoei, "Non-uniform reducing the involved differentiators' orders and lyapunov stability preservation problem in dynamic systems," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 67, no. 4, pp. 735–739, Apr. 2020.

[81] V. N. Phat and N. T. Thanh, "New criteria for finite-time stability of nonlinear fractional-order delay systems: A Gronwall inequality approach," *Appl. Math. Lett.*, vol. 83, pp. 169–175, Sep. 2018.

[82] F. Zhang and Z. Zeng, "Multiple Mittag–Leffler stability of delayed fractional-order Cohen–Grossberg neural networks via mixed monotone operator pair," *IEEE Trans. Cybern.*, early access, Jan. 27, 2020, doi: 10.1109/TCYB.2019.2963034.

[83] J. A. Gallegos, N. Aguila-Camacho, and M. Duarte-Mermoud, "Vector Lyapunov-like functions for multi-order fractional systems with multiple time-varying delays," *Commun. Nonlinear Sci. Numer. Simulat.*, vol. 83, Apr. 2020, Art. no. 105089.

[84] S. Zhang, L. Liu, and X. Cui "LMI-based stability of nonlinear non-autonomous fractional-order systems with multiple time delays," *IEEE Access*, vol. 7, pp. 12016–12026, 2019.

[85] P. Li, L. Chen, R. Wu, J. A. T. Machado, A. M. Lopes, and L. Yuan, "Robust asymptotic stability of interval fractional-order nonlinear systems with time-delay," *J. Franklin Inst.*, vol. 355, pp. 7749–7763, Oct. 2018.

[86] V. Badri and M. S. Tavazoei, "Stability analysis of fractional order time-delay systems: Constructing new Lyapunov functions from those of integer order counterparts," *IET Control Theory Appl.*, vol. 13, no. 15, pp. 2476–2481, Oct. 2019.

[87] N. T. Thanh and V. N. Phat, "Improved approach for finite-time stability of nonlinear fractional-order systems with interval time-varying delay," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 66, no. 8, pp. 1356–1360, Aug. 2019.

[88] A. Oustaloup, "Fractional order sinusoidal oscillators: Optimization and their use in highly linear FM modulation," *IEEE Trans. Circuits Syst.*, vol. 28, no. 10, pp. 1007–1009, Oct. 1981.

[89] A. G. Radwan, A. M. Soliman, and A. S. Elwakil, "Design equations for fractional–order sinusoidal oscillators: Four practical circuit examples," *Int. J. Circuit Theory Appl.*, vol. 36, pp. 473–492, Jun. 2008.

[90] A. G. Radwan, A. S. Elwakil, and A. M. Soliman, "Fractional-order sinusoidal oscillators: Design procedure and practical examples," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 55, no. 7, pp. 2051–2063, Aug. 2008.

[91] A. G. Radwan and K. N. Salama, "Passive and active elements using fractional $L_\beta C_\alpha$ circuit," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 58, no. 10, pp. 2051–2063, Oct. 2011.

[92] M. S. Tavazoei, "Upper and lower bounds for the maximum number of frequencies that can be generated by a class of fractional oscillators," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 66, no. 4, pp. 1584–1593, Apr. 2019.

[93] M. S. Tavazoei, M. Haeri, M. Siami, and S. Bolouki, "Maximum number of frequencies in oscillations generated by fractional order LTI systems," *IEEE Trans. Signal Process.*, vol. 58, no. 8, pp. 4003–4012, Aug. 2010.

[94] M. S. Tavazoei and M. Haeri, "A proof for non existence of periodic solutions in time invariant fractional order systems," *Automatica*, vol. 45, pp. 1886–1890, Aug. 2009.

[95] M. S. Tavazoei, "A note on fractional-order derivatives of periodic functions," *Automatica*, vol. 46, pp. 945–948, May 2010.

[96] M. S. Tavazoei and M. Haeri, "A simplification in the proof presented for non existence of periodic solutions in time invariant fractional order systems," 2012. [Online]. Available: https://arxiv.org/abs/1202.5878.

[97] M. S. Tavazoei, "Toward searching possible oscillatory region in order space for nonlinear fractional-order systems," *J. Comput. Nonlinear Dyn.*, vol. 9, Apr. 2014, Art. no. 021011.

[98] Y. Wang and C. Li, "Does the fractional Brusselator with efficient dimension less than 1 have a limit cycle?" *Phys. Lett. A*, vol. 363, pp. 414–419, Apr. 2007.

[99] J. F. Gomez-Aguilar, "Multiple attractors and periodicity on the Vallis model for El Niño/La Niña-Southern oscillation model," *J. Atmosp. Solar Terrestrial Phys.*, vol. 197, Jan. 2020, Art. no. 105172.

[100] X. Lin, S. Zhou, H. Li, H. Tang, and Y. Qi, "Rhythm oscillation in fractional-order relaxation oscillator and its application in image enhancement," *J. Comput. Appl. Math.*, vol. 339, pp. 69–84, Sep. 2018.

[101] V. Gafiychuk and B. Datsko, "Stability analysis and limit cycle in fractional system with Brusselator nonlinearities," *Phys. Lett. A*, vol. 372, pp. 4902–4904, Jul. 2008.

[102] Z. Guo, A. Y. T. Leung, and H. X. Yang, "Oscillatory region and asymptotic solution of fractional van der Pol oscillator via residue harmonic balance technique," *Appl. Math. Model.*, vol. 35, pp. 3918–3925, Aug. 2011.

[103] Q. X. Liu, J. K. Liu, and Y. M. Chen, "Asymptotic limit cycle of fractional van der Pol oscillator by homotopy analysis method and memory-free principle," *Appl. Math. Model.*, vol. 40, pp. 3211–3220, Feb. 2016.

[104] G. Zhang and Z. Wu, "Approximate limit cycles of coupled nonlinear oscillators with fractional derivatives," *Appl. Math. Model.*, vol. 77, pp. 1294–1309, Jan. 2020.

[105] C. B. Tabi, "Dynamical analysis of the FitzHugh–Nagumo oscillations through a modified Van der Pol equation with fractional-order derivative term," *Int. J. Non-Linear Mech.*, vol. 105, pp. 173–178, Oct. 2018.

[106] M. S. Tavazoei, "Fractional order chaotic systems: History, achievements, applications, and future challenges," *Eur. Phys. J. Spec. Topics*, vol. 229, pp. 887–904, Mar. 2020.

[107] F. Oliva-Palomo, A. J. Muñoz-Vázquez, A. Sánchez-Orta, V. Parra-Vega, C. Izaguirre-Espinosa, and P. Castillo, "A fractional nonlinear PI-structure control for robust attitude tracking of quadrotors," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 55, no. 6, pp. 2911–2920, Dec. 2019.
[108] B. Wang, K. Shi, L. Yang, F. Wu, and D. Chen, "Fuzzy generalised predictive control for a class of fractional-order non-linear systems," *IET Control Theory Appl.*, vol. 12, no. 2, pp. 87–96, Jan. 2018.
[109] S. Song, B. Zhang, J. Xia, and Z. Zhang, "Adaptive backstepping hybrid fuzzy sliding mode control for uncertain fractional-order nonlinear systems based on finite-time scheme," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 50, no. 4, pp. 1559–1569, Apr. 2020.
[110] X. Li, C. Wen, and Y. Zou, "Adaptive backstepping control for fractional-order nonlinear systems with external disturbance and uncertain parameters using smooth control," *IEEE Trans. Syst., Man, Cybern., Syst.*, early access, May 6, 2020, doi: 10.1109/TSMC.2020.2987335
[111] S. Song, J. H. Park, B. Zhang, X. Song, and Z. Zhang, "Adaptive command filtered neuro-fuzzy control design for fractional-order nonlinear systems with unknown control directions and input quantization," *IEEE Trans. Syst., Man, Cybern., Syst.*, early access, Feb. 4, 2020, doi: 10.1109/TSMC.2020.2967425
[112] J. Wei, Y. Zhang, and H. Bao, "An exploration on adaptive iterative learning control for a class of commensurate high-order uncertain nonlinear fractional order systems," *IEEE/CAA J. Autom. Sinica*, vol. 5, no. 2, pp. 618–627, Mar. 2018.
[113] L. Chen, N. Saikumar, and S. H. HosseinNia, "Development of robust fractional-order reset control," *IEEE Trans. Control Syst. Technol.*, vol. 28, no. 4, pp. 1404–1417, Jul. 2020.
[114] Y. H. Lim, K. K. Oh, and H. S. Ahn, "Stability and stabilization of fractional-order linear systems subject to input saturation," *IEEE Trans. Autom. Control*, vol. 58, no. 4, pp. 1062–1067, Apr. 2013.
[115] N. Naderi Soorki and M. S. Tavazoei, "Constrained swarm stabilization of fractional order linear time invariant swarm systems," *IEEE/CAA J. Autom. Sinica*, vol. 3, no. 3, pp. 320–331, Jul. 2016.
[116] S. Kamal, R. K. Sharma, T. N. Dinh, H. MS, and B. Bandyopadhyay, "Sliding mode control of uncertain fractional-order systems: A reaching phase free approach," *Asian J. Control*, to be published, doi: 10.1002/asjc.2223.
[117] M. S. Tavazoei, M. Haeri, S. Jafari, S. Bolouki, and M. Siami, "Some applications of fractional calculus in suppression of chaotic oscillations," *IEEE Trans. Ind. Electron.*, vol. 55, no. 11, pp. 4094–4101, Nov. 2008.
[118] K. Erenturk, "Fractional-Order $PI^{\lambda}D^{\mu}$ and active disturbance rejection control of nonlinear two-mass drive system," *IEEE Trans. Ind. Electron.*, vol. 60, no. 9, pp. 3806–3813, Sep. 2013.
[119] P. Roy and B. K. Roy, "Sliding mode control versus fractional-order sliding mode control: Applied to a magnetic levitation system," *J. Control Autom. Elect. Syst.*, vol. 31, pp. 597–606, Apr. 2020.
[120] J. Luo, "State-feedback control for fractional-order nonlinear systems subject to input saturation," *Math. Problems Eng.*, vol. 2014, Jan. 2014, Art. no. 891639.
[121] M. S. Tavazoei and M. Haeri, "Stabilization of unstable fixed points of fractional-order systems by fractional-order linear controllers and its applications in suppression of chaotic oscillations," *J. Dyn. Syst. Meas. Control*, vol. 132, Mar. 2010, Art. no. 021008.
[122] J. S. H. Tsai, T. H. Chien, S. M. Guo, Y. P. Chang, and L. S. Shieh, "State-space self-tuning control for stochastic fractional-order chaotic systems," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 54, no. 3, pp. 632–642, Mar. 2007.
[123] Y. Cao, Y. Li, W. Ren, and Y. Q. Chen, "Distributed coordination of networked fractional-order systems," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 40, no. 2, pp. 362–370, Apr. 2010.
[124] M. Tiller, *Introduction to Physical Modeling with Modelica*. Boston, MA, USA: Kluwer Acad. Publ., 2001.
[125] P. Gong, Q. L. Han, and W. Lan, "Finite-time consensus tracking for incommensurate fractional-order nonlinear multiagent systems with directed switching topologies," *IEEE Trans. Cybern.*, early access, Mar. 13, 2020, doi: 10.1109/TCYB.2020.2977169.
[126] P. R. Yang, S. Liu, Y. Y. Tan, Y. J. Zhang, and W. Jiang, "Consensus analysis of fractional-order nonlinear multi-agent systems with distributed and input delays," *Neurocomputing*, vol. 329, pp. 46–52, Feb. 2019.
[127] X. Wang, H. Wu, and J. Cao, "Global leader-following consensus in finite time for fractional-order multi-agent systems with discontinuous inherent dynamics subject to nonlinear growth," *Nonlinear Anal. Hybrid Syst.*, vol. 37, Aug. 2020, Art. no. 100888.
[128] W. Zhu, B. Chen, and J. Yang, "Consensus of fractional-order multi-agent systems with input time delay," *Fractional Calculus Appl. Anal.*, vol. 20, pp. 4120–4125, Feb. 2017.
[129] P. Gong, W. Lan, and Q.-L. Han, "Robust adaptive fault-tolerant consensus control for uncertain nonlinear fractional-order multi-agent systems with directed topologies," *Automatica*, vol. 117, Jul. 2020, Art. no. 109011.
[130] P. Gong, "Distributed consensus of non-linear fractional-order multi-agent systems with directed topologies," *IET Control Theory Appl.*, vol. 10, pp. 2515–2525, Dec. 2016.
[131] G. Benito, A. Beatriz, A. Eduardo, L. Antonio, and N. Antonio, "Temperature-dependent thermal capacitance characterization for SOI-MOSFETs," *IEEE Trans. Electron Devices*, vol. 66, no. 10, pp. 4120–4125, Oct. 2019.
[132] A. C. Lynch, "Variation of capacitance with temperature in metallized-mica capacitors," *Proc. IEE IIA Insul. Mater.*, vol. 66, no. 3, pp. 38–40, Mar. 1953.
[133] N. L. Kusters and O. Petersons, "The voltage coefficients of precision capacitors," *IEEE Trans. Commun. Electron.*, vol. 82, no. 5, pp. 612–621, Nov. 1963.
[134] C. Besset, S. Bruyère, S. Blonkowski, S. Crémer, and E. Vincent, "MIM capacitance variation under electrical stress," *Microelectron. Rel.*, vol. 43, pp. 1237–1240, Aug. 2003.
[135] D. Sierociuk, I. Podlubny, and I. Petras, "Experimental evidence of variable-order behavior of ladders and nested ladders," *IEEE Trans. Control Syst. Technol.*, vol. 21, no. 2, pp. 459–466, Mar. 2013.
[136] K. Diethelm, N. J. Ford, and A. D. Freed, "A predictor-corrector approach for the numerical solution of fractional differential equations," *Nonlinear Dyn.*, vol. 11, pp. 3–22, Jul. 2002.
[137] M. Tavakoli-Kakhki, M. Haeri, and M. S. Tavazoei, "Notes on the state space realizations of rational order transfer functions," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 58, no. 5, pp. 1099–1108, May 2011.
[138] H. K. Diethelm, "On the separation of solutions of fractional differential equations," *Fract. Calculus Appl. Anal.*, vol. 11, no. 2, pp. 259–268, 2008.
[139] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ, USA: Prentice-Hall, 2002.
[140] M. S. Tavazoei, M. Haeri, M. Attari, S. Bolouki, and M. Siami, "More details on analysis of fractional-order van der Pol oscillator," *J. Vib. Control*, vol. 15, no. 6, pp. 803–819, 2009.
[141] C. Farges, M. Moze, and J. Sabatier, "Pseudo-state feedback stabilization of commensurate fractional order systems," *Automatica*, vol. 46, pp. 1730–1734, Oct. 2010.
[142] Z. M. Alqahtani, M. El-Shahed, and N. J. Mottram, "Derivative-order-dependent stability and transient behaviour in a predator-prey system of fractional differential equations," *Lett. Biomath.*, vol. 6, pp. 32–49, Dec. 2019.
[143] T. V. Cutsem and C. Vournas, *Voltage Stability of Electric Power Systems*. Boston, MA, USA: Springer, 1998.
[144] J. J. E. Slotine and W. Li, *Applied Nonlinear Control*. Englewood Cliffs, NJ, USA: Prentice-Hall, 1991.
[145] C. M. Kellett, "Classical converse theorems in Lyapunov's second method," *Discrete Continuous Dyn. Syst. B*, vol. 20, pp. 2333–2360, Oct. 2015.
[146] P. Pepe and I. Karafyllis, "Converse Lyapunov–Krasovskii theorems for systems described by neutral functional differential equations in Hale's form," *Int. J. Control*, vol. 86, no. 2, pp. 232–243, 2013.
[147] Y. Lin, E. D. Sontag, and Y. Wang, "A smooth converse Lyapunov theorem for robust stability," *SIAM J. Control Optim.*, vol. 34, no. 1, pp. 124–160, 1996.
[148] A. Chaillet and A. Loria, "A converse Lyapunov theorem for semiglobal practical asymptotic stability and application to cascades-based control," in *Proc. 45th IEEE Conf. Decis. Control*, San Diego, CA, USA, 2006, pp. 4259–4264.
[149] J. Gallegos and M. Duarte-Mermoud, "Converse theorems in Lyapunov's second method and applications for fractional order systems," *Turkish J. Math.*, vol. 43, no. 3, pp. 1626–1639, 2019.

MOHAMMAD SALEH TAVAZOEI (Member, IEEE) received the B.Sc., M.Sc., and Ph.D. degrees in electrical engineering from the Sharif University of Technology, Tehran, Iran, in 2003, 2005, and 2008, respectively, where he is currently an Associate Professor with the Department of Electrical Engineering. His current research interests include fractional order circuits and systems and their applications in engineering. He was a recipient of the Young Investigator Award of the IEEE Iran Section in 2012. He was also selected as the Prominent Young Scientist in electrical and computer engineering by the Academy of Sciences of Iran in 2017.

MAHSAN TAVAKOLI-KAKHKI received the bachelor's and master's degrees in electrical engineering from Ferdowsi University, Iran, in 2004 and 2007, respectively, and the Ph.D. degree in electrical engineering (control) from the Sharif University of Technology, Iran, in 2011. She is currently an Assistant Professor with the K. N. Toosi University of Technology. Her research interests are in robust control systems, time-delay control systems, fractional-order control systems, model-order reduction, and applied fractional calculus.

FEDERICO BIZZARRI (Senior Member, IEEE) was born in Genoa, Italy, in 1974. He received the Laurea (M.Sc.) five-year degree (*summa cum laude*) in electronic engineering and the Ph.D. degree in electrical engineering from the University of Genoa, Genoa, Italy, in 1998 and 2001, respectively. Since October 2018, he has been an Associate Professor with the Electronic and Information Department, Politecnico di Milano, Milan, Italy. He is a Research Fellow with the Advanced Research Center on Electronic Systems for Information and Communication Technologies "E. De Castro" (ARCES), University of Bologna, Italy. He served as an Associate Editor of the IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS I: REGULAR PAPERS from 2012 to 2015 and he was awarded as one of the 2012–2013 Best Associate Editors of this journal.